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Market Structure and Indeterminacy of Stationary Equilibria in a Decentralized Monetary Economy

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Market structure and indeterminacy of stationary equilibria in a decentralized monetary economy*

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Abstract

This study investigates which market structure gives rise to indeterminacy of stationary equilibria in a decentralized economy with non-degenerate distributions of money holdings. I develop a price-posting model with divisible money and then, examine two alternative markets: a pairwise random matching market and a many-to-many exchange. Importantly, the former market balances the number of matched buyers and sellers by definition. As a result, indeterminacy arises under the pairwise matching while a unique equilibrium exists in the many-to-many market. This balancing assumption also leads to the indeterminacy in a Walrasian market.

Keywords: search theory, money, indeterminacy
Journal of Economic Literature Classification Number: D31, D51, D83, E41

1 Introduction

This study analyzes which type of market structure of decentralized monetary trade that gives rise to the real indeterminacy of stationary equilibria initially found by Green and Zhou (1998). Notably, the indeterminacy of equilibrium distribution is of non-degenerate money holdings, which is of growing concern in monetary economics. Theoretically, such indeterminacy is potentially a straightforward consequence of decentralized trades with divisible money. Moreover, recent empirical studies emphasize the influence of heterogeneity of household balance sheets on the effects of inflation (e.g., Doepke and Schneider (2006) and Auclert (2017)). Thus, the present study aims to provide a foundation for one aspect of the non-degenerate money holding distributions, namely, indeterminacy.

A series of works following Green and Zhou (1998) study the circulation of divisible money using a variety of search models and find a specific type of indeterminacy. There is a continuum of stationary equilibria where each equilibrium differs not only in nominal prices but also real terms, such as consumption and production. In addition, the distributions of money holdings and the number of matchings are uncertain. As a result, economic welfare is unpredictable.

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I develop a model of price-posting whereby sellers post unit prices of a consumption good and form submarkets. Then, each buyer visits a submarket and quotes a purchase quantity. Finally, a market structure determines the allocation in every submarket. This price-posting market represents daily shopping, for example, a person decides how much meat to buy after looking at its price by weight in a supermarket. On this model, I examine two alternative market structures: bilateral trade with random pairwise matching and a two-sided many-to-many exchange. The random matching one is a variant of the standard competitive search markets (e.g., Moen (1997); and Rocheteau and Wright (2005)) and the many-to-many matching is a type of market game (Shapley and Shubik (1977)). As a result, the bilateral trade market gives rise to the indeterminacy of the stationary equilibrium distributions of money holdings, but the many-to-many exchange market holds a unique equilibrium.

Pairwise matching equalizes the number of buyers and sellers in both on and off paths. This balanced trade assumption creates one more identity in the system of equations that determine the distribution of the stationary equilibrium money holdings; it makes a continuum of solutions. On the other hand, the many-to-many exchange assumption excludes this identity.

In addition to the decentralized price-posting model, the present study shows that a similar assumption of balancing buyers and sellers may give rise to indeterminacy in a centralized market. I consider a Walrasian market with buyer and seller sides. In the standard case, the Walrasian market holds a unique equilibrium because it allows a many-to-many exchange. However, under a limited participation restriction by which only the same population of buyers and sellers enters the market, the model also leads to indeterminacy.

This study builds on the literature on decentralized trade models with divisible money and indeterminacy. Green and Zhou (1998); Zhou (1999); Green and Zhou (2002); Kamiya and Sato (2004); Kamiya and Shimizu (2006, 2007, 2011); Ishihara (2010) examine many types of money search models with non-degenerated distributions of money holdings and show the similar indeterminacy. It also arises in other models such as market place choice by Matsui and Shimizu (2005), auction market by Kamiya and Shimizu (2013), New Monetarist model with indivisible good by Jean et al. (2010), and Walrasian market with indivisible good by Kamiya et al. (2017).

Wallace (1998) conjectures that the nominal nature of money is the cause of indeterminacy. However, Zhou (2003) questions this finding by showing indeterminacy in a model with commodity money. Another conjecture is that the indivisibility of goods assumed in Green and Zhou (1998) is the source of indeterminacy. On the contrary, Ishihara (2010) and Kamiya and Shimizu (2011) find indeterminacy with divisible goods. This study's price-posting model also reveals indeterminacy with divisible consumption goods. Kamiya and Shimizu (2006) show the kind of system of equations about stationary distributions of money holdings that support indeterminacy but a question remains about the underlying market structure causing indeterminacy[1]. Kamiya and Shimizu (2013) study a centralized auction market and conjecture that the non-Walrasian price determination causes the indeterminacy. However, the present study derives indeterminacy in a Walrasian market too. In a different context, another stationary equilibrium indeterminacy is found in economic growth models with knowledge diffusion (e.g., Luttmer (2007)). Their indeterminacy is associated with fat-tail distributions of firm productivity. Nevertheless, indeterminacy in decentralized monetary models possibly emerges on bounded distributions.

This study is further related to tractable search models of non-degenerate distributions of money holdings. Berentsen et al. (2005) and Rocheteau et al. (2018) study delayed mean-reverting adjustment processes of money holdings using the new-monetarist model and obtain non-degenerate distributions. Menzio et al. (2013) provide a competitive search model with block recursive structure, which results in non-degenerate distribution associated with sorting into different submarkets. While these models incorporate Walrasian markets, the present study analyzes entirely decentralized markets[2]. Interestingly, this study derives similar distributions which hold discrete masses. Other notable approaches to generating non-degenerate distributions in money search models include a countable number of money holdings (Camera and Corbae (1999)) and numerical methods (Molico (2006); Chiu and Molico (2010, 2011)).

The present study's many-to-many exchange trade is a variant of Shapley and Shubik (1977)'s market game. There are other approaches to incorporate the many-to-many exchange in decentralized monetary models: Corbae et al. (2003) consider a type of coalition formation game with indivisible money, Howitt (2005) studies the appearance of trading places, and Julien et al. (2008), Galenianos and Kircher (2008), and Kamiya and Shimizu (2013) analyze auction markets.

The structure of the rest of the paper is as follows. In Section 2, I propose a simple example and provide the intuition for the process by which a bilateral trade assumption makes the equilibrium indeterminate. In Section 3, the environment of the economy and equilibrium concept are introduced. Section 4 introduces a pairwise matching market and obtains indeterminacy. In Section 5, a many-to-many exchange market attains the unique equilibrium. Section 6 considers a Walrasian model with and without the limited participation of buyers and sellers, and studies the emergence of indeterminacy. Section 7 presents the conclusion.

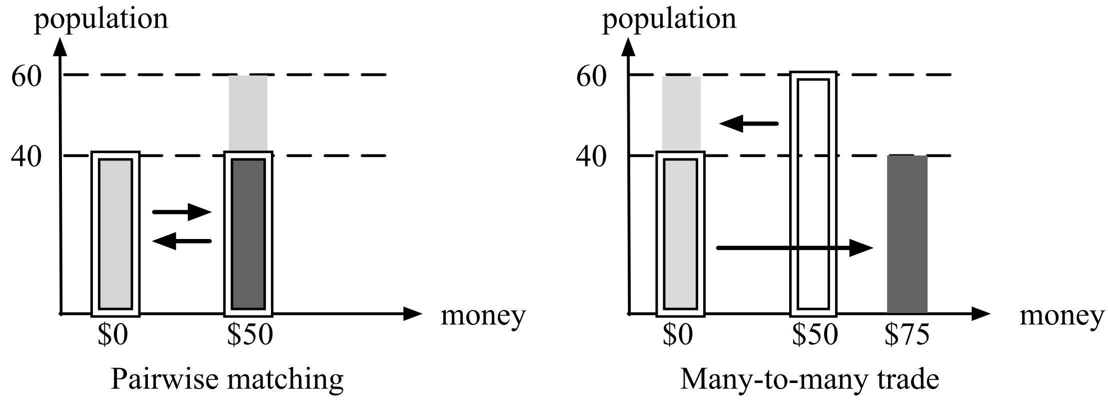

Figure 1: Distribution of money holdings

2 A simple example

This section proposes a simple example and provides the intuition for the main result. The example is presented in a reduced-form version of the models in the later sections. There is a market in which buyers and sellers exchange money and goods every period. The population is 100. The money supply is fixed at $3000. Disregarding consumption, production, and incentives of agents, I assume only a money transfer rule: an agent holding positive units of money enters the buyer side and spends all the money holding. An agent holding no cash enters the seller side and receives money from the buyer side.

I propose two alternative market structures.

1. Pairwise matching: a random draw decides the matchings of one buyer and one seller. Each buyer gives the whole amount of money to the partner. The number of matchings is the smaller of the population of either the buyer or the seller side,

2. Many-to-many exchange: all agents on the buyer side pay all money holdings. The money is distributed equally to agents on the seller side.

Pairwise matching represents a search theory of money with the most efficient matching function, that is, there are no unmatched agents on the smaller population side. Many-to-many exchange is interpreted as Shapley and Shubik (1977)'s market game or Walrasian market.

Suppose there is a money holding distribution in which 60 agents hold $50 each and 40 people hold no cash[3]. Under the pairwise trade assumption, the number of matchings is 40 =

[1]Jean et al. (2010) point out the importance of strategic coordination issues. It is required for multiple equilibria in general. Since indeterminacy following Green and Zhou (1998) arises with a continuum of equilibria, additional conditions provided by Kamiya and Shimizu (2006) are necessary.

[2]These studies do not report indeterminacy, possibly because of Walrasian markets, which indeed allow many-to-many matching. Another difference is that these studies consider only equilibria with continuous and differentiable value functions; indeterminacy often arises with step value functions in the literature whereas the value functions in this study have kinks.

[3]This example considers only distribution with two masses.

$\min\{60, 40\}$. All 40 sellers hold \$50 in the next period. There are 40 among 60 buyers who exhaust their cash. As in the left diagram of Figure 1, the money holding distribution is stationary. In general, all the distributions in which x people hold $\$3000/x$ and $1 - x$ hold nothing can be steady-state equilibria. The pairwise trade market exchanges the same number of buyers and sellers and keeps the distribution unchanged.

However, in the many-to-many trade case, all the buyers spend their money. In the next period, 60 agents hold no cash, and 40 agents accumulate \$75 each. Therefore, the distribution is not stationary as shown in the right diagram of Figure 1. The unique stationary distribution is that 50 agents hold \$60 each, and 50 agents hold no cash[4]. The many-to-many exchange procedure matches different populations of buyers and sellers. It differentiates the inflow and outflow of each point on the distribution and violates the stationarity.

3 Environment and Equilibrium Concept

This section introduces a baseline structure of the price-posting model and the equilibrium concept. The solution is provided with the pairwise trade procedure in Section 4, and with many-to-many matching in Section 5.

3.1 Environment

Time is discrete and infinite. There are two types of goods: a consumption good and money. Both goods are divisible. Money is storable while the consumption good is perishable. Money supply is constant at M through all periods.

There is a unit measure of homogeneous agents. They are anonymous, that is, the history of actions and money holdings are unobserved by other agents[5]. Each agent has a linear utility function uq where q is the unit of the consumption. Each agent also has a production technology that converts her labor input to the consumption good. I assume capacity-constrained production, following the industrial organization literature since Levitan and Shubik (1972). Production of q units yields the following labor disutility:

$$C(q) = \begin{cases} 0 & \text{if } q = 0 \\ c & \text{if } q \in (0, 1] \\ +\infty & \text{if } q > 1 \end{cases} .$$

In other words, each seller produces at most one unit of the consumption good with constant disutility c. The function induces a strong incentive for sellers to lower prices because of the

[4] Note that this study considers a distribution of money holdings to be stationary if it is unchanged every period. Cyclical equilibria may also exist, for example, 60 agents hold \$50 in even periods and 40 agents hold \$75 in odd periods. This study focuses only on the strict concept of stationary equilibrium, because it is the main issue in the existing literature. Cyclical equilibrium is another worthwhile topic to study, for example, Arbuzov et al. (2019)

[5] These assumptions preclude non-monetary equilibrium by keeping history, as in Kocherlakota (1998) and contagion equilibrium, as in Araujo (2004).

zero marginal costs. This ensures a kind of Bertrand–Edgeworth competition among sellers which makes the equilibrium simple. Each agent can either only consume or only produce in each period. The utilities are discounted by a factor β.

Each period consists of four stages.

- Stage 1: each agent decides whether to become a seller or a buyer.
- Stage 2: each seller posts a unit price of the consumption good p in terms of money. The sellers posting the same price p forms submarket p.
- Stage 3: each buyer observes the distribution of the posted prices and chooses one submarket.
- Stage 4: each buyer in each submarket p quotes demand q. In submarket p, given the participating buyers and sellers, either one-to-one matching or multilateral exchange procedure decides the allocations of money and the consumption good.

This study assumes that the consumption good is general, that is, each buyer demands goods produced by any other sellers. The division of buyers and sellers is enough to generate a single coincidence of wants and circulate money, as in Rocheteau and Wright (2005)[6].

3.2 Strategy and Equilibrium

A probability measure λ represents the distribution of money holdings. It satisfies the total population $1 = \int_0^\infty d\lambda(m)$ and the money supply $M = \int_0^\infty m d\lambda(m)$. I assume a symmetric strategy in which all agents holding m units of money choose the same strategy in the equilibrium. The strategy allows the use of money holding m as the index of agents[7].

Definition 1. *A strategy of an agent is a sequence of actions over the four stages* $x(m) = \big(x_k(m)\big)_{k=1,2,3,4}$ *contingent on her money supply* m *at the beginning of stage 1.*

- *The action in stage 1,* $x_1(m)$*, is buying or selling. Let* S *be the set of sellers and* B *be that of buyers.*
- *In stage 2, each seller observes the set of buyers and sellers, and then posts a price* p*. No buyer is active.*

$$x_2(m|x_1, B, S) = \begin{cases} p > 0 & \text{if } \; x_1(m) \in S \\ \emptyset & \text{otherwise.} \end{cases}$$

Let S_p *be the set of sellers in submarket* p*. The set of all the submarkets is* $\mathcal{P} = \{p|S_p \neq \emptyset\}$*.*

[6]Another possible assumption is that there are $K \geq 3$ types of agents and the population of each type is $1/K$. A type k agent consumes the good produced only by type $k-1$. Since this economy deals with a price-posting procedure, a type k buyer can go to a type $k-1$ seller's submarket. Hence, the types can be ignored without loss of generality.

[7]In other words, each agent chooses a pure strategy. The assumption does not lose generality, because only a measure-zero agent at a margin may choose a mixed strategy. Thus, to consider a deviation from an equilibrium, I assume a unilateral deviation. Each agent's state variable can be represented only by m instead of $\lambda(m)$ because $\lambda(m)$ is stationary.

- *In stage 3, each buyer enters one submarket. Each seller is inactive.*

$$x_3\big(m|x_1, B, (S_p)_{p\in\mathcal{P}}\big) = \begin{cases} p \in \mathcal{P} & \text{if } \ x_1(m) \in B \\ \emptyset & \text{otherwise.} \end{cases}$$

  Let B_p be the set of buyers in submarket p.

- *In stage 4, each buyer posts a non-negative demand q given her budget constraint. Each seller is inactive.*

$$x_4\big(m|x_1, x_3, (S_p)_{p\in\mathcal{P}}, (B_p)_{p\in\mathcal{P}}\big) = \begin{cases} q \in [0, m/p] & \text{if } \ x_1(m) \in B \text{ and } p = x_3(m) \\ \emptyset & \text{if } \ x_1(m) \in S. \end{cases}$$

The definition of equilibrium consists of three functions of money holding m at the beginning of the period: strategy $x(m)$, the discounted sum of utilities $V(m)$ and the distribution of money holdings $\lambda(m)$.

Definition 2. *The equilibrium consists of stationary $x(m)$, $\lambda(m)$, and $V(m)$ satisfying the following conditions: (i) $\lambda(m)$ is consistent with $x(m)$, (ii) $V(m)$ is induced by $x(m)$ and $\lambda(m)$, (iii) given $\lambda(m)$ and $V(m)$, $x(m)$ is a subgame perfect equilibrium from stage 1 to stage 4, and (iv) $V(0) > 0$.*

This concept is called *stationary monetary equilibrium* in the literature on the search theory of money. While most studies[8] consider a one-shot game every period, the present study derives the subgame perfect equilibrium given the dynamic game in each period. Given stationary $\lambda(m)$, the discounted value depends only on m. Each agent keeps track of only m, because the history of other agents is anonymous and the agent is atomic. Condition (iv) means that engaging in monetary trade is more attractive than autarky. Then, there exist some buyers and sellers exchange money and goods in the stationary monetary equilibrium. The equilibrium necessary holds the following submarket:

Definition 3. *Submarket p is active if $\lambda(S_p) > 0$ and $\lambda(B_p) > 0$.*

Meanwhile, a submarket may exist that has measure zero agents. It is crucial to consider each seller's incentive because one seller's deviation from the equilibrium makes such a measure zero submarket. To decide each buyer's incentive in stage 3, the model needs to specify what kind of matching and trades appear in such a submarket. This study follows the literature of the large economy with atomless agents (e.g., Dubey et al. (1982)) and assumes an approximate equilibrium. The equilibrium in a measure zero submarket is solved by assuming a small and positive measure of sellers. Then, the equilibrium is obtained when the measure of market participants converges to zero, which this study calls ε-*submarket*.

[8]The models in the literature since Green and Zhou (1998) use a take-it-or-leave-it game, which is generally a two-shot game in general because an offerer takes into account the receiver's response. However, all the models assume unobservable money holding of the trade partner. The offerer cares only about the average actions of all possible receivers. Then, the dynamic decision disappears.

Definition 4. *Suppose that there exists $p \in \mathcal{P}$ such that $S_p \neq \emptyset$ and $\lambda(S_p) = 0$. This is called ε-submarket. The equilibrium is obtained by solving the buyers' actions in stage 3 and 4 given that there is a sufficiently small $\varepsilon > 0$ measure of sellers, and then taking $\varepsilon \to 0$.*

The following two lemmas are straightforward implications of the definition.

Lemma 1. *$V(m)$ is weakly increasing in the equilibrium.*

Proof. Suppose $x = \{x_t, x_{t+1}, \cdots\}$ is the optimal actions of agent m from time t to the future, which induces $V(m)$. Consider agent $m' > m$. Indeed, agent m' can also takes x and acquire the same discounted utility $V(m)$. By the optimality of the strategy of agent m', $V(m') \geq V(m)$. □

Lemma 2. *In the equilibrium, each agent becomes a buyer and spends money in finite periods ahead.*

Proof. By Lemma 1, $V(m) \geq V(0) > 0$ for all $m > 0$ in the equilibrium. In the model, the only way to acquire positive utility is by being a buyer and spending money. If agent m never purchases consumption goods, the discounted sum of utility is never positive. □

4 Pairwise Trade with Random Matching

This section builds a variant of the competitive search market, that is, stage 4 incorporates a pairwise trade with random matching. I derive a set of equilibria differing in both the distribution of money holdings and real allocation.

In submarket p, the number of matchings is determined by the measure of sellers $\lambda(S_p)$ and that of buyers $\lambda(B_p)$. I assume that the number of matchings is $\min\{\lambda(S_p), \lambda(B_p)\}$ without loss of generality. Acemoglu and Shimer (1999) call this matching function *frictionless*. This assumption is for an explicit comparison of the pairwise trade and the many-to-many matching markets. Indeterminacy can be obtained under any constant returns to scale matching function.

In a matching, buyer m optimally decides a purchase quantity q_m under production limit $q_m \leq 1$ and budget constraint $pq_m \leq m$.

4.1 Equilibrium

There is a continuum of stationary equilibria, represented by Proposition 1. In each equilibrium, all sellers post the same price p^*. All buyers participate in submarket p^* and spend all their cash. The single price p^* is indeterminate. Each p^* has an associated two-point equilibrium distribution of money holdings.

There are two main properties that make the equilibrium tractable. The first is a buyer's pay-all strategy in stage 4. Given posted price p in stage 4, each buyer optimally chooses quantity q:

$$V_{4B}(m|p) = \max_q \; uq + \beta V(m - pq), \quad \text{s.t. } pq \le m, \; q \le 1. \tag{1}$$

This equation defines a buyer's discounted sum of utilities evaluated at stage 4 given a price p as $V_{4B}(m|p)$. In the equilibrium, $p = p^*$. The marginal gain from an increase in q is u. However, it yields disutility $\beta p^* V'(m - p^* q)$. The stationarity implies that the current period marginal gain is strictly larger than future one. Saving one unit of money enables the purchase of $1/p^*$ units of good in the future, because p^* is unique and stationary. The utility will increase by u/p^*. By the discount factor $\beta < 1$,

$$\beta p^* V'(m - pq^*) \le \beta p^*(u/p^*) = \beta u < u.$$

Equation (1) has a corner solution in which either $p^* q = m$ or $q = 1$ holds.

The second property making the equilibrium tractable is the single-price equilibrium in stages 2 and 3. Suppose that all the buyers hold $m = p^*$. Then, each seller posts the same price p^* and each buyer necessarily chooses submarket p^*. This is a Nash equilibrium. By the corner solution in Equation (1), each buyer chooses $q = 1$ and $p^* q = m = p^*$. No seller has an incentive to offer a different price. An offer $p > p^*$ does not generate profit because each buyer pays at most p^*. A lower price $p < p^*$ is worse, because the sales amount never increases owing to the capacity limit $q \le 1$.

Given these two properties, indeterminate distributions of money holdings appear. In the two-point distribution[9], both the single price and the buyer's money holdings are p^*. The total number of matchings is $\lambda(0) = \min\{\lambda(0), \lambda(p^*)\}$. By the first property, all matched buyers spend p^*, and the same number of sellers receive p^*. The equilibrium exchanges the same population of agents between two points, and hence, the distribution is stationary. In the case of a slightly higher price $\tilde{p}^* > p^*$, there are fewer buyers. The second property ensures that the new price $\tilde{p}^*$ is an equilibrium in stages 2 and 3. Then, the market again exchanges the same number of buyers and sellers. Another equilibrium with a different distribution arises.

Indeterminacy is real because the amount of production is $\lambda(0)$ in each period. Efficiency and the welfare change according to the single price p^*. The result is unsurprising. While this market is a variant of the competitive search market, the main feature is similar to the random search theories.

Proposition 1. *Suppose $\beta u - c > 0$. For any p^* such that*

$$M \le p^* < M\left(\frac{(1-\beta)c}{\beta(u-c)} + 1\right),$$

an equilibrium exists in which

[9]Some models, such as those of Green and Zhou (1998) and Zhou (1999), derive single-price equilibria with more than two points distributions. Some agents accumulate money several times owing to random matching. However, such a distribution disappears in the present model by the buyer/seller choice in stage 1. Interestingly, the many-to-many matching model presented in the next section causes different incentives for agents and gives rise to more than two point distributions.

1. *$\lambda(p^*) = M/p^*$ and $\lambda(0) = 1 - \lambda(p^*)$, otherwise $\lambda(m) = 0$ where $\lambda(0) < \lambda(p^*)$,*

2. *The strategy is $x(m) = (S, p^*, \emptyset, \emptyset)$ if $m \leq \bar{m}$ and $x(m) = (B, \emptyset, \tilde{p}, \max\{1, m/\tilde{p}\})$ otherwise, where the equilibrium matching probability of a buyer is $\tilde{\lambda} \equiv \lambda(0)/\lambda(p^*)$, the buyer's choice of the submarket is*

$$\tilde{p} = \arg\max_{p} \ \tilde{\lambda} V_{4B}(m|p),$$

and the cutoff amount of money holding in stage 1 is

$$\bar{m} = \frac{p^*[\beta\tilde{\lambda}u - (1 - \beta + \beta\tilde{\lambda})c]}{\tilde{\lambda}u(1 + \beta\tilde{\lambda})},$$

3. *The value function is defined by the following equations:*

$$V(0) = \frac{\beta\tilde{\lambda}u - (1 - \beta + \beta\tilde{\lambda})c}{(1 - \beta)(1 + \beta\tilde{\lambda})}, \tag{2}$$

$$V(m) = V(0) \quad \text{if } m \leq \bar{m} \tag{3}$$

$$V(m) = \tilde{\lambda}\left[\frac{m}{p^*}u + \beta V(0)\right] + \beta(1 - \tilde{\lambda})V(m) \quad \text{if } \bar{m} < m \leq p^* \tag{4}$$

$$V(m) = \tilde{\lambda}\left[u + \beta V(m - p^*)\right] + \beta(1 - \tilde{\lambda})V(m) \quad \text{if } p^* < m \tag{5}$$

The strategy $x(m)$ is defined as history dependent in one period. In the equilibrium, buyers have only one option p^*. The strategy does not depend on the history before the current period, because each agent is anonymous and atomic. Equation (3) represents the discounted value that an agent with a low money holding becomes a seller in the current period. She holds $p^* + m$ units of money in the next period. However, given the unique posted price p^* and the capacity constraint $q \leq 1$, she can pay only p^* in each matching. This means that $m > 0$ units of money are never used. Equation (4) describes an agent who becomes a buyer in the current period. She meets a seller with probability $\tilde{\lambda}$ and spends all her cash. Finally, Equation (5) shows that, if an agent holds $m > p^*$ amount of money, she becomes a buyer and spends only p^* units of production capacity.

Proof. This proof constructs a set of equilibria by the guess and verify method. I first assume a set of $V(m)$ and $\lambda(m)$, which is characterized by the indeterminate single price p^*. Then, given assumption $V(m)$ and $\lambda(m)$, the subgame perfect equilibrium strategy $x(m)$ is derived. Each step is summarized as a lemma.

The best response actions are derived backward from stages 4 to 1.

stage 4

Given an offered unit price p, agent m's problem is

$$\max_{q \in \mathbb{R}_+} \; uq + \beta V(m - pq), \quad \text{s.t. } pq \leq m, \; q \leq 1. \tag{6}$$

Lemma 3. *If $p \leq \left(\frac{1-\beta(1-\tilde{\lambda})}{\tilde{\lambda}}\right) p^*$, the buyer chooses $q = 1$.*

Proof. By Equation (3), (4) and (5), the slope of $V(m)$ is less than or equal to $\frac{\tilde{\lambda}u}{p^*[1-\beta(1-\tilde{\lambda})]}$; hence, $q = 1$ maximizes the utility. □

If $p = p^*$, a buyer chooses $q = 1$ because $[1 - \beta(1 - \tilde{\lambda})]/\tilde{\lambda} \geq 1$.

stage 2 and 3

I consider stages 2 and 3 one at a time and show the single price equilibrium.

Lemma 4. *In stages 2 and 3, an equilibrium exists at which all sellers post p^*.*

Proof. Consider a seller's strategy given that all other sellers post p^*. If the seller posts $p = p^*$, she succeeds in matching with probability 1 and, by Lemma 3, acquires p^* units of money. If the seller posts $p \neq p^*$, an ε-market opens. The matching probability does not increase more than 1. A lower price $p < p^*$ is not profitable. Moreover, a higher price $p > p^*$ does not raise revenue because all buyers spend p^* units of money given the assumed money holding distribution. □

stage 1

Here, I calculate the discounted sum of the utilities of being a seller and a buyer depending on m, and then derive the cutoff $\bar{m}$. By Lemma 4, each seller m posts p^* and matches with probability one. The discounted value is

$$-c + \beta V(m + p^*). \tag{7}$$

Consider a case in which the agent chooses to be a buyer. If $m < p^*$, she meets a seller with probability $\tilde{\lambda}$ and spends all her money holding. Hence, the discounted utility is Equation (4). If $m \geq p^*$, she can spend at most p^* units of money, which means (5).

The following proposition shows the existence of $\bar{m}$ which equates Equations (3) and (4). The former is larger if $m \leq \bar{m}$ and smaller if $\bar{m} < m \leq p^*$. Then, Equation (5) gives a wealthy agent's incentive to choose the buyer side.

Lemma 5. *There exists $\bar{m} \leq p^*$ such that agent m becomes a seller if $m \leq \bar{m}$ and a buyer otherwise.*

Proof. In Appendix. $\square$

The following two lemmas show that $x(m)$ induces $\lambda(m)$ and $V(m)$. The distribution $\lambda(m)$ is clear because $\lambda(0)$ sellers and buyers match and exchange p^* money and goods. Indeed this flow makes $\lambda(m)$ stationary.

Lemma 6. *The strategy $x(m)$ makes $\lambda(m)$ stationary.*

Proof. The $\lambda(0)$ measure of agents choose the seller side and offer p^* because $0 < \bar{m}$. By $p^* > \bar{m}$, the $\lambda(p^*)$ become buyers. The number of matching is $\lambda(0) = \min\{\lambda(p^*), \lambda(0)\}$. In each matching, the buyer quotes $q = 1$ and spends p^* by Lemma 3. Therefore, all the $m = 0$ agents will accumulate p^* and the $\lambda(0)$ among $\lambda(p^*)$ buyers exhaust their cash in the next period. Therefore, λ is stationary under x. $\square$

Lemma 7. *Strategy $x(m)$ is consistent with $V(m)$.*

Proof. In Appendix. $\square$

Finally, the equilibrium requires $V(0) > 0$ to enable sellers to participate in the market. By Equation (2), the condition is rewritten as

$$\tilde{\lambda} > \frac{(1-\beta)c}{\beta(u-c)}.$$

Since $M < p^* \leq 2M$, $0 < \tilde{\lambda} \leq 1$. Such $\tilde{\lambda}$ exists if $\beta u > c$. The right-hand side also yields the lower bound of p^*. The two-point distribution implies $p^* = M(\tilde{\lambda}+1)$, then $p^* > M\left(\frac{(1-\beta)c}{\beta(u-c)} + 1\right)$. $\square$

5 Many-to-many matching market

This section proposes another market to that in Section 4, a many-to-many matching, and obtains the unique equilibrium. Suppose that there are $Q_p^S = \int_{m \in S_p} d\lambda(m)$ measure of sellers in submarket p. Since each seller produces at most one unit, Q_p^S is interpreted as the aggregate supply. An agent m posts a quote $q(m) \in [0, m/p]$. The total amount is $Q_p^B = \int_{m \in B_p} q(m) d\lambda(m)$, which is interpreted as the aggregate demand.

Definition 5. *In submarket p, the many-to-many matching procedure yields an allocation such that every seller exchanges q_p^S consumption good for pq_p^S money and each buyer m trades $pq_p^B(m)$ money for $q_p^B(m)$ good defined as*

$$q_p^S = \min\left\{1, \frac{Q_p^B}{Q_p^S}\right\}, \quad q_p^B(m) = q(m) \cdot \min\left\{1, \frac{Q_p^S}{Q_p^B}\right\}.$$

Call submarket p excess supply if $Q_p^S > Q_p^B$, balanced if $Q_p^S = Q_p^B$, and excess demand otherwise. In an excess supply submarket, each buyer acquires the exact posted quantity $q(m)$. Every seller produces less than 1 amount to meet the demand. In an excess demand submarket, each buyer obtains the posted good reduced by the same proportion Q_p^S/Q_p^B. In contrast to the pairwise matching market, the many-to-many market allows all agents to trade. Instead of the rationing, each quantity of trade is adjusted.

This procedure is related to the market game of Shapley and Shubik (1977). In the market game, the nominal price per unit is determined by the demand/supply ratio Q_p^B/Q_p^S. By contrast, in the present study, the price per unit is predetermined by the sellers' postings. The demand/supply ratio Q_p^B/Q_p^S determines only the allocation of the consumption good[10].

This amount of trade is defined only if $Q_p^S > 0$. Therefore, the ε-submarket assumption in Definition 4 is crucial. To consider one seller's deviation to p, the consequence depends on how many buyers enter submarket p and how much they quote[11].

5.1 Equilibrium

The unique equilibrium is characterized by the single price p^*. Given a set of parameters, p^* and $\lambda(m)$ are uniquely determined. The main difference between the pairwise matching and the many-to-many markets is that the latter possibly generates an equilibrium with more than two masses. Agents hold $0, p^*, 2p^*, \cdots, Np^*$ money. Note that this is not an indeterminacy: a set of parameters uniquely pins down both p^* and N.

To provide intuition, consider the case of $N = 1$, which gives rise to a similar two-point distribution. This case arises under some parameters. In the equilibrium, each agent with 0 units of money becomes a seller and earns p^* units of money. In the next period, the agent becomes a buyer and spends p^*. Because of no rationing, the equilibrium distribution of money holdings is unique. The measure of no-cash agents is $\lambda(0)$ and they hold p^* in the next period. Therefore the distribution is stationary only if $\lambda(0) = \lambda(p^*)$.

The equilibrium is characterized by a similar strategy as the one-to-one matching case. In stage 4, each buyer pays all money holdings. A buyer's problem is written as

$$V_{4B}(m|p) = \max_q \; uq_B + \beta V(m - pq_B), \quad \text{s.t. } pq \le m, \quad q_B = q \cdot \min\left\{1, \frac{Q_p^S}{Q_p^B}\right\}, \tag{8}$$

which is slightly different from the pairwise matching market. The consumption amount is adjusted by the seller/buyer ratio, and the capacity constraint $q \le 1$ does not bind. Nonetheless, $pq = m$ holds at the optimum because of the stationary price p^* and the discount factor β.

[10] In a market game, a good is never allocated to a person who derives no utility from it, such as a seller in my model. However, my model allows the possibility that there remains a good left unsold.

[11] Note that a submarket with a positive measure of sellers and zero-measure buyers is straightforward: it is an excess supply submarket with $q_p^S = 0$ and $q_p^B(m) = q(m)$

The single price equilibrium with the unique balanced submarket holds in stages 2 and 3. In the pairwise trade market, Bertrand–Edgeworth competition gives sellers lower prices and increases the probability of matching. In the many-to-many market, a cheaper offer may attract more buyers and raise revenue. Sellers in an excess supply submarket have this incentive of deviation, because they produce less than 1 unit each. Then, excess supply submarkets disappear. Excess demand submarkets also do not exist. In such a submarket, buyers purchase less amount than they desire. Hence, some sellers may post slightly higher prices. Some buyers may also deviate and choose the new submarkets to purchase more.

In an equilibrium with $N > 1$, the distribution satisfies $\lambda(0) = \lambda(p^*) = \lambda(2p^*) = \cdots = \lambda(Np^*)$. Similarly, there is a unique balanced submarket p^*. An agent $m = 0$ becomes a seller first and earns p^*. Then, she chooses a seller again and saves $2p^*$. She selects the seller side in a row until $m = Np^*$. After that, she becomes a buyer and spends Np^*. In other words, the cutoff level of money holding in stage 1 is between $(N-1)p^*$ and Np^*. This one-way process is similar to Menzio et al. (2013). The submarket is balanced, because, on the seller side, $N/(N+1)$ measure of sellers supplies goods. In the buyer side, $1/(N+1)$ population of buyers pay Np^* amount of money. Hence, the total demand is also $N/(N+1)$.

This $N > 1$ distribution emerges because each buyer is able to purchase more than one unit of good in one period. Consider discounted values from the current period to three periods ahead without matching friction. In the pairwise matching market, a seller obtains $-c+\beta u-\beta^2 c+\beta^3 u$. If she accumulates $N = 2$, then it would be $-c-\beta c+\beta^2 u+\beta^3 u$, which is strictly worse, owing to $\beta < 1$. However, in the many-to-many matching market, she may accumulate $N > 1$, because she is able to spend all the amount in one period. A discounted value $-c - \beta c - \beta^2 c + \beta^3 3u$ is possible. It may be better if u is sufficiently larger than c and that β is close enough to 1.

This uniqueness result can be interpreted in a general framework provided by Kamiya and Shimizu (2006), which mainly considers N-point distribution with the single a price p^*. Stationarity holds by equalities of inflow and outflow at $m = 0, p^*, 2p^*, \ldots, Np^*$. They are written as a system of equations to determine the distribution. Kamiya and Shimizu (2006) show the existence of a hidden identity in the system. It is derived by an assumption that, besides the population flow, the amounts of money transferred among different points are balanced both on and off paths. The framework of Kamiya and Shimizu (2006) does not explicitly depends on search markets; however, this identity still relies on a kind of one-to-one matching assumption. The many-to-many trade market does not share the identity.

Proposition 2. *If an integer N satisfies the following condition:*

$$[(1-\beta^{N+1}) - (1-\beta)N]u > (1-\beta)c \geq [(1-\beta^{N+2}) - (1-\beta)(N+1)]u,$$

then there exists the unique equilibrium where

1. *The distribution of money holdings is $\lambda(m) = 1/(N+1)$ if $m = 0, p^*, 2p^*, \ldots, Np^*$ with $p^* = 2M/N$, otherwise $\lambda(m) = 0$.*

2. *The value function $V(m)$ is recursively defined as*

$$V(0) = \frac{\beta^N Nu - \left(\frac{1-\beta^N}{1-\beta}\right) c}{1 - \beta^{N+1}},$$

$$V(m) = -c + \beta V(m + p^*) \quad \textit{if } m < \bar{m},$$

$$V(m) = \frac{um}{p^*} + \beta V(0) \quad \textit{if } m \geq \bar{m},$$

where

$$\bar{m} = p^* \left[\frac{\beta u - c}{1 - \beta} - V(0)\right],$$

3. *The strategy is $x(m) = (S, p^*, \emptyset, \emptyset)$ if $m < \bar{m}$ and $x(m) = (B, \emptyset, \tilde{p}, m/\tilde{p}\})$ if $m \geq \bar{m}$ where $\tilde{p} = \arg\max_p \ \tilde{\lambda} V_{4B}(m|p)$.*

The proof consists of several necessary conditions of the equilibrium, which are summarized as lemmas. First, the equilibrium value function $V(m)$ is strictly increasing (Lemma 8). An agent accumulating an additional small amount of money has a chance to spend it and yield more utility. It is because the capacity constraint does not bind. The equilibrium does not contain any excess supply submarket (Lemma 9). A seller's deviation to offer a lower price benefits buyers as well as the seller, because lower supply/demand ratio is generated. Similarly, any excess demand submarket disappears because a higher price brings benefits to a seller and improves the quantity adjustment ratio for a buyer (Lemma 11). Therefore, only balanced submarkets may exist in the equilibrium. The balanced submarket must be unique; otherwise, each buyer chooses a lower price submarket (Lemma 10). By the linear utility function, each buyer is incentivized to spend all their money holdings given the unique posted price p^* (Lemma 12). This makes the value function continuous (Lemma 13). No jump on $V(m)$ makes a unique cutoff action in stage 1 such that becoming a buyer if $m \geq \bar{m}$ and a seller otherwise (Lemma 14). Finally, the condition that the supply and demand are balanced in the unique submarket p^* makes the unique distribution of money holdings.

Proof. Lemma 8 to Lemma 15 obtain necessary conditions of the equilibrium. According to these lemmas, if the equilibrium exists, it is unique and holds the single balanced submarket p^*. Lemma 16 assures the existence of this equilibrium.

Lemma 8. *$V(m)$ is strictly increasing in the equilibrium.*

Proof. Suppose $x = \{x_t, x_{t+1}, \cdots\}$ is the optimal actions of agent m from the current period t, which induces $V(m)$. By Lemma 2, an agent becomes a buyer and uses money in finite periods ahead, that is, there exists $T \geq t$, $p > 0$, and $q > 0$, such that $x_T = \{B, \emptyset, p, q\}$. Consider a sequence of actions of agent holding $m + \sigma$ where $\sigma > 0$. Take $\tilde{x} = \{\tilde{x}_t, \tilde{x}_{t+1}, \cdots\}$ such that

$x_\tau = \tilde{x}_\tau$ for all $\tau \neq T$ and $x_T = \{B, \emptyset, p, \tilde{q}\}$, where $\tilde{q} = \big[(m+\sigma)/p\big] \cdot \min\{1, (Q_p^B/Q_p^S)\}$. Let $\tilde{V}$ be the associated discounted sum of utility. $\tilde{x}$ is a feasible action of agent $m+\sigma$. By $\tilde{q} > q$, $\tilde{V} > V(m)$. By the optimality, $V(m+\sigma) \geq \tilde{V}$. Therefore, $V(m+\sigma) > V(m)$. □

Lemma 9. *There is no excess supply submarket in the equilibrium.*

Proof. Suppose there is an excess supply submarket p. This lemma considers a seller's incentive to post $p - \sigma$ where $\sigma > 0$ is sufficiently small. By Definition 4, $Q_{p-\sigma}^S = \varepsilon < Q_p^S$.

In submarket p, there is a buyer m who consumes q and saves $m' = m - pq$. If submarket $p-\sigma$ is excess supply or balanced, while keeping m', the buyer can consume $(m'-m)/(p-\sigma) > q$. Every buyer in submarket p deviates while submarket $p-\sigma$ is excess supply or balanced. In the equilibrium, submarket $p - \sigma$ is excess demand under $\varepsilon \to 0$.

Consider a seller in submarket p. Her revenue is $pQ_p^S/Q_p^B < p$. If she deviates and posts $p-\sigma$, submarket $p-\sigma$ is excess demand. Then, she earns $p-\sigma$, which is higher than pQ_p^S/Q_p^B for sufficiently small σ. Since $V(m)$ is strictly increasing, posting p is not optimal. □

Lemma 10. *The submarket is unique in the equilibrium.*

Proof. By Lemma 9, only excess demand and/or balanced submarkets exist in the equilibrium. Suppose there are two submarkets $p < p'$. Each seller provides 1 unit in both submarkets. The revenue is higher in p'; hence, given strictly increasing $V(m)$, every seller in submarket p moves to p'. This does not satisfy the equilibrium. □

Lemma 11. *The unique submarket is balanced in the equilibrium.*

Proof. By Lemma 9, there is no excess supply submarket. Suppose that the unique submarket p^* is excess demand. Suppose a seller posts $p^*(1+\sigma)$ where $\sigma > 0$ is sufficiently small[12]. If submarket $p^*(1+\sigma)$ is also excess demand, the seller has a strict incentive of deviation. This is because that she sells 1 unit of goods in both submarkets and $V(m)$ is strictly increasing.

Hereafter, I show that submarket $p^*(1+\sigma)$ is excess demand. Suppose $x(m) = \{x_t, x_{t+1}, \cdots\}$ is the optimal actions of a buyer m from the current period t, which induces $V(m)$. Suppose the buyer chooses q_t in period t. By Lemma 2, there is a period $T > t$ when this agent becomes a buyer and spends a positive amount of money again. Let $x_T = \{B, \emptyset, p^*, q_T\}$. This buyer's posts q_t and q_T satisfy the constraints: $q_t \leq (Q_{p^*}^S/Q_{p^*}^B)(m/p^*)$ and $q_T \leq (Q_{p^*}^S/Q_{p^*}^B)(m_T/p^*)$, where m_T is her money holding at the beginning of period T.

On the contrary, assume that submarket $p^*(1+\sigma)$ is excess supply or balanced. Consider a series of actions $\tilde{x} = \{\tilde{x}_t, \tilde{x}_{t+1}, \cdots\}$, where $x_\tau = \tilde{x}_\tau$ for all $\tau \notin \{t, T\}$. Assume $\tilde{x}_t = \{B, \emptyset, p^*(1+\sigma), q(1+\delta)\}$, where

$$\delta = \left[\frac{Q_{p^*}^S}{Q_{p^*}^B} - \frac{1}{1+\sigma}\right]\sigma.$$

[12] I use $p^*(1+\sigma)$ instead of $p^* + \sigma$ for simplifying the equations.

This implies $\delta > 0$ and $\delta \to 0$ as $\sigma \to 0$. Since $\frac{Q^S_{p^*+\sigma}}{Q^B_{p^*+\sigma}} > 1$, there exist $\sigma > 0$ which satisfies the budget constraint, that is, $p^* q_t(1+\sigma)(1+\delta) \leq m$. Then, the agent's money holding at the beginning of period T is $m_T - p^* q_t(\sigma + \delta + \sigma\delta)$. Let $\tilde{x}_T = \{B, \emptyset, p^*, \tilde{q}_T\}$ where

$$\tilde{q}_T = q_T - \frac{Q^S_{p^*}}{Q^B_{p^*}} \frac{p^* q_t(\sigma + \delta + \sigma\delta)}{p^*},$$

which satisfies the budget constraint.

The discounted utility of periods t and T under $x(m)$ is $uq_t + \beta^{T-t} uq_T$. That for $\tilde{x}$ is $uq_t(1+\sigma) + \beta^{T-t} u\tilde{q}_T$. Therefore, the agent has a strict incentive to move to $p^*(1+\sigma)$ if

$$uq_t\sigma > \beta^{T-t} u(q_T - \tilde{q}_T) = \beta^{T-t} uq_t(Q^S_{p^*}/Q^B_{p^*})[\sigma + \delta + \delta\sigma]$$

By the definition of δ, the inequality is $1 > \beta^{T-t}(1+\sigma)$. This is satisfied for sufficiently small $\sigma > 0$. □

Lemma 12. *Every buyer in the unique submarket exhausts cash, that is, $p^* q = m$.*

Proof. Suppose, on the contrary, that buyer m decides $m_{t+1} = m - p^* q > 0$ in period t and chooses a series of actions $x(m)$ thereafter. By Lemma 2, there exists period T in which agent m becomes a buyer and $m_T - m_{T+1} > 0$. Consider $\tilde{x}(m)$ where $\tilde{m}_{t+1} = m_{t+1} + \sigma$ and $\tilde{m}_T - \tilde{m}_{T+1} = m_T - m_{T+1} - \sigma$ for sufficiently small $\sigma > 0$. Indeed it satisfies the budget constraint. The period t and T actions of $x(m)$ yields $u(m_T - m_{T+1})/p^* + \beta^{T-t}(\tilde{m}_T - \tilde{m}_{T+1})/p^*$, while $\tilde{x}(m)$ leads to $u(m_T - m_{T+1} + \sigma)/p^* + \beta^{T-t}(\tilde{m}_T - \tilde{m}_{T+1} - \sigma)/p^*$. $\tilde{x}(m)$ yields strictly higher utility than $x(m)$. □

Lemma 13. *The value function $V(m)$ is continuous. For all $m' > m \geq 0$, $V(m') - V(m) \leq u(m' - m)/p^*$.*

Proof. Consider cases in which agent m becomes a buyer for the first time in n period ahead, since period t for $n = 0, 1, 2, \cdots$. By Lemma 2, there exists an upper bound N such that $n \leq N$. Agent m chooses $m_{t+n+1} = 0$ by Lemma 12. Define $\tilde{V}_n(m)$ as

$$\tilde{V}_0(m) = \frac{um}{p^*} + \beta V(0)$$

and

$$\tilde{V}_n(m) = -c \sum_{i=1}^{n} \beta^{i-1} + \beta^n u(m + np^*) + \beta^{n+1} V(0)$$

for $n = 0, 1, 2, \cdots, N$. Then, $V(m) = \max\{\tilde{V}_0(m), \tilde{V}_1(m), \ldots, \tilde{V}_N(m)\}$. Since $\tilde{V}_n(m)$ for all $n = 0, \ldots, N$ and max function are continuous, $V(m)$ is also continuous. For all $n = 0, \ldots, N$ and $m' > m \geq 0$, $\tilde{V}_n(m') - \tilde{V}_n(m) \leq (u/p^*)(m' - m)$. Therefore, $V(m') - V(m) \leq (u/p^*)(m' - m)$. □

Lemma 14. *There exists $\bar{m}$ such that an agent becomes a buyer if $m > \bar{m}$ and a seller otherwise.*

Proof. Let $V_S(m) = -c + \beta V(m + p^*)$ and $V_B(m) = um/p^* + \beta V(0)$. Then, $V(m) = \max\{V_S(m), V_B(m)\}$. By $V(0) > 0$, $V(0) = V_S(0) > V_B(0)$. Let $\bar{m}$ be the minimum money holding of being a buyer, that is, $V_S(m) > V_B(m)$ for all $m < \bar{m}$. By the continuity of $V(m)$, $V_S(\bar{m}) = V_B(\bar{m})$. Take $m > \bar{m}$. $V_B(m) - V_B(\bar{m}) = u(m - \bar{m})/p^*$. By Lemma 13, $V_S(m) - V_S(\bar{m}) = \beta[V(m + p^*) - V(m)] < u(m - \bar{m})/p^*$. Therefore, $V_B(m) > V_S(m)$ for all $m > \bar{m}$. □

Lemma 15. *Suppose the unique equilibrium posted price p^* satisfies $(N-1)p^* < \bar{m} \leq Np^*$ with a positive integer N. Then, the equilibrium money holding distribution is $\lambda(m) = 1/(N+1)$ if $m = 0, p^*, 2p^*, \ldots, Np^*$ and $\lambda(m) = 0$ otherwise.*

Proof. Given Lemmas 11 and 14, the unique submarket has price p^*, and $N \in \mathbb{N}$ exists such that $(N-1)p^* < \bar{m} \leq Np^*$. Let $I_f(\mathcal{M})$ and $O_f(\mathcal{M})$ denote the inflow to and outflow from the set of money holdings $\mathcal{M} \subset \mathbb{R}$ in terms of the measure of agents. The money holding distribution λ is stationary if $I_f(\mathcal{M}) = O_f(\mathcal{M}) = \lambda(\mathcal{M})$ for all $\mathcal{M}$. By Lemma 14 and Lemma 12, $I_f(0) = O_f(\{m|m > \bar{m}\})$. Each agent $m \leq \bar{m}$ become a seller and earns p^*. Hence, $I_f(\{m|np^* < m < (n+1)p^* \text{ for } n = 0, 1, 2, \ldots, N-1\}) = 0$. Then, $\lambda(0) = \lambda(p^*) = \lambda(2p^*) = \cdots = \lambda\big((N-1)p^*\big) = \lambda\big(\{m|m \geq \bar{m}\}\big)$. By the single price p^* again, $I_f(\{m|m \geq \bar{m}\}) = I_f(Np^*)$. The stationary distribution satisfies $\lambda(0) = \lambda(p^*) = \cdots = \lambda(p^*N) = 1/(N+1)$. The single price satisfies $p^* = 2M/N$, because the total money supply is M. □

The last lemma shows the existence of the unique equilibrium.

Lemma 16. *If a positive integer N satisfies $[(1-\beta^{N+1}) - (1-\beta)N]u > (1-\beta)c \geq [(1-\beta^{N+2}) - (1-\beta)(N+1)]$, then a unique equilibrium with N-point distribution exists.*

Proof. Suppose that the equilibrium has N masses as in Lemma 15. Let $V_N(m)$ be the discounted utilities in the equilibrium with a positive integer N, $V_N^S(m) = -c + \beta V_N(m + p^*)$ be the value of selling goods, and $V_N^B(m) = um/p^* + \beta V_N(0)$ be the value when purchasing goods and spending all money holding. On the equilibrium path, agent $m = 0$ sells goods N times, being a buyer, and spends all money holding. Therefore,

$$V_N(0) = -c\sum_{i=0}^{N-1}\beta^i + \beta^N Nu + \beta^{N+1}V_N(0) \;\Leftrightarrow\; V_N(0) = \frac{\beta^N Nu - \left(\frac{1-\beta^N}{1-\beta}\right)c}{1-\beta^{N+1}}. \tag{9}$$

The equilibrium exists if $V_N^S(p^*n) > V_N^B(p^*n)$ for all $n = 0, 1, \ldots, N-1$ and $V_N^S(p^*N) \leq V_N^B(p^*N)$. The incentive condition for $n \leq N-1$ is

$$V_N(np^*) = V_N^S(np^*) > V_N^B(np^*). \tag{10}$$

By Lemma 13,

$$V_N^S\big(np^*\big) - V_N^S\big((n-1)p^*\big) = V_N\big(np^*\big) - V_N\big((n-1)p^*\big) \le u = V_N^B\big(np^*\big) - V_N^B\big((n-1)p^*\big)$$

for $n = 1, \ldots, N-1$. Therefore, it is sufficient to show $V_N^S\big((N-1)p^*\big) > V_N^B\big((N-1)p^*\big)$ for Equation (10). This is rewritten as

$$-c + \beta V(Np^*) > (N-1)u + \beta V(0).$$

By Equation (9), the condition is rewritten by $f_1(N)$ as

$$f_1(N) \equiv [(1-\beta^{N+1}) - (1-\beta)N]u > (1-\beta)c.$$

Similarly, the incentive condition for $m = N$ is

$$V_N^S\big(Np^*\big) \le V_N^B\big(Np^*\big).$$

It is characterized by $f_2(N)$ as

$$f_2(N) \equiv [\beta(1-\beta^{N+1}) - (1-\beta)N]u \le (1-\beta)c.$$

Then,

$$f_2(N) = \beta(1-\beta^{N+1}) - (1-\beta)N = 1-\beta^{N+2}-1+\beta-(1-\beta)N = 1-\beta^{N+2}-(1-\beta)(N+1) = f_1(N+1)$$

The sequence of $f_1(N)$ and $f_2(N)$ satisfies

$$f_1(1) > f_2(1) = f_1(2) > f_2(2) = f_1(3) > f_2(3) = \cdots.$$

Therefore, if $f_1(N) > (1-\beta)c \ge f_2(N)$ is satisfied, a unique equilibrium with N-point mass distribution exists. $\square$

The proof of Lemma 16 defines the cutoff amount $\bar{m}$ as

$$V^B(\bar{m}) = \frac{u\bar{m}}{p^*} + V(0) = -c + \frac{\beta u\bar{m}}{p^*} + \beta V(0) = V^S(\bar{m})$$

$$\Leftrightarrow \ \bar{m} = p^*\left[\frac{\beta u - c}{1-\beta} - V(0)\right]$$

It also solves the value function $V(m)$ recursively as in this proposition.

$\square$

6 Walrasian market

In this section, I show a similar result of the indeterminacy in a Walrasian market. Consider a unit measure of a continuum of agents who have the same preference as in the price-posting model. Similarly, there are consumption good and fixed amount of money M. There is one competitive market which has two sides: the seller and the buyer. As the price taking case in Rocheteau and Wright (2005), money is still essential. Each agent decides which side to enter at the beginning of each period. I consider two cases about the centralized market: no restriction and limited participation.

6.1 No Restriction

I derive an equilibrium in which the unit price p is stationary. Each agent solves

$$V(m) = \max\{V_S(m), V_B(m)\},$$

$$V_S(m) = \max_{q\in[0,1]} -c + \beta V(m + pq),$$

$$V_B(m) = \max_{q\in[0,m/p]} \frac{uq}{p} + \beta V(m - pq),$$

where $V(m)$ is the ex-ante discounted utility, $V_S(m)$ is the ex-post value of a seller, and $V_B(m)$ is that of a buyer. Since the utility function is linear, no buyer has an incentive to save money; hence, $V_B(m) = um/p + \beta V(0)$. The fixed costs of production allows the seller to produce $q = 1$. Therefore,

$$V(m) = \max\left\{-c + \beta V(m + p),\ \frac{um}{p} + \beta V(0)\right\}.$$

The solution is

$$V(m) = \begin{cases} \frac{\beta u - c}{1-\beta^2} & \text{if } m \le \bar{m}, \\ \frac{um}{p} + \beta\left(\frac{\beta u - c}{1-\beta^2}\right) & \text{if } m > \bar{m}, \end{cases}$$

where $\bar{m} = \frac{p(\beta u - c)}{u(1+\beta)}$. The optimal behavior is that of being a seller if $m \le \bar{m}$ and of being a buyer otherwise. This satisfies $\bar{m} < p$.

Let the money holding distribution be $0 < \lambda(p) < 1$ and $\lambda(0) = 1 - \lambda(p)$. The transition of money is similar: an agent $m = 0$ becomes a seller and earns p, and then, becomes a buyer and exhausts her cash.

The aggregate good demand is $\lambda(p) \cdot p/p = \lambda(p)$, while the supply is $\lambda(0)$. The equilibrium price p^* solves $\lambda(p^*) = \lambda(0) = 1/2$. This is the unique distribution.

6.2 Limited participation

Next, the Walrasian market incorporates a type of pairwise matching assumption: the measures of participating buyers and sellers must be equal. Let $\lambda(B)$ and $\lambda(S)$ be the measures of agents who are willing to enter the buyer side and seller side, respectively. The market randomly chooses $\min\{h(S), h(B)\}$ measure of buyers and sellers to enter and make the others autarky.

Consider a case in which $\lambda(B) \geq \lambda(S)$ and define $\tilde{\lambda} = \lambda(S)/\lambda(B)$. Given a stationary price p, the values of a potential seller and buyer are

$$V_S(m) = \max_{q \in [0,1]} -c + \beta V(m + pq),$$

$$V_B(m) = \tilde{\lambda} \cdot \max_{q \in [0, m/p]} \left[\frac{uq}{p} + \beta V(m - pq) \right] + (1 - \tilde{\lambda}) \beta V(m).$$

Given the similar actions of each buyer and seller, the ex-ante value function is

$$V(m) = \max \left\{ -c + \beta V(m + p),\ \tilde{\lambda} \left[\frac{um}{p} + \beta V(0) \right] + (1 - \tilde{\lambda}) \beta V(m) \right\}.$$

Then, the solution is

$$V(m) = \begin{cases} \frac{\beta\tilde{\lambda}u - [1-\beta(1-\tilde{\lambda})]c}{1-\beta(1-\tilde{\lambda}+\beta\tilde{\lambda})} & \text{if } m \leq \bar{m}, \\ \frac{\tilde{\lambda}um}{p[1-\beta(1-\tilde{\lambda})]} + \beta \left(\frac{\beta\tilde{\lambda}u - [1-\beta(1-\tilde{\lambda})]c}{1-\beta(1-\tilde{\lambda}+\beta\tilde{\lambda})} \right) & \text{if } m > \bar{m}, \end{cases}$$

where $\bar{m} = \frac{p(\beta u - c)}{u(1+\beta)}$.

Assume a similar distribution of money holdings with two masses at 0 and p. The optimal choice is the same as in the no-restriction case. However, the measure of participating buyers in this market is $\lambda(0) = \min\{\lambda(0), \lambda(p)\}$ instead of $\lambda(p)$ in the no-restriction case. Then, the aggregate demand is $\lambda(0) \cdot p/p = \lambda(0)$. Indeed the aggregate supply is $\lambda(0)$; hence, the equilibrium holds with a continuum of stationary prices p^*.

It is worthwhile to compare the result with Kamiya and Shimizu (2013), who show the indeterminacy in an auction market and uniqueness in a Walrasian market in a similar environment. My result indicates that the uniqueness might not depend on Walrasian competitive price determination. From the viewpoint of pairwise matching, their indeterminacy in the auction market seems to depend on the indivisible good assumption, that is, the measure of successful sellers is always the same as that of the buyers. They also conjecture that a uniqueness result might hold in an all-pay auction market, which is one example of many-to-many matching.

7 Conclusion

This study analyzes the indeterminacy of stationary distributions of money holdings in a decentralized economy. A price-posting model with two alternative market structures is considered. The equilibrium is indeterminate with the pairwise trade market but unique in the many-to-many matching. A Walrasian market model also gives arise to a similar result under a similar participation restriction.

The market structures in this study are extremes in opposite directions. On the one hand, the pairwise trade assumption completely excludes agents who fail to make matches in the market.

On the other hand, the many-to-many matching market allows all agents to be involved. An interesting extension would be a mixed case in which some agents match bilaterally, some have multilateral negotiations, and some are excluded. I conjecture that the unique equilibrium may still arise because only the exact equality of the measures of buyer and seller make indeterminate distributions. The mixed case might be useful as a method of equilibrium selection in an indeterminate model, such as introducing the small possibility of many-to-many matching in a search market.

Appendix

Proof of Lemma 5

I check the incentives of three cases: $m < \bar{m}$, $\bar{m} \leq m \leq p^*$, and $m > p^*$. Since the last case is independent from $\bar{m}$, I consider this one first and show that agent m becomes a buyer. This is equivalent to Equation (5) and (7) satisfying

$$-c + \beta V(m+p^*) \leq \tilde{\lambda}\left[u + \beta V\left(m - p^*\right)\right] + (1-\tilde{\lambda})\beta V(m),$$

$$\Leftrightarrow \ \tilde{\lambda}\big[V(m+p^*) - V\left(m-p^*\right)\big] + (1-\tilde{\lambda})\big[V(m+p^*) - V(m)\big] \leq \frac{\tilde{\lambda}u + c}{\beta}.$$

For later use, by Equation 4,

$$V(p^*) - V(0) = \frac{hu - c}{1 + \beta h}. \tag{A.1}$$

The equations of $V(m)$ implies $V(m+p^*) - V(m) \leq V(p^*) - V(0)$ and $V(m+p^*) - V\left(m - p^*\right) \leq V(2p^*) - V(0) \leq 2[V(p^*) - V(0)]$ for all $m \geq 0$. Hence, it is sufficient to show

$$(1+\tilde{\lambda})[V(p^*) - V(0)] \leq \frac{\tilde{\lambda}u + c}{\beta}.$$

By Equation (A.1),

$$\Leftrightarrow \ \frac{(1+\tilde{\lambda})(\tilde{\lambda}u + c)}{1 + \beta\tilde{\lambda}} \leq \frac{\tilde{\lambda}u + c}{\beta}.$$

Since $\beta(1+\tilde{\lambda}) \leq 1 + \beta\tilde{\lambda}$, the condition is satisfied.

The next case is $m < \bar{m}$, in which each agent becomes a seller. First, suppose $m < \bar{m}$. If agent m chooses to be a seller, the discounted value, by Equation (4), is

$$\beta V(m+p^*) = \tilde{\lambda}\left[u + \beta V(m)\right] + \beta(1-\tilde{\lambda})V(m+p^*).$$

By Equation (5),

$$\beta V(p^*) = \tilde{\lambda}\left[u + \beta V(0)\right] + \beta(1-\tilde{\lambda})V(p^*).$$

Since $V(m) = V(0)$, $V(m+p^*) = V(p^*)$. Then,

$$-c + \beta V(p^*) = -c + \beta\left(\frac{\tilde{\lambda}u + c}{1+\beta\tilde{\lambda}} + V(0)\right)$$

$$= \frac{\beta(\tilde{\lambda}u + c) - c(1+\beta\tilde{\lambda})}{1+\beta\tilde{\lambda}} + \beta V(0) = \frac{\beta\tilde{\lambda}u - c(1-\beta+\beta\tilde{\lambda})}{1+\beta\tilde{\lambda}} + \beta V(0).$$

If the agent selects the buyer side, the discounted value is

$$\tilde{\lambda}\left(\frac{mu}{p^*} + \beta V(0)\right) + \beta(1-\tilde{\lambda})V(m) = \frac{\tilde{\lambda}mu}{p^*} + \beta V(0).$$

Therefore, I need to show

$$\frac{\beta\tilde{\lambda}u - c(1-\beta+\beta\tilde{\lambda})}{1+\beta\tilde{\lambda}} > \frac{\tilde{\lambda}mu}{p^*}.$$

This can be rewritten as

$$\frac{p^*[\beta\tilde{\lambda}u - c(1-\beta+\beta\tilde{\lambda})]}{\tilde{\lambda}u(1+\beta\tilde{\lambda})} > m,$$

whose left-hand side is $\bar{m}$.

Finally, I consider $\bar{m} \leq m < p^*$. Equation (4) is rewritten as

$$[1-\beta(1-\tilde{\lambda})]V(m) = \tilde{\lambda}\left[\frac{m}{p^*}u + \beta V(0)\right].$$

This is satisfied if $m = \bar{m}$. Then,

$$\Rightarrow [1-\beta(1-\tilde{\lambda})][V(m) - V(0)] = [1-\beta(1-\tilde{\lambda})][V(m) - V(\bar{m})] = \frac{\tilde{\lambda}u}{p^*}\big(m - \bar{m}\big).$$

Similarly, Equation (5) implies

$$[1-\beta(1-\tilde{\lambda})]V(m+p^*) = \tilde{\lambda}\left[u + \beta V(m)\right],$$

which also holds if $m = 0$. Then,

$$[1-\beta(1-\tilde{\lambda})][V(m+p^*) - V(p^*)] = \frac{\beta\tilde{\lambda}^2u(m-\bar{m})}{[1-\beta(1-\tilde{\lambda})]p^*},$$

$$\Leftrightarrow \; V(m+p^*) = V(p^*) + \frac{\beta\tilde{\lambda}^2u(m-\bar{m})}{[1-\beta(1-\tilde{\lambda})]^2p^*}.$$

If agent $\bar{m} \leq m < p^*$ chooses to be a seller, the expected value is

$$\begin{aligned}
-c + \beta V(m+p^*) &= -c + \beta V(p^*) + \frac{\beta^2\tilde{\lambda}^2u(m-\bar{m})}{[1-\beta(1-\tilde{\lambda})]^2p^*} \\
&= -c + \beta\left[\frac{\tilde{\lambda}u + c}{1+\beta\tilde{\lambda}} + V(0)\right] + \frac{\beta^2\tilde{\lambda}^2u(m-\bar{m})}{p^*[1-\beta(1-\tilde{\lambda})]^2} \\
&= \frac{\beta\tilde{\lambda}u - (1-\beta+\beta\tilde{\lambda})c}{1+\beta\tilde{\lambda}} + \frac{\beta^2\tilde{\lambda}^2u(m-\bar{m})}{p^*[1-\beta(1-\tilde{\lambda})]^2} + \beta V(0)
\end{aligned}$$

If she decides to be a buyer, it is

$$\tilde{\lambda}\left(\frac{mu}{p^*}+\beta V(0)\right)+\beta(1-\tilde{\lambda})V(m)$$

$$=\tilde{\lambda}\left(\frac{mu}{p^*}+\beta V(0)\right)+\beta(1-\tilde{\lambda})\left[\frac{\tilde{\lambda}u(m-\bar{m})}{p^*[1-\beta(1-\tilde{\lambda})]}+V(0)\right]$$

$$=\frac{\tilde{\lambda}mu}{p^*}+\beta(1-\tilde{\lambda})\frac{\tilde{\lambda}u(m-\bar{m})}{p^*[1-\beta(1-\tilde{\lambda})]}+\beta V(0).$$

I need to show

$$-c+\beta V(m+p^*)\leq\tilde{\lambda}\left(\frac{mu}{p^*}+\beta V(0)\right)+\beta(1-\tilde{\lambda})V(m),$$

which is rewritten as

$$\frac{\beta\tilde{\lambda}u-(1-\beta+\beta\tilde{\lambda})c}{1+\beta\tilde{\lambda}}+\frac{\beta^2\tilde{\lambda}^2u(m-\bar{m})}{p^*[1-\beta(1-\tilde{\lambda})]^2}\leq\frac{\tilde{\lambda}mu}{p^*}+\beta(1-\tilde{\lambda})\frac{\tilde{\lambda}u(m-\bar{m})}{p^*[1-\beta(1-\tilde{\lambda})]}$$

$$\Leftrightarrow\ \frac{\beta\tilde{\lambda}u-(1-\beta+\beta\tilde{\lambda})c}{1+\beta\tilde{\lambda}}+\frac{\beta\tilde{\lambda}u(m-\bar{m})}{p^*[1-\beta(1-\tilde{\lambda})]}\left[\frac{\beta\tilde{\lambda}}{1-\beta(1-\tilde{\lambda})}-(1-\tilde{\lambda})\right]\leq\frac{\tilde{\lambda}mu}{p^*}$$

$$\Leftrightarrow\ \frac{p^*[\beta\tilde{\lambda}u-(1-\beta+\beta\tilde{\lambda})c]}{\tilde{\lambda}u(1+\beta\tilde{\lambda})}+\frac{\beta(m-\bar{m})}{1-\beta(1-\tilde{\lambda})}\left[\frac{(1-\tilde{\lambda})(1-\beta)-\tilde{\lambda}^2\beta}{1-\beta(1-\tilde{\lambda})}\right]\leq m$$

$$\Leftrightarrow\ \bar{m}+\frac{\beta(m-\bar{m})}{1-\beta(1-\tilde{\lambda})}\left[\frac{(1-\tilde{\lambda})(1-\beta)-\tilde{\lambda}^2\beta}{1-\beta(1-\tilde{\lambda})}\right]\leq m$$

$$\Leftrightarrow\ \bar{m}\left[1-\frac{\beta(1-\tilde{\lambda})(1-\beta)-\tilde{\lambda}^2\beta^2}{[1-\beta(1-\tilde{\lambda})]^2}\right]\leq m\left[1-\frac{\beta(1-\tilde{\lambda})(1-\beta)-\tilde{\lambda}^2\beta^2}{[1-\beta(1-\tilde{\lambda})]^2}\right]$$

$$\Leftrightarrow\ \bar{m}\leq m.$$

□

Proof of Lemma 7

This proof is divided into several cases. First, suppose an agent holds $m<\bar{m}$. Given the strategy x, she becomes a seller. Then, the Bellman equation satisfies

$$V(m)=-c+\beta V(m+p^*)=-c+\beta\left\{\tilde{\lambda}\left[u+\beta V(m)\right]+(1-\tilde{\lambda})V(m+p^*)\right\}$$

Given $\tilde{\lambda}=[1-\lambda(p^*)]/\lambda(p^*)$ and $\lambda(p^*)=M/p^*$, this equation is independent from m. Therefore, $V(0)=V(m)$ for $m<\bar{m}$.

Then, the expected return is

$$
\begin{aligned}
&-c+\beta V(m+p^*) = -c+\beta V(p^*)\\
&= -c + \frac{\beta\tilde{\lambda}u}{1-\beta(1-\tilde{\lambda})} + \left(\frac{\beta^2\tilde{\lambda}}{1-\beta(1-\tilde{\lambda})}\right) V(0)\\
&= -c + \frac{\beta\tilde{\lambda}u}{1-\beta(1-\tilde{\lambda})} + \left(\frac{\beta^2\tilde{\lambda}}{1-\beta(1-\tilde{\lambda})}\right) \frac{\beta\tilde{\lambda}u-(1-\beta+\beta\tilde{\lambda})c}{(1-\beta)(1+\beta\tilde{\lambda})}\\
&= \left(\frac{(1-\beta)(1+\beta\tilde{\lambda})+\beta^2\tilde{\lambda}}{[1-\beta(1-\tilde{\lambda})](1-\beta)(1+\beta\tilde{\lambda})}\right)\beta\tilde{\lambda}u - \left(\frac{\beta^2\tilde{\lambda}}{(1-\beta)(1+\beta\tilde{\lambda})}+1\right)c\\
&= \frac{\beta\tilde{\lambda}u-(1-\beta+\beta\tilde{\lambda})c}{(1-\beta)(1+\beta\tilde{\lambda})},
\end{aligned}
\tag{A.2}
$$

which yields (3).

Second, if $p^* > m \geq \bar{m}$, the agent becomes a buyer. Given x, the agent trades with probability $\tilde{\lambda}$; then the discounted utility is

$$\tilde{\lambda}\left(\frac{m}{p^*}u+\beta V(0)\right)+\beta(1-\tilde{\lambda})V(m), \tag{A.3}$$

which is (4).

Finally, suppose $m \geq p^*$. The agent becomes a buyer and the expected utility is

$$\tilde{\lambda}[u+\beta V(m-p^*)]+\beta(1-\tilde{\lambda})V(m). \tag{A.4}$$

It is Equation (5).

$\square$

References

Acemoglu, Daron and Robert Shimer (1999), "Efficient unemployment insurance." *The Journal of Political Economy*, 107, 893–928.

Araujo, Luis (2004), "Social norms and money." *Journal of Monetary Economics*, 51, 241–256.

Arbuzov, Vyacheslav, Yu Awaya, Hiroki Fukai, and Makoto Watanabe (2019), "Endogenous repo cycles."

Auclert, Adrien (2017), "Monetary policy and the redistribution channel." Technical report, National Bureau of Economic Research.

Berentsen, Aleksander, Gabriele Camera, and Christopher Waller (2005), "The distribution of money balances and the nonneutrality of money." *International Economic Review*, 46, 465–487.

Camera, Gabriele and Dean Corbae (1999), "Money and price dispersion." *International Economic Review*, 40, 985–1008.

Chiu, Jonathan and Miguel Molico (2010), "Liquidity, redistribution, and the welfare cost of inflation." *Journal of Monetary Economics*, 57, 428–438.

Chiu, Jonathan and Miguel Molico (2011), "Uncertainty, inflation, and welfare." *Journal of Money, Credit and Banking*, 43, 487–512.

Corbae, Dean, Ted Temzelides, and Randall Wright (2003), "Directed matching and monetary exchange." *Econometrica*, 71, 731–756.

Doepke, Matthias and Martin Schneider (2006), "Inflation and the redistribution of nominal wealth." *Journal of Political Economy*, 114, 1069–1097.

Dubey, Pradeep, Andreu Mas-Colell, and Martin Shubik (1982), "Efficiency properties of strategic market games: an axiomatic approach." In *Noncooperative Approaches to the Theory of Perfect Competition*, 225–248, Elsevier.

Galenianos, Manolis and Philipp Kircher (2008), "A model of money with multilateral matching." *Journal of Monetary Economics*, 55, 1054–1066.

Green, Edward J and Ruilin Zhou (1998), "A rudimentary random-matching model with divisible money and prices." *Journal of Economic theory*, 81, 252–271.

Green, Edward J and Ruilin Zhou (2002), "Dynamic monetary equilibrium in a random matching economy." *Econometrica*, 70, 929–969.

Howitt, Peter (2005), "Beyond search: fiat money in organized exchange." *International Economic Review*, 46, 405–429.

Ishihara, Hidehiko (2010), "Existence of a continuum of equilibria in a monetary random-matching model." *Review of Economic Dynamics*, 13, 886–898.

Jean, Kasie, Stanislav Rabinovich, and Randall Wright (2010), "On the multiplicity of monetary equilibria: Green–zhou meets lagos–wright." *Journal of Economic Theory*, 145, 392–401.

Julien, Benoit, John Kennes, and Ian King (2008), "Bidding for money." *Journal of Economic Theory*, 142, 196–217.

Kamiya, Kazuya, So Kubota, and Kayuna Nakajima (2017), "Real indeterminacy of stationary monetary equilibria in centralized economies." *The Japanese Economic Review*, 68, 497–520.

Kamiya, Kazuya and Takashi Sato (2004), "Equilibrium price dispersion in a matching model with divisible money." *International Economic Review*, 45, 413–430.

Kamiya, Kazuya and Takashi Shimizu (2006), "Real indeterminacy of stationary equilibria in matching models with divisible money." *Journal of Mathematical Economics*, 42, 594–617.

Kamiya, Kazuya and Takashi Shimizu (2007), "On the role of tax subsidy scheme in money search models." *International Economic Review*, 48, 575–606.

Kamiya, Kazuya and Takashi Shimizu (2011), "Stationary monetary equilibria with strictly increasing value functions and non-discrete money holdings distributions: An indeterminacy result." *Journal of Economic Theory*, 146, 2140–2150.

Kamiya, Kazuya and Takashi Shimizu (2013), "Dynamic auction markets with fiat money." *Journal of Money, Credit and Banking*, 45, 349–378.

Kocherlakota, Narayana R (1998), "Money is memory." *Journal of Economic Theory*, 81, 232–251.

Levitan, Richard and Martin Shubik (1972), "Price duopoly and capacity constraints." *International Economic Review*, 111–122.

Luttmer, Erzo GJ (2007), "Selection, growth, and the size distribution of firms." *The Quarterly Journal of Economics*, 122, 1103–1144.

Matsui, Akihiko and Takashi Shimizu (2005), "A theory of money and marketplaces." *International Economic Review*, 46, 35–59.

Menzio, Guido, Shouyong Shi, and Hongfei Sun (2013), "A monetary theory with non-degenerate distributions." *Journal of Economic Theory*, 148, 2266–2312.

Moen, Espen R (1997), "Competitive search equilibrium." *Journal of political Economy*, 105, 385–411.

Molico, Miguel (2006), "The distribution of money and prices in search equilibrium." *International Economic Review*, 47, 701–722.

Rocheteau, Guillaume, Pierre-Olivier Weill, and Tsz-Nga Wong (2018), "An heterogeneous-agent new-monetarist model with an application to unemployment." Technical report, National Bureau of Economic Research.

Rocheteau, Guillaume and Randall Wright (2005), "Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium." *Econometrica*, 73, 175–202.

Shapley, Lloyd and Martin Shubik (1977), "Trade using one commodity as a means of payment." *The Journal of Political Economy*, 937–968.

Wallace, Neil (1998), "Introduction to modeling money and studying monetary policy." *Journal of Economic Theory*, 81, 223–231.

Zhou, Ruilin (1999), "Individual and aggregate real balances in a random-matching model." *International Economic Review*, 40, 1009–1038.

Zhou, Ruilin (2003), "Does commodity money eliminate the indeterminacy of equilibrium?" *Journal of Economic Theory*, 110, 176–190.