On Stable and Strategy-Proof Rules in Matching Markets with Contracts

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Abstract

This paper studies stable and (one-sided) strategy-proof rules in many-to-one matching markets with contracts. Not assuming any kind of substitutes condition or the law of aggregate demand, we obtain the following results. First, the number of stable and strategy-proof rules is at most one. Second, the doctor-optimal stable rule, whenever it exists, is the unique candidate for a stable and strategy-proof rule. Third, a stable and strategy-proof rule, whenever it exists, is second-best optimal for doctor welfare, in that no individually rational and strategy-proof rule can dominate it. This last result is further generalized to non-wasteful and strategy-proof rules. Due to the weak assumptions, our analysis covers a broad range of markets, including cases where a (unique) stable and strategy-proof rule is not equal to the one induced by the cumulative offer process or the deferred acceptance algorithm.

Keywords: matching with contracts, stability, strategy-proofness, uniqueness, efficiency, irrelevance of rejected contracts

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1 Introduction

Stability and (one-sided) strategy-proofness are two leading desiderata in two-sided matching market design.[1][2] In the classic setup, it is well-known that a matching rule is stable and strategy-proof if and only if it is the one induced by the *deferred acceptance algorithm* (Gale and Shapley, 1962; Dubins and Freedman, 1981; Roth, 1982; Alcalde and Barberà, 1994). The same result holds true in the generalized *matching with contracts* model, as long as hospitals' choice functions satisfy the *substitutes* condition and the *law of aggregate demand*.[3] With these conditions, Hatfield and Milgrom (2005) verify, among many other things, that the deferred acceptance rule is stable and strategy-proof, and Sakai (2011) further shows that no other rule satisfies both desiderata. Recently, however, several real-world markets that violate the substitutes condition (and the law of aggregate demand) have been found.[4]

To cover such a broader range of markets, we study stable and strategy-proof rules with the only assumption that the choice functions on the hospital side satisfy a common mild requirement, called the *irrelevance of rejected contracts* (henceforth, IRC) condition. This rationality condition requires that if a contract is not chosen from a menu, removing it from the menu should not change the chosen set. It is logically independent of the substitutes condition and the law of aggregate demand, and is (implicitly) assumed

[1]Throughout the paper, we refer to one side of the market as doctors and the other as hospitals, whereas applications of two-sided matching theory are not restricted to medical matches.

[2]It is common in the literature to impose strategy-proofness only for the doctor side, partly because strategy-proofness for both sides is incompatible with stability on the full domain of admissible preferences (Roth, 1982). While the present study also investigates one-sided strategy-proof rules, an alternative approach is to study two-sided strategy-proofness on restricted domains (e.g., Alcalde and Barberà, 1994, Sections 4–5; Sönmez, 1999). See also Section 4.2 for further discussion on the role of the preference domain in the present study.

[3]The substitutes condition requires that if a contract is chosen from a menu, it should be also chosen when other contracts are removed from the menu. The law of aggregate demand requires that the number of chosen contracts be weakly greater when the menu enlarges in the set sense.

[4]The examples include cadet-branch matching in the U.S. Army (Sönmez, 2013; Sönmez and Switzer, 2013), affirmative actions in school choice programs and college admissions (Aygün and Turhan, 2016; Kominers and Sönmez, 2016), and lawyer-court matching in Germany (Dimakopoulos and Heller, 2014).

throughout the literature.[5]

Only with this assumption, we obtain the following results. Theorems 1–2 are on the uniqueness of stable and strategy-proof rules and extend the existing results mentioned above: Theorem 1 states that the number of such rules is at most one, although there may or may not exist one without additional restrictions; and Theorem 2 establishes that the doctor-optimal stable rule is the unique candidate for a stable and strategy-proof rule, whenever it is well-defined, although it may or may not be strategy-proof without additional assumptions. Theorem 3 is on the constrained optimality of a stable and strategy-proof rule. Namely, we show that a stable and strategy-proof rule, if it exists, is never dominated in terms of doctor welfare by any other individually rational and strategy-proof rule. Furthermore, Theorem 4 shows that the same holds true even if stability is weakened to non-wastefulness in the above statement. These latter two theorems generalize similar existing results in the school choice literature (e.g., Abdulkadiroglu et al., 2009; Kesten, 2010; Kesten and Kurino, 2016).

Our approach is novel in the study of matching with contracts without the substitutes condition. The common approach in the literature is, as in Hatfield and Kojima (2010), to introduce weaker conditions of substitutability and investigate the performance of the deferred acceptance algorithm or its variant called the *cumulative offer process*. In contrast, we do not rely on any substitutes condition or the algorithmic properties of a specific rule. As a consequence, a major advantage of our approach is broader applicability. Particularly, our analysis covers the following cases where standard results in the literature fail to hold due to lack of the substitutes condition:

- cases where a (unique) stable and strategy-proof rule exists but differs from the ones induced by the deferred acceptance algorithm and the cumulative offer process;

[5] Aygün and Sönmez (2012, 2013) point out the importance of this condition, which is implicitly assumed in Hatfield and Milgrom (2005) and Hatfield and Kojima (2010).

- cases where the set of stable allocations does not have a lattice structure and/or the doctor-optimal stable allocation does not exist;
- cases where the "rural hospital theorem" fails to hold (i.e., cases where a doctor is matched at one stable allocation but not at another); and
- cases where "embedding" into a simpler model of matching with salaries à la Kelso and Crawford (1982) is impossible.[6]

Relatedly, our weak assumption also necessitates new analytical techniques, because we cannot exploit the above-mentioned standard results even though some of them have been often useful in the related literature. We thus develop a new proof technique that only relies on stability and strategy-proofness (as well as the IRC condition), thereby revealing the direct link between the two desiderata and our conclusions. The transparency of the logic could be another merit of our general approach.

The rest of this paper is organized as follows. Section 2 describes the model and introduces key concepts. Section 3 presents the main results. Section 4 further discusses our approach. Appendix A provides additional examples.

2 Preliminaries

We study the standard setting of a many-to-one matching market with contracts. Let D and H be finite sets of doctors and hospitals, respectively. The finite set of possible contracts is given by $X \subset D \times H \times \Theta$ for some finite Θ.[7] For each contract $x \in X$, let $\mathrm{d}(x)$ and $\mathrm{h}(x)$ be its projections onto D and H, i.e., $x = (\mathrm{d}(x), \mathrm{h}(x), \theta)$ for some $\theta \in \Theta$. In

[6] Under the substitutes condition, Echenique (2012) establishes a one-to-one correspondence between a market with contracts and a market with salaries where doctors are gross substitutes in the standard sense. When such "embedding" is possible, some of the results in the market with contracts would directly follow their counterparts in the market with salaries. See also Schlegel (2015) for the possibility of embedding with weaker substitutes conditions.

[7] For example, Θ can be interpreted as the set of discretized wage levels (Kelso and Crawford, 1982) and/or job descriptions (Roth, 1984).

other words, x is a bilateral contract between $\mathrm{d}(x) \in D$ and $\mathrm{h}(x) \in H$.

A subset $X' \subset X$ of contracts is said to be an *allocation* if it includes at most one contract for each doctor, i.e., if $x, x' \in X'$ and $x \neq x'$ imply $\mathrm{d}(x) \neq \mathrm{d}(x')$.[8] The set of all possible allocations is denoted by $\mathscr{X} \subset 2^X$. For each allocation $X' \in \mathscr{X}$ and doctor $d \in D$, let $\mathrm{x}(d, X')$ denote the contract that X' assigns to d; i.e., $\mathrm{x}(d, X') = x$ if $x \in X'$ and $\mathrm{d}(x) = d$. If there is no such contract in X', doctor d is said to be assigned a *null-contract* and we write $\mathrm{x}(d, X') = \varnothing$. Similarly, let $\mathrm{X}(h, X') = \{x \in X' : \mathrm{h}(x) = h\}$ be the set of (non-null) contracts that X' assigns to hospital $h \in H$.

Each doctor $d \in D$ has a strict preference relation $\succ_d$ over $\{x \in X : \mathrm{d}(x) = d\} \cup \{\varnothing\}$. The domain of all possible preferences for doctor d is denoted by $\mathscr{P}_d$. Given his preference relation $\succ_d$, a non-null contract x is said to be *acceptable* to doctor d if $x \succ_d \varnothing$.[9] The set of acceptable contracts to doctor d, as a function of $\succ_d$, is given by $\mathrm{Ac}(\succ_d) := \{x \in X : x \succ_d \varnothing\}$. The profile of the doctors' preference relations is denoted by $\succ_D = (\succ_d)_{d \in D}$. Let $\mathscr{P}_D := \prod_{d \in D} \mathscr{P}_d$ be the domain of all possible preference profiles. Each hospital $h \in H$ has a choice function $C_h : 2^X \to \mathscr{X}$ such that for all $X' \subset X$, (i) $C_h(X') \in 2^{X'} \cap \mathscr{X}$ and (ii) $\mathrm{h}(x) = h$ for all $x \in C_h(X')$. Throughout the paper, except for Section 4.2, we assume that the choice functions satisfy the following mild requirement: Hospital h's choice function $C_h(\cdot)$ is said to satisfy the *irrelevance of rejected contracts* (henceforth, IRC) condition if $x \notin C_h(X' \cup \{x\})$ implies $C_h(X' \cup \{x\}) = C_h(X')$ for all $X' \subset X$ and $x \in X$.[10] The profile of the hospitals' choice functions is denoted by $C_H(\cdot) = (C_h(\cdot))_{h \in H}$.

Given $\succ_D$ and $C_H(\cdot)$, we define the following concepts on $\mathscr{X}$: An allocation $X' \in \mathscr{X}$ is said to be *individually rational* if (i) $\mathrm{x}(d, X') \succeq_d \varnothing$ for all $d \in D$, and (ii) $C_h(X') = \mathrm{X}(h, X')$ for all $h \in H$. A pair of a hospital $h \in H$ and a subset $X'' \subset X$ of contracts is said to *block*

[8] As a contract is a tuple (d, h, θ), this definition is parallel to defining a matching as a set of married couples in the marriage market.

[9] Note that $x \succ_d \varnothing$ implies $\mathrm{d}(x) = d$ since $\succ_d$ is defined over $\{x \in X : \mathrm{d}(x) = d\} \cup \{\varnothing\}$.

[10] Note that this condition is satisfied if a choice function is induced by a strict preference over subsets of contracts (as in most of the examples below).

an allocation X' if (i) $C_h(X' \cup X'') = X'' \neq C_h(X')$ and (ii) $\mathrm{x}(d, C_h(X' \cup X'')) \succeq_d \mathrm{x}(d, X')$ for all $d \in \{\mathrm{d}(x)\}_{x \in C_h(X' \cup X'')}$. An allocation X' is said to be *stable* if it is individually rational and not blocked by any $(h, X'') \in H \times 2^X$. An allocation X' is said to *strictly dominate* another allocation $X'' \neq X'$ if $\mathrm{x}(d, X') \succeq_d \mathrm{x}(d, X'')$ for all $d \in D$.[11] A stable allocation X^* is said to be *doctor-optimal* if it strictly dominates any other stable allocation.

Given $C_H(\cdot)$ as well as (D, H, X), a matching *rule* is a mapping $f : \mathscr{P}_D \to \mathscr{X}$, which associates each possible preference profile of doctors with an allocation. A rule $f(\cdot)$ is said to be *stable* (resp. *individually rational*) if for all $\succ_D \in \mathscr{P}_D$, its output $f(\succ_D)$ is stable (resp. individually rational) with respect to $(C_H(\cdot), \succ_D)$. Similarly, the *doctor-optimal stable rule*, denoted by $X^*(\cdot)$ if it exists, is a rule such that for all $\succ_D \in \mathscr{P}_D$, its output $X^*(\succ_D)$ is the doctor-optimal stable allocation with respect to $(C_H(\cdot), \succ_D)$. A rule $f(\cdot)$ is said to *strictly dominate* another rule $g(\cdot) \neq f(\cdot)$ if $\mathrm{x}(d, f(\succ_D)) \succeq_d \mathrm{x}(d, g(\succ_D))$ for all $d \in D$ and $\succ_D \in \mathscr{P}_D$.[12] Finally, a rule $f(\cdot)$ is said to be *strategy-proof* if $\mathrm{x}(d, f(\succ_D)) \succeq_d \mathrm{x}\left(d, f\left(\succ'_d, \succ_{D-\{d\}}\right)\right)$ for all $d \in D$, $\succ_D \in \mathscr{P}_D$, and $\succ'_d \in \mathscr{P}_d$, where $\succ_{D-\{d\}} = (\succ_{d'})_{d' \in D-\{d\}}$.

3 Results

To start our analysis, we introduce the following weaker notion of blocking coalitions: We say that a pair $(h, X'') \in H \times 2^X$ *weakly blocks* an allocation X' if (i) $C_h(X' \cup X'') \neq C_h(X')$ and (ii) $\mathrm{x}(d, C_h(X' \cup X'')) \succeq_d \mathrm{x}(d, X')$ for all $d \in \{\mathrm{d}(x)\}_{x \in C_h(X' \cup X'')}$. This definition is weak in that the first part does not require $C_h(X' \cup X'') = X''$. Under the IRC condition, however, it is straightforward to verify that the two blocking concepts are equally effective in the following sense.

Lemma 1. *Suppose that hospital h's choice function $C_h(\cdot)$ satisfies the IRC condition. For any*

[11]Note that $X' \neq X''$ and $\mathrm{x}(d, X') \succeq_d \mathrm{x}(d, X'')$ for all d imply $\mathrm{x}(d', X') \succ_{d'} \mathrm{x}(d', X'')$ for some d'.

[12]Again, $g(\cdot) \neq f(\cdot)$ implies $\mathrm{x}(d', f(\succ_D)) \succ_{d'} \mathrm{x}(d', g(\succ_D))$ for some d' and $\succ_D$ if $f(\cdot)$ strictly dominates $g(\cdot)$.

allocation $X' \in \mathscr{X}$, then, there exists $X'' \subset X$ such that (h, X'') blocks X' if and only if there exists $X''' \subset X$ such that (h, X''') weakly blocks X'.

Proof. The "only if" part is immediate from the definitions. To see the "if" part, suppose that (h, X''') weakly blocks X', and let $X'' := C_h(X' \cup X''')$. Then, the IRC condition implies $C_h(X' \cup X'') = C_h(X' \cup X''') = X''$ and hence, the first requirement for (h, X'') to block X' is satisfied. The second requirement is also trivially satisfied by the assumption that (h, X''') weakly blocks X'. ∎

Lemma 1 leads to the following observation, which will be the key in the proofs of Theorems 1–2.

Lemma 2. *Suppose that every hospital $h \in H$ has a choice function $C_h(\cdot)$ satisfying the IRC condition, and that X' and X'' are two distinct stable allocations at $(C_H(\cdot), \succ_D)$. Then, there exists a doctor $d \in D$ who is assigned distinct non-null contracts by X' and X'', i.e., $\varnothing \neq \mathrm{x}(d, X') \neq \mathrm{x}(d, X'') \neq \varnothing$.*

Proof. The proof is by contraposition. Assume the negation of the consequent, i.e.,

$$\left[\mathrm{x}\left(d, X'\right) \neq \mathrm{x}\left(d, X''\right)\right] \Longrightarrow \left[\varnothing \in \left\{\mathrm{x}\left(d, X'\right), \mathrm{x}\left(d, X''\right)\right\}\right], \text{ for all } d \in D,$$

where X' and X'' are two (possibly identical) stable allocations at $(C_H(\cdot), \succ_D)$. Since X'' is stable (and thus individually rational), this implies for all $d \in D$,

$$\left[\mathrm{x}\left(d, X''\right) \neq \varnothing\right] \Longrightarrow \left[\mathrm{x}\left(d, X''\right) \succ_d \varnothing = \mathrm{x}\left(d, X'\right) \text{ or } \mathrm{x}\left(d, X'\right) = \mathrm{x}\left(d, X''\right)\right],$$

and hence,

$$\left[\mathrm{x}\left(d, X''\right) \neq \varnothing\right] \Longrightarrow \left[\mathrm{x}\left(d, X''\right) \succeq_d \mathrm{x}\left(d, X'\right)\right].$$

For an arbitrary hospital $h \in H$, then, (h, X'') satisfies the second requirement to weakly block X'. Since (h, X'') cannot weakly block X' by stability and Lemma 1, it must violate the first requirement; i.e., $C_h(X' \cup X'') = \mathrm{X}(h, X')$ must hold. As the symmetric arguments also imply $C_h(X' \cup X'') = \mathrm{X}(h, X'')$ for all $h \in H$, it follows that $\mathrm{X}(h, X') = \mathrm{X}(h, X'')$ for all $h \in H$ and thus $X' = X''$. ■

Our first main result generalizes the existing results on the uniqueness of a stable and strategy-proof rule by Alcalde and Barberà (1994, Theorem 3) and Sakai (2011, Theorem 1). While this theorem does not require any substitutes condition or the law of aggregate demand, its proof depends on Lemma 2, which in turn necessitates the IRC condition. See Example 3 in Appendix A for a counterexample in the absence of the IRC condition.

Theorem 1. *Suppose that every hospital $h \in H$ has a choice function $C_h(\cdot)$ satisfying the IRC condition. Then, there exists at most one stable and strategy-proof rule; i.e., if $f(\cdot)$ and $g(\cdot)$ are both stable and strategy-proof, $f(\succ_D) = g(\succ_D)$ for all $\succ_D \in \mathscr{P}_D$.*

Proof. Towards a contradiction, suppose that there exist two distinct stable and strategy-proof rules, $f(\cdot)$ and $g(\cdot)$. Among preference profiles satisfying $f(\succ_D) \neq g(\succ_D)$, let $\succ_D^*$ be a "minimal" one in terms of the total number of acceptable contracts across doctors; that is, $f(\succ_D^*) \neq g(\succ_D^*)$ and

$$\left[f(\succ_D) \neq g(\succ_D) \Longrightarrow \sum_{d \in D} |\mathrm{Ac}(\succ_d)| \geq \sum_{d \in D} |\mathrm{Ac}(\succ_d^*)| \right] \text{ for all } \succ_D \in \mathscr{P}_D.$$ [13]

Then, by Lemma 2, there must exist a doctor d^* such that $\varnothing \neq \mathrm{x}(d^*, f(\succ_D^*)) \neq \mathrm{x}(d^*, g(\succ_D^*)) \neq \varnothing$. Note that this also implies $|\mathrm{Ac}(\succ_{d^*}^*)| \geq 2$.

Now, suppose without loss of generality that $\mathrm{x}(d^*, f(\succ_D^*)) \succ_{d^*}^* \mathrm{x}(d^*, g(\succ_D^*))$, and let $\succ_D^{**} := \left(\succ_{d^*}^{**}, \succ_{D-\{d^*\}}^*\right)$, where $\succ_{d^*}^{**}$ is a preference relation of doctor d^* such that only

[13] Note that $\succ_D^*$ exists because X is assumed to be finite and thus so is $\mathscr{P}_D$.

$\mathrm{x}\left(d^*, f(\succ^*_D)\right)$ is acceptable, i.e., $\mathrm{Ac}(\succ^{**}_{d^*}) = \{\mathrm{x}\left(d^*, f(\succ^*_D)\right)\}$. Notice that

$$\mathrm{x}\left(d^*, f(\succ^{**}_D)\right), \mathrm{x}\left(d^*, g(\succ^{**}_D)\right) \in \Big\{\mathrm{x}\left(d^*, f(\succ^*_D)\right), \varnothing\Big\},$$

since $f(\cdot)$ and $g(\cdot)$ are assumed to be stable (and hence individually rational). Then, the strategy-proofness of $f(\cdot)$ and $g(\cdot)$ implies

$$\begin{aligned}
&\mathrm{x}\left(d^*, f(\succ^*_D)\right) = \mathrm{x}\left(d^*, f(\succ^{**}_D)\right) \neq \varnothing, \text{ and} \\
&\mathrm{x}\left(d^*, g(\succ^*_D)\right) \neq \mathrm{x}\left(d^*, g(\succ^{**}_D)\right) = \varnothing,
\end{aligned}$$

respectively, and hence, $f(\succ^{**}_D) \neq g(\succ^{**}_D)$. This, however, contradicts the definition of $\succ^*_D$, since $\left|\mathrm{Ac}\left(\succ^{**}_{d^*}\right)\right| = 1 < 2 \leq \left|\mathrm{Ac}(\succ^*_{d^*})\right|$ and $\succ^{**}_{D-\{d^*\}} = \succ^*_{D-\{d^*\}}$, and the proof is complete. ■

Following the same line of proof, we can also show that whenever it exists, the doctor-optimal stable rule is the unique candidate for a stable and strategy-proof rule. Note, however, that this unique candidate may or may not be strategy-proof without additional assumptions.

Theorem 2. *Suppose that every hospital $h \in H$ has a choice function $C_h(\cdot)$ satisfying the IRC condition, and that the doctor-optimal stable allocation $X^*(\succ_D)$ exists for all $\succ_D \in \mathscr{P}_D$. If $f(\cdot)$ is a stable and strategy-proof rule, then, $f(\succ_D) = X^*(\succ_D)$ for all $\succ_D \in \mathscr{P}_D$.*

Proof. Towards a contradiction, suppose that the doctor-optimal stable rule $X^*(\cdot)$ is well-defined, and that $f(\cdot) \neq X^*(\cdot)$ is a stable and strategy-proof rule. As in the proof of Theorem 1, let $\succ^*_D \in \mathscr{P}_D$ be a "minimal" preference profile, which exists by assumption, such that $f\left(\succ_D^*\right) \neq X^*\left(\succ_D^*\right)$ and

$$\left[f\left(\succ_D\right) \neq X^*\left(\succ_D\right) \Longrightarrow \sum_{d\in D} |\mathrm{Ac}(\succ_d)| \geq \sum_{d\in D} |\mathrm{Ac}(\succ_d^*)|\right] \text{ for all } \succ_D \in \mathscr{P}_D.$$

Then, by Lemma 2, there must exist a doctor d^* such that $\varnothing \neq \mathrm{x}\left(d^*, f(\succ_D^*)\right) \neq \mathrm{x}\left(d^*, X^*(\succ_D^*)\right) \neq \varnothing$. Note that this also implies $\mathrm{x}\left(d^*, X^*(\succ_D^*)\right) \succ_{d^*}^* \mathrm{x}\left(d^*, f(\succ_D^*)\right) \succ_{d^*}^* \varnothing$.

Now let $\succ_D^{**} := \left(\succ_{d^*}^{**}, \succ_{D-\{d^*\}}^*\right)$, where $\succ_{d^*}^{**}$ is a truncation of $\succ_{d^*}^*$ at $\mathrm{x}\left(d^*, X^*(\succ_D^*)\right)$, i.e., a preference relation such that

$$\mathrm{Ac}\left(\succ_{d^*}^{**}\right) = \left\{x \in X : x \succeq_{d^*}^* \mathrm{x}\left(d^*, X^*(\succ_D^*)\right)\right\},$$

and

$$[x \succ_{d^*}^{**} y \Longleftrightarrow x \succ_{d^*}^* y] \text{ for all } x, y \in \mathrm{Ac}(\succ_{d^*}^{**}).$$

Notice that $X^*(\succ_D^{**}) = X^*(\succ_D^*)$ by construction.[14] Together with the strategy-proofness of $f(\cdot)$, this further implies

$$\mathrm{x}\left(d^*, X^*(\succ_D^{**})\right) = \mathrm{x}\left(d^*, X^*(\succ_D^*)\right) \succ_{d^*}^* \mathrm{x}\left(d^*, f(\succ_D^*)\right) \succeq_{d^*}^* \mathrm{x}\left(d^*, f(\succ_D^{**})\right),$$

and hence, $f(\succ_D^{**}) \neq X^*(\succ_D^{**})$. This, however, contradicts the definition of $\succ_D^*$, since $\mathrm{Ac}\left(\succ_{d^*}^{**}\right) \subset \mathrm{Ac}\left(\succ_{d^*}^*\right) - \left\{\mathrm{x}\left(d^*, f(\succ_D^*)\right)\right\}$ and $\succ_{D-\{d^*\}}^{**} = \succ_{D-\{d^*\}}^*$, and the proof is complete. ■

Compared to the existing uniqueness results, Theorems 1–2 above are technically

[14] To see this, note first that $X^*(\succ_D^*)$ is also stable at $\succ_D^{**}$. For $X^*(\succ_D^{**}) \neq X^*(\succ_D^*)$ to hold, therefore, $X^*(\succ_D^{**})$ cannot be stable at $\succ_D^*$. For some (h, X') to block $X^*(\succ_D^{**})$ at $\succ_D^*$ but not at $\succ_D^{**}$, however, $\mathrm{x}\left(d^*, X^*(\succ_D^{**})\right) = \varnothing$ must hold. Since $\mathrm{x}\left(d^*, X^*(\succ_D^*)\right) \succ_{d^*}^{**} \varnothing$, this means that $X^*(\succ_D^{**})$ cannot dominate $X^*(\succ_D^*)$ at $\succ_D^{**}$, which is a contradiction.

novel for two related reasons. First, our proof of Theorem 1 requires no dominance relation between $f(\cdot)$ and $g(\cdot)$. Consequently, it is applicable even when the doctor-optimal stable allocation does not always exist. In contrast, the uniqueness results by Alcalde and Barberà (1994, Theorem 3) and Sakai (2011, Theorem 1) are established by showing that any stable rule that is strictly dominated by the doctor-optimal stable rule cannot be strategy-proof, and hence, the existence of the doctor-optimal stable rule is critical in their proofs. Second, our proofs do not call for the *rural hospital theorem* either, which states that every agent (i.e., every doctor and every hospital) signs the same number of non-null contracts across all stable allocations. Instead we utilize Lemma 2, which could be seen as a weaker version of the rural hospital theorem but holds true without any restrictions on $C_H(\cdot)$ other than the IRC condition.[15] It is this distinction that makes the proof of Theorem 2 non-trivial, although its statement might look very close to the previous results.

A natural question that stems from Theorem 2 would be whether or not we can replace doctor-optimal stability with a weaker requirement that is applicable even when the doctor optimal stable rule is not well-defined. Specifically, one might wonder if a stable and strategy-proof rule always chooses an allocation that is not dominated by another stable allocation. Actually, the answer to this question is known to be negative.[16]

Fact 1 (Kominers and Sönmez, 2016). *A stable and strategy-proof rule may choose an allocation that is strictly dominated by another stable allocation.*[17]

Given that the outcomes of a stable and strategy-proof rule may be dominated even among stable allocations, it could be of policy interest whether the doctor welfare can be Pareto-improved. Since Theorem 1 implies that such improvement is impossible main-

[15]Note that the conclusion of Lemma 2 immediately follows if the rural hospital theorem holds.

[16]In matching markets without contracts, contrastingly, Pathak and Sönmez (2013, Lemma 1) establish that the dominance in terms of outcomes between two stable rules implies the dominance in terms of manipulability. For an extension of this result, see also Chen et al. (2016).

[17]See Theorem 3 and Example 4 of Kominers and Sönmez (2016).

taining both stability and strategy-proofness, it would be natural to ask if it becomes possible once we weaken the stability requirement. Our next main result, Theorem 3, shows that such improvement is generally impossible. This extends the existing results in the school choice literature that the student-optimal stable rule is second-best optimal among strategy-proof rules (see, Abdulkadiroglu et al., 2009; Kesten, 2010; Kesten and Kurino, 2016).[18]

Theorem 3. *Suppose that every hospital $h \in H$ has a choice function $C_h(\cdot)$ satisfying the IRC condition. Then, no individually rational and strategy-proof rule strictly dominates a stable and strategy-proof rule.*

Proof. Towards a contradiction, suppose that $f(\cdot)$ is individually rational and strategy-proof, $g(\cdot)$ is stable and strategy-proof, and that $f(\cdot)$ strictly dominates $g(\cdot)$. As in the proofs of Theorems 1–2, let $\succ_D^* \in \mathscr{P}_D$ be a preference profile, which exists by assumption, such that $f\left(\succ_D^*\right) \neq g\left(\succ_D^*\right)$ and

$$\left[f\left(\succ_D\right) \neq g\left(\succ_D\right) \Longrightarrow \sum_{d \in D} |\mathrm{Ac}(\succ_d)| \geq \sum_{d \in D} |\mathrm{Ac}(\succ_d^*)|\right] \text{ for all } \succ_D \in \mathscr{P}_D.$$

Then, there must exist $d^* \in D$ such that

$$\mathrm{x}\left(d^*, f(\succ_D^*)\right) \succ_{d^*}^* \mathrm{x}\left(d^*, g(\succ_D^*)\right) \succ_{d^*}^* \varnothing. \tag{$*$}$$

To see this, suppose contrarily that for all $d \in D$, $\mathrm{x}\left(d, f(\succ_D^*)\right) \succ_d^* \mathrm{x}\left(d, g(\succ_D^*)\right)$ implies $\mathrm{x}\left(d, g(\succ_D^*)\right) = \varnothing$. This entails $f(\succ_D^*) \supsetneq g(\succ_D^*)$ and hence, for some $h \in H$,

$$\begin{aligned} C_h\left(f(\succ_D^*) \cup g(\succ_D^*)\right) = C_h\left(f(\succ_D^*)\right) &= \mathrm{X}\left(h, f(\succ_D^*)\right) \\ &\neq \mathrm{X}\left(h, g(\succ_D^*)\right) = C_h\left(g(\succ_D^*)\right), \end{aligned}$$

[18] See also Anno and Kurino (2016) and Erdil (2014) for related results.

where the second and last equalities hold by the individual rationality of $f(\cdot)$ and $g(\cdot)$, respectively. Therefore, $(h, f(\succ^*_D))$ weakly blocks $g(\succ^*_D)$, but by Lemma 1, this contradicts the stability of $g(\cdot)$.

Now, take a new preference relation $\succ^{**}_{d^*}$ of d^* such that $\mathrm{Ac}(\succ^{**}_{d^*}) = \{\mathrm{x}(d^*, f(\succ^*_D))\}$, and let $\succ^{**}_D := \left(\succ^{**}_{d^*}, \succ^*_{D-\{d^*\}}\right)$. Then, the strategy-proofness of $f(\cdot)$ and $g(\cdot)$, along with equation $(*)$, implies

$$\mathrm{x}(d^*, f(\succ^{**}_D)) = \mathrm{x}(d^*, f(\succ^*_D)) \succ^*_{d^*} \mathrm{x}(d^*, g(\succ^*_D)) \succ^*_{d^*} \mathrm{x}(d^*, g(\succ^{**}_D)) = \varnothing,$$

and thus, $f(\succ^{**}_D) \neq g(\succ^{**}_D)$. However, this contradicts the definition of $\succ^*_D$ since $\left|\mathrm{Ac}(\succ^{**}_{d^*})\right| < \left|\mathrm{Ac}(\succ^*_{d^*})\right|$ and $\succ^{**}_{D-\{d^*\}} = \succ^*_{D-\{d^*\}}$, and the proof is complete. ■

In Theorem 3, the individual rationality of the dominating rule is indispensable. To see this, note that the rule that always assigns each doctor his first-best contract is strategy-proof and dominates any other rule. In the classic matching and assignment models, such "first-best" rules are usually precluded by the quotas of hospitals. While we do not have the exact counterpart of quotas in the current framework, it is the requirement of $C_h(X') = \mathrm{X}(h, X')$ that plays a similar role. In other words, the individual rationality of the hospital side encompasses what is usually referred to as the feasibility constraint.

4 Discussion

4.1 Applicability of Our Results

This paper studies the model of many-to-one matching with contracts, and derives a number of properties that a stable and strategy-proof rule must generally satisfy. A notable feature of our approach is that we only impose a minimal structure on hospitals' choice functions, i.e., the IRC condition, and do not rely on any algorithmic properties of a matching rule. This generality allows us to capture the joint implications of stability and strategy-proofness transparently, making our results widely applicable. Particularly, our results are applicable even if a stable and strategy-proof rule is not equal to the ones induced by the cumulative offer process and the deferred acceptance algorithm, and such markets actually exist as we exemplify below. To formally state our claim, we now define the two algorithms with our notation.

Definition 1. Given $(C_H(\cdot), \succ_D)$, the *cumulative offer process* proceeds as follows.[19]

- Initial condition: Let $D_0 = D$ and $M_0 = \emptyset$.
- Step $t \geq 1$: An arbitrarily chosen doctor $d_t \in D_{t-1}$ offers his best contract, $x_t^{d_t}$, among those remaining (i.e., among $X - \bigcup_{\tau=1}^{t-1}\{x_\tau^{d_\tau}\}$). Let $M_t = M_{t-1} \cup \{x_t^{d_t}\}$ be the menu of contracts that have been offered up to this step. Among M_t, each hospital h holds the best combination of contracts, $C_h(M_t)$. Finally, let D_t be the set of doctors for whom (i) no contract is currently held by any hospital and (ii) not all acceptable contracts have been offered yet, i.e.,

$$D_t = \left\{ d \in D : \left[d \notin \left\{ \mathrm{d}\left(x'\right) \right\}_{x' \in C_h(M_t)} \text{ for all } h \in H \right] \text{ and } \left[\mathrm{Ac}(\succ_d) - \bigcup_{\tau=1}^{t} \left\{ x_\tau^{d_\tau} \right\} \neq \emptyset \right] \right\}.$$

  Proceed to step $t+1$ if D_t is non-empty and terminate otherwise.
- Outcome: When the process terminates at step T, its outcome is $\bigcup\limits_{h \in H} C_h(M_T)$.

The *deferred acceptance algorithm* is almost the same as the cumulative offer process, with the only difference being $M_t = \left[\bigcup\limits_{h \in H} C_h(M_{t-1}) \right] \cup \{x_t^{d_t}\}$. That is, at each step of the deferred acceptance algorithm, hospitals can choose only from the newly offered contract

[19] The following is based on the definition by Hatfield and Kojima (2010), which is commonly used in the literature. Strictly speaking, it defines a class of algorithms rather than a single algorithm, because it does not specify how d_t is chosen from D_{t-1}. The algorithms in this class are not outcome-equivalent in general, although they are under certain conditions (Hatfield and Kominers, 2014; Hatfield et al., 2015; Hirata and Kasuya, 2014; Kominers and Sönmez, 2016). For further properties of the cumulative offer process, see also Afacan (2014, 2016).

and those currently being held, but not from those rejected in previous steps. □

Fact 2. *There exists a market (i.e., $(D, H, X, C_H(\cdot))$) where a stable and strategy-proof rule exists but is not induced by the cumulative offer process or the deferred acceptance algorithm.*[20]

Proof. The proof is by example. See Example 1 in Section 4.1.1. ■

This fact would highlight the distinction between the scope of the present paper and of Hatfield et al. (2015), who identify a sufficient condition and an almost necessary condition for the cumulative offer process to be stable and strategy-proof. That is, our results cover a strictly larger domain, whereas those of Hatfield et al. (2015) are stronger in identifying the exact form of a stable and strategy-proof rule.

For another instance, our results are also applicable even if the rural hospital theorem does not hold. Given the arguments presented after Theorem 2, this would be particularly important for the relevance of Theorem 2. That is, the question is if the doctor-optimal stable rule can be strategy-proof even when the rural hospital theorem fails to hold. If there is no such case, Theorem 2 would boil down to previous results and Theorem 1, since the doctor-optimal stable rule can be shown to be strategy-proof whenever the rural hospital theorem holds.[21] In fact, there exist such markets and hence, Theorem 2 applies to a strictly larger domain of choice functions than the previous results.

Fact 3. *There exists a market (i.e., $(D, H, X, C_H(\cdot))$) where the doctor-optimal stable rule exists and is strategy-proof whereas the rural hospital theorem does not hold for all $\succ_D$.*

Proof. The proof is by example. See Example 2 in Section 4.1.1. ■

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[20] More precisely, what we illustrate below is a stable and strategy-proof rule that is not induced by *any* cumulative offer process or deferred acceptance algorithm, even though we can define a class of (non-outcome-equivalent) cumulative offer process*es* and of deferred acceptance algorithm*s*. See also footnote 19 above.

[21] See the proofs of Hatfield and Milgrom (2005, Theorem 11) and Hatfield and Kojima (2010, Theorem 7).

4.1.1 Examples for the Proofs of Facts 2–3

Example 1. Let $D = \{d_1, d_2, d_3\}$, $H = \{h\}$, and $X = \{x_i, y_i\}_{i\in\{1,2,3\}}$, where x_i and y_i denote two distinct contracts between d_i and h. The hospital has a preference relation

$$\succ_h\colon\ \{x_1, y_2, y_3\} \succ_h \{y_1, x_2, y_3\} \succ_h \{x_1, x_2, y_3\}$$
$$\succ_h \{y_1, y_2\} \succ_h \{y_1\} \succ_h \{y_2\} \succ_h \{x_3\} \succ_h \varnothing,$$

which induces its choice function $C_h(\cdot)$.[22] In what follows, we establish that in this market, a stable and strategy-proof rule $f(\cdot)$ exists, whereas neither the cumulative offer process nor the deferred acceptance algorithm induces $f(\cdot)$.[23]

First, define a stable rule $f(\cdot)$ as in Tables 1–2. We can verify the strategy-proofness of $f(\cdot)$ as follows. By checking each column of Tables 1–2 we can make the following observation: Taking $\succ_{d_2}$ and $\succ_{d_3}$ as fixed, doctor d_1 is always assigned his best contracts either from $\{x_1, y_1, \varnothing\}$ or from $\{y_1, \varnothing\}$. In either case, as his "choice set" is independent of $\succ_{d_1}$, it is apparent that doctor d_1 has no incentive to misreport. Similarly, we can see, by checking the rows of Tables 1–2, that doctor d_2 cannot profitably manipulate $f(\cdot)$. Further, although it is more cumbersome, we can also confirm that $f(\cdot)$ is also strategy-proof for doctor d_3. To do so, divide $\mathscr{P}_{d_1} \times \mathscr{P}_{d_2}$ (i.e., all possible $(\succ_{d_1}, \succ_{d_2})$'s) into four cases, as in Table 3. Note first that when $(\succ_{d_1}, \succ_{d_2})$ is in case A, $\mathrm{x}(d_3, f(\succ_D))$ is constant with respect to $\succ_{d_3}$. For the remaining three cases, moreover, d_3 is always assigned his best contract (i) from $\{x_3, y_3, \varnothing\}$ in case B, (ii) from $\{x_3, \varnothing\}$ in case C, and (iii) from $\{y_3, \varnothing\}$ in case D. Thus, doctor d_3 has no incentive to misreport $\succ_{d_3}$ for any $(\succ_{d_1}, \succ_{d_2})$ taken as fixed.

[22] That is, for each $X' \subset X$, $C_h(X')$ is the most preferred subset of $\mathrm{X}(h, X')$ according to $\succ_h$.

[23] While the doctor-optimal stable rule does not exist in this example, it is not necessarily such non-existence that prevents the cumulative offer process from operating well. Example 4 in Appendix A presents a market where the doctor-optimal stable rule exists and is strategy-proof, but not all cumulative offer processes induce it.

[24] Here we abuse notation and identify preferences with ordered lists (up to the null contract). For instance, "$x_1, y_1, \varnothing$" represents $\succ_{d_1}$ such that $x_1 \succ_{d_1} y_1 \succ_{d_1} \varnothing$. The same applies to all tables below.

	$x_2, y_2, \varnothing$	$y_2, x_2, \varnothing$	$x_2, \varnothing$	$y_2, \varnothing$	$\varnothing$
$x_1, y_1, \varnothing$	$\{x_1, x_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\{y_1, x_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\{y_1\}$
$y_1, x_1, \varnothing$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1\}$
$x_1, \varnothing$	$\{x_1, y_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\{x_3\}$	$\{x_1, y_2, y_3\}$	$\{x_3\}$
$y_1, \varnothing$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1\}$
$\varnothing$	$\{y_2\}$	$\{y_2\}$	$\{x_3\}$	$\{y_2\}$	$\{x_3\}$

(a) Case of $x_3 \succ_{d_3} y_3 \succ_{d_3} \varnothing$.

	$x_2, y_2, \varnothing$	$y_2, x_2, \varnothing$	$x_2, \varnothing$	$y_2, \varnothing$	$\varnothing$
$x_1, y_1, \varnothing$	$\{x_1, x_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\{x_1, x_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\{y_1\}$
$y_1, x_1, \varnothing$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1\}$
$x_1, \varnothing$	$\{x_1, x_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\{x_1, x_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\{x_3\}$
$y_1, \varnothing$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1\}$
$\varnothing$	$\{y_2\}$	$\{y_2\}$	$\{x_3\}$	$\{y_2\}$	$\{x_3\}$

(b) Case of $y_3 \succ_{d_3} x_3 \succ_{d_3} \varnothing$.

	$x_2, y_2, \varnothing$	$y_2, x_2, \varnothing$	$x_2, \varnothing$	$y_2, \varnothing$	$\varnothing$
$x_1, y_1, \varnothing$	$\{x_1, x_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\{x_1, x_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\{y_1\}$
$y_1, x_1, \varnothing$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1\}$
$x_1, \varnothing$	$\{x_1, x_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\{x_1, x_2, y_3\}$	$\{x_1, y_2, y_3\}$	$\varnothing$
$y_1, \varnothing$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1, x_2, y_3\}$	$\{y_1, y_2\}$	$\{y_1\}$
$\varnothing$	$\{y_2\}$	$\{y_2\}$	$\varnothing$	$\{y_2\}$	$\varnothing$

(c) Case of $y_3 \succ_{d_3} \varnothing \succ_{d_3} x_3$.

Table 1: Definition of $f(\cdot)$ in Example 1. The rows and columns represent the preferences of doctor d_1 and d_2, respectively.[24]

	$x_2, y_2, \varnothing$	$y_2, x_2, \varnothing$	$x_2, \varnothing$	$y_2, \varnothing$	$\varnothing$
$x_1, y_1, \varnothing$	$\{y_1, y_2\}$	$\{y_1, y_2\}$	$\{y_1\}$	$\{y_1, y_2\}$	$\{y_1\}$
$y_1, x_1, \varnothing$	$\{y_1, y_2\}$	$\{y_1, y_2\}$	$\{y_1\}$	$\{y_1, y_2\}$	$\{y_1\}$
$x_1, \varnothing$	$\{y_2\}$	$\{y_2\}$	$\{x_3\}$	$\{y_2\}$	$\{x_3\}$
$y_1, \varnothing$	$\{y_1, y_2\}$	$\{y_1, y_2\}$	$\{y_1\}$	$\{y_1, y_2\}$	$\{y_1\}$
$\varnothing$	$\{y_2\}$	$\{y_2\}$	$\{x_3\}$	$\{y_2\}$	$\{x_3\}$

(a) Case of $x_3 \succ_{d_3} \varnothing \succ_{d_3} y_3$.

	$x_2, y_2, \varnothing$	$y_2, x_2, \varnothing$	$x_2, \varnothing$	$y_2, \varnothing$	$\varnothing$
$x_1, y_1, \varnothing$	$\{y_1, y_2\}$	$\{y_1, y_2\}$	$\{y_1\}$	$\{y_1, y_2\}$	$\{y_1\}$
$y_1, x_1, \varnothing$	$\{y_1, y_2\}$	$\{y_1, y_2\}$	$\{y_1\}$	$\{y_1, y_2\}$	$\{y_1\}$
$x_1, \varnothing$	$\{y_2\}$	$\{y_2\}$	$\varnothing$	$\{y_2\}$	$\varnothing$
$y_1, \varnothing$	$\{y_1, y_2\}$	$\{y_1, y_2\}$	$\{y_1\}$	$\{y_1, y_2\}$	$\{y_1\}$
$\varnothing$	$\{y_2\}$	$\{y_2\}$	$\varnothing$	$\{y_2\}$	$\varnothing$

(b) Case of $\mathrm{Ac}\left(\succ_{d_3}\right) = \varnothing$.

Table 2: Definition of $f(\cdot)$ in Example 1 (continued). The rows and columns represent the preferences of doctor d_1 and d_2, respectively.

	$x_2, y_2, \varnothing$	$y_2, x_2, \varnothing$	$x_2, \varnothing$	$y_2, \varnothing$	$\varnothing$
$x_1, y_1, \varnothing$	D	D	D	D	A
$y_1, x_1, \varnothing$	D	A	D	A	A
$x_1, \varnothing$	D	D	B	D	C
$y_1, \varnothing$	D	A	D	A	A
$\varnothing$	A	A	C	A	C

Table 3: Partition of $\mathscr{P}_{d_1} \times \mathscr{P}_{d_2}$ in Example 1. The rows and columns represent the preferences of doctor d_1 and d_2, respectively.

To show that the cumulative offer process does not induce $f(\cdot)$, suppose that $\succ_D$ is given by

$$\succ_{d_1}: x_1 \succ_{d_1} y_1 \succ_{d_1} \emptyset,$$

$$\succ_{d_2}: x_2 \succ_{d_2} y_2 \succ_{d_2} \emptyset, \text{ and}$$

$$\succ_{d_3}: x_3 \succ_{d_3} y_3 \succ_{d_3} \emptyset.$$

With this preference profile, $f(\succ_D) = \{x_1, x_2, y_3\}$, as specified in the colored cell in Table 1a. For the cumulative offer process to return this outcome, doctor d_3 needs to propose y_3 and hence, x_3 must be rejected beforehand. This further implies that y_1 or y_2 (or both) must be offered by the end of the process. Then, however, the final pool of offers (i.e., the first definition of M_t in Definition 1) must include $\{x_1, y_2, y_3\}$ or $\{y_1, x_2, y_3\}$ (or both) and thus, the hospital must not pick $f(\succ_D) = \{x_1, x_2, y_3\}$, which is less preferable. That is, the cumulative offer process cannot induce the stable and strategy-proof rule $f(\cdot)$ in this market.

Analogously, the deferred acceptance algorithm cannot induce $f(\cdot)$ either. At $\succ_D$ as specified above, either y_1 or y_2 must be offered by the end of the algorithm for exactly the same reasoning as in the previous paragraph. Since $x_1 \succ_{d_1} y_1$ and $x_2 \succ_{d_2} y_2$, this implies that either x_1 or x_2 is rejected beforehand. At the last step of the algorithm, therefore, the choice set for h (i.e., the second definition of M_t in Definition 1) cannot contain both x_1 and x_2. That is, the deferred acceptance algorithm cannot return $f(\succ_D) = \{x_1, x_2, y_3\}$ at $\succ_D$. □

Example 2. Suppose that $D = \{d_1, d_2, d_3\}$, $H = \{h, h'\}$, and $X = \{x_i, x_i'\}_{i \in \{1,2,3\}}$, where x_i (resp. x_i') represents a contract between doctor d_i and hospital h (resp. h'). The choice functions of the hospitals, $C_h(\cdot)$ and $C_{h'}(\cdot)$, are induced by preference relations

$$\begin{aligned}\succ_h &: \{x_1\} \succ_h \{x_2, x_3\} \succ_h \{x_2\} \succ_h \{x_3\} \succ_h \varnothing, \text{ and} \\ \succ_{h'} &: \{x_2'\} \succ_{h'} \{x_1'\} \succ_{h'} \varnothing,\end{aligned}$$

respectively. With the resulting choice functions, the doctor-optimal stable allocation exists (and coincides with the outcome of the deferred acceptance algorithm) for any $\succ_D$, as summarized in Table 4. In what follows, we check that in this market, the rural hospital theorem does not hold for all $\succ_D$, whereas the doctor-optimal stable rule $X^*(\cdot)$ is strategy-proof.

To see the rural hospital theorem fails to hold in this market, fix a preference profile $\succ_D$ such that

$$\begin{aligned}\succ_{d_1} &: x_1' \succ_{d_1} x_1 \succ_{d_1} \varnothing, \\ \succ_{d_2} &: x_2 \succ_{d_2} x_2' \succ_{d_2} \varnothing, \text{ and} \\ \succ_{d_3} &: x_3 \succ_{d_3} \varnothing.\end{aligned}$$

As shown in the colored cell in Table 4, the doctor-optimal stable allocation at $\succ_D$ is $X^* = \{x_1', x_2, x_3\}$, whereas there exists another stable allocation $X_* = \{x_1, x_2'\}$. Note that doctor d_3 is assigned a non-null contract at X^* but not at X_*, and hence, the rural hospital theorem fails.

It remains to verify that the doctor-optimal stable rule, $X^*(\cdot)$, is strategy-proof in this market. For doctors d_1 and d_2, note that $\mathrm{x}\,(d_1, X^*(\succ_D))$ and $\mathrm{x}\,(d_2, X^*(\succ_D))$ are independent of $\succ_{d_3}$. Hence, the incentives for doctors d_1 and d_2 to manipulate will remain unchanged if d_3 is eliminated from the market. Actually, once d_3 is omitted, the remaining market reduces to a standard one-to-one matching market without contracts and thus,

	$x_2, x_2', \varnothing$	$x_2', x_2, \varnothing$	$x_2, \varnothing$	$x_2', \varnothing$	$\varnothing$
$x_1, x_1', \varnothing$	$\{x_1, x_2'\}$	$\{x_1, x_2'\}$	$\{x_1\}$	$\{x_1, x_2'\}$	$\{x_1\}$
$x_1', x_1, \varnothing$	$\{x_1', x_2, x_3\}$	$\{x_1, x_2'\}$	$\{x_1', x_2, x_3\}$	$\{x_1, x_2'\}$	$\{x_1', x_3\}$
$x_1, \varnothing$	$\{x_1, x_2'\}$	$\{x_1, x_2'\}$	$\{x_1\}$	$\{x_1, x_2'\}$	$\{x_1\}$
$x_1', \varnothing$	$\{x_1', x_2, x_3\}$	$\{x_2', x_3\}$	$\{x_1', x_2, x_3\}$	$\{x_2', x_3\}$	$\{x_1', x_3\}$
$\varnothing$	$\{x_2, x_3\}$	$\{x_2', x_3\}$	$\{x_2, x_3\}$	$\{x_2', x_3\}$	$\{x_3\}$

(a) Case of $x_3 \in \mathrm{Ac}(\succ_{d_3})$.

	$x_2, x_2', \varnothing$	$x_2', x_2, \varnothing$	$x_2, \varnothing$	$x_2', \varnothing$	$\varnothing$
$x_1, x_1', \varnothing$	$\{x_1, x_2'\}$	$\{x_1, x_2'\}$	$\{x_1\}$	$\{x_1, x_2'\}$	$\{x_1\}$
$x_1', x_1, \varnothing$	$\{x_1', x_2\}$	$\{x_1, x_2'\}$	$\{x_1', x_2\}$	$\{x_1, x_2'\}$	$\{x_1'\}$
$x_1, \varnothing$	$\{x_1, x_2'\}$	$\{x_1, x_2'\}$	$\{x_1\}$	$\{x_1, x_2'\}$	$\{x_1\}$
$x_1', \varnothing$	$\{x_1', x_2\}$	$\{x_2'\}$	$\{x_1', x_2\}$	$\{x_2'\}$	$\{x_1'\}$
$\varnothing$	$\{x_2\}$	$\{x_2'\}$	$\{x_2\}$	$\{x_2'\}$	$\varnothing$

(b) Case of $x_3 \notin \mathrm{Ac}(\succ_{d_3})$.

Table 4: Doctor optimal stable allocations in Example 2. The rows and columns represent the preferences of doctor d_1 and d_2, respectively.

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d_1 and d_2 have no incentive to manipulate the doctor-optimal stable rule (Dubins and Freedman, 1981; Roth, 1982). For doctor d_3, observe that $\mathrm{x}(d_3, X^*(\succ_D))$ is either x_3 or $\varnothing$, and that it depends on $\succ_{d_3}$ only through whether or not $x_3 \in \mathrm{Ac}(\succ_{d_3})$. Therefore, d_3 has no incentive to report that x_3 is acceptable when it is not, and vice versa. In sum, the doctor-optimal stable rule is strategy-proof in this market. $\square$

4.2 Applicability of Our Proof Technique

In our analysis, all proofs share a common technique of deriving a contradiction starting from a "minimal" preference profile in terms of the number of acceptable contracts. On the one hand, this requires a sufficiently rich preference domain so that the preference profile remains admissible even after manipulation. Consequently, our results do not directly extend to the cases of restricted preference domains, such as in Kesten (2010) and Kesten and Kurino (2016), who consider the domain where any school seats are necessarily acceptable. On the other hand, as long as the full domain is assumed, our technique can be applicable elsewhere, e.g., to study other solution concepts than stability. For instance, consider the following weakening of stability:

Definition 2. An individually rational allocation X' is *non-wasteful* if there is no other individually rational allocation X'' with $X'' \supsetneq X'$.[25] □

That is, an allocation is non-wasteful if it is maximal in the set sense among the individually rational ones. While it is a part of the standard definition for non-wasteful allocations to be maximal among those both feasible and individually rational, feasibility can be seen as a part of individual rationality in the matching-with-contracts framework, as argued after Theorem 3. In the standard assignment problems, indeed, our definition reduces to the maximality among feasible and individually rational assignments, if we endow each hospital (or object) h with $C_h(\cdot)$ such that $C_h(X') = \mathrm{X}(h, X')$ if and only if $|\mathrm{X}(h, X')| \leq q_h$, where q_h is the "quota" of h.[26] Nevertheless, the present framework

[25]It is immediate to verify that under the IRC condition, stability implies non-wastefulness as defined above. This is not necessarily the case in the absence of the IRC condition; see Example 5 in Appendix A.

Relatedly, Alva and Manjunath (2016) propose a related condition called *participation-maximality*, and establish various results on individually rational and participation-maximal allocations, some of which are close to ours in the present paper. A key distinction between the two papers is that (among individually rational allocations) their participation-maximality is stronger than our non-wastefulness and consequently, the former is not necessarily implied by stability even when the hospitals' choice functions satisfy the IRC condition.

[26]Of course, the quota does not uniquely pin down $C_h(\cdot)$, because it imposes no restriction on $C_h(X')$

allows a richer class of constraints that could be of practical relevance.[27] Based on our proof technique, Theorem 3 can be extended to non-wasteful and strategy-proof rules as follows:

Theorem 4. *No individually rational and strategy-proof rule strictly dominates a non-wasteful and strategy-proof rule.*[28]

Proof. In the proof of Theorem 3, the stability of $g(\cdot)$ and the IRC condition are needed only to guarantee the existence of d^* satisfying equation $(*)$. Hence it suffices to prove this part from non-wastefulness. Indeed, if such d^* does not exist, it follows as in the proof of Theorem 3 that $f(\succ_D^*) \supsetneq g(\succ_D^*)$, which directly contradicts non-wastefulness. ■

Theorem 4 can be slightly further strengthened since, as is apparent from the definition of individual rationality, the choice functions are relevant for non-wastefulness only through whether $C_h(X') = \mathrm{X}(h, X')$ or not. Although it is technically straightforward, this extension allows us to incorporate weak preferences of the hospital side and consequently, includes Theorem 1 of Abdulkadiroglu et al. (2009) as a special case.[29]

Corollary 5. *Suppose that $C_H(\cdot)$ and $C'_H(\cdot)$ are such that $C_h(X') = \mathrm{X}(h, X') \Rightarrow C'_h(X') = \mathrm{X}(h, X')$ for all $h \in H$ and $X' \subset X$. Then, no strategy-proof rule that is individually rational with respect to $C_H(\cdot)$ strictly dominates a strategy-proof rule that is non-wasteful with respect to $C'_H(\cdot)$.*

when $|\mathrm{X}(h, X')|$ exceeds the quota. However, this additional degree of freedom is irrelevant for the current purpose (see Corollary 5 below).

[27]For example, suppose that a school offers two distinct programs, and they require some common resources at different factor intensity (e.g., one is mathematics-teacher intensive while the other is English-teacher intensive). Then, the total number of students that those programs can accommodate would be non-constant and depend on its composition. Such a constraint cannot be represented as either the quota of the school or the quotas of the programs.

[28]Note that the IRC condition is unnecessary in this Theorem (and Corollary 5 below).

[29]In the school choice problem, $C_h(X') = \mathrm{X}(h, X')$ holds only when the number of students is no more than the quota of h, or equivalently, only when no tie-breaking is necessary. If $C_h(\cdot)$ and $C'_h(\cdot)$ are two choice functions resulting from different tie-breakings but the same weak priority structure, thus, they satisfy the condition in Corollary 5, $C_h(X') = \mathrm{X}(h, X') \Rightarrow C'_h(X') = \mathrm{X}(h, X')$.

Proof. The proof is exactly the same as of Theorems 3–4 and thus omitted. ■

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A Additional Examples

This appendix provides the examples that are referred to but omitted in the main body. The first example illustrates that Lemmas 1–2 do not generally hold true without the IRC condition. As a consequence, multiple stable and strategy-proof rules exist in this example.

Example 3. Let $D = \{d_1, d_2\}$, $H = \{h\}$, and $X = \{x_1, x_2\}$, where for each $i \in \{1, 2\}$, x_i is a contract between d_i and h. Hospital h's choice function is given by $C_h(\{x_1\}) = x_1$, $C_h(\{x_2\}) = x_2$, and $C_h(\{x_1, x_2\}) = \varnothing$. It is immediate to check that $C_h(\cdot)$ violates the IRC condition. Now suppose that each doctor d_i has a preference relation with $x_i \succ_{d_i} \varnothing$. At this $(C_H(\cdot), \succ_D)$, then, neither allocation $\{x_1\}$ nor $\{x_2\}$ is blocked by any coalition, although both are weakly blocked by $(h, \{x_1, x_2\})$.[30] That is, the conclusion of Lemma 1 fails to hold. Consequently, both $\{x_1\}$ and $\{x_2\}$ are stable at this profile, and the conclusion of Lemma 2 also fails. Lastly, define for each $i \in \{1,2\}$ a rule $f^i(\cdot)$ by

$$f^i(\succ'_D) = \begin{cases} \{x_i\} & \text{if } x_i \succ'_{d_i} \varnothing, \\ \{x_j\} & \text{if } \varnothing \succ'_{d_i} x_i \text{ and } x_j \succ'_{d_j} \varnothing, \\ \varnothing & \text{otherwise,} \end{cases}$$

where $j \in \{1,2\} - \{i\}$. In words, $f^i(\cdot)$ is the serial dictatorship where doctor i has priority over j. Further, note also that stability reduces to individual rationality in this market, because both $\{x_1\}$ and $\{x_2\}$ are stable at $\succ_D$ as defined above. Given these observations, we can check both $f^1(\cdot)$ and $f^2(\cdot)$ are stable and strategy-proof. □

The next example shows that in general, the cumulative offer process may not lead to the doctor-optimal stable rule, even if it exists and is strategy-proof.

Example 4. Let $D = \{d_1, d_2\}$, $H = \{h\}$, and $X = \{x_i, y_i\}_{i \in \{1,2\}}$, where x_i and y_i denote two distinct contracts between d_i and h. The hospital's choice function is induced by

$$\succ_h : \{x_1, x_2\} \succ_h \{y_1, x_2\} \succ_h \varnothing.$$

In this market, the doctor-optimal stable rule $X^*(\cdot)$ exists:

$$X^*(\succ_D) = \begin{cases} \{x_1, x_2\} & \text{if } x_1 = \max\{x_1, y_1, \varnothing\} \text{ and } x_2 \in \mathrm{Ac}(\succ_{d_2}), \\ \{y_1, x_2\} & \text{if } y_1 = \max\{x_1, y_1, \varnothing\} \text{ and } x_2 \in \mathrm{Ac}(\succ_{d_2}), \\ \varnothing & \text{otherwise,} \end{cases}$$

[30] The second requirement of weak blocking is vacuously satisfied since $C_h(\{x_1, x_2\}) = \varnothing$.

where the maximums are taken with respect to $\succ_{d_1}$. Thus, our Theorem 2 tells that $X^*(\cdot)$ is the unique candidate for a stable and strategy-proof rule, and indeed, it is strategy-proof. To see this, note that $X^*(\cdot)$ selects the best allocation for doctor d_1 subject to the individual rationality constraints for d_2 and h. Hence, doctor d_1 has no incentive to misreport. Doctor d_2 cannot manipulate $X^*(\cdot)$ either, since (i) it depends on $\succ_{d_2}$ only through whether $x_2 \in \mathrm{Ac}(\succ_{d_2})$ or not, and (ii) $X^*(\succ_D) \ni x_2$ only when $x_2 \in \mathrm{Ac}(\succ_{d_2})$.

While the doctor-optimal stable rule exists and is strategy-proof, not all the cumulative offer processes induce $X^*(\cdot)$.[31] To see this, consider the process such that doctor d_1 makes an offer whenever $d_1 \in D_{t-1}$, where D_{t-1} is as defined in Definition 1. Then, when $\succ_D$ is given by

$$\succ_{d_1}: y_1 \succ_{d_1} x_1 \succ_{d_1} \varnothing, \text{ and}$$

$$\succ_{d_2}: y_2 \succ_{d_2} x_2 \succ_{d_2} \varnothing,$$

this cumulative offer process returns $\{x_1, x_2\} \neq X^*(\succ_D) = \{y_1, x_2\}$. □

The last example shows that non-wastefulness as in Definition 2 does not necessarily follow from stability in the absence of the IRC condition.

Example 5. Let $D = \{d_1, d_2\}$, $H = \{h_1, h_2\}$, and $X = \{x_1, x_2\}$, where for each $i \in \{1, 2\}$, x_i is a contract between d_i and h_i. Suppose that each d_i has a preference relation such that $x_i \succ_{d_i} \varnothing$, and each h_i has a choice function $C_{h_i}(\cdot)$ such that $C_{h_i}(X) = x_i$ and $C_{h_i}(X') = \varnothing$ for all $X' \subsetneq X$. Allocation $X' = \varnothing$ is then wasteful. At the same time it is also stable, because there is no blocking coalition although both $(h_1, \{x_1, x_2\})$ and $(h_2, \{x_1, x_2\})$ *weakly* block $X' = \varnothing$. □

[31] In this market, there are multiple non-outcome-equivalent cumulative offer processes. Depending on how to specify who makes an offer at each step, some of them induce $X^*(\cdot)$ while others do not.