# Some Asymptotic Results for the Yard-Sale Model of Asset Exchange

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**Abstract**

Concerning the yard-sale model of asset exchange, some refinements of the results in the literature are given in both discrete- and continuous-time settings. The wealth concentration result is extended to possibly unfair games. In a fair case, an asymptotic behavior of every player's wealth process is derived. In the continuous-time settings, the effect of a proportional capital tax is also investigated. It is not assumed that the frequency of transactions is homogeneous for every pair of players in the discrete-time settings, or that the intensity of transactions is homogeneous in the continuous-time settings.

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# 1 Introduction

The yard-sale model of asset exchange was introduced by Chakraborti [3], and Hayes [5] was the first to refer to it as the *yard-sale model.* In this model, two or more players play the following series of zero-sum games. At the beginning each player is endowed with some wealth, possibly heterogeneous, and at each step, two of the players are randomly chosen, they play some kind of zero-sum fair game, where the loser gives to the winner a fraction of the minimum of the two players' amounts of wealth at that time. The fraction is always less than 1 and is typically modeled to be a constant or uniformly distributed. This is a fair game, and none of the players go bankrupt, but Hayes [5] showed by computer simulation an oligopoly phenomenon, where the wealth gradually concentrates in the hands of a very small number of players. He also showed that the wealth distribution among the players after many steps has a power-law tail when the number of players is large.

Boghosian [1] introduced a continuous-time analogue of the model where the cumulative number of transactions follows a Poisson process. In Section IV of his paper, he also considered a limit case of small and high-frequency transactions. For those models, Boghosian *et al.* [2] showed that the wealth concentrates asymptotically to a single individual.

Chorro [4] considers the model from a viewpoint of stochastic calculus. He gives a martingale proof to the wealth concentration property in the discrete-time settings. In a continuous-time setting, he defines each player's wealth process as the solution to a stochastic differential equation driven by Wiener processes. This corresponds to the limit case of small and high-frequency transactions considered in Boghosian [1].

From a viewpoint of economics, the yard-sale model is just a toy model, since it does not involve decision making on production, consumption, or investment. The model might be relevant, however, to the growing wealth disparities between economic entities in the real world.

The aim of this paper is to give, in both discrete- and continuous-time settings, a generalization and a refinement of some results in the literature. We generalize the wealth concentration result to (possibly) unfair games where there is a difference in ability between the players. In a fair case, we also derive an asymptotic property of every player's wealth process and show that wealth disparities grow between every pair of two players. The idea of the proof is to consider the logarithm of the ratio of the two players' wealth processes. In addition, in the continuous-time settings, we show that the introduction of the slightest proportional capital tax completely changes the asymptotic behaviors of the wealth processes. For any of the propositions, we do not assume a homogeneous frequency of transactions for every pair of players in the discrete-time settings or a homogeneous intensity of transactions in the continuous-time settings.

Our discrete- and continuous-time results are given in Sections 2 and 3, respectively.

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# 2 Discrete-time Setting

The following formulation is essentially due to Boghosian [1]. The random time when the $t$-th transaction occurs in his paper corresponds to the fixed time $t$ in this section of the present paper.

## 2.1 Definition and a Known Result

**Definition 2.1** *Define the set $I = \{1, \ldots, n\}$, where $n \geq 2$. Let $x_1, \ldots, x_n$ be $n$ positive numbers such that $\sum_{i \in I} x_i = 1$*

*Consider $n(n-1)$ real-valued stochastic processes on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us denote each of them by $Z_{ij} = \{Z_{ij}(t)\}_{t=1,2,\ldots}$, where $(i,j) \in I^2$ with $i \neq j$. We assume that, for every $t \geq 1$ :*

- $-1 < Z_{ij}(t) < 1$ *a.s. for every pair $(i,j) \in I^2$ with $i \neq j$;*
- $Z_{ji}(t) = -Z_{ij}(t)$ *a.s. for every pair $(i,j) \in I^2$ with $i \neq j$;*
- $Z_{ij}(t) Z_{k\ell}(t) = 0$ *a.s. if $\{i,j\} \neq \{k,\ell\}$.*

*Let $\{\mathcal{F}(t)\}_{t \geq 0}$ be the filtration generated by all the process $Z_{ij}$'s.*

*The $\mathbb{R}^n$-valued adapted process $X = \{X(t)\}_{t \geq 0}$, whose $i$-th component is denoted by $X_i = \{X_i(t)\}_{t \geq 0}$, is defined recursively as follows: $X_i(0) := x_i$, and for $t \geq 1$,*

$$
\begin{aligned}
X_i(t) \quad &:= \quad \begin{cases} X_i(t-1) + \min\{X_i(t-1), X_j(t-1)\}\, Z_{ij}(t) \\ \qquad\qquad \text{if } Z_{ij}(t) \neq 0 \text{ for a } j \in I \backslash \{i\}, \\ \\ X_i(t-1) \quad \text{if } Z_{ij}(t) = 0 \text{ for every } j \in I \backslash \{i\} \end{cases} \\
&= \quad X_i(t-1) + \sum_{j \in I \backslash \{i\}} \Big\{ \min\{X_i(t-1), X_j(t-1)\}\, Z_{ij}(t) \Big\}. \qquad (1)
\end{aligned}
$$

*Remark 1.* The interpretation of our definition is as follows. We interpret $I$ as the set of the $n$ players of the game. For every player $i \in I$, the stochastic process $X_i(\cdot)$ represents his/her relative wealth process with initial value $x_i$. If $Z_{ij}(t) \neq 0$, then players $i$ and $j$ are chosen for the game at time $t$ and player $i$ gains $\min\{X_i(t-1), X_j(t-1)\}\, Z_{ij}(t)$, where the signed fraction $Z_{ij}(t)$ is positive if player $i$ wins and is negative if he/she loses. The assumption $Z_{ij} = -Z_{ji}$ means that this is a zero-sum game. Only one pair of players is chosen for the game at each time $t$, which corresponds to the assumption that $Z_{ij}(t)Z_{k\ell}(t) = 0$ a.s. if $\{i, j\} \neq \{k, \ell\}$.

*Remark 2.* It follows immediately from our definition that, for every $t \geq 0$, $\sum_{i \in I} X_i(t) = 1$ and $X_i(t) > 0$ for every $i \in I$, a.s.

The wealth concentration result for the yard-sale model in the literature is stated as follows.

**Proposition 2.2 (Boghosian *et al.* [2] and Chorro [4])** *For every $t \geq 1$, we assume that the random variables $Z_{ij}(t)$ are independent of $\mathcal{F}(t-1)$ and that each $Z_{ij}(t)$ is equal in distribution to the product of the following two independent random variables:*

- *a Bernoulli random variable with parameter $\frac{2}{n(n-1)}$;*
- *a random variable with fixed distribution $\nu$ on the interval $(-1, 1)$, where $\nu(\{0\}) < 1$ and $\nu$ is symmetric w.r.t. the origin,* i.e., $\forall z \in (-1, 1)$, $\nu\big((-1, z]\big) = \nu\big([-z, 1)\big)$.

*Then we have*

$$\mathbb{P}\left[\exists i \in I \;\; \text{such that} \;\; \lim_{t \to \infty} X_i(t) = 1\right] = 1$$

*and*

$$\forall i \in I, \quad \mathbb{P}\left[\lim_{t \to \infty} X_i(t) = 1\right] = x_i.$$

## 2.2 Our Results

The two propositions and the corollary in this subsection will be proved in Subsection 2.4. Except for the corollary, we do not assume that each pair of players is chosen with equal probability.

Our first result generalizes Proposition 2.2 to possibly unfair games where there is a difference in ability between the players. In the following statement, the expression $i \prec j$ is interpreted as 'player $j$ is at least as competent as player $i$.'

**Proposition 2.3** *Assume the following properties.*

- *There exists a positive constant $\varepsilon$ such that, for every $t \geq 1$ and for every pair $(i, j) \in I^2$ with $i \neq j$,*

$$\mathbb{E}\big[\, |Z_{ij}(t)| \,\big|\, \mathcal{F}(t-1) \,\big] = \mathbb{E}\big[\, |Z_{ji}(t)| \,\big|\, \mathcal{F}(t-1) \,\big] > \varepsilon \quad \text{a.s.} \tag{2}$$

- *The set $I$ is totally ordered, where the deterministic order $\preceq$ does not need to be identical to the usual ordinal of $I = \{1, \ldots, n\}$. For every $t \geq 1$ and every pair $(i, j) \in I^2$ with $i \prec j$,*

$$\mathbb{E}\big[\, Z_{ij}(t) \,\big|\, \mathcal{F}(t-1) \,\big] = -\mathbb{E}\big[\, Z_{ji}(t) \,\big|\, \mathcal{F}(t-1) \,\big] \leq 0 \quad \text{a.s.} \tag{3}$$

*Then we have*

$$\mathbb{P}\Big[\,\exists i\in I \ \ \textit{such that} \ \ \lim_{t\to\infty} X_i(t)=1\Big] \;=\; 1 \tag{4}$$

*and*

$$\forall i\in I,\quad \mathbb{P}\Big[\,\exists j\in I \ \ \textit{such that} \ \ j\succeq i \ \ \textit{and} \ \ \lim_{t\to\infty} X_j(t)=1\Big] \;\geq\; \sum_{j\succeq i} x_j. \tag{5}$$

*Furthermore, if*

$$\mathbb{E}\big[\,Z_{ij}(t)\,\big|\,\mathcal{F}(t-1)\,\big] \;=\; \mathbb{E}\big[\,Z_{ji}(t)\,\big|\,\mathcal{F}(t-1)\,\big] \;=\; 0 \qquad a.s. \tag{6}$$

*for every* $t\geq 1$ *and for every pair* $(i,j)\in I^2$ *with* $i\neq j$*, then we have*

$$\forall i\in I,\quad \mathbb{P}\Big[\lim_{t\to\infty} X_i(t)=1\Big] \;=\; x_i. \tag{7}$$

Our second result refines Proposition 2.2 by showing, in a fair case, an asymptotic behavior of the wealth processes of the players other than the single individual with concentrated wealth.

**Proposition 2.4** *For every* $t\geq 1$*, we assume that the random variables* $Z_{ij}(t)$ *are independent of* $\mathcal{F}(t-1)$ *and that each* $Z_{ij}(t)$ *is equal in distribution to the product of the following two independent random variables:*

- *a Bernoulli random variable with parameter* $p_{ij}$*, where the positive constants* $p_{ij}$ *satisfy* $p_{ij}=p_{ji}$ *and, when* $n\geq 4$*, we further assume that*

$$\textit{either}\quad p_{ij} > \sum_{k\in I\setminus\{i,j\}} (p_{ik}-p_{jk})^+ \quad \textit{or} \quad p_{ij} > \sum_{k\in I\setminus\{i,j\}} (p_{jk}-p_{ik})^+ \tag{8}$$

  *holds for each pair* $(i,j)\in I^2$ *with* $i\neq j$*;*

- *a random variable with fixed distribution* $\nu$ *on the interval* $(-1,1)$*, where* $\nu(\{0\})<1$ *and* $\nu$ *is symmetric w.r.t. the origin,* i.e., $\forall z\in(-1,1)$, $\nu\big((-1,z]\big)=\nu\big([-z,1)\big)$.

*Then the assertions of Proposition 2.2 hold. Furthermore, if the measure* $\nu$ *also satisfies the integrability condition*

$$\sigma := \sqrt{\int_{-1}^{1}\big\{\log(1+z)\big\}^2\nu(dz)} \;<\; \infty, \tag{9}$$

*then we have, for every pair* $(i,j)\in I^2$ *with* $i\neq j$*,*

$$\mathbb{P}\left[\lim_{t\to\infty}\frac{X_i(t)}{X_j(t)}=0\right] \;+\; \mathbb{P}\left[\lim_{t\to\infty}\frac{X_i(t)}{X_j(t)}=\infty\right] \;=\; 1. \tag{10}$$

*Moreover,*

$$\forall i \in I, \quad \lim_{t\to\infty} \frac{\log X_i(t)}{t} \;=\; -\beta \sum_{j\in I\setminus\{i\}} p_{ij}\, \mathbb{I}_{\{\lim_{t\to\infty} \frac{X_i(t)}{X_j(t)}=0\}} \quad a.s., \tag{11}$$

*where* $\beta := -\int_{-1}^{1} \log(1+z)\,\nu(dz) > 0$*. In particular, if all the constants* $p_{ij}$ *are equal to* $\frac{2}{n(n-1)}$*, then*

$$\forall i \in I, \quad \lim_{t\to\infty} \frac{\log X_i(t)}{t} \;=\; -\frac{2\beta\, \sharp\big\{ j \in I\setminus\{i\} \,\big|\, \lim_{t\to\infty} \frac{X_i(t)}{X_j(t)} = 0 \big\}}{n(n-1)} \quad a.s. \tag{12}$$

*Remark 3.* The assertion (10) implies that each player has a random 'limit rank,' and the random variable $\sharp\big\{ j \in I\setminus\{i\} \,\big|\, \lim_{t\to\infty} \frac{X_i(t)}{X_j(t)} = 0 \big\}$ in (12) represents player $i$'s limit rank minus 1. It should also be noted that, even when each player has a limit rank, not all patterns of limit ranks are realized, since the above assertion (11) implies that

$$\sum_{k\in I\setminus\{i\}} p_{ik}\, \mathbb{I}_{\{\lim_{t\to\infty} \frac{X_i(t)}{X_k(t)}=0\}} \;\le\; \sum_{k\in I\setminus\{i\}} p_{jk}\, \mathbb{I}_{\{\lim_{t\to\infty} \frac{X_j(t)}{X_k(t)}=0\}}$$

if the limit rank of player $i$ is higher than that of player $j$. For example, suppose $n = 3,\ I = \{1, 2, 3\},$ and $p_{12} > p_{13} + p_{23}$. Then player 3's limit rank can never be realized as the lowest.

*Remark 4.* When $n \ge 4$, the assumption (8) holds if

$$\frac{n-2}{n-1} < \frac{\min_{(i,j)} p_{ij}}{\max_{(i,j)} p_{ij}} \le 1.$$

Indeed, for this case we have

$$\begin{aligned} p_{ij} - \sum_{k\in I\setminus\{i,j\}} (p_{ik} - p_{jk})^+ \quad &\ge \quad \min_{(k,\ell)} p_{k\ell} \,-\, (n-2)\Big\{ \max_{(k,\ell)} p_{k\ell} - \min_{(k,\ell)} p_{k\ell} \Big\} \\ &= \quad (n-1) \min_{(k,\ell)} p_{k\ell} \,-\, (n-2) \max_{(k,\ell)} p_{k\ell} \\ &> \quad 0. \end{aligned}$$

Also, when $n = 3$, the assumption (8) holds automatically for all positive constants $p_{ij}$. Indeed, by denoting $I = \{i, j, k\}$, we see that

- if $p_{ik} \le p_{jk}$ then $p_{ij} > 0 = (p_{ik} - p_{jk})^+$;
- if $p_{ik} \ge p_{jk}$ then $p_{ij} > 0 = (p_{jk} - p_{ik})^+$.

When $n = 2$, the assumption holds automatically with the right-hand sides of the two expressions of (8) defined to be zero.

*Remark 5.* The uniform distribution on the interval $(-1, 1)$ satisfies the above assumptions for $\nu$, where the constants $\sigma$ and $\beta$ are $\sqrt{(\log 2)^2 - 2\log 2 + 2}$ and $1 - \log 2$, respectively.

The following corollary of Proposition 2.4 is concerned with the Lorenz curve and the Gini coefficient.

**Corollary 2.5** *Under the assumptions of Proposition 2.4 with all the constants* $p_{ij}$ *being equal to* $\frac{2}{n(n-1)}$*, the following two asymptotic properties hold regardless of the initial wealth* $x_i$*,* $i \in I$*.*

**(i)** *Denoting by* $L_k(t)$ *the sum of the* $k$ *lowest values among the* $n$ *players' wealth* $X_i(t)$*,* $i \in I$*, at time* $t$*, we have that*

$$\forall k \in \{1, \ldots, n\}, \quad \lim_{t\to\infty} \frac{\log L_k(t)}{t} \;=\; -\frac{2\beta\,(n-k)}{n(n-1)} \quad a.s.$$

**(ii)** *The Gini coefficient at time* $t$*, denoted by* $Gini(t)$*, satisfies*

$$\lim_{t\to\infty} \frac{\log\big\{(1-\frac{1}{n}) - Gini(t)\big\}}{t} \;=\; -\frac{2\beta}{n(n-1)} \quad a.s.$$

## 2.3 Technical Lemmas

The following three lemmas are rather technical but useful for the proof of Proposition 2.4. All the proofs will be given in Subsection 2.5.

The first lemma is on submartingales and will be used for the second lemma.

**Lemma 2.6** *Let* $S = \{S(t)\}_{t=0,1,\ldots}$ *be a real-valued square-integrable submartingale, with* $S_0 = 0$*, on a stochastic basis. Assume the existence of two positive constants* $\sigma_{max}$ *and* $\mu_{min}$ *such that, for every* $t \geq 1$*,*

$$Var\big[\,\Delta S(t)\,\big|\,\mathcal{F}(t-1)\,\big] \;\leq\; \sigma_{max}^2 \quad \textit{and} \quad \mathbb{E}\big[\,\Delta S(t)\,\big|\,\mathcal{F}(t-1)\,\big] \;\geq\; \mu_{min} \quad a.s.,$$

*where* $\Delta S(t) := S(t) - S(t-1)$*. It then holds that* $\lim_{t\to\infty} S(t) = \infty$ *a.s. and*

$$\mathbb{P}\left[\,\inf_{t\geq 0} S(t) < -\Big(\frac{12\sigma_{max}^2}{\mu_{min}} + \frac{\mu_{min}}{2}\Big)\right] \;\leq\; \frac{1}{2}. \tag{13}$$

The second lemma is about an asymptotic behavior of some stochastic processes and will be applied to Proposition 2.4 with $Y(t) := \log\frac{X_i(t)}{X_j(t)}$.

**Lemma 2.7** *Let* $Y = \{Y(t)\}_{t=0,1,\ldots}$ *be a real-valued adapted process on a stochastic basis.*

**(i)** *Assume the existence of two positive constants* $\varepsilon$ *and* $p$ *such that*

$$\mathbb{P}\big[\,\Delta Y(t) > \varepsilon\,\big|\,\mathcal{F}(t-1)\,\big] \;\geq\; p \quad \textit{and} \quad \mathbb{P}\big[\,\Delta Y(t) < -\varepsilon\,\big|\,\mathcal{F}(t-1)\,\big] \;\geq\; p \tag{14}$$

*a.s. for every* $t \geq 1$*, where* $\Delta Y(t) := Y(t) - Y(t-1)$*. It then holds that*

$$\begin{aligned} &\mathbb{P}\left[\,\lim_{t\to\infty} Y(t) = \infty\,\right] + \mathbb{P}\left[\,\lim_{t\to\infty} Y(t) = -\infty\,\right] \\ &+\mathbb{P}\left[\,\limsup_{t\to\infty} Y(t) = \infty \;\; \textit{and} \;\; \liminf_{t\to\infty} Y(t) = -\infty\,\right] \;=\; 1. \end{aligned} \tag{15}$$

**(ii)** *In addition to (14), assume further that $Y$ is square-integrable and there exist three constants $c \in \mathbb{R}$, $\sigma_{max} > 0$, and $\mu_{min} > 0$ such that, for every $t \geq 1$,*

$$Var\big[\,\Delta Y(t)\,\big|\,\mathcal{F}(t-1)\,\big] \;\leq\; \sigma^2_{max} \tag{16}$$

*and*

$$\mathbb{E}\big[\,\Delta Y(t)\,\big|\,\mathcal{F}(t-1)\,\big] \;\geq\; \mu_{min} \tag{17}$$

*a.s. on the event $\big\{Y(t-1) > c\big\}$. We then have*

$$\mathbb{P}\left[\lim_{t\to\infty} Y(t) = \infty\right] + \mathbb{P}\left[\lim_{t\to\infty} Y(t) = -\infty\right] \;=\; 1. \tag{18}$$

The third lemma is concerned with the measure $\nu$ in the statement of Proposition 2.4.

**Lemma 2.8** *Let $\nu$ be a probability measure on $\big((-1,1), \mathcal{B}((-1,1))\big)$ satisfying the following properties:*

- $\nu(\{0\}) < 1$;
- $\int_{-1}^{1} z\,\nu(dz) \leq 0$;
- $\int_{-1}^{1} \big\{\log(1+z)\big\}^2 \nu(dz) < \infty$.

*Then the function $f : [0,1] \to \mathbb{R}$ defined by*

$$f(x) := \int_{-1}^{1} \log(1+xz)\nu(dz)$$

*is continuous, strictly decreasing, strictly concave, and $f(0) = 0$. Also, the function $g : [0,1] \to \mathbb{R}$ defined by*

$$g(x) := \int_{-1}^{1} \big\{\log(1+xz)\big\}^2 \nu(dz)$$

*is strictly increasing.*

## 2.4 Proofs of the Propositions and the Corollary

*Proof of Proposition 2.3.* For this possibly unfair case, we extend the proof by Chorro [4] of Proposition 2.2. We first show the existence of the almost sure limit $X_i(\infty) := \lim_{t\to\infty} X_i(t)$ for each process $X_i$. If we assume the condition (6), *i.e.* if the games are fair, then each process $X_i(\cdot)$ is a bounded martingale with respect to the filtration $\mathcal{F}(\cdot)$ and, as Chorro [4] mentions, its almost sure convergence follows readily from the martingale convergence theorem (see e.g. Chapter 11 of Williams [8]). For the general case, we proceed as follows. For $i \in I$, we define the process

$$S_i(t) := \sum_{j \succeq i} X_j(t).$$

The process $S_i$ is then a bounded submartingale. Indeed, for two distinct elements $j$ and $k$ of $I$, the following properties on $\Delta S_i(t)$ hold a.s.:

- if $j \prec i$ and $k \prec i$, then $\mathbb{I}_{\{Z_{jk}(t)\neq 0\}}\,\Delta S_i(t) = 0$;
- if $j \succeq i$ and $k \succeq i$, then

$$\begin{aligned}
& \mathbb{I}_{\{Z_{jk}(t)\neq 0\}}\,\Delta S_i(t) \\
=\; & \mathbb{I}_{\{Z_{jk}(t)\neq 0\}}\big\{\Delta X_j(t) + \Delta X_k(t)\big\} \\
=\; & \mathbb{I}_{\{Z_{jk}(t)\neq 0\}}\,\Big(\min\{X_j(t-1), X_k(t-1)\}\Big)\,\big\{Z_{jk}(t) + Z_{kj}(t)\big\} \\
=\; & 0;
\end{aligned}$$

- if $j \succeq i$ and $k \prec i$, then

$$\begin{aligned}
& \mathbb{E}\big[\,\mathbb{I}_{\{Z_{jk}(t)\neq 0\}}\,\Delta S_i(t)\,\big|\,\mathcal{F}(t-1)\,\big] \\
=\; & \mathbb{E}\big[\,\mathbb{I}_{\{Z_{jk}(t)\neq 0\}}\,\Delta X_j(t)\,\big|\,\mathcal{F}(t-1)\,\big] \\
=\; & \min\{X_j(t-1), X_k(t-1)\}\;\mathbb{E}\big[\,Z_{jk}(t)\,\big|\,\mathcal{F}(t-1)\,\big] \\
\geq\; & 0.
\end{aligned}$$

The martingale convergence theorem again yields the almost sure convergence of each submartingale $S_i$, which in turn implies the almost sure convergence of each process $X_i$.

The rest of the proof is the same as Chorro [4]. For the sake of completeness, we present a proof. By the definition (1) of $X_i(t)$, we have

$$\begin{aligned}
|\Delta X_i(t)| \;&=\; \begin{cases} \min\{X_i(t-1), X_j(t-1)\}\,|Z_{ij}(t)| \\ \qquad\qquad \text{if } Z_{ij}(t) \neq 0 \text{ for a } j \in I\backslash\{i\}, \\ 0 \quad \text{if } Z_{ij}(t) = 0 \text{ for every } j \in I\backslash\{i\} \end{cases} \\
&=\; \sum_{j\in I\backslash\{i\}} \Big\{\min\{X_i(t-1), X_j(t-1)\}\,|Z_{ij}(t)|\Big\}. \qquad (19)
\end{aligned}$$

The almost sure convergence of each process $X_i$ implies $\lim_{t\to\infty} |\Delta X_i(t)| = 0$ a.s. and, since the increments are also bounded, the dominated convergence theorem yields $\lim_{t\to\infty} \mathbb{E}\big[|\Delta X_i(t)|\big] = 0$. Therefore,

$$\begin{aligned}
0 &= \lim_{t\to\infty} \mathbb{E}\big[|\Delta X_i(t)|\big] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\sum_{j\in I\setminus\{i\}} \min\{X_i(t-1), X_j(t-1)\}\,|Z_{ij}(t)|\right] \quad \text{by (19)} \\
&= \lim_{t\to\infty} \mathbb{E}\left[\sum_{j\in I\setminus\{i\}} \min\{X_i(t-1), X_j(t-1)\}\,\mathbb{E}\big[\,|Z_{ij}(t)|\,\big|\,\mathcal{F}(t-1)\,\big]\right] \\
&\qquad \text{by the tower property} \\
&\geq \varepsilon \limsup_{t\to\infty} \mathbb{E}\left[\sum_{j\in I\setminus\{i\}} \min\{X_i(t-1), X_j(t-1)\}\right] \quad \text{by (2)} \\
&= \varepsilon\, \mathbb{E}\left[\sum_{j\in I\setminus\{i\}} \min\{X_i(\infty), X_j(\infty)\}\right],
\end{aligned}$$

which implies that, for every pair $(i, j) \in I^2$ with $i \neq j$,

$$\min\{X_i(\infty), X_j(\infty)\} = 0 \quad \text{a.s.}$$

and therefore

$$\sharp\big\{\, i \in I \,\big|\, X_i(\infty) > 0 \,\big\} \;\leq\; 1 \quad \text{a.s.}$$

It also holds that $\sum_{i\in I} X_i(\infty) = 1$ a.s., which completes the proof of our assertion (4). Furthermore, the assertion (5) follows from the fact that $S_i(\cdot)$ is a bounded submartingale □

*Proof of Proposition 2.4.* We divide our argument into three steps.

*Step 1.* If the first two assumptions of the proposition are satisfied, then the assumptions of the proposition 2.3 are all satisfied as well, so the assertions (4) and (7) hold.

*Step 2.* Next we assume (9) as well and prove (10). Fix a pair $(i, j) \in I^2$ with $i \neq j$. Assume without loss of generality that $p_{ij} > \sum_{k\in I\setminus\{i,j\}} (p_{ik} - p_{jk})^+$ : for the case $n \leq 3$ see Remark 4 after the statement of the proposition. It suffices to show that the process $Y_{ij}(t) := \log \frac{X_i(t)}{X_j(t)}$ satisfies the assumptions of Lemma 2.7.

We have that, for every $t \geq 1$,

$$
\begin{aligned}
\Delta Y_{ij}(t) &:= Y_{ij}(t) - Y_{ij}(t-1) \\
&= \log\Big(\frac{X_i(t)}{X_j(t)}\Big) - \log\Big(\frac{X_i(t-1)}{X_j(t-1)}\Big) \\
&= \log\Big(\frac{X_i(t)}{X_i(t-1)}\Big) - \log\Big(\frac{X_j(t)}{X_j(t-1)}\Big) \\
&= \begin{cases}
\log\Big[1+\min\Big\{1, \frac{X_j(t-1)}{X_i(t-1)}\Big\} Z_{ij}(t)\Big] & \\
-\log\Big[1+\min\Big\{1, \frac{X_i(t-1)}{X_j(t-1)}\Big\} Z_{ji}(t)\Big] & \text{if } Z_{ij}(t) \neq 0; \\
\log\Big[1+\min\Big\{1, \frac{X_k(t-1)}{X_i(t-1)}\Big\} Z_{ik}(t)\Big] & \text{if } Z_{ik}(t) \neq 0 \text{ and } k \neq j\,; \\
-\log\Big[1+\min\Big\{1, \frac{X_k(t-1)}{X_j(t-1)}\Big\} Z_{jk}(t)\Big] & \text{if } Z_{jk}(t) \neq 0 \text{ and } k \neq i\,; \\
0 & \text{otherwise.}
\end{cases}
\end{aligned}
\tag{20}
$$

For the case $Z_{ij}(t) \neq 0$, the signs of $\log\Big[1+\min\big\{1, \frac{X_j(t-1)}{X_i(t-1)}\big\} Z_{ij}(t)\Big]$ and $-\log\Big[1+\min\big\{1, \frac{X_i(t-1)}{X_j(t-1)}\big\} Z_{ji}(t)\Big]$ are the same, and either $\min\big\{1, \frac{X_j(t-1)}{X_i(t-1)}\big\}$ or $\min\big\{1, \frac{X_i(t-1)}{X_j(t-1)}\big\}$ is equal to 1. It follows that, for every $0 < z < 1$,

$$
\mathbb{P}\big[\,\Delta Y_{ij}(t) \geq \log(1+z) \,\big|\, \mathcal{F}(t-1)\,\big] \;\geq\; p_{ij}\,\nu([z,1))
$$

and

$$
\mathbb{P}\big[\,\Delta Y_{ij}(t) \leq -\log(1+z) \,\big|\, \mathcal{F}(t-1)\,\big] \;\geq\; p_{ij}\,\nu([z,1))
$$

a.s. By choosing $z$ small enough so that $\nu([z,1)) > 0$, we see that the assumption (14) of Lemma 2.7 is satisfied for $Y_{ij}$.

It follows from (20) and the last statement of Lemma 2.8 that, for every pair $(k,\ell) \in I^2$ with $k \neq \ell$,

$$
\frac{\mathbb{E}\big[\,\mathbb{I}_{\{Z_{k\ell}\neq 0\}}\{\Delta Y_{ij}(t)\}^2 \,\big|\, \mathcal{F}(t-1)\,\big]}{p_{k\ell}} \leq \begin{cases}
4\sigma^2 & \text{if } \{i,j\} = \{k,\ell\}; \\
\sigma^2 & \text{if } \sharp\big(\{i,j\} \cap \{k,\ell\}\big) = 1; \\
0 & \text{if } \{i,j\} \cap \{k,\ell\} = \emptyset
\end{cases}
$$

a.s., where the constant $\sigma$ is defined in (9), and thus

$$
\mathbb{E}\big[\,\{\Delta Y_{ij}(t)\}^2 \,\big|\, \mathcal{F}(t-1)\,\big] \leq \Big(\max_{(k,\ell)} p_{k\ell}\Big)\Big\{4\sigma^2 + 2(n-2)\sigma^2\Big\} \;\text{ a.s.}
$$

Since $Var\big[\,\Delta Y_{ij}(t) \,\big|\, \mathcal{F}(t-1)\,\big] \leq \mathbb{E}\big[\,\{\Delta Y_{ij}(t)\}^2 \,\big|\, \mathcal{F}(t-1)\,\big]$, the assumption (16) of Lemma 2.7 is satisfied for $Y_{ij}$.

It also follows from (20) that

$$
\begin{aligned}
&\frac{\mathbb{E}\big[\,\mathbb{I}_{\{Z_{k\ell}\neq 0\}}\,\Delta Y_{ij}(t)\,\big|\,\mathcal{F}(t-1)\,\big]}{p_{k\ell}}\\
=&\left\{\begin{array}{l}
f\Big(\min\big\{1,\frac{X_j(t-1)}{X_i(t-1)}\big\}\Big)-f\Big(\min\big\{1,\frac{X_i(t-1)}{X_j(t-1)}\big\}\Big)\quad \text{if } \{i,j\}=\{k,\ell\};\\
f\Big(\min\big\{1,\frac{X_k(t-1)}{X_i(t-1)}\big\}\Big)\quad \text{if } i=\ell \text{ and } j\neq k;\\
-f\Big(\min\big\{1,\frac{X_k(t-1)}{X_j(t-1)}\big\}\Big)\quad \text{if } j=\ell \text{ and } i\neq k;\\
0\quad \text{if } \{i,j\}\cap\{k,\ell\}=\emptyset
\end{array}\right.
\end{aligned}
$$

a.s., where the deterministic function $f$ is defined in Lemma 2.8, and therefore

$$
\begin{aligned}
&\mathbb{E}\big[\,\Delta Y_{ij}(t)\,\big|\,\mathcal{F}(t-1)\,\big]\\
=\;& p_{ij}\,\Big\{f\Big(\min\big\{1,\frac{X_j(t-1)}{X_i(t-1)}\big\}\Big)-f\Big(\min\big\{1,\frac{X_i(t-1)}{X_j(t-1)}\big\}\Big)\Big\}\\
&+\sum_{k\in I\setminus\{i,j\}}\Big\{p_{ik}\,f\Big(\min\big\{1,\frac{X_k(t-1)}{X_i(t-1)}\big\}\Big)-p_{jk}\,f\Big(\min\big\{1,\frac{X_k(t-1)}{X_j(t-1)}\big\}\Big)\Big\}
\end{aligned}
\tag{21}
$$

a.s. Since $f$ is continuous, strictly decreasing, and $f(0)=0$, it is possible to choose a constant $0<\epsilon<1$ such that

$$p_{ij}\,|f(\epsilon)|\;<\;\Big\{p_{ij}-\sum_{k\in I\setminus\{i,j\}}(p_{ik}-p_{jk})^{+}\Big\}\,|f(1)|.$$

On the set $\{\epsilon X_i(t-1)>X_j(t-1)\}=\{Y_{ij}(t-1)>\log\frac{1}{\epsilon}\}$, the right-hand side of (21) is larger than

$$
\begin{aligned}
& p_{ij}\,\big\{f(\epsilon)-f(1)\big\}\;+\sum_{k\in I\setminus\{i,j\}}(p_{ik}-p_{jk})\,f\Big(\min\big\{1,\frac{X_k(t-1)}{X_i(t-1)}\big\}\Big)\\
\geq\;& p_{ij}\,\big\{|f(1)|-|f(\epsilon)|\big\}\;-\sum_{k\in I\setminus\{i,j\}}(p_{ik}-p_{jk})^{+}\,|f(1)|\\
>\;& 0.
\end{aligned}
$$

The assumption (17) of Lemma 2.7 is thus satisfied for $Y_{ij}$.

*Step 3.* In this step we prove the final assertion (11). First, the positivity of the constant $\beta=-f(1)$ follows from Lemma 2.8. Next, for every $t\geq 1$ we have

$$
\begin{aligned}
\frac{\log X_i(t)}{t}\;&=\;\frac{\log X_i(0)+\sum_{s=1}^{t}\log\frac{X_i(s)}{X_i(s-1)}}{t}\\
&=\;\frac{\log X_i(0)+\sum_{s=1}^{t}\sum_{j\in I\setminus\{i\}}\log\Big[1+\min\big\{1,\frac{X_j(s-1)}{X_i(s-1)}\big\}Z_{ij}(s)\Big]}{t}.
\end{aligned}
$$

For the proof of (11), therefore, it suffices to show that, for each pair $(i,j) \in I^2$ with $i \neq j$,

$$\lim_{t\to\infty} \frac{\sum_{s=1}^{t} \log\Big[1+\min\big\{1, \frac{X_j(s-1)}{X_i(s-1)}\big\} Z_{ij}(s)\Big]}{t} = p_{ij} f(1)\, \mathbb{I}_{\big\{\lim_{t\to\infty} \frac{X_j(t)}{X_i(t)}=\infty\big\}} \quad \text{a.s.} \tag{22}$$

Fix the pair $(i,j)$ and consider the event

$$A := \left\{ \omega \in \Omega \;\middle|\; \lim_{t\to\infty} \frac{\sum_{s=1}^{t} \log\{1+Z_{ij}(s,\omega)\}}{t} = p_{ij}\, f(1) \right\}.$$

Since the random variables $\{Z_{ij}(t)\}_{t=1,2,\dots}$ are i.i.d. with $\mathbb{E}\big[\log\{1+Z_{ij}(t)\}\big] = p_{ij} f(1)$, the strong law of large numbers gives $\mathbb{P}(A) = 1$. For each $\omega \in A \cap \big\{\lim_{t\to\infty} \frac{X_j(t)}{X_i(t)} = \infty\big\}$, there exists a positive integer $\tau(\omega)$ such that $\frac{X_j(s-1,\omega)}{X_i(s-1,\omega)} \geq 1$ for every $s \geq \tau(\omega)$, and therefore

$$\begin{aligned}
& \lim_{t\to\infty} \frac{\sum_{s=1}^{t} \log\Big[1+\min\big\{1, \frac{X_j(s-1,\omega)}{X_i(s-1,\omega)}\big\} Z_{ij}(s,\omega)\Big]}{t} \\
= \; & \lim_{t\to\infty} \frac{\sum_{s=\tau(\omega)}^{t} \log\Big[1+\min\big\{1, \frac{X_j(s-1,\omega)}{X_i(s-1,\omega)}\big\} Z_{ij}(s,\omega)\Big]}{t} \\
= \; & \lim_{t\to\infty} \frac{\sum_{s=\tau(\omega)}^{t} \log\{1+Z_{ij}(s,\omega)\}}{t} \\
= \; & \lim_{t\to\infty} \frac{\sum_{s=1}^{t} \log\{1+Z_{ij}(s,\omega)\}}{t} \\
= \; & p_{ij}\, f(1).
\end{aligned}$$

We also consider the event

$$B := \bigcap_{k\in\mathbb{N}} \left\{ \omega \in \Omega \;\middle|\; \begin{array}{l} \lim_{t\to\infty} \frac{\sum_{s=1}^{t} \log\left\{1+\frac{1}{k}|Z_{ij}(s,\omega)|\right\}}{t} = p_{ij} f_+(\frac{1}{k}) \\ \text{and } \lim_{t\to\infty} \frac{\sum_{s=1}^{t} \log\left\{1-\frac{1}{k}|Z_{ij}(s,\omega)|\right\}}{t} = p_{ij} f_-(\frac{1}{k}) \end{array} \right\},$$

where $f_\pm(x) := \int_{-1}^{1} \log(1 \pm x|z|)\, \nu(dz)$. The strong law of large numbers again gives $\mathbb{P}(B) = 1$. For each $\omega \in B \cap \big\{\lim_{t\to\infty} \frac{X_j(t)}{X_i(t)} = 0\big\}$, the same argument as above yields that

$$\forall k \in \mathbb{N}, \quad \limsup_{t\to\infty} \frac{\sum_{s=1}^{t} \log\Big[1+\min\big\{1, \frac{X_j(s-1,\omega)}{X_i(s-1,\omega)}\big\} Z_{ij}(s,\omega)\Big]}{t} \leq p_{ij} f_+(\tfrac{1}{k})$$

$$\text{and} \quad \liminf_{t\to\infty} \frac{\sum_{s=1}^{t} \log\Big[1+\min\big\{1, \frac{X_j(s-1,\omega)}{X_i(s-1,\omega)}\big\} Z_{ij}(s,\omega)\Big]}{t} \geq p_{ij} f_-(\tfrac{1}{k}),$$

which implies that

$$\lim_{t\to\infty} \frac{\sum_{s=1}^{t} \log\Big[1+\min\big\{1, \frac{X_j(s-1,\omega)}{X_i(s-1,\omega)}\big\} Z_{ij}(s,\omega)\Big]}{t} = 0.$$

This completes the proof of (22) and, consequently, that of (11). $\square$

*Proof of Corollary 2.5.* Let $\widetilde{L}_k(t)$ denote the $k$-th lowest value among the $n$ players' wealth $X_i(t)$, $i \in I$, at time $t$. It then follows from (12) of Proposition 2.4 that

$$\forall k \in \{1, \ldots, n\}, \quad \lim_{t\to\infty} \frac{\log \widetilde{L}_k(t)}{t} = -\frac{2\beta\,(n-k)}{n(n-1)} \quad \text{a.s.}$$

Since $L_k(t) = \sum_{i=1}^{k} \widetilde{L}_i(t)$, we have $\widetilde{L}_k(t) \le L_k(t) \le k\,\widetilde{L}_k(t)$, and our assertion on the Lorenz curve holds. For the Gini coefficient, we proceed as follows. Since

$$\begin{aligned}
\mathrm{Gini}(t) &= \frac{2\sum_{k=1}^{n} k\,\widetilde{L}_k(t) \,-\, (n+1)}{n} \\
&= \frac{2\bigl\{n\,\widetilde{L}_n(t) + \sum_{k=1}^{n-1} k\,\widetilde{L}_k(t)\bigr\} \,-\, (n+1)}{n} \\
&= \frac{2\Bigl[n\,\bigl\{1 - \sum_{k=1}^{n-1} \widetilde{L}_k(t)\bigr\} + \sum_{k=1}^{n-1} k\,\widetilde{L}_k(t)\Bigr] \,-\, (n+1)}{n} \\
&= \Bigl(1 - \frac{1}{n}\Bigr) \,-\, \frac{2}{n}\sum_{k=1}^{n-1} (n-k)\,\widetilde{L}_k(t),
\end{aligned}$$

we have

$$\mathrm{Gini}(t) \ge \Bigl(1 - \frac{1}{n}\Bigr) - 2\sum_{k=1}^{n-1} \widetilde{L}_k(t) = \Bigl(1 - \frac{1}{n}\Bigr) - 2L_{n-1}(t)$$

and

$$\mathrm{Gini}(t) \le \Bigl(1 - \frac{1}{n}\Bigr) - \frac{2}{n}\sum_{k=1}^{n-1} \widetilde{L}_k(t) = \Bigl(1 - \frac{1}{n}\Bigr) - \frac{2}{n}L_{n-1}(t).$$

Hence

$$\lim_{t\to\infty} \frac{\log\bigl\{(1-\frac{1}{n}) - \mathrm{Gini}(t)\bigr\}}{t} = \lim_{t\to\infty} \frac{\log L_{n-1}(t)}{t} = -\frac{2\beta}{n(n-1)} \quad \text{a.s.} \qquad \square$$

## 2.5 Proofs of the Lemmas

*Proof of Lemma 2.6.* We divide our argument into two steps.

*Step 1.* We first show that $\lim_{t\to\infty} S(t) = \infty$ a.s. Let $S = M + A$ be the Doob decomposition of $S$, *i.e.,* define the two real-valued stochastic processes $A = \{A(t)\}_{t=0,1,\ldots}$ and $M = \{M(t)\}_{t=0,1,\ldots}$ as follows:

$$\begin{aligned}
A(0) &:= 0; \\
A(t) &:= \sum_{u=1}^{t} \mathbb{E}\bigl[\,\Delta S(u) \,\big|\, \mathcal{F}(u-1)\,\bigr] \qquad \text{for } t \ge 1; \\
M &:= S - A.
\end{aligned}$$

The process $M$ is then a square-integrable martingale. As $t \to \infty$, the martingale $M(t)$ converges a.s. on the event $\{\langle M\rangle(\infty) < \infty\}$, and the strong law of large numbers for martingales yields that $\lim_{t\to\infty} \frac{M(t)}{\langle M\rangle(t)} = 0$ a.s. on the event $\{\langle M\rangle(\infty) = \infty\}$ (see e.g. Sections 12.13 and 12.14 of Williams [8]). We also have that, for every $t \geq 1$,

$$\begin{aligned}
\langle M\rangle(t) &= \sum_{u=1}^{t} \mathbb{E}\big[\,\{\Delta M(u)\}^2 \,\big|\, \mathcal{F}(u-1)\,\big] \\
&= \sum_{u=1}^{t} Var\big[\,\Delta S(u) \,\big|\, \mathcal{F}(u-1)\,\big] \\
&\leq \sigma_{max}^2\, t \quad \text{a.s.}
\end{aligned}$$

by our assumption (16). Consequently, $\lim_{t\to\infty} \frac{M(t)}{t} = 0$ a.s. It also follows from our assumption (16) that $A(t) \geq \mu_{min}\, t$ and therefore $\lim_{t\to\infty} S(t) = \infty$ a.s.

*Step 2.* In this step, we prove (13). Let $y$ be a positive constant and $\{t_k\}_{k=0,1,\ldots}$ be a strictly increasing sequence of nonnegative integers with $t_0 = 0$. We note that, if an $\omega \in \Omega$ satisfies

$$\forall k \geq 0, \qquad \sup_{t_k \leq s \leq t_{k+1}} \big|M(s,\omega) - M(t_k,\omega)\big| \;\leq\; \Big(y + \frac{\mu_{min}}{2} t_k\Big) \wedge \Big\{\frac{\mu_{min}}{2}(t_{k+1} - t_k)\Big\}, \tag{23}$$

then $\inf_{t\geq 0} S(t,\omega) \geq -y$. Indeed, if the condition (23) is satisfied, then,

$$\begin{aligned}
\forall k \geq 1, \quad S(t_k,\omega) &= M(t_k,\omega) + A(t_k,\omega) \\
&= \Big\{\sum_{i=1}^{k} \big(M(t_i,\omega) - M(t_{i-1},\omega)\big)\Big\} + A(t_k,\omega) \\
&\geq -\Big\{\sum_{i=1}^{k} \frac{\mu_{min}}{2}(t_i - t_{i-1})\Big\} + \mu_{min}\, t_k \\
&= \frac{\mu_{min}}{2}\, t_k,
\end{aligned}$$

and

$$\begin{aligned}
\forall k \geq 0, \qquad \inf_{t_k \leq s \leq t_{k+1}} S(s,\omega) &\geq S(t_k,\omega) + \inf_{t_k \leq s \leq t_{k+1}} \big\{M(s,\omega) - M(t_k,\omega)\big\} \\
&\geq \frac{\mu_{min}}{2}\, t_k - \Big(y + \frac{\mu_{min}}{2} t_k\Big) \\
&= -y.
\end{aligned}$$

Consequently, the probability $\mathbb{P}\big[\inf_{t\geq 0} S(t) < -y\big]$ is dominated by

$$\begin{aligned}
& \sum_{k=0}^{\infty} \mathbb{P}\left[\sup_{t_k \leq s \leq t_{k+1}} \big|M(s,\omega) - M(t_k,\omega)\big| \;>\; \Big(y + \frac{\mu_{min}}{2} t_k\Big) \wedge \Big\{\frac{\mu_{min}}{2}(t_{k+1}-t_k)\Big\}\right] \\
\leq\; & \sum_{k=0}^{\infty} \frac{\mathbb{E}\big[\{M(t_{k+1}) - M(t_k)\}^2\big]}{\big\{(y + \frac{\mu_{min}}{2} t_k) \wedge (\frac{\mu_{min}}{2}(t_{k+1}-t_k))\big\}^2} \qquad \text{by Doob's maximal inequality} \\
\leq\; & \sum_{k=0}^{\infty} \frac{\sigma_{max}^2 (t_{k+1}-t_k)}{\big\{(y + \frac{\mu_{min}}{2} t_k) \wedge (\frac{\mu_{min}}{2}(t_{k+1}-t_k))\big\}^2} \\
=\; & \frac{\sigma_{max}^2\, t_1}{(y \wedge \frac{\mu_{min}}{2} t_1)^2} \;+\; \sum_{k=1}^{\infty} \frac{\sigma_{max}^2 (t_{k+1}-t_k)}{\big\{(y + \frac{\mu_{min}}{2} t_k) \wedge (\frac{\mu_{min}}{2}(t_{k+1}-t_k))\big\}^2}. \qquad (24)
\end{aligned}$$

We choose $y$ and $t_k$'s such that $t_k := \ell\, 2^{k-1}$ for $k \geq 1$, where $\ell := \lceil \frac{24\sigma_{max}^2}{\mu_{min}^2} \rceil$, and $y > \frac{\mu_{min}}{2}\ell$. It then follows that

$$\begin{aligned}
(24) \;&=\; \frac{\sigma_{max}^2\, \ell}{(y \wedge \frac{\mu_{min}}{2}\ell)^2} \;+\; \sum_{k=1}^{\infty} \frac{\sigma_{max}^2\, \ell\, 2^{k-1}}{\big\{(y + \frac{\mu_{min}}{2}\ell\, 2^{k-1}) \wedge (\frac{\mu_{min}}{2}\ell\, 2^{k-1})\big\}^2} \\
&=\; \frac{\sigma_{max}^2\, \ell}{(\frac{\mu_{min}}{2}\ell)^2} \;+\; \sum_{k=1}^{\infty} \frac{\sigma_{max}^2\, \ell\, 2^{k-1}}{(\frac{\mu_{min}}{2}\ell\, 2^{k-1})^2} \\
&=\; \frac{12\,\sigma_{max}^2}{\mu_{min}^2\, \ell} \;\leq\; \frac{1}{2}.
\end{aligned}$$

In addition, we have that

$$\frac{\mu_{min}}{2}\ell < \frac{\mu_{min}}{2}\Big(\frac{24\sigma_{max}^2}{\mu_{min}^2} + 1\Big) = \frac{12\sigma_{max}^2}{\mu_{min}} + \frac{\mu_{min}}{2}.$$

Thus, by setting $y := \frac{12\sigma_{max}^2}{\mu_{min}} + \frac{\mu_{min}}{2}$, the proof of (13) is complete. □

*Proof of Lemma 2.7 (i).* We note that the assertion (15) is equivalent to

$$\mathbb{P}\left[\Big(\limsup_{t\to\infty} Y(t) = \infty \text{ or } -\infty\Big) \text{ and } \Big(\liminf_{t\to\infty} Y(t) = \infty \text{ or } -\infty\Big)\right] = 1.$$

In what follows we assume only $\mathbb{P}\big[\,\Delta Y(t) > \varepsilon \,\big|\, \mathcal{F}(t-1)\,\big] \geq p$ and prove $\mathbb{P}\big[\limsup_{t\to\infty} Y(t) = \infty \text{ or } -\infty\big] = 1$. The liminf version can be proved similarly. We divide our argument into two steps.

*Step 1.* We first show that $\mathbb{P}[\exists \lim Y(t) \in \mathbb{R}] = 0$. For every $\omega \in \Omega$ such that $\exists \lim Y(t,\omega) \in \mathbb{R}$, we have

$$\exists t(\omega) \geq 1, \quad \forall s \geq t(\omega), \quad \big|\Delta Y(s,\omega)\big| \leq \epsilon.$$

It follows that

$$\Big\{\exists \lim_{t\to\infty} Y(t) \in \mathbb{R}\Big\} \subset \bigcup_{t=1}^{\infty}\bigcap_{s=t}^{\infty} \{|\Delta Y(s)| \leq \epsilon\}.$$

It thus suffices to show that $\mathbb{P}\left[\bigcap_{s=t}^{\infty}\left\{|\Delta Y(s)| \le \epsilon\right\}\right] = 0$ for every $t \ge 1$. We have, for every $u \ge t$,

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{s=t}^{u}\left\{|\Delta Y(s)| \le \epsilon\right\}\right] &= \mathbb{E}\left[\prod_{s=t}^{u}\mathbb{I}_{\{|\Delta Y(s)|\le\epsilon\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\Big[\prod_{s=t}^{u}\mathbb{I}_{\{|\Delta Y(s)|\le\epsilon\}}\Big|\mathcal{F}(u-1)\Big]\right] \\
&= \mathbb{E}\left[\mathbb{E}\Big[\mathbb{I}_{\{|\Delta Y(u)|\le\epsilon\}}\Big|\mathcal{F}(u-1)\Big]\prod_{s=t}^{u-1}\mathbb{I}_{\{|\Delta Y(s)|\le\epsilon\}}\right] \\
&= \mathbb{E}\left[\mathbb{P}\Big[|\Delta Y(u)| \le \epsilon\Big|\mathcal{F}(u-1)\Big]\prod_{s=t}^{u-1}\mathbb{I}_{\{|\Delta Y(s)|\le\epsilon\}}\right] \\
&\le (1-p)\,\mathbb{E}\left[\prod_{s=t}^{u-1}\mathbb{I}_{\{|\Delta Y(s)|\le\epsilon\}}\right] \qquad \text{by (14)} \\
&\;\vdots \\
&\le (1-p)^{u-t+1}. \qquad (25)
\end{aligned}$$

Taking the limit $u \to \infty$, we are done.

*Step 2.* We will prove our assertion (15). First, we note the following implications:

$$\begin{aligned}
& \mathbb{P}\left[\limsup_{t\to\infty} Y(t) = \infty \;\text{ or }\; -\infty\right] < 1 \\
\Rightarrow\;& \mathbb{P}\left[\limsup_{t\to\infty} Y(t) \in \mathbb{R}\right] > 0 \\
\Rightarrow\;& \mathbb{P}\left[\limsup_{t\to\infty} Y(t) \in \mathbb{R} \;\text{ and }\; \limsup_{t\to\infty} Y(t) > \liminf_{t\to\infty} Y(t)\right] > 0 \quad \text{by Step 1} \\
\Rightarrow\;& \exists\delta > 0,\quad \mathbb{P}\left[\limsup_{t\to\infty} Y(t) \in \mathbb{R} \;\text{ and }\; \limsup_{t\to\infty} Y(t) - \liminf_{t\to\infty} Y(t) > 2\delta\right] > 0 \\
\Rightarrow\;& \exists\delta > 0,\; \exists a \in \mathbb{R}, \\
& \mathbb{P}\left[a < \limsup_{t\to\infty} Y(t) < a+\delta \;\text{ and }\; \limsup_{t\to\infty} Y(t) - \liminf_{t\to\infty} Y(t) > 2\delta\right] > 0 \\
\Rightarrow\;& \exists\delta > 0,\; \exists a \in \mathbb{R}, \\
& \mathbb{P}\left[a < \limsup_{t\to\infty} Y(t) < a+\delta \;\text{ and }\; \liminf_{t\to\infty} Y(t) < a-\delta\right] > 0. \qquad (26)
\end{aligned}$$

Thus, in order to prove (15), it suffices to show $\mathbb{P}(B_{a,\delta}) = 0$ for every $a \in \mathbb{R}$ and $\delta > 0$, where

$$B_{a,\delta} := \left\{a < \limsup_{t\to\infty} Y(t) < a+\delta \;\text{ and }\; \liminf_{t\to\infty} Y(t) < a-\delta\right\}.$$

Choose three constants $b^{(l)} \in \mathbb{R}$, $b^{(m)} \in \mathbb{R}$ and $\eta \in \mathbb{N}$ such that

$$a - \delta < b^{(l)} < b^{(m)} < a < a + \delta < b^{(m)} + \eta\varepsilon =: b^{(h)},$$

and define recursively the following three sequences of stopping times: for $k \geq 1$,

$$\begin{aligned} \tau_k^{(m)} &:= \inf\left\{ t \geq \tau_{k-1}^{(l)} \,\middle|\, Y(t) > b^{(m)} \right\} \quad \text{where } \tau_0^{(l)} := 0, \\ \tau_k^{(l)} &:= \inf\left\{ t \geq \tau_k^{(m)} \,\middle|\, Y(t) < b^{(l)} \right\}, \\ \tau_k^{(h)} &:= \inf\left\{ t \geq \tau_k^{(m)} \,\middle|\, Y(t) > b^{(h)} \right\}, \end{aligned}$$

where $\inf \emptyset := \infty$. On the event $B_{a,\delta}$, the stopping times $\tau_k^{(m)}$ and $\tau_k^{(l)}$ are finite for every $k \geq 1$ but $\tau_k^{(h)}$ becomes eventually infinite, and thus $\tau_k^{(l)} < \tau_k^{(h)}$ for all sufficiently large $k$. Thus

$$\mathbb{P}(B_{a,\delta}) \leq \mathbb{P}\left[ \tau_k^{(l)} < \tau_k^{(h)} \text{ for all sufficiently large } k \right].$$

Since our assumption $\mathbb{P}\left[ \Delta Y(t) > \varepsilon \,\middle|\, \mathcal{F}(t-1) \right] \geq p$ implies that

$$\mathbb{P}\left[ \tau_k^{(l)} < \tau_k^{(h)} \,\middle|\, \mathcal{F}(\tau_k^{(m)}) \right] < 1 - p^{\eta} \quad \text{a.s. on} \quad \left\{ \tau_k^{(m)} < \infty \right\}$$

for every $k \geq 1$, a similar argument as in (25) yields

$$\mathbb{P}\left[ \tau_k^{(l)} < \tau_k^{(h)} \text{ for all sufficiently large } k \right] = 0. \qquad \Box$$

*Proof of Lemma 2.7 (ii).* In order to prove (18), it suffices to show that

$$\mathbb{P}\left[ \liminf_{t\to\infty} Y(t) < \limsup_{t\to\infty} Y(t) = \infty \right] = 0.$$

Choose two real numbers $c^{(l)}$ and $c^{(h)}$ such that $c < c^{(l)} < c^{(l)} + \left( \frac{12\sigma_{max}^2}{\mu_{min}} + \frac{\mu_{min}}{2} \right) < c^{(h)}$, and define recursively the following two sequences of stopping times: for $k \geq 1$,

$$\begin{aligned} \widetilde{\tau}_k^{(h)} &:= \inf\left\{ t \geq \widetilde{\tau}_{k-1}^{(l)} \,\middle|\, Y(t) > c^{(h)} \right\} \quad \text{where } \widetilde{\tau}_0^{(l)} := 0, \\ \widetilde{\tau}_k^{(l)} &:= \inf\left\{ t \geq \widetilde{\tau}_k^{(h)} \,\middle|\, Y(t) < c^{(l)} \right\}, \end{aligned}$$

where $\inf \emptyset := \infty$. On the event $\{ \liminf Y(t) < c^{(l)} < \infty = \limsup Y(t) \}$, the stopping times $\widetilde{\tau}_k^{(h)}$ and $\widetilde{\tau}_k^{(l)}$ are finite for every $k \geq 1$. It follows from Lemma 2.6 that

$$\mathbb{P}\left[ \widetilde{\tau}_k^{(l)} < \infty \,\middle|\, \mathcal{F}(\widetilde{\tau}_k^{(h)}) \right] \leq \frac{1}{2} \quad \text{a.s. on} \quad \left\{ \widetilde{\tau}_k^{(h)} < \infty \right\}$$

for every $k \geq 1$, and therefore a similar argument as in (25) yields

$$\mathbb{P}\left[ \liminf_{t\to\infty} Y(t) < c^{(l)} < \infty = \limsup_{t\to\infty} Y(t) \right] \leq \mathbb{P}\left[ \forall k \geq 1,\ \widetilde{\tau}_k^{(l)} < \infty \right] = 0.$$

Since $c^{(l)}$ can be taken as large as possible, the proof of (18) is complete. $\square$

*Proof of Lemma 2.8.* The continuity of $f$ follows from Lebesgue's dominated convergence theorem. The strict concavity of $f$ follows from the fact that, for each fixed $z \in (-1,1)\backslash\{0\}$, the function $x \mapsto \log(1+xz)$ is strictly concave. The function $f$ is twice differentiable on the interval $[0,1)$ and

$$f'(0) = \int_{-1}^{1} z\,\nu(dz) \le 0,$$

which implies that $f$ is strictly decreasing.

The function $g$ is strictly increasing because, for each fixed $z \in (-1,1)\backslash\{0\}$, the function $x \mapsto \{\log(1+xz)\}^2$ is strictly increasing. $\square$

# 3 Continuous-time Setting

## 3.1 Definition and a Known Result

The following formulation is a slight generalization of that of Chorro [4].

**Definition 3.1** *Define the set $I = \{1,\dots,n\}$, where $n \ge 2$. Let $x_1,\dots,x_n$ be $n$ positive numbers such that $\sum_{i\in I} x_i = 1$.*

*Consider $\frac{n(n-1)}{2}$ independent one-dimensional Wiener processes, starting from the origin, on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let us denote each of them by $W_{ij} = \{W_{ij}(t)\}_{t\ge 0}$, where $(i,j) \in I^2$ with $i<j$.*

*For each pair $(i,j) \in I^2$ with $i<j$, we also consider two constants $\sigma_{ij} > 0$ and $\mu_{ij} \in \mathbb{R}$, and set*

$$W_{ji}(\cdot) := -W_{ij}(\cdot), \qquad \sigma_{ji} := \sigma_{ij}, \qquad \mu_{ji} := -\mu_{ij}.$$

*Let $\{\mathcal{F}(t)\}_{t\ge 0}$ be the filtration generated by all the Wiener processes.*

*The $\mathbb{R}^n$-valued adapted process $X = \{X(t)\}_{t\ge 0}$, whose $i$-th component is denoted by $X_i = \{X_i(t)\}_{t\ge 0}$, is defined as the unique solution of the following stochastic differential equation:*

$$\begin{cases} X_i(0) &=& x_i, \\ dX_i(t) &=& \displaystyle\sum_{j\in I\backslash\{i\}} \min\big\{X_i(t),\, X_j(t)\big\}\big(\sigma_{ij}\,dW_{ij}(t) + \mu_{ij}dt\big). \end{cases} \tag{27}$$

*Remark 1.* This $n$-dimensional stochastic differential equation satisfies the Lipschitz condition, so it has the unique strong solution. It follows immediately from our definition that $\sum_{i\in I} X_i(t) = 1$ for every $t \ge 0$. The positivity of each component process $X_i(t)$ is shown as follows: it is possible to take the logarithm of $X_i(t)$ up to the first hitting time of the process to zero and, if $X_i$ hit zero at a finite time, then $\log X_i$ would explode to $-\infty$. It follows from Itô's formula that

$$\begin{aligned} d\,\log X_i(t) &= \frac{d\,X_i(t)}{X_i(t)} - \frac{d\langle X_i(t)\rangle}{2\,X_i^2(t)} \\ &= \sum_{j\in I\setminus\{i\}} \min\left\{1,\ \frac{X_j(t)}{X_i(t)}\right\} \big(\sigma_{ij}\,dW_{ij}(t) + \mu_{ij}dt\big) \\ &\quad -\frac{1}{2}\sum_{j\in I\setminus\{i\}} \sigma_{ij}^2\,\min\left\{1,\ \Big(\frac{X_j(t)}{X_i(t)}\Big)^2\right\}\,dt. \end{aligned}$$

In the last expression, the coefficients of $dW_{ij}(t)$'s and $dt$'s are all bounded, and therefore $\log X_i(t)$ never explodes at a finite time.

**Proposition 3.2 (Boghosian *et al.* [2])** *Assume that all constants $\sigma_{ij}$ are equal and all constants $\mu_{ij}$ are zero. Then we have*

$$\mathbb{P}\left[\exists i\in I \ \ \textit{such that} \ \ \lim_{t\to\infty} X_i(t) = 1\right] \,=\, 1$$

*and*

$$\forall i\in I,\quad \mathbb{P}\left[\lim_{t\to\infty} X_i(t) = 1\right] \,=\, x_i.$$

## 3.2 Our Results

The first two propositions and the corollary we give in this subsection are continuous-time analogues of the results in Subsection 2.2. Our third proposition is only for this continuous-time setting and investigates the effect of a proportional capital tax. All the propositions will be proved in Subsection 3.4.

Our first result generalizes Proposition 3.2 to possibly unfair games.

**Proposition 3.3** *Suppose that the set $I$ is totally ordered, where the deterministic order $\preceq$ does not need to be identical to the usual ordinal of $I = \{1,\dots,n\}$. For every pair $(i,j)\in I^2$ with $i\prec j$, we assume that $\mu_{ij} = -\mu_{ji} \le 0$.*

*Then we have*

$$\mathbb{P}\left[\exists i\in I \ \ \textit{such that} \ \ \lim_{t\to\infty} X_i(t) = 1\right] \,=\, 1 \tag{28}$$

*and*

$$\forall i\in I,\quad \mathbb{P}\left[\exists j\in I \ \ \textit{such that} \ \ j\succeq i \ \ \textit{and} \ \ \lim_{t\to\infty} X_j(t) = 1\right] \,\ge\, \sum_{j\succeq i} x_j. \tag{29}$$

*Furthermore, if all constants $\mu_{ij}$ are zero, then we have*

$$\forall i\in I,\quad \mathbb{P}\left[\lim_{t\to\infty} X_i(t) = 1\right] \,=\, x_i. \tag{30}$$

Our second proposition shows growing wealth disparities between every pair of two players.

**Proposition 3.4** *Assume that all constants $\mu_{ij}$ are zero. When $n \geq 4$, we further assume that*

$$\text{either} \quad \sigma_{ij}^2 > \sum_{k \in I \setminus \{i,j\}} (\sigma_{ik}^2 - \sigma_{jk}^2)^+ \quad \text{or} \quad \sigma_{ij}^2 > \sum_{k \in I \setminus \{i,j\}} (\sigma_{jk}^2 - \sigma_{ik}^2)^+ \tag{31}$$

*holds for each pair $(i,j) \in I^2$ with $i \neq j$. It then holds that*

$$\mathbb{P}\left[\lim_{t \to \infty} \frac{X_i(t)}{X_j(t)} = 0\right] + \mathbb{P}\left[\lim_{t \to \infty} \frac{X_i(t)}{X_j(t)} = \infty\right] = 1 \tag{32}$$

*for every pair $(i,j) \in I^2$ with $i \neq j$. Moreover,*

$$\forall i \in I, \quad \lim_{t \to \infty} \frac{\log X_i(t)}{t} = -\frac{1}{2} \sum_{j \in I \setminus \{i\}} \sigma_{ij}^2 \, \mathbb{I}_{\{\lim_{t \to \infty} \frac{X_i(t)}{X_j(t)} = 0\}} \quad a.s. \tag{33}$$

*In particular, if all the constants $\sigma_{ij}$ are equal with common value $\sigma$, then*

$$\forall i \in I, \quad \lim_{t \to \infty} \frac{\log X_i(t)}{t} = -\frac{\sigma^2 \, \sharp\left\{j \in I \setminus \{i\} \,\middle|\, \lim_{t \to \infty} \frac{X_i(t)}{X_j(t)} = 0\right\}}{2} \quad a.s. \tag{34}$$

*Remark 2.* As for the discrete-time settings, the assertion (32) implies that each player has his/her limit rank: see Remark 3 after the statement of Proposition 2.4.

*Remark 3.* When $n \geq 4$, the assumption (31) holds if

$$\sqrt{\frac{n-2}{n-1}} < \frac{\min_{(i,j)} \sigma_{ij}}{\max_{(i,j)} \sigma_{ij}} \leq 1.$$

Indeed, for this case we have

$$\begin{aligned} \sigma_{ij}^2 - \sum_{k \in I \setminus \{i,j\}} (\sigma_{ik}^2 - \sigma_{jk}^2)^+ \quad &\geq \quad \min_{(k,\ell)} \sigma_{k\ell}^2 - (n-2)\left\{\max_{(k,\ell)} \sigma_{k\ell}^2 - \min_{(k,\ell)} \sigma_{k\ell}^2\right\} \\ &= \quad (n-1) \min_{(k,\ell)} \sigma_{k\ell}^2 - (n-2) \max_{(k,\ell)} \sigma_{k\ell}^2 \\ &> \quad 0. \end{aligned}$$

Also, when $n = 3$, the assumption (31) holds automatically for all positive constants $\sigma_{ij}$. Indeed, by denoting $I = \{i,j,k\}$, we see that

- if $\sigma_{ik} \leq \sigma_{jk}$ then $\sigma_{ij}^2 > 0 = (\sigma_{ik}^2 - \sigma_{jk}^2)^+$;
- if $\sigma_{ik} \geq \sigma_{jk}$ then $\sigma_{ij}^2 > 0 = (\sigma_{jk}^2 - \sigma_{ik}^2)^+$.

When $n = 2$, the assumption holds automatically with the right-hand sides of the two expressions of (31) defined to be zero.

The following corollary can be proved in the same way as the discrete-time Corollary 2.5, so its proof is omitted. Part **(ii)** refines Boghosian *et al.* [2].

**Corollary 3.5** *Under the assumptions of Proposition 3.4 with the constants $\sigma_{ij}$ being all equal with common value $\sigma$, the following two asymptotic properties hold regardless of the initial wealth $x_i$, $i \in I$.*

**(i)** *Denoting by $L_k(t)$ the sum of the $k$ lowest values among the $n$ players' wealth $X_i(t)$, $i \in I$, at time $t$, we have that*

$$\forall k \in \{1, \ldots, n\}, \quad \lim_{t\to\infty} \frac{\log L_k(t)}{t} \;=\; -\frac{\sigma^2\,(n-k)}{2} \qquad a.s.$$

**(ii)** *The Gini coefficient at time $t$, denoted by $Gini(t)$, satisfies*

$$\lim_{t\to\infty} \frac{\log\left\{(1-\frac{1}{n}) - Gini(t)\right\}}{t} \;=\; -\frac{\sigma^2}{2} \qquad a.s.$$

For the continuous-time settings, we also give the following proposition on the effect of a capital tax. With the slightest proportional capital tax, each player's wealth process oscillates between 0 and 1 regardless of the constants $\sigma_{ij}$ and $\mu_{ij}$.

**Proposition 3.6** *Let $\alpha$ be a positive constant. The $\mathbb{R}^n$-valued process $\widetilde{X} = \{\widetilde{X}(t)\}_{t\geq 0}$, whose $i$-th component is denoted by $\widetilde{X}_i = \{\widetilde{X}_i(t)\}_{t\geq 0}$, is defined as the unique solution of the following stochastic differential equation:*

$$\begin{cases} \widetilde{X}_i(0) &=\; x_i, \\ d\widetilde{X}_i(t) &=\; \alpha\left\{\dfrac{1}{n} - \widetilde{X}_i(t)\right\}dt + \displaystyle\sum_{j\in I\setminus\{i\}} \min\left\{\widetilde{X}_i(t),\, \widetilde{X}_j(t)\right\}\left(\sigma_{ij}\, dW_{ij}(t) + \mu_{ij} dt\right), \end{cases} \tag{35}$$

*where the constants $x_i$, $\sigma_{ij}$, $\mu_{ij}$, and the Wiener processes $W_{ij}(\cdot)$ are as in Definition 3.1.*

*It then holds that*

$$\forall i \in I, \quad \mathbb{P}\left[\limsup_{t\to\infty} \widetilde{X}_i(t) = 1 \quad and \quad \liminf_{t\to\infty} \widetilde{X}_i(t) = 0\right] \;=\; 1. \tag{36}$$

*Remark 4.* For the PDF approach to the taxation issue concerning the yard-sale model, see §V.D of Boghosian [1]. For discrete-time capital taxation, see also Chorro [4].

## 3.3 Technical Lemmas

The first lemma is a known result, so the proof is omitted.

**Lemma 3.7 (see e.g. Proposition 8.2 of Steele [7])** *Let $S$ be a real-valued Itô process of the form*

$$dS(t) = dW(t) + \mu\, dt, \qquad S(0) = 0$$

*on a stochastic basis, where $W$ is a one-dimensional Wiener process and $\mu$ is a positive constant. For $x \in \mathbb{R}$, define the stopping time*

$$\tau_x \;:=\; \inf \big\{ t \geq 0 \;\big|\; S(t) = x \big\},$$

*where* $\inf \emptyset := \infty$*. Then $\lim_{t\to\infty} S(t) = \infty$ a.s., and we have*

$$\forall y > 0, \qquad \mathbb{P}\big[\tau_{-y} < \infty\big] \;=\; \exp\big(-2\mu\, y\big)$$

*and*

$$\forall x > 0,\ \forall y > 0, \qquad \mathbb{P}\big[\tau_{-y} < \tau_x\big] \;=\; \frac{1 - \exp(-2\mu\, x)}{\exp(2\mu\, y) - \exp(-2\mu\, x)}.$$

The following lemma is a continuous-time analogue of Lemma 2.7 and will be used for proving Propositions 3.4 and 3.6. Its proof will be given in Subsection 3.5

**Lemma 3.8** *Let $Y$ be a real-valued Itô process of the form*

$$dY(t) = dM(t) + \mu(t)dt$$

*on a stochastic basis, where both $\sigma(t) := \sqrt{\frac{d\langle M\rangle(t)}{dt}}$ $\left( = \sqrt{\frac{d\langle Y\rangle(t)}{dt}}\right)$ and $\mu(t)$ are continuous processes.*

**(i)** *Assume the existence of two positive constants $\sigma_{min}$ and $\sigma_{max}$ such that*

$$\sigma_{min} \leq \sigma(t) \leq \sigma_{max}, \qquad \forall t \geq 0, \quad a.s. \tag{37}$$

*It is also assumed that, for each $K \in \mathbb{N}$, there exists some positive constant $\mu_K^{(abs)}$ such that*

$$|\mu(t)| \leq \mu_K^{(abs)} \quad \text{on the event } \big\{|Y(t)| \leq K\big\}, \quad \forall t \geq 0, \quad a.s. \tag{38}$$

*It then holds that*

$$\begin{aligned} &\mathbb{P}\left[\lim_{t\to\infty} Y(t) = \infty\right] + \mathbb{P}\left[\lim_{t\to\infty} Y(t) = -\infty\right] \\ &+\mathbb{P}\left[\limsup_{t\to\infty} Y(t) = \infty \ \text{ and } \ \liminf_{t\to\infty} Y(t) = -\infty\right] \;=\; 1. \end{aligned} \tag{39}$$

**(ii)** *In addition to (37) and (38), assume further the existence of two positive constants $c_1$ and $\mu_1$ such that*

$$\mu(t) > \mu_1 \quad \text{on the event } \big\{Y(t) > c_1\big\}, \quad \forall t \geq 0, \quad a.s. \tag{40}$$

*We then have*

$$\mathbb{P}\left[\lim_{t\to\infty} Y(t)=\infty\right]+\mathbb{P}\left[\lim_{t\to\infty} Y(t)=-\infty\right] \;=\; 1. \tag{41}$$

**(iii)** *In addition to (37) and (38), assume further the existence of two positive constants $c_2$ and $\mu_2$ such that*

$$\mu(t) < -\mu_2 \quad \text{on the event } \big\{Y(t) > c_2\big\} \tag{42}$$

*and*

$$\mu(t) > \mu_2 \quad \text{on the event } \big\{Y(t) < -c_2\big\} \tag{43}$$

*for all $t \geq 0$, a.s. We then have*

$$\mathbb{P}\left[\limsup_{t\to\infty} Y(t)=\infty \;\text{ and }\; \liminf_{t\to\infty} Y(t)=-\infty\right] \;=\; 1. \tag{44}$$

## 3.4 Proofs of the Propositions

*Proof of Proposition 3.3.* The following proof has the same spirit as the discrete-time proof of Chorro [4]. We divide our argument into two steps.

*Step 1.* We first prove the proposition when all constants $\mu_{ij}$ are zero. In this case each process $X_i(\cdot)$ is a bounded martingale, and its almost sure convergence follows from the martingale convergence theorem (see e.g. Section 1.3 B of Karatzas and Shreve [6]). A time-change technique (see e.g. Section 3.4 B of Karatzas and Shreve [6]) shows that, for every continuous martingale, the almost sure convergence is equivalent to the almost sure finiteness of the quadratic variation at time infinity, and thus

$$\langle X_i\rangle(\infty) = \sum_{j\in I\setminus\{i\}} \sigma_{ij}^2 \int_0^\infty \Big(\min\big\{X_i(t),\, X_j(t)\big\}\Big)^2\, dt \;<\; \infty \qquad \text{a.s.}$$

It follows that, for every pair $(i,j)\in I^2$ with $i\neq j$,

$$\min\big\{X_i(\infty),\, X_j(\infty)\big\} \;=\; 0 \qquad \text{a.s.}$$

and therefore

$$\sharp\big\{\, i\in I \;\big|\; X_i(\infty)>0 \,\big\} \;\leq\; 1 \quad \text{a.s.}$$

It also holds that $\sum_{i\in I} X_i(\infty) = 1$ a.s., so the proof of (28) is complete. The assertion (30) follows from the fact that $X_i(\cdot)$ is a bounded martingale.

*Step 2.* We next consider the general case. For $i\in I$, we define the process

$$S_i(t) := \sum_{j\succeq i} X_j(t).$$

Each process $S_i$ is then a bounded submartingale. Indeed,

$$\begin{aligned}
dS_i(t) &= \sum_{j\succeq i} dX_j(t) \\
&= \sum_{j\succeq i} \sum_{k\in I\setminus\{j\}} \min\big\{X_j(t),\, X_k(t)\big\}\big(\sigma_{jk}\, dW_{jk}(t) + \mu_{jk}dt\big) \\
&= \sum_{j\succeq i} \left\{ \begin{array}{l} \displaystyle\sum_{k\succeq i,\, k\neq j} \min\big\{X_j(t),\, X_k(t)\big\}\big(\sigma_{jk}\, dW_{jk}(t) + \mu_{jk}dt\big) \\ \displaystyle +\sum_{k\prec i} \min\big\{X_j(t),\, X_k(t)\big\}\big(\sigma_{jk}\, dW_{jk}(t) + \mu_{jk}dt\big) \end{array} \right\} \\
&= \sum_{j\succeq i} \sum_{k\prec i} \min\big\{X_j(t),\, X_k(t)\big\}\big(\sigma_{jk}\, dW_{jk}(t) + \mu_{jk}dt\big),
\end{aligned}$$

and $\mu_{jk} \geq 0$ for each pair $(j,k) \in I^2$ with $j \succ k$. The martingale convergence theorem again yields the almost sure convergence of each submartingale $S_i$, which in turn implies the almost sure convergence of each process $X_i$.

Let $S_i = S_i(0)+M_i+A_i$ be the Doob-Meyer decomposition of the submartingale $S_i$. Since the submartingale is bounded, its martingale part $M_i$ is uniformly integrable (see e.g. Section 1.4 of Karatzas and Shreve [6]) and therefore $M_i$ also converges almost surely. Moreover,

$$\langle M_i\rangle(\infty) = \langle S_i\rangle(\infty) = \sum_{j\succeq i}\sum_{k\prec i}\int_0^\infty \sigma_{jk}^2 \Big(\min\big\{X_j(t),\, X_k(t)\big\}\Big)^2\, dt.$$

We can thus show (28) in a similar way as in Step 1. The assertion (29) follows from the fact that $S_i(\cdot)$ is a bounded submartingale. □

*Proof of Proposition 3.4.* We divide our argument into two steps.

*Step 1.* In this step, we show (32). Fix a pair $(i,j) \in I^2$ with $i \neq j$. Assume without loss of generality that $\sigma_{ij}^2 > \sum_{k\in I\setminus\{i,j\}}(\sigma_{ik}^2 - \sigma_{jk}^2)^+$ : for the case $n \leq 3$ see Remark 3 after the statement of the proposition. It suffices to show that the process $Y_{ij}(t) := \log \frac{X_i(t)}{X_j(t)}$ satisfies the assumptions of **(i)** and **(ii)** of Lemma 3.8.

We first note that

$$dY_{ij}(t) = d\,\log X_i(t) - d\,\log X_j(t)$$

is equal to

$$\begin{aligned}
&\sum_{k\in I\setminus\{i\}} \sigma_{ik}\, \min\left\{1,\, \frac{X_k(t)}{X_i(t)}\right\} dW_{ik}(t) - \frac{1}{2}\sum_{k\in I\setminus\{i\}} \sigma_{ik}^2\, \min\left\{1,\, \left(\frac{X_k(t)}{X_i(t)}\right)^2\right\} dt \\
- &\sum_{k\in I\setminus\{j\}} \sigma_{jk}\, \min\left\{1,\, \frac{X_k(t)}{X_j(t)}\right\} dW_{jk}(t) + \frac{1}{2}\sum_{k\in I\setminus\{j\}} \sigma_{jk}^2\, \min\left\{1,\, \left(\frac{X_k(t)}{X_j(t)}\right)^2\right\} dt
\end{aligned} \tag{45}$$

by Itô's formula. Here the coefficients of $dW$'s and $dt$'s are all bounded. Moreover, the sum of the $dW_{ij}(t)$ and $dW_{ji}(t)$ terms in (45) is

$$\begin{aligned} & \sigma_{ij} \min\left\{1, \frac{X_j(t)}{X_i(t)}\right\} dW_{ij}(t) - \sigma_{ji} \min\left\{1, \frac{X_i(t)}{X_j(t)}\right\} dW_{ji}(t) \\ = \;& \sigma_{ij} \left( \min\left\{1, \frac{X_j(t)}{X_i(t)}\right\} + \min\left\{1, \frac{X_i(t)}{X_j(t)}\right\} \right) dW_{ij}(t), \end{aligned}$$

the coefficient of which is always larger than the constant $\sigma_{ij}$. Therefore,

$$\sqrt{\frac{d\langle Y_{ij}\rangle(t)}{dt}} \;\geq\; \sigma_{ij},$$

and the assumptions (37) and (38) of Lemma 3.8 is satisfied for $Y_{ij}$.

Furthermore, the sum of the coefficients of the $dt$ terms in (45) is

$$\begin{aligned} & -\frac{1}{2} \sum_{k \in I \setminus \{i\}} \sigma_{ik}^2 \min\left\{1, \left(\frac{X_k(t)}{X_i(t)}\right)^2\right\} + \frac{1}{2} \sum_{k \in I \setminus \{j\}} \sigma_{jk}^2 \min\left\{1, \left(\frac{X_k(t)}{X_j(t)}\right)^2\right\} \\ = \;& \frac{\sigma_{ij}^2}{2} \left( \min\left\{1, \left(\frac{X_i(t)}{X_j(t)}\right)^2\right\} - \min\left\{1, \left(\frac{X_j(t)}{X_i(t)}\right)^2\right\} \right) \\ & + \frac{1}{2} \sum_{k \in I \setminus \{i,j\}} \left( \sigma_{jk}^2 \min\left\{1, \left(\frac{X_k(t)}{X_j(t)}\right)^2\right\} - \sigma_{ik}^2 \min\left\{1, \left(\frac{X_k(t)}{X_i(t)}\right)^2\right\} \right). \end{aligned} \tag{46}$$

Choose a constant $\epsilon$ such that

$$0 \;<\; \epsilon \;<\; \sqrt{1 - \frac{\sum_{k \in I \setminus \{i,j\}} (\sigma_{ik}^2 - \sigma_{jk}^2)^+}{\sigma_{ij}^2}}.$$

On the set $\{\epsilon X_i(t-1) > X_j(t-1)\} = \{Y_{ij}(t-1) > \log \frac{1}{\epsilon}\}$, the right-hand side of (46) is larger than

$$\begin{aligned} & \frac{\sigma_{ij}^2}{2}\,(1-\epsilon^2) \;+\; \frac{1}{2} \sum_{k \in I \setminus \{i,j\}} (\sigma_{jk}^2 - \sigma_{ik}^2) \min\left\{1, \left(\frac{X_k(t)}{X_i(t)}\right)^2\right\} \\ \geq \;& \frac{\sigma_{ij}^2}{2}\,(1-\epsilon^2) \;-\; \frac{1}{2} \sum_{k \in I \setminus \{i,j\}} (\sigma_{ik}^2 - \sigma_{jk}^2)^+ \\ > \;& 0. \end{aligned}$$

The assumption (40) of Lemma 3.8 is therefore satisfied for $Y_{ij}$.

*Step 2.* We next show (33). It holds that

$$\begin{aligned} & d \log X_i(t) \\ = \;& \frac{d\,X_i(t)}{X_i(t)} - \frac{d\langle X_i(t)\rangle}{2\,X_i^2(t)} \\ = \;& \sum_{j \in I \setminus \{i\}} \sigma_{ij} \min\left\{1, \frac{X_j(t)}{X_i(t)}\right\} dW_{ij}(t) - \frac{1}{2} \sum_{j \in I \setminus \{i\}} \sigma_{ij}^2 \min\left\{1, \left(\frac{X_j(t)}{X_i(t)}\right)^2\right\} dt, \end{aligned}$$

and thus

$$
\begin{aligned}
\log X_i(t) \quad &= \quad \log X_i(0) + \sum_{j \in I \setminus \{i\}} \sigma_{ij} \int_0^t \min\left\{1,\ \frac{X_j(u)}{X_i(u)}\right\} dW_{ij}(u) \\
&\quad\ -\frac{1}{2} \sum_{j \in I \setminus \{i\}} \sigma_{ij}^2 \int_0^t \min\left\{1,\ \Big(\frac{X_j(u)}{X_i(u)}\Big)^2\right\} du \\
&=: \quad \log X_i(0) + N_i(t) + A_i(t). \qquad (47)
\end{aligned}
$$

A time-change technique (see e.g. Section 3.4 B of Karatzas and Shreve [6]) shows that every real-valued continuous martingale $N_i(t)$ converges a.s. as $t \to \infty$ exactly on the event $\{\langle N_i\rangle(\infty) < \infty\}$. The same technique together with the strong law of large numbers for the Wiener process (see e.g. Section 2.9 A of Karatzas and Shreve [6]) yields that $\lim_{t\to\infty} \frac{N_i(t)}{\langle N_i\rangle(t)} = 0$ a.s. on the event $\{\langle N_i\rangle(\infty) = \infty\}$. For the martingale $N_i$ defined in (47), we also have

$$
\langle N_i\rangle(t) = \sum_{j \in I \setminus \{i\}} \sigma_{ij}^2 \int_0^t \min\left\{1,\ \Big(\frac{X_j(u)}{X_i(u)}\Big)^2\right\} du \le \Big(\sum_{j \in I \setminus \{i\}} \sigma_{ij}^2\Big)\, t
$$

and therefore $\lim_{t\to\infty} \frac{N_i(t)}{t} = 0$ a.s. Furthermore, since

$$
\lim_{t\to\infty} \min\left\{1,\ \Big(\frac{X_j(t)}{X_i(t)}\Big)^2\right\} = \mathbb{I}_{\left\{\lim_{t\to\infty} \frac{X_j(t)}{X_i(t)} = \infty\right\}} \quad \text{a.s.,}
$$

the above defined process $A_i$ satisfies

$$
\begin{aligned}
\lim_{t\to\infty} \frac{A_i(t)}{t} \quad &= \quad -\frac{1}{2} \sum_{j \in I \setminus \{i\}} \sigma_{ij}^2\, \mathbb{I}_{\left\{\lim_{t\to\infty} \frac{X_j(t)}{X_i(t)} = \infty\right\}} \\
&= \quad -\frac{1}{2} \sum_{j \in I \setminus \{i\}} \sigma_{ij}^2\, \mathbb{I}_{\left\{\lim_{t\to\infty} \frac{X_i(t)}{X_j(t)} = 0\right\}} \quad \text{a.s.} \quad \square
\end{aligned}
$$

*Proof of Proposition 3.6.* It suffices to show that, for each fixed $i \in I$, the process $Y_i(t) := \log \frac{\widetilde{X}_i(t)}{1-\widetilde{X}_i(t)}$ satisfies the assumptions of **(i)** and **(iii)** of Lemma

3.8. We first note that

$$
\begin{aligned}
& dY_i(t) \\
= \;& d\log \widetilde{X}_i(t) - d\log\left\{1-\widetilde{X}_i(t)\right\} \\
= \;& \alpha\,\frac{\frac{1}{n}-\widetilde{X}_i(t)}{\widetilde{X}_i(t)}\,dt + \sum_{j\in I\backslash\{i\}} \min\left\{1,\,\frac{\widetilde{X}_j(t)}{\widetilde{X}_i(t)}\right\}\big(\sigma_{ij}\,dW_{ij}(t)+\mu_{ij}dt\big) \\
& -\frac{1}{2}\sum_{j\in I\backslash\{i\}} \sigma_{ij}^2\,\min\left\{1,\,\Big(\frac{\widetilde{X}_j(t)}{\widetilde{X}_i(t)}\Big)^2\right\}dt \\
& +\alpha\,\frac{\frac{1}{n}-\widetilde{X}_i(t)}{1-\widetilde{X}_i(t)}\,dt + \sum_{j\in I\backslash\{i\}} \min\left\{\frac{\widetilde{X}_i(t)}{1-\widetilde{X}_i(t)},\,\frac{\widetilde{X}_j(t)}{1-\widetilde{X}_i(t)}\right\}\big(\sigma_{ij}\,dW_{ij}(t)+\mu_{ij}dt\big) \\
& +\frac{1}{2}\sum_{j\in I\backslash\{i\}} \sigma_{ij}^2\,\min\left\{\Big(\frac{\widetilde{X}_i(t)}{1-\widetilde{X}_i(t)}\Big)^2,\,\Big(\frac{\widetilde{X}_j(t)}{1-\widetilde{X}_i(t)}\Big)^2\right\}dt \\
= \;& \alpha\left\{\frac{\frac{1}{n}}{\widetilde{X}_i(t)} - \frac{\frac{n-1}{n}}{1-\widetilde{X}_i(t)}\right\}dt \\
& + \sum_{j\in I\backslash\{i\}}\left[\min\left\{1,\,\frac{\widetilde{X}_j(t)}{\widetilde{X}_i(t)}\right\} + \min\left\{\frac{\widetilde{X}_i(t)}{1-\widetilde{X}_i(t)},\,\frac{\widetilde{X}_j(t)}{1-\widetilde{X}_i(t)}\right\}\right]\big(\sigma_{ij}\,dW_{ij}(t)+\mu_{ij}dt\big) \\
& +\frac{1}{2}\sum_{j\in I\backslash\{i\}}\sigma_{ij}^2\left[\min\left\{\Big(\frac{\widetilde{X}_i(t)}{1-\widetilde{X}_i(t)}\Big)^2,\,\Big(\frac{\widetilde{X}_j(t)}{1-\widetilde{X}_i(t)}\Big)^2\right\} - \min\left\{1,\,\Big(\frac{\widetilde{X}_j(t)}{\widetilde{X}_i(t)}\Big)^2\right\}\right]dt.
\end{aligned}
\tag{48}
$$

Here, in the second and third lines on the right-hand side of (48), the coefficients of $dW$'s and $dt$'s are all bounded, since $\frac{\widetilde{X}_j(t)}{1-\widetilde{X}_i(t)} < 1$ for every $j \in I\backslash\{i\}$. Moreover, the coefficient of the $dW_{ij}(t)$ term is bounded away from zero, which ensures the assumption (37). Indeed, if there exists some $k \in I\backslash\{i\}$ such that $\widetilde{X}_k(t) \geq \widetilde{X}_i(t)$, then

$$
\begin{aligned}
& \sum_{j\in I\backslash\{i\}} \sigma_{ij}\left[\min\left\{1,\,\frac{\widetilde{X}_j(t)}{\widetilde{X}_i(t)}\right\} + \min\left\{\frac{\widetilde{X}_i(t)}{1-\widetilde{X}_i(t)},\,\frac{\widetilde{X}_j(t)}{1-\widetilde{X}_i(t)}\right\}\right] \\
\geq \;& \sigma_{ik}\,\min\left\{1,\,\frac{\widetilde{X}_k(t)}{\widetilde{X}_i(t)}\right\} \geq \min_{j\in I\backslash\{i\}}\sigma_{ij}.
\end{aligned}
$$

If $\widetilde{X}_j(t) < \widetilde{X}_i(t)$ for every $j \in I\backslash\{i\}$, on the other hand, then $\widetilde{X}_i(t) > \frac{1}{n}$ and there exists some $k \in I \backslash \{i\}$ such that $\frac{\widetilde{X}_k(t)}{1-\widetilde{X}_i(t)} \geq \frac{1}{n}$, and thus

$$\begin{aligned} &\sum_{j \in I \backslash \{i\}} \sigma_{ij} \left[ \min\left\{1, \frac{\widetilde{X}_j(t)}{\widetilde{X}_i(t)}\right\} + \min\left\{\frac{\widetilde{X}_i(t)}{1-\widetilde{X}_i(t)}, \frac{\widetilde{X}_j(t)}{1-\widetilde{X}_i(t)}\right\}\right] \\ \geq\ & \sigma_{ik} \min\left\{\frac{\widetilde{X}_i(t)}{1-\widetilde{X}_i(t)}, \frac{\widetilde{X}_k(t)}{1-\widetilde{X}_i(t)}\right\} \geq \frac{1}{n} \min_{j \in I \backslash \{i\}} \sigma_{ij}. \end{aligned}$$

By the first term on the right-hand side of (48), the assumptions (38), (42) and (43) of Lemma 3.8 are satisfied. □

## 3.5 Proof of Lemma 3.8

We first show the assertions for the special case $\sigma(t) \equiv 1$, and after that we prove them for the general case.

*Proof of Lemma 3.8 for the case* $\sigma(t) \equiv 1$. In this special case, the process $Y$ is of the form $Y(t) = Y(0) + W(t) + \int_0^t \mu(s)ds$, where $W$ is a one-dimensional Wiener process with $W(0) = 0$. We divide our argument into four steps.

*Step 1.* We first show that $\mathbb{P}\big[\exists \lim_{t\to\infty} Y(t) \in \mathbb{R}\big] = 0$. It suffices to show that

$$\forall K \in \mathbb{N}, \quad \mathbb{P}\Big[\exists \lim_{t\to\infty} Y(t) \in [-K, K]\Big] = 0. \tag{49}$$

For every $\omega \in \Omega$ such that $\exists \lim_{t\to\infty} Y(t,\omega) \in [-K, K]$, we have

$$\exists s(\omega) \in \mathbb{N}, \quad \forall u \geq s(\omega), \quad \Big(\big|Y(u,\omega)\big| \leq K+1 \ \text{ and } \ \big|Y(u+1,\omega) - Y(u,\omega)\big| \leq 1\Big)$$

and thus $\big|W(u+1,\omega) - W(u,\omega)\big| \leq 1 + \mu_{K+1}^{(abs)}$. It follows that

$$\Big\{\exists \lim_{t\to\infty} Y(t) \in [-K,K]\Big\} \subset \bigcup_{s=1}^{\infty} \bigcap_{u=s}^{\infty} \Big\{\big|W(u+1) - W(u)\big| \leq 1 + \mu_{K+1}^{(abs)}\Big\}.$$

The same argument as (25) in the discrete-time setting yields (49).

*Step 2.* We will prove our assertion (39). First, we note that the assertion is equivalent to

$$\mathbb{P}\left[\Big(\limsup_{t\to\infty} Y(t) = \infty \text{ or } -\infty\Big) \text{ and } \Big(\liminf_{t\to\infty} Y(t) = \infty \text{ or } -\infty\Big)\right] = 1.$$

In what follows we prove $\mathbb{P}\left[\limsup_{t\to\infty} Y(t) = \infty \text{ or } -\infty\right] = 1$. The liminf version can be proved similarly. The same argument as (26) in the discrete-time setting makes it sufficient to show $\mathbb{P}(B_{a,\delta}) = 0$ for every $a \in \mathbb{R}$ and $\delta > 0$, where

$$B_{a,\delta} := \left\{a < \limsup_{t\to\infty} Y(t) < a + \delta \ \text{ and } \ \liminf_{t\to\infty} Y(t) < a - \delta\right\}.$$

Choose three constants $b^{(l)}$, $b^{(m)}$, and $b^{(h)}$ such that

$$a - \delta < b^{(l)} < b^{(m)} < a < a + \delta < b^{(h)},$$

and define recursively the following three sequences of stopping times: for $k \geq 1$,

$$\begin{aligned} \tau_k^{(m)} &:= \inf\left\{ t \geq \tau_{k-1}^{(l)} \,\middle|\, Y(t) > b^{(m)} \right\} \quad \text{where } \tau_0^{(l)} := 0, \\ \tau_k^{(l)} &:= \inf\left\{ t \geq \tau_k^{(m)} \,\middle|\, Y(t) < b^{(l)} \right\}, \\ \tau_k^{(h)} &:= \inf\left\{ t \geq \tau_k^{(m)} \,\middle|\, Y(t) > b^{(h)} \right\}, \end{aligned}$$

where $\inf \emptyset := \infty$. On the event $B_{a,\delta}$, the stopping times $\tau_k^{(m)}$ and $\tau_k^{(l)}$ are finite for every $k \geq 1$ but $\tau_k^{(h)}$ becomes eventually infinite, and thus $\tau_k^{(l)} < \tau_k^{(h)}$ for all sufficiently large $k$. Thus

$$\mathbb{P}(B_{a,\delta}) \;\leq\; \mathbb{P}\big[\, \tau_k^{(l)} < \tau_k^{(h)} \text{ for all sufficiently large } k \,\big].$$

Choose a $K \in \mathbb{N}$ such that $K > \max\left\{ |b^{(h)}|, |b^{(l)}| \right\}$. Then

$$Y(t + \tau_k^{(m)}) - Y(\tau_k^{(m)}) \geq W(t + \tau_k^{(m)}) - W(\tau_k^{(m)}) - \mu_K^{(abs)} t$$

for every $0 \leq t \leq \min\left\{ \tau_k^{(h)}, \tau_k^{(l)} \right\} - \tau_k^{(m)}$. It then follows from Lemma 3.7 that there exists a constant $0 < q < 1$ such that

$$\mathbb{P}\big[\, \tau_k^{(l)} < \tau_k^{(h)} \,\big|\, \mathcal{F}(\tau_k^{(m)}) \,\big] \;<\; q \quad \text{a.s. on} \quad \left\{ \tau_k^{(m)} < \infty \right\}$$

for every $k \geq 1$. The same argument as (25) in the discrete-time setting then yields

$$\mathbb{P}\big[\, \tau_k^{(l)} < \tau_k^{(h)} \text{ for all sufficiently large } k \,\big] \;=\; 0.$$

*Step 3.* With (40) assumed, the assertion (41) can be proved in a similar way as the discrete-time assertion (18). When considering the upper bound on the conditional probability

$$\mathbb{P}\big[\, \widetilde{\tau}_k^{(l)} < \infty \,\big|\, \mathcal{F}(\widetilde{\tau}_k^{(h)}) \,\big],$$

we use Lemma 3.7.

*Step 4.* Finally we prove the assertion (44) by assuming (42) and (43). It suffices to show $\mathbb{P}\left[\lim_{t\to\infty} Y(t) = \infty\right] = 0$ with (42) assumed. We have

$$\begin{aligned} \left\{ \lim_{t\to\infty} Y(t) = \infty \right\} &= \bigcup_{m\in\mathbb{N}} \left\{ \lim_{t\to\infty} Y(t) = \infty \text{ and } \forall t \geq m,\; Y(t) > c_2 \right\} \\ &\subset \bigcup_{m\in\mathbb{N}} \left\{ \lim_{t\to\infty} Y(t) = \infty \text{ and } \forall t \geq m,\; \mu(t) < -\mu_2 \right\} \end{aligned}$$

and

$$\forall m \in \mathbb{N}, \quad \mathbb{P}\Big[ \lim_{t\to\infty} Y(t) = \infty \text{ and } \forall t \geq m,\; \mu(t) < -\mu_2 \Big] = 0$$

by (42) and Lemma 3.7. This completes the proof. $\square$

*Proof of Lemma 3.8 for the general case.* We will use a time-change technique and reduce the problem to the case $\sigma(t) \equiv 1$. Let $\tau(t)$ be the inverse of the strictly increasing process $\langle M \rangle(u) = \int_0^u \sigma^2(s)\, ds$ :

$$\tau(t) := \inf \left\{ u \geq 0 \,\middle|\, \langle M \rangle(u) \geq t \right\}$$

Since we have assumed $\sigma_{min} \leq \sigma(t) \leq \sigma_{max}$, we note that $\tau(t) < \infty$ and $\lim_{t\to\infty} \tau(t) = \lim_{u\to\infty} \langle M \rangle(u) = \infty$. Moreover,

$$\frac{d}{dt} \int_0^{\tau(t)} \mu(s)\, ds \;=\; \frac{d\tau(t)}{dt}\, \mu(\tau(t)) \;=\; \frac{\mu(\tau(t))}{\sigma^2(\tau(t))}.$$

Defining

$$\widehat{Y}(t) := Y(\tau(t)), \qquad \widehat{M}(t) := M(\tau(t)), \qquad \widehat{\mu}(t) := \frac{\mu(\tau(t))}{\sigma^2(\tau(t))},$$

we see that

$$\widehat{Y}(t) = Y(0) + \widehat{M}(t) + \int_0^t \widehat{\mu}(s)\, ds.$$

In addition, $\widehat{M}$ is a Wiener process (see e.g. Section 3.4 B of Karatzas and Shreve [6]), and

$$\frac{\mu(\tau(t))}{\sigma^2_{max}} \leq \widehat{\mu}(t) \leq \frac{\mu(\tau(t))}{\sigma^2_{min}}.$$

It is therefore possible to apply to $\widehat{Y}$ the argument for the case $\sigma(t) \equiv 1$, and $Y$ is its time-changed process. $\square$

# 4 Conclusion

Concerning the yard-sale model of asset exchange, we have given a generalization and a refinement of some results in the literature, in both discrete- and continuous-time settings. We have generalized the wealth concentration result to some possibly unfair games. In a fair case, we have also investigated an asymptotic behavior of every player's wealth process. Wealth disparities grow between every pair of two players, and each player has his/her 'limit rank.' Moreover, in the continuous-time setting we have proved that, with the slightest proportional capital tax, each player's relative wealth process has a completely different asymptotic behavior and oscillates between 0 and 1.

Currently the authors have the following four questions, which we hope will turn into future research topics.

1) To what extent is it possible to generalize the wealth concentration results of Propositions 2.3 and 3.3?

2) In the fair case of Propositions 2.4 and 3.4, what is the probability that each player ends up with a specific limit rank?

3) In the fair case of Propositions 2.4 and 3.4, what is the asymptotic behavior of the players' wealth processes for the general case where there is no restriction on the probabilities $p_{ij}$ or the diffusion coefficients $\sigma_{ij}$. It is the authors' conjecture that each player has his/her limit rank in general.

4) Suppose that the number of players follows a branching process, where no player dies. Then, for each initial player, what is the asymptotic behavior of the total wealth of his/her descendants? The authors conjecture that, if the population growth is fast enough, then the extent of wealth disparities is kept limited.

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