The Lucas Imperfect Information Model with Imperfect Common Knowledge*

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Abstract

In the Lucas Imperfect Information model, output responds to unanticipated monetary shocks. We incorporate more general information structures into the Lucas model and demonstrate that output also responds to (dispersedly) anticipated monetary shocks if the information is imperfect common knowledge. Thus, the real effects of money consist of the unanticipated part and the anticipated part, and we decompose the latter into two effects, an imperfect common knowledge effect and a private information effect. We then consider an information structure composed of public and private signals. The real effects disappear when either signal reveals monetary shocks as common knowledge. However, when the precision of private information is fixed, the real effects are small not only when a public signal is very precise but also when it is very imprecise. This implies that a more precise public signal can amplify the real effects and make the economy more volatile.

Keywords: real effects, neutrality of money, iterated expectations, the Lucas model, imperfect common knowledge.

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1 Introduction

In the Lucas Imperfect Information model (Lucas, 1972, 1973), which formalizes the idea of Phelps (1970), markets are decentralized and agents in each market have only limited information about prices in other markets. As a consequence, output responds to unanticipated monetary shocks; that is, imperfect information about prices generates real effects of money. However, if monetary shocks are anticipated, no real effects arise. This implies that monetary shocks cannot have lasting effects, which is considered to be a serious shortcoming of the Lucas model.

This shortcoming is attributed to the assumption that all agents share information about the economy except the prices in their own markets (Townsend, 1983; Phelps, 1983); that is, the information is common knowledge. Based on this observation, Woodford (2003) introduces a model departing from the Lucas model in three respects: (i) agents have idiosyncratic private information, (ii) they adopt a monopolistically-competitive pricing framework, and (iii) information processing is dynamic. Then, he demonstrates the persisting real effects numerically. In his model, the problem to find a solution is complicated, but the solution can be obtained numerically.

In this paper, we introduce a model departing from the Lucas model in one aspect: agents are allowed to have any combination of information as imperfect common knowledge. More specifically, we study the Lucas model with arbitrary Gaussian information structures. Thus, the underlying model is simpler than that of Woodford (2003), but the information structure is more general. Our approach has two advantages. First, we follow the standard formulation of the Lucas model, also appearing in textbooks such as Romer (2019), and thus the role of information in our model is more plainly understood when it is compared to that in the original model. Next, we can provide a closed-form solution for each Gaussian information structure, which enables us to study the role of information analytically. On the other hand, the Lucas model has no dynamics, which is the limitation of our approach. This paper focuses on the impact of imperfect common knowledge on the real effects in one period rather than that on the lasting effects over multiple periods.

First, we demonstrate that the real effects consist of the unanticipated shock effect and the (dispersedly) anticipated shock effect. The expected value of the unanticipated shock effect equals zero, but that of the anticipated shock effect does not. Thus, the anticipated shock effect is indeed anticipated, but the expected value differs across different markets. We also show that the anticipated effect is composed of a private information effect, which depends upon the prediction in each market, and an imperfect common knowledge effect, which depends upon the aggregate prediction.

Next, we focus on the information structure in which agents receive a combination of a (common) public signal and a (idiosyncratic) private signal. In this case, the private information effect is a linear combination of a private signal and an unobserved monetary shock, and the imperfect common knowledge effect is a linear combination of a public signal and an unobserved monetary shock. The real effects are largest when either signal does not convey any information about monetary shocks, and they disappear when either signal reveals monetary shocks as common knowledge. However, when the precision of private information is fixed, the real effects are small not only when a public signal contains very precise information but also when it contains very imprecise information. Thus, a more precise public signal can amplify the real effects and make the economy more volatile. The intuition behind this result can be understood in terms of the law of large numbers. When a public signal is very imprecise, agents mainly use idiosyncratic private signals in predicting monetary shocks, and thus the aggregate prediction is very precise by the law of large numbers, thereby reducing the aggregate uncertainty and the real effects. In contrast, when a public signal is not so imprecise, agents use a public signal as well, and thus the aggregate prediction correlated with a public signal is less precise by the failure of the law of large numbers, amplifying the aggregate uncertainty and the real effects.

Finally, we show that the predicted aggregate price level in each market in the Lucas model is mathematically equivalent to a Bayesian Nash equilibrium in a Bayesian game studied by Radner (1962). This observation is useful because the existence and uniqueness of the market equilibrium is implied by those in the corresponding Bayesian game. When the number of separated markets is infinite, the corresponding Bayesian game is given by a beauty contest game studied by Morris and Shin (2002). Morris and Shin (2002) also point out a similar connection: they consider a modified version of the Lucas model which requires each separated market to clear and show that the price in each market corresponds to an equilibrium action in a beauty contest game. Thus, our discussion complements the discussion of Morris and Shin (2002) by showing that, in the case of the standard Lucas model which requires the aggregate market to clear, the predicted aggregate price level in each market corresponds to an equilibrium action in a beauty contest game. Note the difference between the predicted aggregate price level in each market and the price in each market: this difference determines the level of supply in each market in the Lucas model.

The impact of public information discussed in this paper is also independently found by Hellwig (2002) and Amato and Shin (2003) for the model of Woodford (2003), as noted in Hellwig (2008). It is also related to the main finding of Morris and Shin (2002). Morris and Shin (2002) examine the mean squared error of the equilibrium action of each player from the state as a measure of a welfare loss in a beauty contest game and show that it can increase with the precision of public information. In contrast, we examine the variance of output in the Lucas model as a measure of the real effects, which is shown to be equivalent to the mean squared error of the *aggregate* equilibrium action from the state in a beauty contest game, and show that it can increase with the precision of public information.

The organization of this paper is as follows. We introduce the model in Section 2. Assuming symmetric Gaussian information structures, we obtain the market equilibrium in a closed form in Section 3. Then, we study the impact of public information on the real effects in Section 4. Section 5 is devoted to the connection between the Lucas model and Bayesian games studied by Radner (1962) and Morris and Shin (2002).

2 The Lucas model with imperfect common knowledge

We consider the Lucas Imperfect Information model (Lucas, 1972, 1973), which also appears in textbooks such as Blanchard and Fischer (1989, p. 356) and Romer (2019, p. 293). We follow the standard formulation, but we drop the assumption that suppliers share information about the economy.

The economy consists of n separate competitive markets, each of which is called an island. Let y_i and p_i denote the logarithms of output and the nominal price of the output, respectively, in island $i \in \{1, \dots, n\}$. The supply function in island i is

$$y_i = b(p_i - E[p|I_i, p_i]), \tag{1}$$

where $b > 0$ is a constant, $p = n^{-1} \sum_{i=1}^{n} p_i$ is the aggregate price level in logarithm, I_i is the information about the economy available in island i which is independent of the relative price $p_i - p$, and $E[\,\cdot\,|I_i, p_i]$ is the expectation operator conditional on I_i and p_i. The equation (1) implies that suppliers increase output when they perceive an increase in the expected relative price $p_i - E[p|I_i, p_i]$. In the standard formulation, I_i is the same for all i, but we allow I_i to differ across i.

Given I_i, the conditional joint distribution of p and p_i is normal with the same expected value, i.e, $E[p_i - p|I_i] = 0$. Then, by the property of normal distributions, there exists $\theta \in (0, 1)$ such that

$$E[p|I_i, p_i] = (1 - \theta)E[p|I_i] + \theta p_i. \tag{2}$$

Thus, (1) is rewritten as

$$y_i = b(1 - \theta)(p_i - E[p|I_i]) = \beta(p_i - E[p|I_i]) = \beta(p_i - E_i p), \tag{3}$$

where $\beta = b(1 - \theta) > 0$ and $E_i = E[\,\cdot\,|I_i]$. We assume that θ does not depend upon i and I_i.[1]

We can obtain the following aggregate supply function by taking the aggregation of (3) over $i \in \{1, \dots, n\}$:

$$y = \beta(p - \bar{E}p), \tag{4}$$

where $y = n^{-1} \sum_i y_i$ and $\bar{E} = n^{-1} \sum_i E_i$. If I_i is the same for all i, then $E_i = E \equiv E[\cdot|I]$ with $I_i = I$, and thus (4) is reduced to

$$y = \beta(p - Ep), \tag{5}$$

which is the famous Lucas supply function. Output in (5) is increasing in the price surprise $p - Ep$. In contrast, output in (4) is increasing in the aggregation of the price surprise in each island $p - E_i p$.

The aggregate demand function is

$$y = m - p, \tag{6}$$

where m is nominal money in logarithm. This is one of the simplest possible ways to model aggregate demand, which is derived form the quantity equation in logarithm $m + v = p + y$ together with the assumption that the velocity in logarithm v equals zero.

Then, the market clearing condition is

$$m - p = \beta(p - \bar{E}p) \tag{7}$$

by (4) and (6), which is rewritten as

$$p = \frac{1}{1+\beta} m + \frac{\beta}{1+\beta} \bar{E}p = (1 - r)m + r\bar{E}p, \tag{8}$$

[1]For example, assume that $z_i = p - p_i$ is normally distributed with mean zero and variance $\sigma_z^2 = c\sigma_p^2$, where σ_p^2 is the endogenously determined variance of p and $c > 0$ is constant. That is, prices in separated markets are more volatile and dispersed when the aggregate price level is more volatile. Then, $\theta = 1/(1 + c)$.

where $r = \beta/(1+\beta)$. Operating $\bar{E}$ on both sides in (8) gives

$$\bar{E}p = (1-r)\bar{E}m + r\bar{E}^{(2)}p,$$

where $\bar{E}^{(2)} = \bar{E}\bar{E}$. Plugging this into (8) gives

$$p = (1-r)m + (1-r)r\bar{E}m + r^2\bar{E}^{(2)}p.$$

By repeating this, we obtain the following representation of p:

$$p = (1-r)\left(m + \sum_{k=1}^{\infty} r^k \bar{E}^{(k)} m\right), \tag{9}$$

where $\bar{E}^{(k+1)} = \bar{E}\bar{E}^{(k)}$ for $k \geq 1$. Moreover, by plugging (9) into (6), we obtain the following representation of y:

$$y = r\left(m - (1-r)\sum_{k=0}^{\infty} r^k \bar{E}^{(k+1)} m\right). \tag{10}$$

We will establish the existence (i.e. the convergence) of (9) and (10) in Section 5.

When I_i is the same for all i and thus $E_i = E$, the law of iterated expectations holds, i.e., $\bar{E}^{(k)}m = Em$ for all $k \geq 1$, and thus (10) is reduced to

$$y = r\left(m - Em\right).$$

This equation shows the key implication of the standard Lucas model: an unanticipated monetary shock has real effects on output, whereas an anticipated monetary shock has no such effects.

In contrast, when I_i differs across i, the law of iterated expectations does not hold, i.e., $\bar{E}^{(k)}m \neq \bar{E}^{(k')}m$ for $k \neq k'$, as will be calculated in a closed form in the next section. This implies that not only an unanticipated monetary shock but also an anticipated monetary shock has real effects. To demonstrate it, we decompose (10) into the following three terms for each i:

$$y = r\left(m - E_i m\right) + r\left(E_i m - \bar{E}m\right) + r\left(\bar{E}m - (1-r)\sum_{k=0}^{\infty} r^k \bar{E}^{(k+1)} m\right). \tag{11}$$

We call the first term an unanticipated shock effect because $E_i(m - E_i m) = 0$. We call the second term a private information effect because it is the difference between the expectation $E_i m$ in island i and the aggregate expectation $\bar{E}m$. We call the last term an imperfect common knowledge effect because it is attributed to imperfect common knowledge of information and the resulting failure of the law of iterated expectations. It should be noted that the real effects induced by the second and third terms are in fact anticipated in the sense that the expected values of these terms are not equal to zero.

3 Gaussian information structures

In this section, we calculate the real effects of money on output in a closed form. To this end, we assume that information available in island i is a random vector $\mathbf{x}_i \in \mathbb{R}^L$ and that $(m, \mathbf{x}_1, \dots, \mathbf{x}_n)$ is normally distributed with mean zero, i.e., $E[m] = 0$ and $E[\mathbf{x}_i] = \mathbf{0}$ for all i. We also assume that the covariance structure is symmetric with respect to the exchange of islands: for all $i, i', j, j' \in \{1, \dots, n\}$ with $i \neq i'$ and $j \neq j'$,

$$\mathrm{var}(\mathbf{x}_i) = \mathrm{var}(\mathbf{x}_{i'}),\ \mathrm{cov}(m, \mathbf{x}_i) = \mathrm{cov}(m, \mathbf{x}_{i'}),\ \mathrm{cov}(\mathbf{x}_i, \mathbf{x}_j) = \mathrm{cov}(\mathbf{x}_{i'}, \mathbf{x}_{j'}).$$

Then, for a $1 \times L$ matrix $M = \mathrm{cov}(m, \mathbf{x}_i)\mathrm{var}(\mathbf{x}_i)^{-1}$ and an $L \times L$ matrix $A = \mathrm{cov}(\mathbf{x}_j, \mathbf{x}_i)\mathrm{var}(\mathbf{x}_i)^{-1}$, it holds that $E_i m = M\mathbf{x}_i$ and $E_i \mathbf{x}_j = A\mathbf{x}_i$ for all $i \neq j$.

Using M and A, we calculate (9) and (10). Let $\bar{\mathbf{x}} = n^{-1} \sum_i \mathbf{x}_i$ and $A_n = (n-1)n^{-1}A + n^{-1}I$, where I is a unit matrix. Then,

$$\bar{E}m = n^{-1} \sum_i E_i m = n^{-1} \sum_i M\mathbf{x}_i = M\bar{\mathbf{x}},$$

$$\bar{E}\bar{\mathbf{x}} = n^{-1} \sum_i E_i \bar{\mathbf{x}} = n^{-1} \sum_i \left((n-1)n^{-1}A\mathbf{x}_i + n^{-1}\mathbf{x}_i\right) = A_n\bar{\mathbf{x}}.$$

Thus,

$$\bar{E}^{(2)}m = \bar{E}M\bar{\mathbf{x}} = n^{-1} \sum_i E_i M\bar{\mathbf{x}} = Mn^{-1} \sum_i E_i \bar{\mathbf{x}} = MA_n\bar{\mathbf{x}},$$

$$\vdots$$

$$\bar{E}^{(k)}m = MA_n^{k-1}\bar{\mathbf{x}} \tag{12}$$

for all $k \geq 2$. By plugging (12) into (9) and (10), we obtain

$$p = (1-r)(m + rM(I - rA_n)^{-1}\bar{\mathbf{x}}), \tag{13}$$

$$y = r\left(m - (1-r)M(I - rA_n)^{-1}\bar{\mathbf{x}}\right) \tag{14}$$

$$= r\,(m - M\mathbf{x}_i) + r\,(M\mathbf{x}_i - M\bar{\mathbf{x}}) + r\left(M - (1-r)M(I - rA_n)^{-1}\right)\bar{\mathbf{x}}, \tag{15}$$

where $r\,(m - M\mathbf{x}_i)$ is the unanticipated shock effect, $r\,(M\mathbf{x}_i - M\bar{\mathbf{x}})$ is the private information effect, and $r\left(M - (1-r)M(I - rA_n)^{-1}\right)\bar{\mathbf{x}}$ is the imperfect common knowledge effect. The last two effects are anticipated because the expected value of the private information effect in island i is

$$E_i r\,(M\mathbf{x}_i - M\bar{\mathbf{x}}) = rM\,(I - A_n)\,\mathbf{x}_i, \tag{16}$$

and that of the imperfect common knowledge effect is

$$E_i r \left(M - (1-r)M(I - rA_n)^{-1} \right) \bar{\mathbf{x}} = r \left(M - (1-r)M(I - rA_n)^{-1} \right) A_n \mathbf{x}_i, \tag{17}$$

which are nonzero as long as the coefficient matrices in (16) and (17) are not zero matrices.

4 Public and private signals

In this section, we focus on the Gaussian information structure with $\mathbf{x}_i$ consisting of public and private signals and study the impact of each signal on the real effects of money. In so doing, we consider the limiting case as the number of islands goes to infinity, which substantially simplifies the calculation, but we can also obtain a similar result in the finite case.

The signal in island i is

$$\mathbf{x}_i = \begin{bmatrix} v_i \\ w \end{bmatrix} = \begin{bmatrix} m + \varepsilon_i \\ m + \varepsilon_0 \end{bmatrix},$$

where v_i is a private signal and w is a public signal. Random variables m, ε_i, and ε_0 are independently normally distributed with mean zero and variance τ_m^{-1}, τ_v^{-1}, and τ_w^{-1}, respectively. The reciprocal of the variance of a private signal τ_v is referred to as the precision of private information, and that of a public signal τ_w is referred to as the precision of public information.

Note that

$$\begin{aligned}
M &= \mathrm{cov}(m, \mathbf{x}_i)\mathrm{var}(\mathbf{x}_i)^{-1} = \begin{bmatrix} \dfrac{\tau_v}{\tau_m + \tau_v + \tau_w} & \dfrac{\tau_w}{\tau_m + \tau_v + \tau_w} \end{bmatrix}, \\
A &= \mathrm{cov}(\mathbf{x}_j, \mathbf{x}_i)\mathrm{var}(\mathbf{x}_i)^{-1} = \lim_{n\to\infty} A_n = \begin{bmatrix} \dfrac{\tau_v}{\tau_m + \tau_v + \tau_w} & \dfrac{\tau_w}{\tau_m + \tau_v + \tau_w} \\ 0 & 1 \end{bmatrix}, \\
\bar{\mathbf{x}} &= \lim_{n\to\infty} n^{-1} \sum_i \mathbf{x}_i = \begin{bmatrix} m \\ w \end{bmatrix}.
\end{aligned}$$

Thus, (13), (14), and (15) are rewritten as

$$\begin{aligned}
p &= \frac{(1-r)(\tau_m + \tau_v + \tau_w)m + r\tau_w w}{\tau_m + (1-r)\tau_v + \tau_w}, \\
y &= \frac{r((\tau_m + \tau_w)m - \tau_w w)}{\tau_m + (1-r)\tau_v + \tau_w} \\
&= r(m - E_i m) + \frac{r\tau_v(v_i - m)}{\tau_m + \tau_v + \tau_w} + \frac{r^2\tau_v((\tau_m + \tau_w)m - \tau_w w)}{(\tau_m + \tau_v + \tau_w)(\tau_m + (1-r)\tau_v + \tau_w)}.
\end{aligned}$$

The private information effect in the second term is a linear combination of v_i and m, and the imperfect common knowledge effect in the third term is a linear combination of w and m. Note that, when a private signal contains no information about m (i.e., $\tau_v = 0$), the second and third terms vanish, in which case the model is reduced to the standard Lucas model.

As a measure of the real effects, we adopt the variance of y and represent it as a function of $\tau = (\tau_v, \tau_w)$:

$$\begin{aligned} V(\tau) &\equiv E[y^2] \\ &= E[((\tau_m + \tau_w)m - \tau_w w)^2] \cdot r^2/(\tau_m + (1-r)\tau_v + \tau_w)^2 \\ &= E[(\tau_m m - \tau_w \varepsilon_0)^2] \cdot r^2/(\tau_m + (1-r)\tau_v + \tau_w)^2 \\ &= r^2(\tau_m + \tau_w)/(\tau_m + (1-r)\tau_v + \tau_w)^2. \end{aligned} \tag{18}$$

Note that money has no real effects if $V(\tau) = 0$ because y is a constant for any m in this case. It is straightforward to show that

$$\lim_{\tau_v \to \infty} V(\tau) = \lim_{\tau_w \to \infty} V(\tau) = 0 \leq V(\tau) \leq r^2/\tau_m = V(0,0).$$

Thus, money has the largest real effects if there is no information about m (i.e., $\tau_v = \tau_w = 0$). In contrast, money has no real effects if either signal reveals the true value of m (i.e., $\tau_v = \infty$ or $\tau_w = \infty$). However, when the precision of private information is fixed, money has very small real effects not only when a public signal is very precise but also when it is very imprecise, as the next proposition shows. That is, $V(\tau)$ is not a monotone function of τ_w, whereas it is a decreasing function of τ_v.

Proposition 1. *The variance of y is decreasing in τ_v; that is,*

$$\partial V(\tau)/\partial \tau_v < 0.$$

The variance of y is increasing in τ_w if $\tau_w < (1-r)\tau_\epsilon - \tau_m$ and decreasing in τ_w if $\tau_w > (1-r)\tau_\epsilon - \tau_m$; that is,

$$\partial V(\tau)/\partial \tau_v \gtreqless 0 \iff \tau_w \lesseqgtr (1-r)\tau_\epsilon - \tau_m,$$

which implies that

$$\max_{\tau_w} V(\tau_v, \tau_w) = V(\tau_v, \min\{0, (1-r)\tau_v - \tau_m\}).$$

Proof. A direct calculation yields

$$\begin{aligned} \partial V(\tau)/\partial \tau_v &= -2(1-r)r^2(\tau_m + \tau_w)/(\tau_m + (1-r)\tau_v + \tau_w)^3, \\ \partial V(\tau)/\partial \tau_w &= -r^2(\tau_m + \tau_w - (1-r)\tau_v)/(\tau_m + (1-r)\tau_v + \tau_w)^3, \end{aligned}$$

which implies the proposition. □

Fix $\tau_v > 0$. If $\tau_v < \tau_m/(1-r)$ (i.e., the precision of private information is sufficiently low), $V(\tau)$ is decreasing in τ_w, and thus $V(\tau)$ is maximized when $\tau_w = 0$ (i.e., a public signal contains no information about m). However, if $\tau_v > \tau_m/(1-r)$ (i.e., the precision of private information is sufficiently high), $V(\tau)$ is increasing in τ_w for $\tau_w < (1-r)\tau_v - \tau_m$, and thus $V(\tau)$ is maximized when $\tau_w = (1-r)\tau_v - \tau_m$ (i.e., a public signal contains noisy information about m).

To explain the intuition behind the above result, note that

$$V(\tau) = \beta^2 E[(\bar{E}p - p)^2] \tag{19}$$

by (4). Thus, $V(\tau)$ equals the mean squared error of the aggregate expectation $\bar{E}p$ from the true value of p (times a constant β^2). If the precision of public information is very low, the correlation between $E_i p$ and $E_j p$ is also very low, so $\bar{E}p$ is close to p by the law of large numbers, which results in small real effects. However, as the precision of public information increases, the correlation between $E_i p$ and $E_j p$ also increases, so $\bar{E}p$ is not necessarily close to p by the failure of the law of large numbers, which results in larger real effects.

We can formally discuss the above intuition in the limiting case as $\tau_m \to 0$; that is, m has an improper prior. In this case, the expected value of m in island i is

$$E_i m = \frac{\tau_v}{\tau_v + \tau_w} v_i + \frac{\tau_w}{\tau_v + \tau_w} w.$$

Now assume that $\tau_w = 0$; that is, a public signal contains no information. Then, $E_i m = v_i$ and $\bar{E}m = m$ by the law of large numbers. This implies that $p = m$ by (9), $y = 0$ by (10), and thus $V(\tau) = 0$, which implies the following proposition.

Proposition 2. *Assume that $\tau_m = 0$; that is, m has an improper prior. Then, $V(\tau) = 0$ if and only if $\tau_w = \infty$ or $\tau_w = 0$.*

That is, money has no real effects not only when a public signal reveals m but also when it conveys no information about m. In other words, the real effects arise if and only if a public signal conveys noisy information about m. This is because, in the absence of a public signal, there is no uncertainty in the aggregate expectation $\bar{E}m = m$, thus eliminating the volatility of the economy. A similar argument applies as long as τ_m is sufficiently small compared to τ_v. This is why a more precise public signal can amplify the real effects and make the economy more volatile.

Finally, we evaluate the anticipated real effects. By direct calculation, the expected value of

y in island i is

$$E_i y = \frac{r\tau_v((\tau_m + \tau_w)v_i - \tau_w w)}{(\tau_m + \tau_v + \tau_w)(\tau_m + (1-r)\tau_v + \tau_w)},$$

and its variance is

$$V_E(\tau) \equiv E[(E_i y)^2] = \frac{r^2\tau_v(\tau_m + \tau_w)}{(\tau_m + \tau_v + \tau_w)(\tau_m + (1-r)\tau_v + \tau_w)^2}.$$

Because

$$V_E(0, \tau_w) = \lim_{\tau_v \to \infty} V_E(\tau_v, \tau_w) = \lim_{\tau_w \to \infty} V_E(\tau_v, \tau_w) = 0,$$

money has no anticipated real effects if and only if there is no private signal, as in the standard Lucas model, or if either signal reveals m, in which case money does not have the unanticipated real effects as well. In other words, money has the anticipated real effects if and only if a private signal conveys noisy information about m and a public signal does not perfectly reveal m. We can obtain the value of τ which maximizes $V_E(\tau)$, but it is a solution to a system of quadratic equations, so we do not calculate it here.

5 The Lucas model and Radner's Bayesian game

In this section, we demonstrate that the market equilibrium in the Lucas model is mathematically equivalent to a Bayesian Nash equilibrium in a Bayesian game studied by Radner (1962). This observation is useful because the existence and uniqueness of the market equilibrium is implied by those in the corresponding Bayesian game established by Radner (1962). When once the market equilibrium is interpreted as a Bayesian Nash equilibrium, Proposition 1 is closely related to the findings of Morris and Shin (2002) and Ui and Yoshizawa (2015).

5.1 The market equilibrium as a Bayesian Nash equilibrium

Let $a_i = E_i p$ be the expected aggregate price level in island i. By (8), a_i is a solution to

$$a_i = (1-r)E_i m + rn^{-1}\sum_{j=1}^{n} E_i a_j,$$

which is rewritten as

$$a_i = \alpha_1 \sum_{j \neq i} E_i a_j + \alpha_2 E_i m, \tag{20}$$

where $\alpha_1 = rn^{-1}/(1 - rn^{-1})$ and $\alpha_2 = (1 - r)/(1 - rn^{-1})$. This equation can be interpreted as the first order condition for the problem to maximize

$$E_i \left[-a_i^2 + 2\alpha_1 \sum_{j \neq i} a_i a_j + 2\alpha_2 m a_i \right]$$

with respect to a_i. In other words, $(a_i)_{i \in \{1,\ldots,n\}}$ is a Bayesian Nash equilibrium of a Bayesian game with a quadratic payoff function

$$-a_i^2 + 2\alpha_1 \sum_{j \neq i} a_i a_j + 2\alpha_2 m a_i, \tag{21}$$

where $a_i \in \mathbb{R}$ is player i's action and $m \in \mathbb{R}$ is a payoff state. As pointed out by Ui (2009), this game is a Bayesian potential game (Monderer and Shapley, 1996; van Heumen et al., 1996) which has the same best correspondence as that of a Bayesian game with an identical payoff function

$$-\sum_i a_i^2 + 2\alpha_1 \sum_{i,j:i<j} a_i a_j + 2\alpha_2 m \sum_i a_i. \tag{22}$$

Radner (1962) studies a Bayesian game with an identical payoff function, which is referred as a team, and shows that it has a unique Bayesian Nash equilibrium if the identical payoff function is strictly concave, which is true no matter what the information structure is. Moreover, Radner (1962) obtains the unique Bayesian Nash equilibrium in a closed form in the case of Gaussian information structures.[2]

It is straightforward to show that (22) is strictly concave if and only if $-1 < \alpha_1 < 1/(n-1)$. Note that $\alpha_1 = rn^{-1}/(1 - rn^{-1}) < 1/(n-1)$. Thus, a Bayesian Nash equilibrium $(E_i p)_{i \in \{1,\ldots,n\}}$ exists, which also guarantees the convergence of (9) and (10). In the case of Gaussian information structures, we can also obtain $E_i p$ in a closed form using the results of Radner (1962) to calculate $y = m - p = r(m - \bar{E}p)$, which coincides with (14).

The above discussion is summarized in the following proposition.

Proposition 3. *The expected aggregate price in island i coincides with the equilibrium action of player i in a Bayesian game given by* (21) *or* (22)*. Moreover, under any information structure, a Bayesian Nash equilibrium exists, and it is unique.*

[2]Ui (2009, 2016) extends these results of Radner (1962) to more general Bayesian games. Ui (2016) also provides an elementary proof for the uniqueness.

5.2 Morris and Shin (2002)

In the limit as $n \to \infty$,[3] (20) is reduced to

$$a_i = rE_i\bar{a} + (1-r)E_i m, \tag{23}$$

where $\bar{a} = \lim_{n\to\infty} n^{-1} \sum_i a_i$, because

$$\lim_{n\to\infty} \alpha_1 \sum_{j\neq i} E_i a_j = \lim_{n\to\infty} rn^{-1}/(1-rn^{-1}) \sum_{j=1}^{n} E_i a_j = rE_i\bar{a},$$
$$\lim_{n\to\infty} \alpha_2 E_i m = (1-r)E_i m.$$

Note that (23) can be interpreted as the first order condition of a Bayesian Nash equilibrium in a Bayesian game with a payoff function

$$-a_i^2 + 2ra_i\bar{a} + 2(1-r)E_i m + f(\bar{a}, m),$$

where $f(\bar{a}, m)$ is a function of $(\bar{a}, m)$. A Bayesian game with the above payoff function is referred to as a beauty contest game because player i's best response given by (23) is the expected value of the weighted average of the opponents' aggregate action $\bar{a}$ and the payoff state m.

Morris and Shin (2002) consider a beauty contest game with the same Gaussian information structure as that discussed in Section 4 assuming that $\tau_m = 0$ (i.e., an improper prior). They also point out the equivalence of a Bayesian Nash equilibrium in a beauty contest game and the market equilibrium in the Lucas model, but their discussion is based upon a modified version of the Lucas model. Instead of the aggregate demand (6), Morris and Shin (2002) assume that island i has its own demand function $y_i = E_i m - p_i$ and require that demand should equal supply in each island, i.e., $E_i m - p_i = \beta(p_i - E_i p)$ for all i. This is rewritten as $p_i = rE_i p + (1-r)E_i m$, which coincides with (23) when $p_i = a_i$ and $p = \lim_{n\to\infty} n^{-1} \sum_i p_i$. Therefore, the price p_i in island i coincides with the action a_i of player i in a beauty contest game.

In contrast to Morris and Shin (2002), we follow the standard formulation of the Lucas model; that is, we require the market clearing condition $m - p = \beta(p - \bar{E}p)$ in the aggregate market. By Proposition 3, the expected aggregate price level $E_i p$ in island i coincides with the action a_i of player i in a beauty contest game. Note the difference between the price p_i in island i and the expected aggregate price level $E_i p$ in island i, which determines the supply in island i through $y_i = \beta(p_i - E_i p)$ in (3).

[3] On the application of Radner's theorem in the case of infinite number of players, see Ui and Yoshizawa (2013).

In their main result, Morris and Shin (2002) study the property of the mean squared error of the equilibrium action a_i from the payoff state m

$$MSE[a_i] \equiv E[(a_i - m)^2]$$

as a measure of a welfare loss, which is the same for all i by the symmetry of payoff and information structures, and show that

$$\begin{aligned} &\partial MSE[a_i]/\partial \tau_v < 0, \\ &\partial MSE[a_i]/\partial \tau_w \gtrless 0 \iff \tau_w \lessgtr (1-r)(2r-1)\tau_\epsilon. \end{aligned} \tag{24}$$

In Proposition 1, we study the property of the variance of y given by $V(\tau)$. Note that

$$V(\tau) = E[(m-p)^2] = rE[(\bar{E}p - m)^2] \tag{25}$$

by (8), which implies that $V(\tau)$ equals a constant times the mean squared error of the aggregate equilibrium action[4] $\bar{a} = \bar{E}p$ from the payoff state m

$$MSE[\bar{a}] \equiv E[(\bar{a} - m)^2].$$

Proposition 1 in the special case of $\tau_m = 0$ is summarized as follows:

$$\begin{aligned} &\partial MSE[\bar{a}]/\partial \tau_v < 0, \\ &\partial MSE[\bar{a}]/\partial \tau_w \gtrless 0 \iff \tau_w \lessgtr (1-r)\tau_\epsilon. \end{aligned} \tag{26}$$

It is interesting to compare (24) and (26) in our context of the Lucas model, where $a_i = E_i p$ is the prediction of the aggregate price level in island i, and $\bar{a} = \bar{E}p$ is the aggregate prediction of the aggregate price level. The mean squared error of the prediction in each island and that of the aggregate prediction share the following properties.

- $MSE[a_i]$ and $MSE[\bar{a}]$ are minimized and equal to zero when $\tau_v = \infty$ or $\tau_w = \infty$.
- $MSE[a_i]$ and $MSE[\bar{a}]$ are decreasing in τ_v.
- $MSE[a_i]$ and $MSE[\bar{a}]$ are increasing in τ_w if $\tau_w < (1-r)(2r-1)\tau_v$ and decreasing in τ_w if $\tau_w > (1-r)\tau_v$.

[4] Morris and Shin (2002) discuss $E[(\bar{a} - m)^2]$, but they do not calculate its value in the equilibrium of a beauty contest game.

However, if $(1-r)(2r-1)\tau_v < \tau_w < (1-r)\tau_v$, $MSE[\bar{a}]$ is increasing in τ_w, but $MSE[a_i]$ is decreasing in τ_w. That is, even when $MSE[\bar{a}]$ increases with τ_w, $MSE[a_i]$ can decrease with τ_w, and whenever $MSE[a_i]$ increases with τ_w, $MSE[\bar{a}]$ also increases with τ_w. In particular, if $r < 1/2$, $MSE[a_i]$ necessarily decreases with τ_w, whereas $MSE[\bar{a}]$ can increase with τ_w. This difference is also attributed to the law of large numbers. Suppose that $\tau_w = 0$. As discussed in Section 4, the aggregate prediction $\bar{a} = \bar{E}p$ equals m because $\bar{E}p = p = m$ by the law of large numbers, and thus $MSE[\bar{a}] = 0$. This implies that a more precise public signal increases $MSE[\bar{a}]$. In contrast, the law of large numbers does not work in an unaggregated prediction $a_i = E_i p$, and a more precise public signal reduces $MSE[a_i] > 0$.

5.3 Ui and Yoshizawa (2015)

Ui and Yoshizawa (2015) also consider a Bayesian Nash equilibrium given by (20) under the same Gaussian information structure as that discussed in Section 4 and study the property of the following function:

$$W(\tau) = E\left[c_1 \sum_i a_i^2 + c_2 \sum_{i<j} a_i a_j + c_3 \sum_i m a_i + c_4 \sum_i a_i + c_5\right], \tag{27}$$

which is the expected value of a quadratic function of the equilibrium actions and the payoff state. They show that there exist constants $\zeta, \eta, \xi \in \mathbb{R}$ such that

$$W(\tau) = \zeta(\mathrm{var}[a_i] - \mathrm{cov}[a_i, a_j]) + \eta\mathrm{cov}[a_i, a_j] + \xi, \tag{28}$$

where $\mathrm{var}[a_i]$ is the variance of an action and $\mathrm{cov}[a_i.a_j]$ is the covariance of actions of different players. This representation is useful because the ratio of the coefficients ζ/η together with the signs of ζ and η determines whether $W(\tau)$ is increasing, decreasing, or otherwise. We can explain the intuition as follows by focusing on the precision of public information. The covariance $\mathrm{cov}[a_i, a_j]$ necessarily increases with the precision of public information because more precise information causes more correlated actions. In contrast, the difference between the variance and the covariance $\mathrm{var}[a_i] - \mathrm{cov}[a_i, a_j]$ necessarily decreases with the precision of public information because a higher correlation of actions brings the variance and the covariance closer. The property of $W(\tau)$ is determined by the combination of the above properties of $\mathrm{cov}[a_i, a_j]$ and $\mathrm{var}[a_i] - \mathrm{cov}[a_i, a_j]$.

We can regard $V(\tau)$ is a special case of (27) with $a_i = E_i p$ and obtain the following representation of $V(\tau)$.

Lemma 1. *For $\zeta = -2r/(1-r)$, $\eta = -r$, and $\xi = r\tau_m^{-1}$, $V(\tau)$ has a representation* (28) *with $a_i = E_i p$.*

This representation is useful in understanding why $V(\tau)$ can increase with τ_w. Note that both ζ and η are negative. Thus, we can explain the property of $V(\tau)$ in terms of the properties of $-(\mathrm{var}[a_i] - \mathrm{cov}[a_i, a_j])$ and $-\mathrm{cov}[a_i, a_j]$. When τ_w is small, $\mathrm{cov}[a_i, a_j]$ is also small, so $-(\mathrm{var}[a_i] - \mathrm{cov}[a_i, a_j])$ is dominant in $V(\tau)$, and it is increasing in τ_w. When τ_w is large, $-(\mathrm{var}[a_i] - \mathrm{cov}[a_i, a_j])$ is close to zero, so $-\mathrm{cov}[a_i, a_j]$ is dominant in $V(\tau)$, and it is decreasing in τ_w. Therefore, $V(\tau)$ is increasing in τ_w when τ_w is small and decreasing in τ_w when τ_w is large.[5]

For completeness, we give a proof for Lemma 1.

Proof of Lemma 1. Let $a_i = E_i p$ and $\bar{a} = \bar{E} p$. Then, by (25),

$$V(\tau) = E[r(\bar{a} - m)^2] = rE[\bar{a}^2 - 2m\bar{a} + m^2] = rE[\bar{a}^2] - 2rE[m\bar{a}] + r\tau_m^{-1}. \tag{29}$$

First, because $E[a_i] = 0$, we obtain

$$\begin{aligned} E[\bar{a}^2] &= \lim_{n\to\infty} E\left[\left(n^{-1}\sum_i a_i\right)^2\right] \\ &= \lim_{n\to\infty} n^{-2} E\left[\sum_i a_i^2 + 2\sum_{i<j} a_i a_j\right] \\ &= \lim_{n\to\infty} n^{-2} E\left[n a_i^2 + n(n-1) a_i a_j\right] \\ &= \mathrm{cov}[a_i, a_j]. \end{aligned} \tag{30}$$

Next, note that

$$E[m\bar{a}] = \lim_{n\to\infty} n^{-1} \sum_i E[m a_i] = E[m a_i] = E[E_i m a_i] = E[a_i E_i m]. \tag{31}$$

Because (23) is rewritten as

$$a_i = rE_i\bar{a} + (1-r)E_i m = r \lim_{n\to\infty} n^{-1} \sum_j E_i a_j + (1-r)E_i m = rE_i a_j + (1-r)E_i m,$$

and it implies that

$$a_i E_i m = (a_i^2 - r a_i E_i a_j)/(1-r) = E_i(a_i^2 - r a_i a_j)/(1-r),$$

[5] Using Lemma 1 together with the results of Ui and Yoshizawa (2015), we can also obtain Proposition 1 because Ui and Yoshizawa (2015) characterize $W(\tau)$ for all ζ and η.

(31) is reduced to

$$E[m\bar{a}] = E[(a_i^2 - ra_ia_j)/(1-r)] = (\mathrm{var}[a_i] - r\mathrm{cov}[a_i, a_j])/(1-r). \tag{32}$$

Therefore, by (29), (30), and (32),

$$\begin{aligned} V(\tau) &= r\mathrm{cov}[a_i, a_j] - 2r(\mathrm{var}[a_i] - r\mathrm{cov}[a_i, a_j])/(1-r) + r\tau_m^{-1} \\ &= -2r(\mathrm{var}[a_i] - \mathrm{cov}[a_i, a_j])/(1-r) - r\mathrm{cov}[a_i, a_j] + r\tau_m^{-1}, \end{aligned}$$

which establishes the lemma. □

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