

RCNE Discussion Paper series

No. 4

Characterizations of Social Choice Correspondences  
Based on Equality of Capabilities in a Pure Exchange Economy

Norihito Sakamoto

2017年11月

**Research Center for Normative Economics**  
Institute of Economic Research  
Hitotsubashi University  
<http://www.ier.hit-u.ac.jp/rcne/>

# Characterizations of Social Choice Correspondences Based on Equality of Capabilities in a Pure Exchange Economy

Norihito Sakamoto\*

March, 2012

revised October, 2017

Keywords: capability approach, equality of capabilities, common capability

## Abstract

This paper examines theoretical properties of Amartya Sen's capability approach in a formal model which describes a simple exchange economy with unequal abilities. Specifically, we define and axiomatically characterize the following two classes of social choice correspondences (SCCs) based on the notion of "*equality of capabilities*": (1) SCCs which assign egalitarian and efficient allocations in terms of a social preference ordering defined on capability sets; (2) SCCs which maximize an intersection of all individuals' capability sets with respect to the relation of set inclusion. Our main results show that in a single-good economy, two SCCs can be characterized by a similar combination of three requirements: principles of equal treatment, Pareto efficiency, and rank preservation. However, in a two or more goods economy, a class of SCCs maximizing an intersection of capabilities cannot be characterized by the above three principles, while they are still necessary conditions.

JEL codes: D60, D63, I30, I31, I32

---

\*Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda, Chiba, Japan, 278-8510. E-mail: n-sakamoto@rs.tus.ac.jp; Tel: +81-4-7124-1501; Fax: +81-4-7122-1560. The author would like to thank Kotaro Suzumura, Yuko Mori, Koichi Tadenuma, Naoki Yoshihara, and Reiko Gotoh for their helpful comments and suggestions. The first draft of this paper was written while the author was staying the Oxford Poverty and Human Development Initiative (OPHI) of the University of Oxford as a visiting research fellow. The author is grateful for its hospitality. Financial support from Ministry of Education, Culture, Sports, Science and Technology is gratefully acknowledged.

# 1 Introduction

The capability approach (CA) is one of the most important approaches to evaluating human well-being. This approach evaluates one's well-being by measuring an extent of states which a person can be or abilities to which a person can do. Since CA avoids difficulties with the adaptive preference problem and diversity among person's abilities to utilize characteristics from consuming commodities, CA have many advantages over basic methods to measure human well-being by utility, income, happiness, and life satisfaction.

In his celebrated works, Sen (1980; 1985) criticized traditional approaches to interpret income or utility as a surrogate indicator of well-being because of problems with adaptive preferences and income fetishism. Then, he claimed CA could properly reflect human well-being in contrast to the above approaches and there is a good reason to care for the equality of capabilities on the problem of distributive justice. However, some problems must occur on making capabilities equal among people. In the case of the equality of welfare or income, it is easy to achieve the equality because both utilities and incomes are real-numbers and there must be proper resource transfers equalizing these single-dimensional indicators. However, since a capability is a set of what a person can do or be, we cannot equalize them among people with unequal abilities to utilize characteristics from their consumption bundles. Hence, we must consider what the equality of capabilities exactly means. Historically, there are at least two methods to make capabilities equal among people.

The first method to achieve the equality of capabilities, which was proposed and examined by many studies (Herrero 1996; Herrero, et al. 1997; Gotoh and Yoshihara 2003), is to maximize an intersection of all persons' capability sets – we call the intersection a *common capability*– in terms of the relation of set inclusion. Following this approach, making the set of functionings that all people enjoys maximal is considered as achieving the equality of capabilities. The second method is to equalize a value of each person's capability in terms of a social preference ordering on capabilities which

is given exogenously (Yoshihara and Xu 2006; 2009). Following this approach, making a value of each person's capability set the same one is considered as achieving the equality of capabilities. These two methods are both reasonable and rational. Then, the following simple example illustrates a difference between these two methods.

Consider a single-good and two-functionings economy. For simplification, suppose that a single-good is money and the amount of money is given by  $\omega \in \mathbb{R}_+$ . Let two functionings be “moving toward places that one wants to go to” and “communicating with people who speak only oral languages.” In this economy, there are three persons. Assume that individual 1 has no physical disability, individual 2 has a hearing impairment, and individual 3 has a visual impairment. Then, each person's capability is represented by the following equations:

$$C_1(x_1) = \{(f_{11}, f_{12}) \in \mathbb{R}_+^2 | f_{11} + f_{12} \leq x_1\},$$

$$C_2(x_2) = \{(f_{21}, f_{22}) \in \mathbb{R}_+^2 | f_{21} + 9f_{22} \leq x_2\},$$

$$C_3(x_3) = \{(f_{31}, f_{32}) \in \mathbb{R}_+^2 | 9f_{31} + f_{32} \leq x_3\},$$

where for all  $i \in \{1, 2, 3\}$ ,  $C_i$  is an individual  $i$ 's capability set,  $x_i$  is  $i$ 's money,  $f_{i1}$  and  $f_{i2}$  are  $i$ 's functionings, that is,  $f_{i1}$  means an individual  $i$ 's degree of “moving toward places one wants to go to” and a functioning  $f_{i2}$  means an individual  $i$ 's degree of “communicating with people who speak only oral languages.”

Since it is difficult for a person with a hearing impairment (resp. a visually impairment) to communicate with people who speak only oral languages (resp. to move toward places one want to go to), individual 1's capability set includes the other's one whenever all persons have the same resource. Let a social preference ordering  $\succsim$  on capability sets be an ordering proposed by Xu (2002; 2003) and Gaertner and Xu (2008) as follows: for all  $i, j \in \{1, 2, 3\}$  and all capability sets  $C_i(x_i)$ ,  $C_j(x_j)$ ,

$$C_i(x_i) \succsim C_j(x_j) \Leftrightarrow \max_{f_i \in C_i(x_i)} f_{i1}^{\frac{1}{2}} f_{i2}^{\frac{1}{2}} \geq \max_{f_j \in C_j(x_j)} f_{j1}^{\frac{1}{2}} f_{j2}^{\frac{1}{2}}.$$

Then,  $(x_1, x_2, x_3) = (\frac{1}{7}\omega, \frac{3}{7}\omega, \frac{3}{7}\omega)$  is the unique solution that maximizes values of all persons' capability sets provided that every capability set has the same value based on the above social preference ordering. On the other hand,  $(x_1, x_2, x_3) = (\frac{1}{11}\omega, \frac{5}{11}\omega, \frac{5}{11}\omega)$

is one of solutions that maximize the intersection of all persons' capability sets in terms of the relation of set inclusion.

Figures 1 and 2 show how two egalitarian views for CA would give different solutions. In the method of maximizing an intersection of all persons' capability sets, the value of 1's capability set is the worst in terms of the social preference ordering. On the other hand, in the method of equalizing the value of each person's capability set, all people have the same value of capability sets in terms of the social preference ordering, but the intersection of them is not maximal in terms of the relation of set inclusion.<sup>1</sup> Generally, if a common capability is maximal, then there is no individual whose capability set includes the common capability. Therefore, as seen in the above example, maximizing a common capability tends to give more resources to those with disabilities rather than equalizing the value of each person's capability set. Thus, the solutions of distributive justice for CA critically depend on the concepts of equality of capabilities.

This paper investigates theoretical properties of social choice correspondences that embody the above two concepts of equality of capabilities and axiomatically characterizes them in traditional pure exchange economies. As a result, we show that two social choice correspondences can be characterized by three categories of axioms in a single-good economy. The first category is a class of *equal treatment* axioms that requires a same capability set for a person with a same ability. The second category is a class of *Pareto efficiency* axioms that requires no feasible allocation to make all persons' capability sets better off. The third category is a class of *rank preservation* axioms that requires people with the same position among their capability assignments in some situations to be still in the same position in different situations. The difference between two social choice correspondences arises from differences of binary relations used in Pareto efficiency axioms and of relative rankings on capability assignments in

---

<sup>1</sup>It can be shown that the common capability is not maximal as follows: In figure 2, individual 1's capability set includes the common capability. Then, a transfer from individual 1 to individuals 2 and 3 makes 2 and 3's capability sets better off, and this improvement makes the common capability larger.

rank preservation axioms. Finally, in a two or more goods economy, we will provide an example which shows that a class of SCCs maximizing common capabilities cannot be characterized by the above three categories of axioms that are still necessary conditions for them.

The structure of the paper is as follows. The next section explains our basic notations and definitions. Section 3 provides our axioms and shows characterization theorems. Furthermore, we give a counter example to see how our axioms fail to characterize social choice correspondences maximizing an intersection of all individuals' capabilities in a simple two-goods economy. Finally, Section 4 concludes the paper with some remarks.

## 2 Basic Notations and Definitions

Consider an  $n$ -individuals,  $m$ -goods, and  $k$ -functionings model in the canonical division economy<sup>2</sup>. For all natural numbers  $l$ , let  $\mathbb{R}_+^l$  (resp.  $\mathbb{R}_{++}^l$ ) be the non-negative (resp. positive)  $l$ -dimensional Euclidean space. A society is consisted of  $n$  individuals. Let  $N = \{1, \dots, n\}$  denote the set of individuals. We consider  $m$ -goods exchange economies and  $\Omega = \mathbb{R}_{++}^m$  denotes the set of initial endowments. An allocation is a vector  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^{nm}$  where each  $x_i = (x_{i1}, \dots, x_{im}) \in \mathbb{R}_+^m$  is a consumption bundle of individual  $i \in N$ . For all initial endowments  $\omega \in \Omega$ , an allocation  $x$  is *feasible* if the total amount of all individuals' consumption bundles is smaller than the initial endowment, that is,  $\omega \geq \sum_{i \in N} x_i$ .<sup>3</sup> For all  $\omega \in \Omega$ , the set of feasible allocations is represented by  $F(\omega)$ . Each individual is assumed to be have his/her *capability correspondence*  $C_i$  which assigns the set of his/her available functionings to each consumption bundle. Then, all capability correspondences satisfy the following properties:

*Compactness:*  $\forall x_i \in \mathbb{R}_+^m$ ,  $C_i(x_i)$  is a bounded and closed set in  $\mathbb{R}_+^k$ .

---

<sup>2</sup>Alternatively, our economy can be interpreted as an  $n$ -individuals,  $m$ -goods, and  $k$ -characteristics model *a la* Gorman (1956) and Lancaster (1966).

<sup>3</sup>Given two vectors  $x$  and  $y$  in  $\mathbb{R}^m$ ,  $x \geq y$  if and only if  $x_k \geq y_k$  for all  $k$  in  $\{1, 2, \dots, m\}$ ;  $x > y$  if and only if  $x \geq y$  and  $x \neq y$ ;  $x > y$  if and only if  $x_k > y_k$  for all  $k$  in  $\{1, 2, \dots, m\}$ .

*Comprehensiveness:*  $\forall x_i \in \mathbb{R}_+^m$ , if  $f_i \in C_i(x_i)$  and  $f_i \geq f'_i$ , then  $f'_i \in C_i(x_i)$ .

*Continuity:*  $C_i$  is continuous in  $\mathbb{R}_+^m$ .

*Strict Monotonicity:*  $\forall x_i, x'_i \in \mathbb{R}_+^m$ , if  $x_i \geq x'_i$ , then  $C_i(x_i) \supset C_i(x'_i)$ . Moreover, if  $x_i > x'_i$ , then  $\text{int}C_i(x_i) \supset C_i(x'_i)$ .

In addition, suppose that each capability correspondence satisfies  $C_i(0, \dots, 0) = (0, \dots, 0)$ . If a capability correspondence  $C_i$  satisfies all above requirements, then it holds  $C_i(x_i) \cap \mathbb{R}_{++}^k \neq \emptyset$  for all  $i \in N$  and  $x_i \in \mathbb{R}_+^m \setminus \{\mathbf{0}\}$ . This means common capabilities must be the subset of positive  $k$ -dimensional Euclidean space for all allocations in  $\mathbb{R}_+^{nm}$ . The set of all capability correspondences is denoted by  $\mathcal{C}$ . Let the set of all compact and comprehensive *capability sets* be  $\mathcal{K}$ . Then, we assume that there is a *social preference ordering*  $\succsim$  on  $\mathcal{K}$ .<sup>4</sup> Let  $\succ$  and  $\sim$  be respectively asymmetric and symmetric part of  $\succsim$ . For all capability sets  $K, K' \in \mathcal{K}$ ,  $K \succsim K'$  means that a capability set  $K$  is at least as good as a capability set  $K'$ . In addition, suppose that all social preference orderings satisfy the properties of *continuity* and *set dominance*, that is, for all  $K \in \mathcal{K}$ ,  $\{K' \in \mathcal{K} | K \succsim K'\}$  and  $\{K' \in \mathcal{K} | K' \succsim K\}$  are both closed, and for all  $K, K' \in \mathcal{K}$ ,  $K \supseteq K'$  implies  $K \succsim K'$ . Then, let  $\mathcal{R}_{\mathcal{K}}$  denote the set of social preference orderings on  $\mathcal{K}$  satisfying the properties of continuity and set dominance. For simplicity of our analysis, a social preference ordering  $\succsim \in \mathcal{R}_{\mathcal{K}}$  is given and fixed throughout this paper.<sup>5</sup> In our setting of the  $n$ -individuals,  $m$ -goods, and  $k$ -functionings model, we can describe an economy by seeing two variables: profiles of capability correspondences and initial endowments. We write an economy as  $\mathbf{e} = (C_N, \omega)$ . Let  $E$  denote the Cartesian product of  $\mathcal{C}^n \times \Omega$ . A social choice correspondence is a mapping  $S$  which assigns a non-empty subset of *feasible allocations* to each economy. That is, for all  $\mathbf{e} = (C_N, \omega) \in E$ ,  $S(\mathbf{e}) \subseteq F(\omega)$  and  $S(\mathbf{e}) \neq \emptyset$ .

---

<sup>4</sup>A social preference ordering may be constructed by aggregating each individual's preference relation defined on the set of capabilities. Alternatively, following Herrero, *et al.* (1998), a social preference ordering can be interpreted as a deduced relation based on social value judgments on the set of *uniform* capability assignments, *i.e.* for all  $K, K' \in \mathcal{K}$ ,  $K \succsim K'$  if and only if  $(K, \dots, K) \succsim^* (K', \dots, K')$  where the binary relation  $\succsim^*$  is an ordering defined on  $\mathcal{K}^n$ .

<sup>5</sup>Note that our social preference orderings are only required to belong to the set  $\mathcal{R}_{\mathcal{K}}$ . Hence, for all  $\succsim \in \mathcal{R}_{\mathcal{K}}$ , our results are robust.

Our purpose is to investigate the properties of some reasonable social choice correspondences and characterize them. In order to define a class of social choice correspondences based on the notion of “equality of capabilities,” consider two social choice correspondences as follows:

**Definition 1:** A social choice correspondence  $S^E$  is an *egalitarian rule* if and only if  $\forall \mathbf{e} = (C_N, \omega) \in E$ ,  $S^E(\mathbf{e}) \subseteq \{x \in F(\omega) | \forall i, j \in N, C_i(x_i) \sim C_j(x_j) \ \& \ \nexists x' \in F(\omega), \forall i \in N, C_i(x'_i) \succ C_i(x_i)\}$ .

**Definition 2:** A social choice correspondence  $S^{CM}$  is a *common capability maximin rule* if and only if  $\forall \mathbf{e} = (C_N, \omega) \in E$ ,  $S^{CM}(\mathbf{e}) \subseteq \{x \in F(\omega) | \nexists x' \in F(\omega), \bigcap_{i \in N} C_i(x'_i) \supset \bigcap_{i \in N} C_i(x_i)\}$ .

An egalitarian rule, which was proposed by Xu and Yoshihara (2006; 2009), assigns egalitarian and efficient allocations in the sense that all individuals’ capability sets have the equal value and no allocation improves all individuals’ capability sets given a social preference ordering  $\succsim$ . On the other hand, a common capability maximin rule, which was proposed by Sen (1985)<sup>6</sup>, also assigns egalitarian and efficient allocations in the sense that it focuses on the set of functionings all individuals can enjoy and chooses feasible allocations that make a common capability set maximal with respect to set inclusion.

Note that our social choice correspondences are *well-defined* if a class of social preference orderings satisfies continuity. That is, given a continuous social preference

---

<sup>6</sup>A variation of this rule was analyzed by previous studies (Gotoh and Yoshihara 2003; Gotoh, Suzumura and Yoshihara 2010) in the setting of simple production economies. Gotoh and Yoshihara (2003) defined the following rule:

**Definition 3:** A social choice correspondence  $S^J$  is a *J-based capability maximin rule* if and only if, given a social preference ordering  $\succsim_J$ ,  $\forall \mathbf{e} = (C_N, \omega) \in E$ ,  $S^J(\mathbf{e}) \subseteq \{x \in F(\omega) | \nexists x' \in F(\omega), \bigcap_{i \in N} C_i(x'_i) \succ_J \bigcap_{i \in N} C_i(x_i)\}$ .

By definition, the J-based capability maximin rule is a refinement of the common capability maximin rule whenever a class of social preference orderings satisfy the property of set dominance. That is,  $\forall \succsim_J \in \mathcal{R}_{\mathcal{X}}, \forall \mathbf{e} \in E, S^J(\mathbf{e}) \subseteq S^{CM}(\mathbf{e})$ .



ordering, these social choice correspondences can assign a non-empty subset of feasible allocations for all exchange economies<sup>7</sup>.

### 3 Axiomatic Characterization

In the above setting of exchange economies, we axiomatically characterize two social choice correspondences by using three classes of reasonable axioms.

First, let us define the set of Pareto assignments applied to the capability approach. There are at least two classes of the Pareto set because evaluation methods of capability sets can be defined in terms of two binary relations: a social preference ordering  $\succsim$  and the relation of set inclusion  $\supseteq$ . Therefore, for all  $\mathbf{e} \in E$ , we can define two Pareto sets  $P(\mathbf{e})$  and  $P^{SI}(\mathbf{e})$  as follows:

$$P(\mathbf{e}) = \{x \in F(\omega) \mid \nexists x' \in F(\omega), \forall i \in N, C_i(x'_i) \succsim C_i(x_i) \& \exists j \in N, C_j(x'_j) \succ C_j(x_j)\},$$

$$P^{SI}(\mathbf{e}) = \{x \in F(\omega) \mid \nexists x' \in F(\omega), \forall i \in N, C_i(x'_i) \supseteq C_i(x_i) \& \exists j \in N, C_j(x'_j) \supset C_j(x_j)\}.$$

Since a social preference ordering  $\succsim$  satisfies the property of set dominance,  $P(\mathbf{e}) \subseteq P^{SI}(\mathbf{e})$  for all  $\mathbf{e} \in E$ . Then, consider the following two Pareto efficiency axioms based on the capability approach.

**Pareto Efficiency of Capability Assignments (PECA):** A social choice correspondence  $S$  satisfies *Pareto Efficiency of Capability Assignments* if and only if  $\forall \mathbf{e} \in E$ ,  $S(\mathbf{e}) \subseteq P(\mathbf{e})$ .

This axiom requires that capability assignments be efficient in terms of social preference orderings defined on  $\mathcal{K}$ .

The following axiom, which has the same spirit of *Pareto Efficiency with respect to Capabilities* proposed by Gotoh and Yoshihara (2003), requires no allocation improve capability assignments with respect to set inclusion.

---

<sup>7</sup>Precisely, a common capability maximin rule always assigns a non-empty subset of feasible allocations for all situations because of continuity of capability correspondences. But a class of egalitarian social choice correspondences might assign an empty set if a social preference ordering could not satisfy continuity.

**Pareto Efficiency of Capability Assignments w.r.t. Set Inclusion (PESI):**

A social choice correspondence  $S$  satisfies *Pareto Efficiency of Capability Assignments w.r.t. Set Inclusion* if and only if  $\forall \mathbf{e} \in E$ ,  $S(\mathbf{e}) \subseteq P^{SI}(\mathbf{e})$ .

Obviously, PECA implies PESI but not vice versa.

The second class of our axioms is an equal treatment axiom for capability assignments.

**Equal Capability for Reference Ability (ECRA):** A social choice correspondence  $S$  satisfies *Equal Capability for Reference Ability* if and only if  $\exists \tilde{C} \in \mathcal{C}$ ,  $\forall \mathbf{e} = (C_N, \omega) \in E$ ,  $\forall x \in S(\mathbf{e})$ , if  $\forall i \in N$ ,  $C_i = \tilde{C}$ , then  $\forall i, j \in N$ ,  $C_i(x_i) = C_j(x_j)$ .

The axiom ECRA is essentially the same one as *Equal Resource for Reference Talent* (Fleurbaey 1995). This axiom requires that individuals should have the same capability set if all individuals have the reference capability correspondence. Obviously, our two social choice correspondences satisfy this axiom.<sup>8</sup>

Next, we introduce two rankings on capability assignments to define a class of rank preservation axioms.

Given a profile of capability sets  $K_N \in \mathcal{K}^n$  and a social preference ordering  $\succsim \in \mathcal{R}_{\mathcal{K}}$ , consider the following two rankings that measure relative positions among capability assignments:

$$\text{rank}_i^C(K_N) = \#\{j \in N \mid K_j \succsim K_i\},$$

---

<sup>8</sup>Our social choice correspondences can be characterized by using stronger versions of ECRA as follows:

**Equal Capability for Equal Ability (ECEA):** A social choice correspondence  $S$  satisfies *Equal Capability for Equal Ability* if and only if  $\forall \mathbf{e} = (C_N, \omega) \in E$ ,  $\forall x \in S(\mathbf{e})$ ,  $\forall i, j \in N$ , if  $C_i = C_j$ , then  $\forall i, j \in N$ ,  $C_i(x_i) = C_j(x_j)$ .

**Equal Capability for Uniform Ability (ECUA):** A social choice correspondence  $S$  satisfies *Equal Capability for Uniform Ability* if and only if  $\forall \mathbf{e} = (C_N, \omega) \in E$ ,  $\forall x \in S(\mathbf{e})$ , if  $\forall i, j \in N$ ,  $C_i = C_j$ , then  $\forall i, j \in N$ ,  $C_i(x_i) = C_j(x_j)$ .

Note that these axioms have the same spirit of *Equal Resource for Equal Talent* and *Equal Resource for Uniform Talent*, which were respectively proposed by Fleurbaey (1994; 1995) and Bossert (1995). In addition, Gotoh and Yoshihara (2003) proposed *Equal Attainable Sets for Equal Handicaps* that applied Equal Resource for Equal Talent to the capability framework.

where for all sets  $A$ ,  $\#A$  is a cardinality of  $A$ .

$$\text{rank}_i^f(K_N) = \max_{f_i \in \partial K_i} \#\{j \in N | f_j \in K_j\},$$

where for all sets  $A$ ,  $\partial A$  is an undominated boundary of  $A$ , that is,  $\partial A = \{a \in A | \nexists a' \in A, a' \geq a\}$ .<sup>9</sup>

The first ranking evaluates individual  $i$ 's capability set in terms of the number of individuals with his/her capability set that is socially preferred to  $i$ 's capability set. By definition, individual  $i$ 's rank equals 1 in this ranking if  $i$ 's capability set is the only greatest element among all individuals' capability sets w.r.t.  $\succsim$ . On the contrary,  $i$ 's rank equals  $n$  if  $i$ 's capability set is the smallest element among all individuals' capability sets w.r.t.  $\succsim$ .

The second ranking evaluates individual  $i$ 's capability set in terms of the maximum number of individuals with his/her functionings that dominate a functioning on the undominated boundary of  $i$ 's capability set. Generally, individual  $i$ 's rank equals 1 in this ranking if  $i$ 's capability set is a superset of all individuals' capability sets. On the contrary,  $i$ 's rank equals  $n$  if  $i$ 's capability set includes functionings on the boundary that all individuals enjoys. Note that if  $i$ 's rank equals  $n$ , then a subset of the undominated boundary of  $i$ 's capability set consists an undominated boundary of a common capability set.

Then, we introduce the following rank preservation axioms which require individuals with the same rank never change their relative positions for different situations of initial endowments or capability correspondences.

**Preserving Relative Ranking among Initial Endowments w.r.t. Rank\***  
**(PRRIE-R\*):** A social choice correspondence  $S$  satisfies *Preserving Relative Ranking among Initial Endowments w.r.t. Rank\** if and only if  $\forall \mathbf{e} = (C_N, \omega), \mathbf{e}' = (C_N, \omega') \in E, \forall x \in S(\mathbf{e}), \forall x' \in S(\mathbf{e}'),$  if  $\exists i, j \in N, \text{rank}_i^*(K_N) = \text{rank}_j^*(K_N),$  then  $\text{rank}_i^*(K'_N) =$

---

<sup>9</sup>We can get the same results if we would modify these rankings as follows:

$$\text{rank}_i^C(K_N) = \#\{j \in N | K_j \succ K_i\} + 1,$$

$$\text{rank}_i^f(K_N) = \max_{f_i \in \partial K_i} \#\{j \in N | f_j \in \text{int}K_j\} + 1.$$

$rank_j^*(K'_N)$ , where  $K_N = (C_1(x_1), \dots, C_n(x_n))$  and  $K'_N = (C_1(x'_1), \dots, C_n(x'_n))$ .

**Preserving Relative Ranking among Capability Profiles w.r.t. Rank\***  
**(PRRCP-R\*):** A social choice correspondence  $S$  satisfies *Preserving Relative Ranking among Capability Profiles w.r.t. Rank\** if and only if  $\forall \mathbf{e} = (C_N, \omega), \mathbf{e}' = (C'_N, \omega) \in E, \forall x \in S(\mathbf{e}), \forall x' \in S(\mathbf{e}')$ , if  $\exists i, j \in N, rank_i^*(K_N) = rank_j^*(K_N)$ , then  $rank_i^*(K'_N) = rank_j^*(K'_N)$ , where  $K_N = (C_1(x_1), \dots, C_n(x_n))$  and  $K'_N = (C'_1(x'_1), \dots, C'_n(x'_n))$ .

In the above axioms, if  $rank_i^*(\bullet) = rank_i^C(\bullet)$  (resp.  $rank_i^*(\bullet) = rank_i^f(\bullet)$ ), then we simply write the axioms as PRRIE-R<sup>C</sup> and PRRIE-R<sup>C</sup> (resp. PRRCP-R<sup>f</sup> and PRRCP-R<sup>f</sup>).

Now our social choice correspondences can be characterized by using the above axioms.

**Theorem 1:** *A social choice correspondence  $S$  satisfies PECA, ECRA, PRRIE-R<sup>C</sup> and PRRIE-R<sup>C</sup> if and only if  $S = S^E$ .*

[Proof] It is easy to prove the necessary part of the statement, so we omit it. We have to show only the sufficiency. Let  $S$  satisfy PESI, ECRA, PRRIE-R<sup>C</sup> and PRRCP-R<sup>C</sup>. Given  $\mathbf{e} = (C_N, \omega) \in E$ , suppose that  $C_i = \tilde{C}$  for all  $i \in N$ . Then, ECRA implies that for all  $i, j \in N, C_i(S_i(\mathbf{e})) = C_j(S_j(\mathbf{e}))$ . By combining PECA with ECRA, we have that  $\forall x \in S(\mathbf{e}), \forall i \in N, C_i(x_i) = \tilde{C}(\omega/n)$ . That is, every individual has the same rank  $n$ .

By repeating applications of the axioms of PRRIE-R<sup>C</sup> and PRRCP-R<sup>C</sup>, the above fact implies that for all  $\mathbf{e} = (C_N, \omega) \in E$ , all  $x \in S(\mathbf{e})$ , all  $i, j \in N, rank_i^f(K_N) = rank_j^f(K_N)$ , where  $K_N = (C_1(x_1), \dots, C_n(x_n))$ . Then, completeness of  $\succsim$  implies that every individual's rank equals  $n$  for all situations. Therefore, we have that  $\forall \mathbf{e} = (C_N, \omega) \in E, \forall x \in S(\mathbf{e}), \forall i, j \in N, C_i(x_i) \sim C_j(x_j)$ . In addition, PECA implies that  $\forall \mathbf{e} = (C_N, \omega) \in E, S(\mathbf{e}) \subseteq P(\mathbf{e})$ .

Hence, if a social choice correspondence  $S$  satisfies PECA, ECRA, PRRIE-R<sup>C</sup> and

PRRCP- $R^C$ , then it is an egalitarian rule.||

The next theorem shows that for economies with single-good, a class of common capability maximin rules can be characterized by the similar axioms used in Theorem 1.

**Theorem 2:** *Suppose  $m = 1$ . Then, a social choice correspondence  $S$  satisfies PESI, ECRA, PRRIE- $R^f$  and PRRCP- $R^f$  if and only if  $S = S^{CM}$ .*

[Proof] It is easy to prove the necessary part of the statement, so we omit it. We have to show only the sufficiency. Let  $S$  satisfy PESI, ECRA, PRRIE- $R^f$  and PRRCP- $R^f$ . Given  $\mathbf{e} = (C_N, \omega) \in E$ , suppose that  $C_i = \tilde{C}$  for all  $i \in N$ . Then, ECRA implies that for all  $i, j \in N$ ,  $C_i(S_i(\mathbf{e})) = C_j(S_j(\mathbf{e}))$ . By combining PECA with ECRA, we have that  $\forall x \in S(\mathbf{e}), \forall i \in N, C_i(x_i) = \tilde{C}(\omega/n)$ . In this case, every individual has the same rank  $n$ .

By repeating applications of the axioms of PRRIE- $R^f$  and PRRCP- $R^f$ , the above fact implies that for all  $\mathbf{e} = (C_N, \omega) \in E$ , all  $x \in S(\mathbf{e})$ , all  $i, j \in N$ ,  $rank_i^f(K_N) = rank_j^f(K_N)$ , where  $K_N = (C_1(x_1), \dots, C_n(x_n))$ . Then, since there is an individual  $k \in N$  such that the subset of an undominated boundary of his/her capability set equals the subset of an undominated boundary of the common capability, it is impossible that  $\forall \mathbf{e} = (C_N, \omega) \in E, \forall x \in S(\mathbf{e}), \forall i \in N, rank_i^f(K_N) < n$ . Therefore, every individual's rank equals  $n$  for all situations. This means that for all  $\mathbf{e} = (C_N, \omega) \in E$ , all  $x \in S(\mathbf{e})$ , all  $i \in N$ ,  $(\partial C_i(x_i) \cap \partial \bigcap_{j \in N} C_j(x_j)) \neq \emptyset$ .

Then, we will show that there does not exist feasible allocation  $x'$  such that  $\bigcap_{j \in N} C_j(x'_j) \supset \bigcap_{j \in N} C_j(x_j)$  for all  $\mathbf{e} = (C_N, \omega) \in E$ , all  $x \in S(\mathbf{e})$ . Suppose that  $\exists \mathbf{e} = (C_N, \omega) \in E, \exists x \in S(\mathbf{e}), \exists x' \in F(\omega), \bigcap_{j \in N} C_j(x'_j) \supset \bigcap_{j \in N} C_j(x_j)$ . Since  $(\partial C_i(x_i) \cap \partial \bigcap_{j \in N} C_j(x_j)) \neq \emptyset$  for all  $i \in N$ , there exists  $k \in N$  such that  $\forall f_k \in (\partial C_k(x_k) \cap \partial \bigcap_{j \in N} C_j(x_j)), \exists f' \in \partial \bigcap_{j \in N} C_j(x'_j), f' \geq f_k$  and for all  $i \neq k, \forall f_i \in (\partial C_i(x_i) \cap \partial \bigcap_{j \in N} C_j(x_j)), \exists f' \in \partial \bigcap_{j \in N} C_j(x'_j), f' \geq f_i$ . Then, Strict Monotonicity of capability correspondences and PESI imply that  $x'_k > x_k$  and  $x'_i \geq x_i$  for all  $i \neq k$ .

However, since  $x \in P^{SI}(\mathbf{e})$ , the allocation  $x'$  is not feasible. So we have  $\nexists x' \in F(\omega)$ ,  $\bigcap_{j \in N} C_j(x'_j) \supset \bigcap_{j \in N} C_j(x_j)$  for all  $\mathbf{e} = (C_N, \omega) \in E$ , all  $x \in S(\mathbf{e})$ .

Hence, if a social choice correspondence  $S$  satisfies PESI, ECRA, PRRIE- $R^f$  and PRRCP- $R^f$ , then it is a common capability maximin rule.||

Theorems 1 and 2 show that basic differences between the two rules are due to the formulations in axioms of Pareto efficiency and rank preservation. In general, solutions of the two rules do not match since the egalitarian rule assigns allocations from  $P(\mathbf{e}) \subseteq P^{SI}(\mathbf{e})$  but the common capability maximin rule assigns allocations from both  $P^{SI}(\mathbf{e}) \setminus P(\mathbf{e})$  and  $P(\mathbf{e})$ . Furthermore, a difference in the rank preservation axioms also leads to difference solutions. In the egalitarian rule, the ranking of each individual's capability set is based on a social preference ordering, whereas in the common capability maximum rule, it is based on a functioning on each individual's upper boundary of a capability set. Since each capability set is comprehensive, letting a ranking depend on functionings on an upper boundary leads to paying attention to an upper boundary of a common capability set. On the other hand, letting a ranking depend on a social preference ordering is generally unrelated to a common capability set. For example, suppose that a social preference ordering is given by the volume of a capability set. If rankings of individuals 1 and 2 are the same in a particular economy, the rank preservation axiom requires that the volumes of individuals 1 and 2's capabilities should be the same for all economic environments. However, this is independent from that a common capability of individuals 1 and 2 is maximal in terms of the relation of set inclusion. Therefore, it turns out that the differences of these axioms lead to different solutions.

Theorem 2 shows that in a single-good economy, necessary and sufficient conditions for common capability maximin rules have the similar spirit of egalitarian rules. However, in a two or more goods economy, these axioms cannot be sufficient conditions for maximizing common capabilities, while they are still necessary conditions for common capability maximin rules. To explain this problem, we provide a simple example in a

two-individuals, two-goods, and two-functionings economy. Let an initial endowment be  $\omega = (4, 4)$ . The economy has two individuals who are characterized by the following capability correspondences<sup>10</sup>:

$$C_1(x_1) = \{(f_{11}, f_{12}) | x_{11}f_{11} + x_{12}f_{12} \leq x_{11}x_{12}\},$$

$$C_2(x_2) = \{(f_{21}, f_{22}) | 0.99x_{21}f_{21} + x_{22}f_{22} \leq x_{21}x_{22}\}.$$

Consider two feasible allocations  $x = (x_1, x_2) = ((3, 1), (1, 3)), x' = (x'_1, x'_2) = ((2, 2), (2, 2)) \in F(\omega)$ . By definition, both allocations  $x$  and  $x'$  belong to  $PSI(\mathbf{e})$ . Then, Figure 3 shows the common capability set of  $x$  is implied by that of  $x'$ , *i.e.*  $(C_1(x'_1) \cap C_2(x'_2)) \supset (C_1(x_1) \cap C_2(x_2))$ . However, an allocation  $x$  belongs to the Pareto set and both 1 and 2's rank equal 2. Therefore, in the exchange economies have 2 or more commodities, the axioms of Theorem 2 fails to characterize a class of common capability maximin rules. Hence, we have to need additional axioms for characterizing common capability rules.

## 4 Concluding Remarks

This paper investigates implications of two concepts of the equality of capabilities. Using three types of axioms –principles of equal treatment, Pareto efficiency and rank preservation–, we can characterize a class of social choice correspondences which formalize notions of equality and efficiency based on the capability approach. Moreover, we provide a counterexample where common capability maximin rules cannot be characterized by the above three types of axioms in a simple exchange economy with 2 goods.

Now we discuss the further implication.

Firstly, our social choice correspondences are well-defined in the setting of the *unique* social preference ordering  $\succ \in R_{\mathcal{X}}$ , but in general, they are not well-defined in the setting where all individuals have their own preference orderings  $\succ_i \in R_{\mathcal{X}}$ . Indeed, if each individual has each value judgment on  $\mathcal{X}$ , then variations of the egalitarian rule

---

<sup>10</sup>These capability correspondences satisfy all properties defined in Section 2.

could not be well-defined. To see this problem, consider the following two variations of the egalitarian rule.

**Definition 4:** A social choice correspondence  $S^{EF}$  is a *envy-free and efficient rule* if and only if  $\forall \mathbf{e} = (C_N, \omega, \{\succsim_i\}_{i \in N}) \in E$ ,  $S^{EF}(\mathbf{e}) \subseteq \{x \in F(\omega) | \forall i, j \in N, C_i(x_i) \succsim_i C_j(x_j)\} \cap P^*(\mathbf{e})$ .

**Definition 5:** A social choice correspondence  $S^{EE}$  is an *egalitarian equivalent rule* if and only if  $\forall \mathbf{e} = (C_N, \omega, \{\succsim_i\}_{i \in N}) \in E$ ,  $S^{EE}(\mathbf{e}) \subseteq \{x \in F(\omega) | \exists \tilde{K} \in \mathcal{K}, \forall i \in N, C_i(x_i) \sim_i \tilde{K}\} \cap P^*(\mathbf{e})$ .

In the above definitions, the Pareto set  $P^*(\mathbf{e})$  is applied to the framework where all individuals have their own preference orderings on capability sets. That is,  $P^*(\mathbf{e}) = \{x \in F(\omega) | \nexists x' \in F(\omega), \forall i \in N, C_i(x'_i) \succsim_i C_i(x_i) \ \& \ \exists j \in N, C_j(x'_j) \succ_j C_j(x_j)\}$ . Obviously, the envy-free and efficient rule is not well-defined while the egalitarian equivalent rule is well-defined whenever all preference orderings are continuous. It is an interesting question to show a necessary and sufficient condition for the envy-free and efficient rule to assign a non-empty subset of feasible allocations for all situations.

Secondly, consider a problem of rationalizing social choice correspondences based on the capability approach. In general, there exist rankings such that their greatest elements equals the set of allocations that the common capability maximin rule assigns for all situations. For example, rankings on  $\mathcal{K}$  proposed by Herrero *et al.* (1998) or Echavarri and Permanyer (2008) are rationalizations of the common capability maximin rule. On the other hand, if an Efficiency-first ranking *a la* Tadenuma (2002), or a Pazner-Schmeidler function proposed by Fleurbaey (1996) is applied to our framework based on the capability approach, then the greatest elements of these rankings equals the set of allocations that the egalitarian rule assigns for all situations.

Thirdly, when social preference orderings belong to the set of value judgments that evaluate capability sets in terms of functionings on their undominated boundary, solutions of the egalitarian rule is a subset of that of the common capability maximin



rule. For example, if a social preference ordering is either a ranking based on functionings that maximize some real-valued functions (Xu 2002; 2003) or a ranking based on functionings of the boundary set that cross on some reference ray (Miyagishima 2010), then  $S^E(\bullet) \subset S^{CM}(\bullet)$  for all situations. On the contrary, if a social preference ordering belong to the set of rankings based on the volume of a capability set (Pattanaik and Xu 2000; Xu 2004; Savaglio and Vannucci 2009) or rankings based on the functionings that are not in an undominated boundary of a capability set (Gaertner and Xu 2006; 2008; 2011; Gaertner 2012), then  $S^E(\bullet) \cap S^{CM}(\bullet) = \emptyset$  for some situations. Then, a necessary and sufficient condition for  $S^E(\bullet) \subset S^{CM}(\bullet)$  is an important and open question since we would like to know how different notions of the equality of capabilities will work for various situations.

Finally, the egalitarian rule may be able to realize allocations reflecting individual diversity. For example, suppose that individual 1 with a hearing impairment can easily increase a level of functioning 1 but has a difficulty to increase a level of functioning 2. On the contrary, individual 2 with a visual impairment can easily increase a level of functioning 2 but has a difficulty to increase a level of functioning 1. Then, the common capability maximin rule eliminates allocations where individual 1 enjoys more functioning 1 and individual 2 enjoys more functioning 2 to maximize a common capability. In contrast, depending on a social preference ordering, the egalitarian rules dose not rule out such an allocation. Although it is not easy to decide which allocation is desirable, the egalitarian rule might be appealing as the rule could make it possible to flexibly consider diversities of individual characteristics. In order to deepen our understanding on the capability approach, we need more ethical consideration on this point.

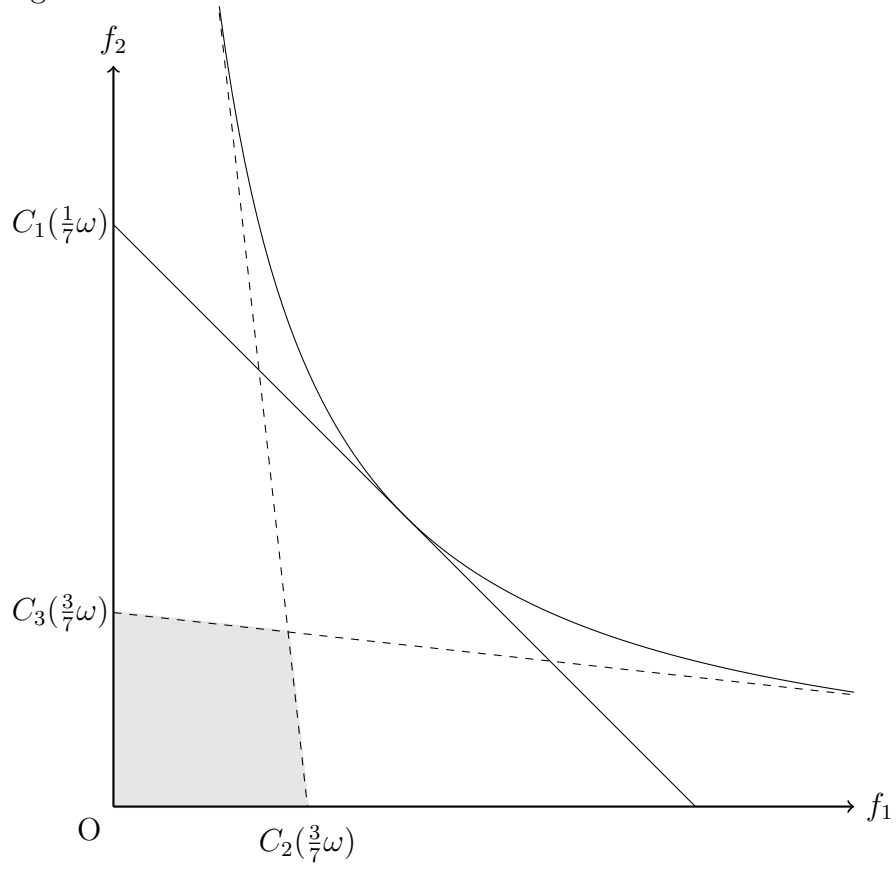
## References

- [1] Bossert, W. (1995) “Redistribution Mechanisms Based on Individual Characteristics,” *Mathematical Social Sciences*, Vol. 29, pp. 1-17.

- [2] Echavarri, R. and I. Permanyer (2008) “Ranking Profiles of Capability Sets,” *Social Choice and Welfare*, Vol. 31, pp. 521-535.
- [3] Fleurbaey, M. (1994) “On Fair Compensation,” *Theory and Decision*, Vol. 36, pp. 277-307.
- [4] Fleurbaey, M. (1995) “Three Solutions for the Compensation Problem,” *Journal of Economic Theory*, Vol. 65, pp. 505-521.
- [5] Gaertner, W. (2012) “Evaluating Sets of Objects in Characteristics Space,” *Social Choice and Welfare*, Vol. 39, pp. 303-321.
- [6] Gaertner, W. and Y, Xu (2006) “Capability Sets as the Basis of a New Measure of Human Development,” *Journal of Human Development*, Vol. 7, pp. 311-321.
- [7] Gaertner, W. and Y, Xu (2008) “A New Class of Measures of the Standard of Living Based on Functionings,” *Economic Theory*, Vol. 35, pp. 201-215.
- [8] Gaertner, W. and Y, Xu (2011) “Reference-Dependent Rankings of Sets in Characteristics Space,” *Social Choice and Welfare*, Vol. 37, pp. 717-728.
- [9] Gorman, W. M. (1956) “A Possible Procedure for Analysing Quality Differentials in the Egg Market,” *Review of Economic Studies*, Vol. 47, pp. 843-856.
- [10] Gotoh, R. and N. Yoshihara (2003) “A Class of Fair social choice correspondences à la Rawls and Sen,” *Economic Theory*, Vol. 22, pp. 63-88.
- [11] Gotoh, R., K. Suzumura, and N. Yoshihara (2010) “Extended Social Ordering Functions for Rationalizing Fair Allocation Rules as Game Forms in the Sense of Rawls and Sen,” *International Journal of Economic Theory*, Vol. 1, pp. 21-41.
- [12] Herrero, C., I. Iturbe-Ormaetxe, J. Nieto (1998) “Ranking Opportunity Profiles on the Basis of the Common Opportunities,” *Mathematical Social Sciences*, Vol. 35, pp. 273-289.

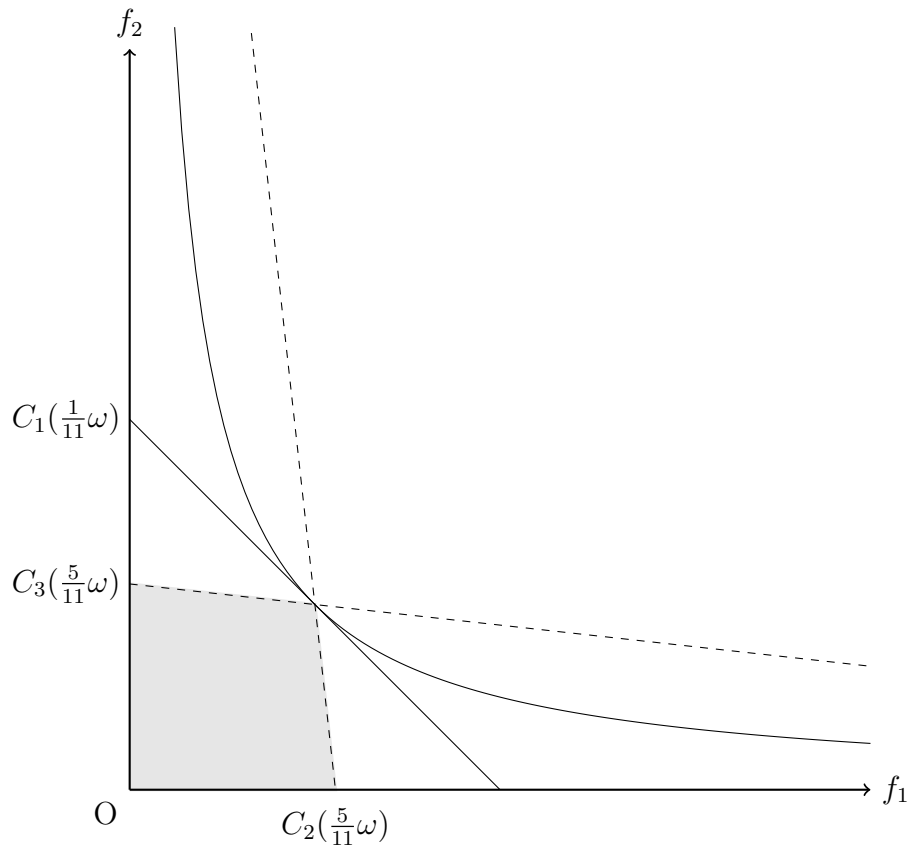
- [13] Lancaster, K. J. (1966) "A New Approach to Consumer Theory," *Journal of Political Economy*, Vol. 74, pp. 132-157.
- [14] Miyagishima, K. (2010) "Ranking Linear Budget Sets," *Social Choice and Welfare*, Vol. 35, pp. 163-173.
- [15] Pattanaik P. K. and Y. Xu (2000) "On Ranking Opportunity Sets in Economic Environments," *Journal of Economic Theory*, Vol. 93, pp. 48-71.
- [16] Savaglio, E. and S. Vannucci (2009) "On the Volume-Ranking of Opportunity Sets in Economic Environments," *Social Choice and Welfare*, Vol. 33, pp. 1-24.
- [17] Sen, A. K. (1985) *Commodities and Capabilities*, Amsterdam: North-Holland.
- [18] Tadenuma, K. (2002) "Efficiency First or Equity First? Two Principles and Rationality of Social Choice," *Journal of Economic Theory*, Vol. 104, pp. 462-472.
- [19] Xu, Y. (2002) "Functioning, Capability and the Standard of Living: An Axiomatic Approach," *Economic Theory*, Vol. 20, pp. 387-399.
- [20] Xu, Y. (2003) "On Ranking Compact and Comprehensive Opportunity Sets," *Mathematical Social Sciences*, Vol. 45, pp. 109-119.
- [21] Xu, Y. (2004) "On Ranking Linear Budget Sets in Terms of Freedom of Choice," *Social Choice and Welfare*, Vol. 22, pp. 281-289.
- [22] Xu, Y. and N. Yoshihara (2006) "Axiomatic Bargaining Theory on Opportunity Assignments," *IER Discussion Paper Series A, No.473*, Institute of Economic Research, Hitotsubashi University.
- [23] Xu, Y. and N. Yoshihara (2009) "Bargaining Theory over Opportunity Assignments and the Egalitarian Solution," *IER Discussion Paper Series A, No.517*, Institute of Economic Research, Hitotsubashi University.

Figure 1:



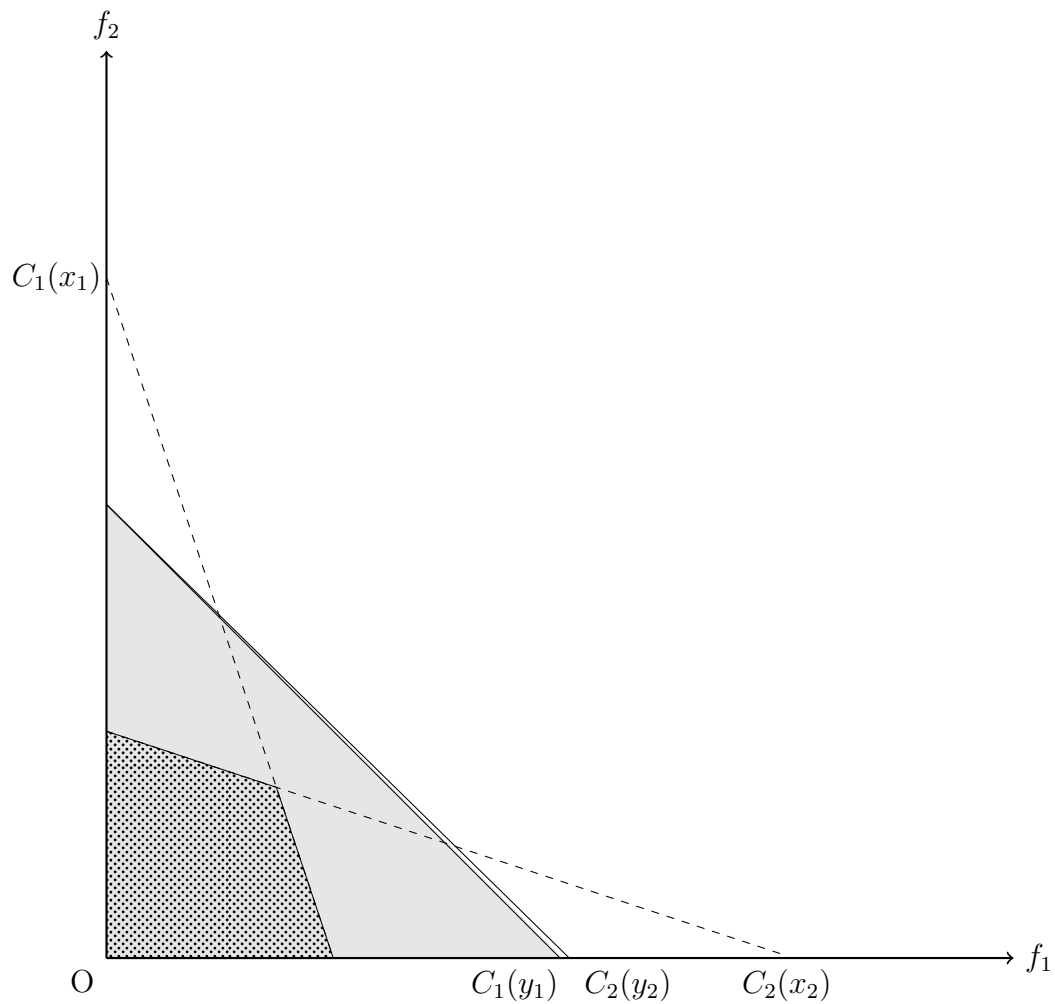
Each capability set has the same value but the intersection of them is not maximal.

Figure 2:



Each capability set has different value but the intersection of them is maximal.

Figure 3:



A common capability of  $y$ , which represents for shaded area, is a superset of that of  $x$ , which represents for dotted area.