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Abstract

This paper proposes normative consequentialist criteria for voting rules under Knightian uncertainty about individual preferences to characterize a weighted majority rule (WMR). The criteria stress the significance of responsiveness, i.e., the probability that the social outcome coincides with the realized individual preferences. A voting rule is said to be robust if, for any probability distribution of preferences, responsiveness of at least one individual is greater than one-half. Our main result establishes that a voting rule is robust if and only if it is a WMR without ties. This characterization of a WMR avoiding the worst possible outcomes complements the well-known characterization of a WMR achieving the optimal outcomes, i.e., efficiency regarding responsiveness.

JEL classification numbers: D71, D81.

Keywords: majority rule, weighted majority rule, responsiveness, Knightian uncertainty.

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1 Introduction

Consider a choice of a voting rule on a succession of two alternatives (such as “yes” or “no”) by a group of individuals who are uncertain about their future preferences. Each individual presumes that the gain from the passage of a favorable issue equals the loss from the passage of an unfavorable issue. Imagine that someone proposes a voting rule such that the expected loss of every individual is greater than the expected gain. Then, the group will not agree to adopt it. In fact, such a voting rule is problematic because the probability that the outcome agrees with an individual’s preference, i.e., responsiveness, is less than one-half for all individuals. This means that a group decision reflects minority preferences on average and that the decision can be eventually not only unfair ex post facto but also more likely incorrect.

To evaluate the expected net gain, individuals must know the true probability distribution of their preferences. However, in reality, they face Knightian uncertainty and have little confidence regarding the true probabilities.[1] This makes it difficult for them to figure out whether the expected net gain is positive or negative, which raises the following questions. Does there exist a voting rule such that the expected net gain of every individual is never negative whatever the underlying probability distribution is? If the answer is yes, what is it?

This paper proposes two normative criteria for voting rules under Knightian uncertainty and provides answers to the above questions. Our criteria require that a voting rule should avoid the following worst-case scenarios. The first worst-case scenario is that the true responsiveness of every individual is less than or equal to one-half, or equivalently, the expected net gain of every individual is nonpositive. By replacing “less than or equal to” with “strictly less than" in this scenario, we obtain a slightly more severe scenario. The second worst-case scenario is that the true responsiveness of every individual is strictly less than one-half, or equivalently, the expected net gain of every individual is strictly negative. A voting rule is said to be *robust*[2] if, for any probability distribution of preferences, it avoids the first worst-case scenario. A voting rule is said to be *weakly robust* if, for any probability distribution of preferences, it avoids the second worst-case scenario. Because the first scenario is less severe than the second one, robustness is a stronger requirement than weak robustness: under robust rules, responsiveness

[1] Knight (1921) distinguishes risky situations, where a decision maker knows the probabilities of all events, and uncertain situations, where a decision maker does not know them.

[2] We borrow the term “robustness” from robust statistics, statistics with good performance for data drawn from a wide range of probability distributions (Huber, 1981).

of at least one individual must be strictly greater than one-half, whereas, under weakly robust rules, responsiveness of every individual can be less than or equal to one-half, as long as responsiveness of at least one individual is equal to one-half, in which case a collective decision is at best neutral to each individual's choice on average.

In the main result, we show that a voting rule is robust if and only if it is a weighted majority rule (WMR) without ties. We also show that a voting rule is weakly robust if and only if it is a WMR allowing ties with an arbitrary tie-breaking rule. The proofs of both results are based upon the theorem of alternatives due to von Neumann and Morgenstern (1944), which is also known as a corollary of Farkas' lemma. Because Farkas' lemma is mathematically equivalent to the fundamental theorem of asset pricing (cf. Dybvig and Ross, 2003, 2008), the proofs can be understood in terms of the following imaginary asset for each individual i: one unit of asset i yields $+1$ if individual i's preference agrees with the collective decision and -1 otherwise. Using the fundamental theorem of asset pricing, we can show that there exists a portfolio with nonnegative weights in all assets (i.e. no short selling) yielding a strictly positive payoff in each state if and only if, for any arbitrage-free price vector, the price of at least one asset is strictly positive. The former condition is true if and only if the voting rule is a WMR and the latter condition is true if and only if responsiveness of at least one individual is greater than one-half, thus implying the equivalence of a robust rule and a WMR.

We apply the above result to anonymous rules, which are considered to be fair, and obtain the following characterization of robust anonymous rules. A simple majority rule (SMR) is a unique robust anonymous rule when the number of individuals is odd, whereas no anonymous rule is robust when the number of individuals is even. In the latter case, however, a SMR with an anonymous tie-breaking rule is a weakly robust anonymous rule. This implies that we face a trade-off between robustness and anonymity when the number of individuals is even: we must be content with a nonanonymous rule if we require robustness and we must be content with a weakly robust rule if we require anonymity.

To illustrate the difference between robustness and weak robustness as well as the trade-off between robustness and anonymity, assume that the number of individuals is even and consider SMRs. A SMR with some tie-breaking rule is robust if and only if it is represented as a WMR allowing no ties. For example, a SMR with a casting (tie-breaking) vote is a robust nonanonymous rule. On the other hand, a SMR with any tie-breaking rule is weakly robust. In particular, a SMR with the status quo tie-breaking rule (i.e., the status quo is followed whenever there is a tie) is a weakly robust anonymous rule. Because no anonymous rule is robust when the number of individuals is even, the choice between the two SMRs depends upon which criterion to prioritize, and notice that, in the real world, both rules are widely used. A SMR with a casting vote is used by legislatures such as the United States Senate, the Australian House of Representatives, and the National Diet of Japan. A SMR with the status quo tie-breaking rule is used by legislatures such as the New Zealand House of Representatives, the British House of Commons, and the Australian Senate.

Our characterization of a WMR avoiding the worst possible outcomes complements the well-known characterization of a WMR achieving the optimal outcomes due to Rae (1969), Taylor (1969), and Fleurbaey (2008). Their result, which we call the Rae-Taylor-Fleurbaey (RTF) theorem, states that a voting rule is a WMR if and only if it maximizes the corresponding weighted sum of responsiveness over all individuals. The RTF theorem can be understood as the following characterization of a WMR in terms of efficiency.[3] First, a voting rule is efficient with respect to responsiveness if and only if it is a WMR with positive weights (i.e., every individual's weight is positive). Next, a voting rule is weakly efficient if and only if it is a WMR with nonnegative weights (i.e., some individual's weight can be zero). In both WMRs, ties are allowed with any tie-breaking rules.

Although the RTF theorem and our result have in common that both examine WMRs using responsiveness, the former characterizes WMRs as efficient rules and the latter as robust rules. We can summarize the connections between them as follows. A weakly efficient rule achieves the optimal outcomes, whereas a weakly robust rule avoids the worst outcomes. However, both of them constitute the same class of WMRs. In this respect, weak robustness together with weak efficiency gives a dual characterization of a WMR in terms of responsiveness. On the other hand, a robust rule is a WMR with nonnegative weights allowing no ties, which can be represented as a WMR with positive weights, whereas an efficient rule is a WMR with positive weights allowing ties. Therefore, robustness is a stronger requirement than efficiency. In particular, when the number of individuals is even, a SMR with an anonymous tie-breaking rule is efficient but not robust.

This paper not only contributes to the literature on the axiomatic foundations of a SMR or a WMR (May, 1952; Fishburn, 1973), but also joins a recently growing literature on economic design with worst-case objectives. Most studies in the latter literature, however, have focused

[3] See also Schmitz and Tröger (2012) and Azrieli and Kim (2014).

on mechanism design. For example, Chung and Ely (2007) consider a revenue maximization problem in a private value auction where the auctioneer does not know agents' belief structures exactly[4] and show that the optimal auction rule is a dominant-strategy mechanism when the auctioneer evaluates rules by their worst-case performance. On the other hand, Carroll (2015) considers a moral hazard problem where the principal does not know the agent's set of possible actions exactly and shows that the optimal contract is linear when the principal evaluates contracts by their worst-case performance.[5] In contrast to these papers, we consider a choice of voting rules with the worst-case objective to characterize WMRs, where the constitution-maker does not know the probability distribution of preferences, thus demonstrating that this approach is also useful in the study of voting and social choice.

The rest of this paper is organized as follows. Section 2 summarizes properties of WMRs which will be used in the subsequent sections. Section 3 introduces the concepts of robustness and weak robustness and Section 4 characterizes robust rules and weakly robust rules. Section 5 compares our result and the RTF theorem. We conclude the paper in Section 6.

2 Weighted majority rules

Consider a group of individuals $N = \{1, \cdots, n\}$ that faces a choice between two alternatives (such as "yes" or "no"). The choice of individual $i \in N$ is represented by a decision variable $x_i \in \{-1, 1\}$. The choices of the group members are summarized by a decision profile $x = (x_i)_{i\in N}$. Let $\mathcal{X} \subseteq \{-1, 1\}^N$ denote the set of all possible profiles.

A voting rule is a mapping $\phi : \mathcal{X} \to \{-1, 1\}$. Let Φ denote the set of all voting rules. A voting rule $\phi \in \Phi$ is a weighted majority rule (WMR)[6] if there exists a nonzero weight vector[7] $w = (w_i)_{i\in N} \in \mathbb{R}^N$ satisfying

$$\phi(x) = \begin{cases} 1 & \text{if } \sum_{i\in N} w_i x_i > 0, \\ -1 & \text{if } \sum_{i\in N} w_i x_i < 0. \end{cases}$$

[4]This is a standard assumption in robust mechanism design (Bergemann and Morris, 2005).

[5]Other recent examples of economic design with worst-case objectives include Bergemann and Schlag (2011), Yamashita (2015), Bergemann et al. (2016), and Carroll (2017) among others. In decision theory, Gilboa and Schmeidler (1989) is a seminal paper.

[6]For an axiomatic foundation of weighted majority rules, see Fishburn (1973), Fishburn and Gehrlein (1977), Nitzan and Paroush (1981), Einy and Lehrer (1989), and Taylor and Zwicker (1992) among others.

[7]We allow negative weights, which appears in Proposition 5 and Appendix A.

A simple majority rule (SMR) is a special case with positive equal weights, i.e., $w_i = w_j > 0$ for all $i, j \in N$. When there is a tie, i.e. $\sum_{i \in N} w_i x_i = 0$, a tie-breaking rule is used to determine a voting rule. For example, a SMR requires a tie-breaking rule if n is even.

A voting rule ϕ is anonymous if it is symmetric in its n variables; that is, $\phi(x) = \phi(x^\pi)$ for each $x \in \mathcal{X}$ and each permutation $\pi : N \to N$, where $x^\pi = (x_{\pi(i)})_{i \in N}$. A SMR is anonymous if n is odd or if n is even and its tie-breaking rule is anonymous, i.e., symmetric in its n variables. A WMR with nonnegative weights is anonymous if and only if it is an anonymous SMR.

The following characterization of WMRs, which is immediate from the definition, plays an important role in the subsequent analysis.

Lemma 1. *A voting rule $\phi \in \Phi$ is a WMR with a weight vector $w \in \mathbb{R}^N$ if and only if*

$$\phi(x) \sum_{i \in N} w_i x_i \geq 0 \textit{ for all } x \in \mathcal{X}.$$

A voting rule $\phi \in \Phi$ is a WMR with a weight vector $w \in \mathbb{R}^N$ allowing no ties if and only if

$$\phi(x) \sum_{i \in N} w_i x_i > 0 \textit{ for all } x \in \mathcal{X}.$$

This lemma states that ϕ is a WMR allowing no ties (allowing ties) if and only if the corresponding weighted sum of $\phi(x)x_i$ over $i \in N$ is positive (nonnegative) for all $x \in \mathcal{X}$ because the left-hand side of the inequality is $\sum_{i \in N} w_i(\phi(x)x_i)$. Note that $\phi(x)x_i$ equals $+1$ if i's choice agrees with the collective decision and -1 otherwise. Thus, we can regard $\phi(x)x_i$ as individual i's payoff by assuming that the gain from the passage of a favorable issue and the loss from the passage of an unfavorable issue are equal and normalized to one.

Note that a weight vector w representing a WMR is a solution to a system of linear inequalities in Lemma 1. This observation leads us to the next lemma, which shows that the set of WMRs with nonnegative weights (i.e., $w_i \geq 0$ for all $i \in N$) coincides with that with positive weights (i.e., $w_i > 0$ for all $i \in N$) if there are no ties.

Lemma 2. *A WMR with nonnegative weights allowing no ties is represented as a WMR with positive weights allowing no ties.*

Proof. Let ϕ be a WMR with nonnegative weights allowing no ties. Consider the set of all weight vectors representing ϕ, which is $W \equiv \{w \in \mathbb{R}^N : \sum_{i \in N} w_i(\phi(x)x_i) > 0 \text{ for all } x \in \mathcal{X}\}$ by Lemma 1. Note that W is an open convex polyhedron containing a nonnegative vector. This implies that W also contains a positive vector. □

3 Voting under uncertainty

Assume that $x \in \mathcal{X}$ is randomly drawn according to a probability distribution $p \in \Delta(\mathcal{X}) \equiv \{p \in \mathbb{R}_+^{\mathcal{X}} : \sum_{x \in \mathcal{X}} p(x) = 1\}$. Let

$$p(\phi(x) = x_i) \equiv p(\{x \in \mathcal{X} : \phi(x) = x_i\}) = \sum_{x \in \mathcal{X} : \phi(x) = x_i} p(x)$$

be the probability that i's choice agrees with the collective decision, which is referred to as responsiveness or the Rae index (Rae, 1969). It is calculated as

$$p(\phi(x) = x_i) = (E_p[\phi(x)x_i] + 1)/2, \tag{1}$$

where $E_p[\phi(x)x_i] \equiv \sum_{x \in \mathcal{X}} p(x)\phi(x)x_i$ is the expected value of individual i's payoff $\phi(x)x_i$, because

$$E_p[\phi(x)x_i] = \sum_{x:\phi(x)=x_i} p(x) - \sum_{x:\phi(x)\neq x_i} p(x) = 2p(\phi(x) = x_i) - 1. \tag{2}$$

Imagine that the individuals agree not to adopt a voting rule if the responsiveness of every individual is less than or equal to one-half, or equivalently, the expected payoff of every individual is nonpositive. However, if they have no information about the true probability distribution of their preferences facing Knightian uncertainty, they cannot evaluate the responsiveness. Under these circumstances, the worst-case scenario is that the true responsiveness of every individual is less than or equal to one-half. We say that a voting rule is robust if, for any probability distribution of preferences, it avoids this worst-case scenario.

Definition 1. A voting rule $\phi \in \Phi$ is *robust* if, for each $p \in \Delta(\mathcal{X})$, responsiveness of at least one individual is strictly greater than one-half:

$$\max_{i \in N} p(\phi(x) = x_i) > 1/2 \text{ for all } p \in \Delta(\mathcal{X}). \tag{3}$$

For example, a WMR with nonnegative weights is robust if there are no ties. In fact, by Lemma 1, $\sum_{i \in N} w_i E_p[\phi(x)x_i] > 0$ for all $p \in \Delta(\mathcal{X})$, so there exists $i \in N$ such that $w_i E_p[\phi(x)x_i] > 0$, i.e., $p(\phi(x) = x_i) > 1/2$.

By replacing "less than or equal to" with "strictly less than" in the above worst-case scenario, we obtain a slightly more severe worst-case scenario in which the responsiveness of every individual is strictly less than one-half. We say that a voting rule is weakly robust if, for any probability distribution of preferences, it avoids this worst-case scenario.

Definition 2. A voting rule $\phi \in \Phi$ is *weakly robust* if, for each $p \in \Delta(\mathcal{X})$, responsiveness of at least one individual is greater than or equal to one-half:

$$\max_{i \in N} p(\phi(x) = x_i) \geq 1/2 \text{ for all } p \in \Delta(\mathcal{X}). \tag{4}$$

For example, a WMR with nonnegative weights is weakly robust even if there are ties. In fact, by Lemma 1, $\sum_{i \in N} w_i E_p[\phi(x)x_i] \geq 0$ for all $p \in \Delta(\mathcal{X})$, so there exists $i \in N$ such that $w_i > 0$ and $E_p[\phi(x)x_i] \geq 0$, i.e., $p(\phi(x) = x_i) \geq 1/2$.

If $\phi \in \Phi$ is weakly robust but not robust, there exists $p \in \Delta(\mathcal{X})$ such that

$$\max_{i \in N} p(\phi(x) = x_i) = 1/2.$$

If such p is the true probability distribution, a collective decision is at best neutral to each individual's choice on average, which is never the case with robust rules.

4 Main results

In this section, we present our main result characterizing robust and weakly robust rules. The result is stated and discussed in Section 4.1. The proofs are given in Section 4.2.

4.1 Characterizations

First, we characterize robust rules. A WMR with nonnegative weights allowing no ties is robust as discussed in Section 3. Our first main result establishes that every robust rule must be such a WMR.

Proposition 1. *A voting rule is robust if and only if it is a WMR with nonnegative weights such that there are no ties.*

Next, we characterize weakly robust rules. A WMR with nonnegative weights is weakly robust even if there are ties as discussed in Section 3. Our second main result establishes that every weakly robust rule must be such a WMR.

Proposition 2. *A voting rule is weakly robust if and only if it is a WMR with nonnegative weights.*

As a corollary of Proposition 1, we characterize robust anonymous rules. If n is odd, a SMR is the unique rule that is both robust and anonymous. In Appendix A, we give another characterization of a SMR with odd n using a stronger version of robustness. However, if n is even, no anonymous rule is robust. That is, there is a trade-off between robustness and anonymity.

Corollary 3. *Suppose that n is odd. Then, a voting rule is robust and anonymous if and only if it is a SMR. Suppose that n is even. Then, no voting rule is both robust and anonymous.*

Proof. A voting rule is robust and anonymous if and only if it is an anonymous WMR with nonnegative weights allowing no ties, which is a SMR with odd n. □

Corollary 3 implies that an anonymous rule is not robust if n is even or if it is not a SMR.[8] In particular, a supermajority rule is not robust because it is anonymous. To illustrate it by a numerical example, consider a two-thirds rule with a very large number of individuals. Suppose that $x_i = 1$ with probability $p \in (1/2, 2/3)$ independently and identically for each $i \in N$. By the law of large numbers, the group decision is -1 with probability close to one, so responsiveness of each individual is close to $1 - p < 1/2$, which implies that this rule is not robust.[9]

As a corollary of Proposition 2, we characterize weakly robust anonymous rules. Although no anonymous rule is robust when n is even, there exists a weakly robust anonymous rule regardless of n, which is an anonymous SMR.

Corollary 4. *A voting rule is weakly robust and anonymous if and only if it is an anonymous SMR.*

Proof. A voting rule is weakly robust and anonymous if and only if it is an anonymous WMR with nonnegative weights, which is an anonymous SMR. □

To illustrate the difference between robustness and weak robustness as well as the trade-off between robustness and anonymity, suppose that n is even and consider SMRs. By Proposition 1, a SMR with some tie-breaking rule is robust if and only if it is represented as a WMR allowing no ties and, by Corollary 3, such a SMR is nonanonymous. For example, a SMR with a casting (tie-breaking) vote is a robust nonanonymous rule. To see this, we consider two cases. First,

[8]We can also show that a SMR rule with an anonymous random tie-breaking rule is not robust.

[9]Even if n is not so large, we can find $p \in \Delta(\mathcal{X})$ such that the responsiveness of every individual is less than or equal to one-half.

assume that the presiding officer with a casting vote is a member of the group of n individuals. This rule is equivalent to a WMR such that the presiding officer's weight is slightly greater than the others' weights. Next, assume that the presiding officer is not a member of the group of n individuals and that he or she votes only when there is a tie. This rule is equivalent to a WMR with $n + 1$ individuals including the presiding officer such that the presiding officer's weight is very close to zero. Each of these WMRs does not have ties and is robust

On the other hand, a SMR with any tie-breaking rule is weakly robust by Proposition 2. In particular, a SMR with the status quo tie-breaking rule (i.e., the status quo is followed whenever there is a tie) is a weakly robust anonymous rule, but it is not robust by Corollary 3.

4.2 Proofs

This section provides the proofs of Propositions 1 and 2. In the proofs, we use the following inequality symbols. For vectors ξ and η, we write $\xi \geq \eta$ if $\xi_i \geq \eta_i$ for each i, $\xi > \eta$ if $\xi_i \geq \eta_i$ for each i and $\xi \neq \eta$, and $\xi \gg \eta$ if $\xi_i > \eta_i$ for each i.

We enumerate elements in $\mathcal{X}$ as $\{x^j\}_{j\in M}$, where $M \equiv \{1, \dots, m\}$ is an index set with $m = 2^n$. Consider an $n \times m$ matrix

$$L = [l_{ij}]_{n\times m} = \left[\phi(x^j)x_i^j\right]_{n\times m}.$$

Note that l_{ij} equals $+1$ if i's choice agrees with the collective decision and -1 otherwise. Using this matrix, we can restate the conditions in Proposition 1 as follows.

(a) By Lemma 1, a voting rule ϕ is a WMR with nonnegative weights allowing no ties if and only if there exists $w = (w_i)_{i\in N} \geq 0$ such that

$$\sum_{i\in N} w_i l_{ij} = \sum_{i\in N} w_i(\phi(x^j)x_i^j) > 0$$

for each $j \in M$, or equivalently, $w^\top L \gg 0$.

(b) By definition, a voting rule is *not* robust if and only if there exists $p = (p_j)_{j\in M} > 0$ such that

$$\sum_{j\in N} l_{ij}p_j = \sum_{j:\phi(x^j)=x_i^j} p_j - \sum_{j:\phi(x^j)\neq x_i^j} p_j \leq 0$$

for each $i \in N$, or equivalently, $Lp \leq 0$.

Proposition 1 states that exactly one of (a) and (b) holds. The following theorem of alternatives due to von Neumann and Morgenstern (1944)[10] guarantees that this is true. The

[10]von Neumann and Morgenstern (1944) use this result to prove the minimax theorem.

same result also appears in the work of Gale (1960, Theorem 2.10) as a corollary of Farkas' lemma.

Lemma 3. *Let A be an $n \times m$ matrix. Exactly one of the following alternatives holds.*

- *There exists $\xi \in \mathbb{R}^n$ satisfying*
$$\xi^\top A \gg 0,\ \xi \geq 0.$$
- *There exists $\eta \in \mathbb{R}^m$ satisfying*
$$A\eta \leq 0,\ \eta > 0.$$

Proof of Proposition 1. Plug L, w, and p into A, ξ, and η in Lemma 3, respectively. Then, Lemma 3 implies that exactly one of (a) and (b) holds. □

We can interpret Lemma 3 as a corollary of the fundamental theorem of asset pricing,[11] which is equivalent to Farkas' lemma. Thus, we can explain why Proposition 1 is true in terms of arbitrage-free pricing in an imaginary asset market as discussed in the introduction, which is elaborated in Appendix B.

We can prove Proposition 2 similarly by restating the conditions in the proposition as follows.

(a′) By Lemma 1, a voting rule ϕ is a WMR with nonnegative weights if and only if there exists $w = (w_i)_{i \in N} > 0$ such that
$$\sum_{i \in N} w_i l_{ij} = \sum_{i \in N} w_i (\phi(x^j) x_i^j) \geq 0$$
for each $j \in M$, or equivalently, $w^\top L \geq 0$.

(b′) By definition, a voting rule is *not* weakly robust if and only if there exists $p = (p_j)_{j \in M} > 0$ such that
$$\sum_{j \in N} l_{ij} p_j = \sum_{j : \phi(x^j) = x_i^j} p_j - \sum_{j : \phi(x^j) \neq x_i^j} p_j < 0$$
for each $i \in N$, or equivalently, $Lp \ll 0$.

Proposition 2 states that exactly one of (a′) and (b′) holds. To prove it, we use Lemma 3 again, but in another way.

Proof of Proposition 2. Plug $-L^\top$, p, and w into A, ξ, and η in Lemma 3, respectively, where we replace (n, m) with (m, n). Then, Lemma 3 implies that exactly one of (a′) and (b′) holds. □

[11] For details on the fundamental theorem of asset pricing, see Dybvig and Ross (2003, 2008) and references therein.

5 Robustness vs. efficiency

Rae (1969) and Taylor (1969) were the first to use responsiveness to characterize voting rules, followed by Straffin (1977) and Fleurbaey (2008). In this section, we discuss their results in comparison to our result.

Note that, by Lemma 1, $\phi \in \Phi$ is a WMR with a weight vector $w \in \mathbb{R}^N$ if and only if $\phi(x)\sum_{i\in N} w_i x_i = |\phi(x)\sum_{i\in N} w_i x_i|$ for all $x \in \mathcal{X}$, which is equivalent to the following inequality: for all $\phi' \in \Phi$ and $x \in \mathcal{X}$,

$$\phi(x)\sum_{i\in N} w_i x_i = \left|\phi(x)\sum_{i\in N} w_i x_i\right| = \left|\phi'(x)\sum_{i\in N} w_i x_i\right| \geq \phi'(x)\sum_{i\in N} w_i x_i. \tag{5}$$

This is true if and only if, for all $p \in \Delta(\mathcal{X})$,

$$\sum_{i\in N} w_i E_p[\phi(x)x_i] = \max_{\phi'\in\Phi}\sum_{i\in N} w_i E_p[\phi'(x)x_i], \tag{6}$$

or equivalently,

$$\sum_{i\in N} w_i p(\phi(x) = x_i) = \max_{\phi'\in\Phi}\sum_{i\in N} w_i p(\phi'(x) = x_i). \tag{7}$$

That is, a necessary and sufficient condition for a voting rule to be a WMR is that it maximizes the corresponding weighted sum of responsiveness over all voting rules for each $p \in \Delta(\mathcal{X})$. This result is summarized in the following proposition due to Fleurbaey (2008),[12] where the sufficient condition is weaker.[13]

Proposition 5. *If ϕ is a WMR with a weight vector w, then* (7) *holds for each $p \in \Delta(\mathcal{X})$. For fixed $p \in \Delta(\mathcal{X})^\circ \equiv \{p \in \Delta(\mathcal{X}) : p(x) > 0 \text{ for each } x \in \mathcal{X}\}$, where every x is possible, if* (7) *holds, then ϕ is a WMR with a weight vector w.*

We call the above result the Rae-Taylor-Fleurbaey (RTF) theorem because it generalizes the Rae-Taylor theorem[14] which focuses on a SMR. Note that a WMR in the RTF theorem can have negative weights. In Appendix A, we characterize a WMR with possibly negative weights by introducing a further weaker version of robustness.

[12]See also Brighouse and Fleurbaey (2010), who discuss the implication of this result for democracy.

[13]To see why a weaker condition suffices, suppose that ϕ is not a WMR. Then, (5) does not hold for some $\phi' \in \Phi$ and $x \in \mathcal{X}$, which contradicts (6) and (7) for each $p \in \Delta(\mathcal{X})^\circ$.

[14]See Rae (1969), Taylor (1969), Straffin (1977), and references in Fleurbaey (2008).

The normative implication of the RTF theorem is efficiency and weak efficiency of WMRs.[15] To give the formal definitions of efficiency and weak efficiency in this context, we consider the set of random voting rules. For $\phi, \phi' \in \Phi$ and $\lambda \in [0, 1]$, the convex combination $\lambda\phi + (1-\lambda)\phi' : \mathcal{X} \to [-1, 1]$ is given by $(\lambda\phi + (1-\lambda)\phi')(x) = \lambda\phi(x) + (1-\lambda)\phi'(x)$ for each $x \in \mathcal{X}$. The convex hull of Φ is denoted by $\mathrm{co}(\Phi)$. We regard $\phi \in \mathrm{co}(\Phi)$ as the following random voting rule: the collective decision is $+1$ with probability $(1 + \phi(x))/2$ and -1 with probability $(1 - \phi(x))/2$. For each $\phi \in \mathrm{co}(\Phi)$, the responsiveness of individual i is calculated as $(E_p[\phi(x)x_i] + 1)/2$.[16] Taking responsiveness (or equivalently, $E_p[\phi(x)x_i]$) as an individual's utility, we can define efficiency and weak efficiency as follows.

Definition 3. Fix $p \in \Delta(\mathcal{X})$. A voting rule $\phi \in \Phi$ is efficient if there is no $\phi' \in \mathrm{co}(\Phi)$ such that $E_p[\phi'(x)x_i] \geq E_p[\phi(x)x_i]$ for all $i \in N$ and $E_p[\phi'(x)x_i] > E_p[\phi(x)x_i]$ for at least one $i \in N$. A voting rule $\phi \in \Phi$ is weakly efficient if there is no $\phi' \in \mathrm{co}(\Phi)$ such that $E_p[\phi'(x)x_i] > E_p[\phi(x)x_i]$ for all $i \in N$.

The RTF theorem can be understood as the following normative characterizations of WMRs.[17]

Corollary 6. *Fix $p \in \Delta(\mathcal{X})^\circ$. A voting rule is efficient if and only if it is a WMR with positive weights. A voting rule is weakly efficient if and only if it is a WMR with nonnegative weights.*

Proof. See Appendix D. □

What the RTF theorem and our result have in common is that both characterize a WMR using responsiveness, but the former considers efficiency and the latter considers robustness. In the remainder of this section, we summarize the relationship between these concepts.

First, compare weak robustness and weak efficiency. A weakly efficient rule achieves the optimal outcomes, whereas a weakly robust rule avoids the worst outcomes. However, both of them constitute the same class of voting rules, WMRs with nonnegative weights. Thus, weak

[15]This issue is not formally discussed in Fleurbaey (2008). Instead, Fleurbaey (2008) considers the optimality of a WMR by assuming that w_i is proportional to i's utility, where the weighted sum of responsiveness is the total sum of expected utilities.

[16]When $x \in \mathcal{X}$ is given, the conditional probability that i's decision agrees with the collective decision is $(1 + \phi(x))/2$ if $x_i = 1$ and $(1 - \phi(x))/2$ if $x_i = -1$. Thus, the conditional probability is equal to $(\phi(x)x_i + 1)/2$.

[17]Another normative implication of Proposition 5 is optimality of WMRs in terms of Paretian social preferences, which is immediate from Harsanyi's utilitarianism theorem (Harsanyi, 1955). See Appendix C.

robustness together with weak efficiency gives a dual characterization of WMRs in terms of responsiveness.

Next, compare robustness and efficiency. An efficient rule is a WMR with positive weights and a robust rule is a WMR with nonnegative weights allowing no ties, which can be represented as a WMR with positive weights allowing no ties by Lemma 2. Therefore, the set of robust WMRs is a proper subset of that of efficient WMRs; that is, robustness is a stronger requirement than efficiency. For example, when n is even, a SMR with any anonymous tie-breaking rule is efficient but not robust.

In summary, we obtain the following relationship between efficiency and robustness.

Corollary 7. *A voting rule is weakly efficient if and only if it is weakly robust. A robust rule is efficient, but an efficient rule is not necessarily robust.*

Finally, we emphasize that our characterization of WMRs is not a restatement of the RTF theorem. As discussed in Section 3, it is obvious that a WMR is robust. Thus, the RTF theorem directly implies that an efficient rule without ties is robust. However, the RTF theorem does not say anything about whether a robust rule is efficient. Our contribution is to identify the set of all robust rules, which is not implied by the RTF theorem.

6 Conclusion

The justification of WMRs and, in particular, SMRs based on efficiency arguments or axiomatic characterizations has yielded some of the celebrated contributions to the social choice and voting literature. The two paramount examples rationalizing a SMR within a dichotomous setting are Condorcet's jury theorem and May's theorem,[18] where the rationalization of a voting rule is based on asymptotic (i.e., infinite-individual) probabilistic criteria or deterministic criteria. An alternative approach based on non-asymptotic (i.e., finite-individual) probabilistic criteria was pioneered by Rae (1969), who suggested aggregate expected net gain or aggregate responsiveness as a meaningful criterion for evaluating the performance of a voting rule in the constitutional stage, namely, where the veil of ignorance prevails.

This paper contributes to the latter literature by joining the recently growing literature on economic design with worst-case objectives discussed in the introduction. That is, we introduce normative criteria for voting rules under Knightian uncertainty about individuals' preferences,

[18] See May (1952), Fishburn (1973), and Dasgupta and Maskin (2008).

robustness and weak robustness. Robustness requires that a voting rule should avoid the worst-case scenario in which the true responsiveness of every individual is less than or equal to one-half, and weak robustness requires that a voting rule should avoid the worst-case scenario in which the true responsiveness of every individual is less than one-half. We establish that a voting rule is robust if and only if it is a WMR without any ties and that a voting rule is weakly robust if and only if it is a WMR with any tie-breaking rule when there are ties. We also find that we face a trade-off between robustness and anonymity when the number of individuals is even: we must be content with a nonanonymous rule if we require robustness and we must be content with a weakly robust rule if we require anonymity.

Our result and the RTF theorem (Rae, 1969; Taylor, 1969; Fleurbaey, 2008) have in common that both examine WMRs using responsiveness. However, the RTF theorem characterizes WMRs as efficient or weakly efficient rules achieving the optimal outcomes, whereas our result characterizes WMRs as robust or weakly robust rules avoiding the worst outcomes. Hence, our result complements the renowned RTF theorem by providing a dual characterization of WMRs and, in particular, of SMRs in terms of responsiveness under Knightian uncertainty.

Appendix

A Other robustness concepts

In this appendix, we consider two other robustness concepts.

Strong robustness and a SMR

Even if a voting rule is robust and responsiveness of at least one individual is strictly greater than one-half, the arithmetic mean of responsiveness of all individuals can be less than one-half, which implies that a collective decision reflects minority preferences on average. To avoid this scenario, a voting rule must satisfy the following stronger requirement.

Definition A. A voting rule $\phi \in \Phi$ is *strongly robust* if, for each $p \in \Delta(\mathcal{X})$, the arithmetic mean of responsiveness is strictly greater than one-half:

$$\sum_{i \in N} p(\phi(x) = x_i)/n > 1/2 \text{ for all } p \in \Delta(\mathcal{X}). \tag{A1}$$

Clearly, a strongly robust rule is robust. In the next proposition, we show that a voting rule is strongly robust if and only if it is robust and anonymous; that is, it is a SMR with odd n.

Proposition A. *Suppose that n is odd. Then, a voting rule is strongly robust if and only if it is a SMR. Suppose that n is even. Then, no voting rule is strongly robust.*

Proof. Note that, by (1), (A1) is equivalent to

$$\sum_{i \in N} E_p[\phi(x)x_i] > 0 \text{ for all } p \in \Delta(\mathcal{X}). \tag{A2}$$

Suppose that n is odd and that ϕ is a SMR. Then, ϕ satisfies (A2) by Lemma 1, so a SMR is strongly robust.

Suppose that ϕ is not a SMR. Then, there exist $y \in \mathcal{X}$ and $S \subsetneq N$ such that $|S| < n/2$ and $\phi(y) = y_i$ if and only if $i \in S$. Let $p \in \Delta(\mathcal{X})$ be such that $p(y) = 1$. Then, $\sum_{i \in N} E_p[\phi(x)x_i] = |S| - |N \setminus S| < 0$, violating (A2). Thus, a voting rule is not strongly robust unless it is a SMR.

Suppose that n is even. Let $y \in \mathcal{X}$ be such that $y_i = 1$ for $i \leq n/2$ and $y_i = -1$ for $i \geq n/2+1$. For $p \in \Delta(\mathcal{X})$ with $p(y) = 1$ and any $\phi \in \Phi$, it holds that $\sum_{i \in N} E_p[\phi(x)x_i] = 0$, violating (A2). Thus, no voting rule is strongly robust when n is even. □

Semi-robustness and a WMR

We consider the following weaker version of robustness to characterize a WMR with possibly negative weights.

Definition B. A voting rule $\phi \in \Phi$ is *semi-robust* if, for each $p \in \Delta(\mathcal{X})$, responsiveness of at least one individual is not equal to one-half.

To understand the implication of semi-robustness, imagine that some individuals are more likely to have correct choices and other individuals are more likely to have wrong choices. However, if the responsiveness of every individual is equal to one-half, then it is difficult to extract information from individuals in order to arrive at a correct group decision. A semi-robust rule does not face this problem for any probability distribution.

The following proposition establishes the equivalence of a semi-robust rule and a WMR with possibly negative weights allowing no ties.

Proposition B. *A voting rule is semi-robust if and only if it is a WMR such that there are no ties.*

To prove Proposition B, we use the following theorem of alternatives, which is referred to as Gordan's theorem. This result also appears in the work of Gale (1960, Theorems 2.9) as a corollary of Farkas' lemma.

Lemma A. *Let A be an $n \times m$ matrix. Exactly one of the following alternatives holds.*

- *There exists $\xi \in \mathbb{R}^n$ satisfying*

$$\xi^\top A \gg 0.$$

- *There exists $\eta \in \mathbb{R}^m$ satisfying*

$$A\eta = 0, \ \eta > 0.$$

Proof of Proposition B. We can restate the conditions in Proposition B as follows.

(a″) By Lemma 1, a voting rule ϕ is a WMR allowing no ties if and only if there exists $w = (w_i)_{i \in N} \neq 0$ such that

$$\sum_{i \in N} w_i l_{ij} = \sum_{i \in N} w_i (\phi(x^j) x_i^j) > 0$$

for each $j \in M$, or equivalently, $w^\top L \gg 0$.

(b″) By definition, a voting rule is *not* semi-robust if and only if there exists $p = (p_j)_{j \in M} > 0$ such that

$$\sum_{j \in N} l_{ij} p_j = \sum_{j : \phi(x^j) = x_i^j} p_j - \sum_{j : \phi(x^j) \neq x_i^j} p_j = 0$$

for each $i \in N$, or equivalently, $Lp = 0$.

Proposition B states that exactly one of (a″) and (b″) holds, which is true by Lemma A. In fact, by plugging L, w, and p into A, ξ, and η in Lemma A, respectively, we can conclude that exactly one of (a″) and (b″) holds. □

B An imaginary asset market

We can explain why Proposition 1 is true in terms of arbitrage-free pricing in an imaginary asset market because we can interpret Lemma 3 as a corollary of the fundamental theorem of asset pricing.

Let M and N be the set of states and the set of assets, respectively. One unit of asset $i \in N$ yields a payoff l_{ij} when state $j \in M$ is realized. Recall that l_{ij} equals $+1$ if i's choice agrees with the collective decision and -1 otherwise. The matrix L is referred to as the payoff matrix. We denote by $q = (q_i)_{i \in N}$ the vector of prices of the n assets.

A portfolio defined by a vector $w = (w_i)_{i \in N}$ consists of w_i units of asset i for each $i \in N$. It yields a payoff $\sum_{i \in N} w_i l_{ij}$ when state $j \in M$ is realized, which is summarized in $w^\top L = (\sum_{i \in N} w_i l_{ij})_{j \in M}$. The price of the portfolio is $w^\top q = \sum_{i \in N} q_i w_i$.

We say that a price vector q is arbitrage-free if $w^\top L \geq 0$ implies $w^\top q \geq 0$; that is, the price of any portfolio yielding a nonnegative payoff in each state is nonnegative. We say that a price vector q is determined by a nonnegative linear pricing rule if there exists a nonnegative vector $p = (p_j)_{j \in M} > 0$, which is referred to as a state price, such that $q = Lp$. The fundamental theorem of asset pricing establishes the equivalence of an arbitrage-free price and the existence of a nonnegative linear pricing rule, which is immediate from Farkas' lemma.

Claim A. *A price vector q is arbitrage-free if and only if it is determined by a nonnegative linear pricing rule. That is, the set of all arbitrage-free price vectors is $\{q : q = Lp,\ p > 0\}$.*

The fundamental theorem of asset pricing has the following corollary, which is immediate from Lemma 3 (a corollary of Farkas' lemma).

Claim B. *There exists a portfolio with nonnegative weights in all assets (i.e. no short selling) yielding a strictly positive payoff in each state if and only if, for any arbitrage-free price vector, the price of at least one asset is strictly positive.*

The former condition is restated as $w^\top L \gg 0$ for some $w \geq 0$ and the latter condition is restated as $Lp \nleq 0$ for all $p > 0$. Therefore, Claim B implies the equivalence of a robust rule and a WMR with nonnegative weights allowing no ties.

This paper does not discuss how to find optimal weights of WMRs, whereas it is a central topic in modern portfolio theory to determine optimal weights. Thus, modern portfolio theory could be useful to find optimal weights of WMRs.

C Robustness vs. optimality

In this appendix, we demonstrate that the RTF theorem has another normative implication, optimality in terms of Paretian social preferences.

Definition C. Fix $p \in \Delta(\mathcal{X})$. A voting rule $\phi \in \Phi$ is optimal with respect to a Paretian von Neumann-Morgenstern (vNM) welfare function if there exists a linear welfare function

$v : \mathrm{co}(\Phi) \to \mathbb{R}$ such that (i) $E_p[\phi'(x)x_i] \geq E_p[\phi''(x)x_i]$ for each $i \in N$ implies $v(\phi') \geq v(\phi'')$ for $\phi', \phi'' \in \mathrm{co}(\Phi)$, and (ii) $v(\phi) \geq v(\phi')$ for all $\phi' \in \mathrm{co}(\Phi)$.

Harsanyi's utilitarianism theorem (Harsanyi, 1955) states that if a linear welfare function $v : \mathrm{co}(\Phi) \to \mathbb{R}$ satisfies the condition (i), then there exists a nonnegative vector $w > 0$ such that $v(\phi) = \sum_{i \in N} w_i E[\phi(x)x_i]$ for each $\phi \in \mathrm{co}(\Phi)$ (cf. Domotor, 1979; Weymark, 1993; Mandler, 2005). Thus, Proposition 5 can be understood as the following normative characterization of WMRs.

Corollary C. *Fix $p \in \Delta(\mathcal{X})^\circ$. A voting rule is optimal with respect to a Paretian vNM welfare function if and only if it is a WMR with nonnegative weights, where the weights of a Paretian vNM welfare function are the same as those of a WMR.*

It should be emphasized that an optimal rule depends upon a vNM welfare function. That is, a WMR with a weight vector w is optimal only when we adopt a vNM welfare function with the same weight vector w. This is in sharp contrast to an efficient rule which does not presuppose any welfare function.

This corollary and Propositions 1 and 2 reveal the following relationship between robustness, weak robustness, and optimality.

Corollary D. *A voting rule is optimal with respect to a Paretian vNM welfare function if and only if it is weakly robust. A robust rule is optimal, but an optimal rule is not necessarily robust.*

D Proof of Corollary 6

Proof. Proposition 5 states that a voting rule is a WMR with a weight vector w if and only if (6) holds. Mathematically, (6) is equivalent to expected utility maximization, where Φ is the set of actions, N is the set of states, and $w_i / \sum_j w_j$ is a probability of state $i \in N$. Therefore, we can apply the theorem of Wald (1950) on admissible decision functions or that of Pearce (1984) on undominated strategies. In particular, Theorems 5.2.1 and 5.2.5 in Blackwell and Girshick (1954) are useful. Theorem 5.2.1 implies that a voting rule is weakly efficient if and only if there exists a weight vector $w > 0$ such that (6) holds. Theorem 5.2.5 implies that a voting rule is efficient if and only if there exists a weight vector $w \gg 0$ such that (6) holds. Therefore, this corollary holds by Proposition 5. $\square$

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