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Multilateral Sato-Vartia Index for International Comparisons of Prices and Real Expenditures[1]

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Abstract

The Sato-Vartia (SV) index for bilateral price comparisons has impressive analytical properties and is used intensively in recent international trade and macroeconomic analyses. In this paper we propose several ways of constructing transitive multilateral version of the SV index. We show that the SV index is only one of many logarithmic indices that satisfy the factor reversal test discussed in index number theory. We derive closed form expressions for the generalized SV indices and empirically implement the new indices for making cross-country price comparison using World Bank data from the 2011 International Comparison Program.

JEL Codes: C13; C83; E01; E31
Key words: Sato-Vartia Index; Multilateral comparisons; Transitivity; Factor Reversal Test

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1. Introduction

The Sato (1976)-Vartia (1976) (SV) index is a logarithmic index to measure changes in prices and quantities which is specifically designed to satisfy factor reversal test. This index satisfies several other test properties including time reversal test, has the same properties as the Fisher (1922) ideal index. It is the first log-change index found to possess this property. Commonly used Törnqvist-Theil and geometric-young indices fail the factor reversal test. The SV index has been shown to be exact cost-of-living (COLI) index for constant elasticity of substitution (CES) utility function ((Sato, 1976 and Feenstra, 1994) and pseudo-superlative (Barnett and Choi, 2008). The SV index features prominently in trade and macroeconomic analyses (Feenstra, 1994; Reading and Weinstein, 2015; and Rao, Rambaldi and Balk, 2015).

The SV index has not been used in multilateral international price and real income comparisons as the index does not satisfy transitivity condition.[2] The Gini- Eltető - Köves -Szulc (GEKS) approach is used to construct transitive multilateral index that is closest (in the least squares sense) to a desired binary index such as the Fisher Index. In this paper we show that the SV index offers a rich set of alternative methods of generating transitive comparisons. We prove several important results including non-uniqueness of the SV index except for the two-country two-commodity case. Empirical results based on data from the 2011 International Comparison Program (ICP) (World Bank, 2015) are presented and contrasted.

2. Notation and the Sato-Vartia Index

Let $\{p_{ik}, q_{ik}\}_{i=1,2,...,N}^{k=1,2,...,M}$ respectively represent the price and quantity of *i-th* commodity in country *k*. The price and quantity index numbers for country *k* with country *j* be denoted by P_{jk} and Q_{jk} . The value or expenditure and the expenditure shares are denoted, respectively, by $E_k \equiv \sum_{i=1}^{N} p_{ik} q_{ik}$ and $w_{ik} \equiv p_{ik} q_{ik} \Big/ \sum_{i=1}^{N} p_{ik} q_{ik}$ $(i = 1, 2, ..., N)$. The value index is defined as $E_{jk} \equiv E_k / E_j$.

An index number satisfies the *factor reversal test* if and only if

$$P_{jk} \cdot Q_{jk} = E_{jk} \Leftrightarrow \ln P_{jk} + \ln Q_{jk} = \ln E_{jk} \Leftrightarrow \ln E_{jk} - \ln P_{jk} - \ln Q_{jk} = 0 \qquad (1)$$

Log-change index number formulae are essentially geometric averages of price and quantity changes which are additive in logarithmic form. Following Sato (1976), log-change price and quantity index numbers have the following generic form:

$$\ln P_{jk} = \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{p_{ik}}{p_{ij}}\right), \ln Q_{jk} = \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{q_{ik}}{q_{ij}}\right), \sum_{i=1}^{N} \phi_i^{jk} = 1, \phi_i^{jk} \geq 0. \qquad (2)$$

where ϕ_i^{jk} are weights which are usually functions of the observed expenditure shares in countries *j* and *k*. The Sato-Vartia log-change index, P_{jk}^{SV} , is defined as in (2) with the weights;[3]

$$\phi_i^{jk} = \frac{\frac{w_{ik} - w_{ij}}{\ln(w_{ik}) - \ln(w_{ij})}}{\sum_{i=1}^{N} \frac{w_{ik} - w_{ij}}{\ln(w_{ik}) - \ln(w_{ij})}}, i = 1,2,...,N. \qquad (3)$$

We establish the following result regarding uniqueness of the SV index.

[2] See Balk (2008) for details of transitivity in multilateral comparisons..

[3] The SV quantity index is similarly represented. Easy to show that the SV index satisfies the factor reversal test. Weights are defined using logarithmic averages of expenditure shares in countries *j* and *k*.

Proposition 1: Log-change index number satisfying the factor reversal test is not unique. It is unique up to a factor of proportionality if and only if the number of commodities is equal to two, in this case the log-change index is unique and equals the SV index.

Proof: A log-change index with weights, $\left(\phi_i^{jk} : i = 1, 2, ..., N\right)$ satisfies factor reversal test if and only the following condition is satisfied:

$$ln\left(\frac{E_k}{E_j}\right) - \sum_{i=1}^{N} \phi_i^{jk}\, ln\left(\frac{p_{ik}}{p_{ij}}\right) - \sum_{i=1}^{N} \phi_i^{jk}\, ln\left(\frac{q_{ik}}{q_{ij}}\right) = 0 \Leftrightarrow \sum_{i=1}^{N} \phi_i^{jk} ln\left(\frac{w_{ik}}{w_{ij}}\right) = 0 \qquad (4)$$

Any index number formula with weights ϕ_i^{jk} satisfying (4) is a log-change index number with factor reversal test property. Equation (4) is one linear homogeneous equation in *N-1* unknown weights since the weights add up to unity. Therefore, there are infinitely many log-change index numbers that satisfy the factor reversal test. Sato-Vartia index is only one from this class.

In the two commodity case (*N*=2) it can be shown by simple algebra that (4) has unique solution given by:

$$\phi_1^{jk} \ln\left(\frac{w_{1k}}{w_{1j}}\right) + \phi_2^{jk} \ln\left(\frac{w_{2k}}{w_{2j}}\right) = 0;\ and\ \phi_1^{jk} + \phi_2^{jk} = 1 \ \Rightarrow \phi_i^{jk} = \frac{\left(w_{i2} - w_{i1}\right)/\left(\ln w_{i2} - \ln w_{i1}\right)}{\sum_{i=1}^{2}\left(w_{i2} - w_{i1}\right)/\left(\ln w_{i2} - \ln w_{i1}\right)} \quad i = 1, 2 \qquad (5)$$

This proves the "if" part. The only if part is established using the following three commodity example.

Table 1: Example of Weights Passing Factor Reversal Test

		Expenditure Share		Sato-Vartia	Altenative Weights
		country j	country k	Weights (equation 3)	Satisfying equation (4)
Commodity	1	0.3	0.2	0.2508	0.2000
	2	0.2	0.4	0.2934	0.2833
	3	0.5	0.4	0.4557	0.5167

Our Proposition is not inconsistent with Fattore (2009) who proves that the SV index is the only log-change index number that satisfies the factor reversal test when observed quantities follow a special structure. Otherwise , SV index is not unique. The choice of index satisfying (4) must depend on other properties. For example, SV index can be justified since it is exact for the CES utility function (Sato, 1976 and Feenstra, 1994).

3. Sato-Vartia Index for multilateral comparisons and Transitivity

Section 2 focused on SV index for bilateral comparison between two countries *j* and *k*. In the case of multilateral comparisons with *M* countries, we are interested in price and quantity comparisons between all pairs of countries represented by $\left\{P_{jk}, Q_{jk}\right\}_{j=1,2,...,M}^{K=1,2,...,M}$. These indices satisfy transitivity if and only if

$$P_{jl} \cdot P_{lk} = P_{jk}\ \ or\ \ln P_{jl} + \ln P_{lk} = \ln P_{lk} \quad \forall j, k, l = 1, 2, ..., M \qquad (6)$$

The SV price index with weights in (3) does not satisfy transitivity property except in the case considered below.

Proposition 2: The SV log-change index number satisfies transitivity if and only if the weights satisfy the condition: $\phi_i^{jk} = \phi_i \ \ \forall\ j, k = 1, 2, ..., M; \mathrm{i} = 1, 2, ..., \mathrm{N}$.

Proof: See Appendix

When the condition in Proposition 2 holds, the resulting index is:

$$P_{jk} = \prod_{i=1}^{N} \left[\frac{p_{ik}}{p_{ij}} \right]^{\phi_i} \quad for\ all\ \ j,k = 1,2,...,M \tag{7}$$

This result is consistent with an important theorem by Funke, Hecker and Voeller (1979) which proves that the index in (7) is the only index that satisfies identity, commensurability and transitivity properties.

SV Based Gini- Eltető - Köves -Szulc (GEKS) Index

The GEKS method is a technique to generate transitive indices that are closest to the binary comparisons that lack transitivity property.

Proposition 3: Let $\left\{ P_{jk}^{SV}, P_{jk}^{SV-GEKS} \right\}_{j=1,2,....M}^{k=1,2,...,M}$ denote the SV and SV-based GEKS indices, then solution to the following minimization problem

$$\min_{P_{jk}} \sum_{j=1}^{M} \sum_{k=1}^{M} \left[\ln P_{jk}^{S-V} - \ln P_{jk} \right]^2 \ subject\ to\ P_{jl} \cdot P_{lk} = P_{jl}\ \ \forall j,k,l = 1,2,...,M$$

is :

$$P_{jk}^{SV-GEKS} = \prod_{l=1}^{M} \left[P_{jl}^{SV} \cdot P_{lk}^{SV} \right]^{1/M} \tag{8}$$

Derivation of (8) can be found in Rao and Banerjee (1985).

The GEKS approach is the only way to generate transitive indices from Fisher binary indices. In the case of SV indices, we propose three additional methods to generate transitive indices that retain the general character of the SV index.

Proposition 2 implies that we need to generate a set of weights $\left\{ \phi_i \right\}_{i=1}^{N}$. We propose three different criteria for generating transitive weights, ϕ_i's, that are as close as possible (maintain characteristicity) to the observed non-transitive SV weights $\left\{ \phi_i^{jk} \right\}_{j,k=1}^{j,k=M}$.

Proposition 4: Suppose we have observed weights, $\phi_i^{jk};\ \forall i\ \ and\ \ j,k$,which are non-negative and add up to unity for any *j* and *k*, $\sum_{i=1}^{N} \phi_i^{jk} = 1$. Then the optimum set of weights, $\left\{ \hat{\phi}_i \right\}_{i=1}^{i=N}$ that

$$\min_{\phi_i} \frac{1}{2} \sum_{j=1}^{M} \sum_{k=1}^{M} \sum_{i=1}^{N} \left(\phi_i^{jk} - \phi_i \right)^2 \quad s.t. \sum_{i=1}^{N} \phi_i = 1 \tag{9}$$

is given by

$$\hat{\phi}_i = \overline{\phi}_i = \frac{\sum_{j=1}^{M} \sum_{k=1}^{M} \left(\phi_i^{jk} \right)}{M^2} \quad for\ all\ i \tag{10}$$

Proof is straightforward.

The log-change price index with weights in (10) is an alternative to the SV-GEKS index in (8). The weights in (10) are obtained when discrepancies for all commodities are considered to be of equal importance. It is possible to accord differential weights reflecting relative importance of items through weighted optimization below.

Proposition 5: Suppose we have observed weights, $\phi_i^{jk};\ \forall i\ \ and\ \ j,k$, which are non-negative and add up to unity for any j and k, $\sum_{i=1}^{N}\phi_i^{jk}=1$. Let $\left\{\alpha_i^{jk}\right\}_{i=1;\,j,k=1}^{i=N;\,j,k=M}$ be a set of non-negative set of weights with $\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{i=1}^{N}\alpha_i^{jk}=1$. Then the optimum set of weights, $\left\{\tilde{\phi}_i\right\}_{i=1}^{i=N}$ that minimizes

$$\frac{1}{2}\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{i=1}^{N}\alpha_i^{jk}\left(\phi_i^{jk}-\phi_i\right)^2 \quad s.t. \sum_{i=1}^{N}\phi_i=1 \tag{11}$$

is given by

$$\tilde{\phi}_i=\frac{1}{N}+\bar{\phi}_i^{\alpha}-\bar{\phi}^{\alpha} \quad for\ all\ i \tag{12}$$

where

$$\bar{\phi}_i^{\alpha}=\frac{1}{M}\sum_{j=1}^{M}\sum_{k=1}^{M}\alpha_i^{jk}\cdot\phi_i^{jk} \quad for\ all\ i;\ \bar{\phi}^{\alpha}=\frac{1}{N}\sum_{i=1}^{N}\bar{\phi}_i^{\alpha} \tag{13}$$

Proof: Follows from first order conditions. The optimal weights $\left\{\tilde{\phi}_i\right\}_{i=1}^{i=N}$ in (13) are non-negative and add up to 1.

Corollary: If weights, $\left\{\alpha_i^{jk}\right\}_{i=1;\,j,k=1}^{i=N;\,j,k=M}$, in Proposition 5 satisfy the condition $\alpha_{1jk}=\alpha_{2jk}=...=\alpha_{Njk};\ \forall j,k$, then weights $\left\{\tilde{\phi}_i\right\}_{i=1}^{i=N}$ in equation (12) simplify to $\left\{\bar{\phi}_i^{\alpha}\right\}_{i=1}^{i=N}$ in equation (13).

The third approach is to construct optimal shares ϕ_i which minimize the magnitude by which the factor reversal test is not satisfied. This is equivalent to the following quadratic optimization minimization problem:

$$\min_{\phi_i}\sum_{j=1}^{M}\sum_{k=1}^{M}\left[\sum_{i=1}^{N}\phi_i\ln\left(\frac{w_{ik}}{w_{il}}\right)\right]^2 \ subject\ to\ 0<\phi_i<1\ and\ \sum_{i=1}^{N}\phi_i=1. \tag{14}$$

No closed form solutions similar to (8), (10) and (12) are available and hence not considered further.

Our multilateral SV indices of the form (7) with weights in (10) and (13) have the additional advantage that they satisfy the strong identity test (see Balk, 2008 for details) which is not satisfied by the Fisher and SV-based GEKS indices.

4. Empirical Results

Results reported here make use of Household Consumption data drawn from the 2011 ICP.[4] We compute the following multilateral indices: (i) GEKS based on Fisher index; (ii) GEKS based on bilateral SV index (equation 8); (iii) SV index with optimal weights in (10) associated with the unweighted objective function (equation 9); (iv) SV index with optimal weights in (12) with three different specification of weights:

$$(a)\ \alpha_{ijk}=\frac{1}{|w_{ik}-w_{ij}|};\ (b)\ \alpha_{ijk}=\frac{(w_{ik}+w_{ij})}{2};\ (c)\ \alpha_{ijk}=\frac{PL_{jk}}{N\times|PL_{jk}-PP_{jk}|}, \tag{15}$$

where PL_{jk} and PP_{jk} are bilateral Laspeyres and Paasche indices, respectively.

A complete table of transitive multilateral price indices with USA as base are in Appendix Tables. We find that the Fisher-based GEKS and SV-based GEKS are similar in magnitude. These are to be expected as the

[4] The authors gratefully acknowledge the data made available by the Global Office of the ICP.

Fisher and SV binary indices tend to be numerically close. The four transitive indices proposed in the paper are also close to the Fisher and SV-based GEKS.

The relative performances various transitive SV indices can be examined using the following two distance measures,

$$\Delta_1 = \text{SQRT}\left\{\frac{1}{\frac{M(M-1)}{2}} \sum_{j=1,j\neq k}^{M} \sum_{k=1}^{M} \left[\ln\left(P_{jk}^{SV-T}\right) - \ln\left(P_{jk}^{SV}\right)\right] \right\} \tag{16}$$

$$\Delta_2 = \text{SQRT}\{\frac{1}{\frac{M(M-1)}{2}} \sum_{j=1,j\neq k}^{M} \sum_{k=1}^{M} \left[\ln\left(E_{jk}\right) - \ln\left(P_{jk}^{SV-T}\right) - \ln\left(Q_{jk}^{SV-T}\right)\right] \} \tag{17}$$

Δ_1 and Δ_2 show the distance from the binary SV index and factor reversal, respectively.[5] Table 2 summarizes the results. The index with weights (3) based on the Laspeyres-Paashce gap (column c) is the best performing index.

Table 2: Relative Performances of Four Transitive Sato-Vartia Indices

Weight	Unweighted	(a)	(b)	(c)
Distance from the Binary S-V	0.0471	0.0481	0.0476	0.0464
Distance from the Factor Reversal	4.9742	4.8890	4.8889	4.2187

Correlations between the six transitive logged price indices (four transitive SV and two GEKS) are greater than 0.999.

5. Conclusions

In this paper we propose extensions of the Sato-Vartia index, dubbed as the ideal log-change index number, for making multilateral price and quantity comparisons. We show that it is possible to construct transitive indices by generalizing the SV index in several directions thereby demonstrating versatility SV index framework. Based on the analysis, we recommend the use of index of the form (7) with optimal weights (c) in equation (15. Our proposed index preserves the core philosophy of the SV index in generating transitive multilateral indices and unlike Fisher and SV-based GEKS indices it satisfies the strong identity test.

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[5] To calculate Δ_2, we need to construct quantity index that requires strictly positive quantity. Thus, we drop observations with zero nominal expenditures when calculating Δ_2.

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The Appendix

1 Proof of Proposition 1

Suppose Q is a quantity index corresponding to the price index, P. Also denote the value index as E.Then, the definition of the factor reversal test is,

$$PQ/E = 1$$
$$\ln P + \ln Q - \ln E = 0.$$

If the price and quantity indices of a log change class, by denoting the total Expenditure of country k by E^k and the expenditure for commodity i in country k by E_{ik}, we can obtain

$$\begin{aligned}\ln P_{jk} + \ln Q_{jk} &= \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{p_{ik}}{p_{ij}}\frac{q_{ik}}{q_{ij}}\right)\\ &= \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{E_{ik}}{E_{ij}}\right).\end{aligned}$$

Then, the definition of the factor reversal can be written as

$$\ln P_{jk} + \ln Q_{jk} - \ln E^k + \ln E^j = 0.$$

This condition can be written in terms of the products of the ratios of the expenditure shares and the weights, that is,

$$\begin{aligned}\sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{E_{ik}}{E_{ij}}\right) - \ln E^k + \ln E^j &= 0\\ \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{E_{ik}}{E^k}\right) - \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{E_{ij}}{E^j}\right) &= 0\\ \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{w_{ik}}{w_{ij}}\right) &= 0,\end{aligned}$$

where

$$w_{ik} = \frac{E_{ik}}{E^k}.$$

If we impose the adding up conditions for the weights, that is,

$$\sum_{i=1}^{N} \phi_i^{jk} = 1 \text{ or all } j, k$$

then, the necessary and sufficient condition for the log change price index to pass the factor reversal test are

$$\sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{w_{ik}}{w_{ij}}\right) = 0 \text{ for all } j, k \tag{A1}$$

$$\sum_{i=1}^{N} \phi_i^{jk} = 1 \text{ or all } j, k \tag{A2}$$

As Sato (1976) shows, the Sato-Vartia index satisfies the factor reversal test for cases with $N > 1$ and $M > 1$, it is easy to show that if $N = 2$ and $M = 2$, the Sato-Vartia index satisfies the factor reversal test. Then, consider the converse. Suppose that a log change price index satisfies the factor reversal test, that is, $(A1)$. Note that the number of conditions $(A1)$ is $M \times (M-1) + M$. On the other hand, the number of parameters, ϕ_i^{jk}, of the price index is $N \times M \times (M-1)$. To exactly identify the price index, both number should be the same, that is

$$\begin{aligned} M \times (M-1) + M &= N \times M \times (M-1) \\ N &= \frac{M \times (M-1) + M}{M \times (M-1)} \\ &= 1 + \frac{1}{(M-1)} \end{aligned}$$

Since N should be an integer, $1/((M-1))$ must be an integer. This holds only when $M = 2$ and $N = 2$. Therefore, for the factor reversal condition to identify the price index, we must have $M = 2$ and $N = 2$.

Finally we need to show that if $M = 2$ and $N = 2$, the index must be the Sato-Vartia index.

Now, using adding up condition, rewrite $(A1)$ as

$$\phi^{jk} \ln\left(\frac{w_{1k}}{w_{1j}}\right) + \left(1 - \phi^{jk}\right) \ln\left(\frac{w_{2k}}{w_{2j}}\right) = 0 \ .$$

The minimum can be attained when

$$\begin{aligned} \phi\alpha_2 + (1-\phi)\beta_2 &= 0, \\ \phi &= \frac{-\beta_2}{\alpha_2 - \beta_2} \\ &= \frac{\ln(w_{22}) - \ln(w_{21})}{\ln(w_{11}) - \ln(w_{12}) - \ln(w_{21}) + \ln(w_{22})}. \end{aligned}$$

Recall that the S.V. weight is defined as

$$sv = \frac{w_{11} - w_{12}}{\ln(w_{11}) - \ln(w_{12})} / \left(\frac{w_{11} - w_{12}}{\ln(w_{11}) - \ln(w_{12})} + \frac{w_{21} - w_{22}}{\ln(w_{21}) - \ln(w_{22})} \right).$$

The denominator for the normalization is,

$$\begin{aligned} &\frac{w_{11} - w_{12}}{\ln(w_{11}) - \ln(w_{12})} + \frac{w_{21} - w_{22}}{\ln(w_{21}) - \ln(w_{22})} \\ &= \frac{w_{11} - w_{12}}{\ln(w_{11}) - \ln(w_{12})} + \frac{1 - w_{11} - (1 - w_{12})}{\ln(1 - w_{11}) - \ln(1 - w_{12})}. \end{aligned}$$

Combine the two terms:

$$\begin{aligned} &\frac{w_{11} - w_{12}}{\ln(w_{11}) - \ln(w_{12})} + \frac{(1 - w_{11}) - (1 - w_{12})}{\ln(1 - w_{11}) - \ln(1 - w_{12})} \\ &= \frac{w_{11} - w_{12}}{(\ln w_{11} - \ln w_{12})(\ln(1 - w_{11}) - \ln(1 - w_{12}))} (\ln(1 - w_{11}) - \ln(1 - w_{12}) - \ln w_{11} + \ln w_{12}). \end{aligned}$$

Therefore,

$$\begin{aligned} sv &= \frac{w_{11} - w_{12}}{\ln(w_{11}) - \ln(w_{12})} * \frac{(\ln w_{11} - \ln w_{12})(\ln(1 - w_{11}) - \ln(1 - w_{12}))}{(w_{11} - w_{12})(\ln(1 - w_{11}) - \ln(1 - w_{12}) - \ln w_{11} + \ln w_{12})} \\ &= \frac{(\ln(1 - w_{11}) - \ln(1 - w_{12}))}{(\ln(1 - w_{11}) - \ln(1 - w_{12}) - \ln w_{11} + \ln w_{12})} \\ &= \frac{\ln(w_{21}) - \ln(w_{22})}{\ln(w_{21}) - \ln(w_{22}) - \ln w_{11} + \ln w_{12}}. \end{aligned}$$

Then we have just proved that

$$\phi = sv.$$

2 Proof of Proposition 2

By definition, for all $j, k, l,$ a transitive price index, P_{jk} and P_{kl} must satisfy the equation.

$$\forall j, k, l, : \ln P_{jk} + \ln P_{kl} = \ln P_{jl}.$$

Suppose the price index is a log-change index number, then, the price index can be written as

$$\ln P_{jk} = \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{p_{ik}}{p_{ij}}\right), \sum_{i=1}^{N} \phi_i^{jk} = 1, \phi_i^{jk} \geq 0,$$

then, by expressing the price index in terms of price relatives at commodity level, we can rewrite the above as

$$
\begin{aligned}
\ln P_{jk} + \ln P_{kl} &= \sum_{i=1}^{N} \phi_i^{jk} \ln\left(\frac{p_{ik}}{p_{ij}}\right) + \sum_{i=1}^{N} \phi_i^{kl} \ln\left(\frac{p_{il}}{p_{ik}}\right) \\
&= \sum_{i=1}^{N} \left(\phi_i^{jk} \ln\left(\frac{p_{ik}}{p_{ij}}\right) + \phi_i^{kl} \ln\left(\frac{p_{il}}{p_{ik}}\right)\right) \\
&= \sum_{i=1}^{N} \left(\phi_i^{jk} \ln\left(\frac{p_{ik}}{p_{ij}}\right) - \phi_i^{kl} \ln\left(\frac{p_{ik}}{p_{il}}\right)\right) \\
&= \sum_{i=1}^{N} \left(\ln(p_{ik})\left(\phi_i^{jk} - \phi_i^{kl}\right) - \phi_i^{jk} \ln(p_{ij}) + \phi_i^{kl} \ln(p_{il})\right) \\
&= \ln P_{jl} \\
&= \sum_{i=1}^{N} \phi_i^{jl} \ln\left(\frac{p_{il}}{p_{ij}}\right).
\end{aligned}
$$

That is, if the price index is of a log-change index number class, the following equation is the necessary and sufficient condition of transitivity;

$$
\ln P_{jk} + \ln P_{kl} = \sum_{i=1}^{N} \phi_i^{jl} \ln\left(\frac{p_{il}}{p_{ij}}\right) \text{ for all } j, k, l.
$$

A simple calculation leads us to the following relations.

$$
\begin{aligned}
&\ln P_{jk} + \ln P_{kl} - \ln P_{jl} \\
&= \sum_{i=1}^{N} \left(\ln(p_{ik})\left(\phi_i^{jk} - \phi_i^{kl}\right) - \phi_i^{jk} \ln(p_{ij}) + \phi_i^{kl} \ln(p_{il}) - \phi_i^{jl} \ln\left(\frac{p_{il}}{p_{ij}}\right)\right) \\
&= \sum_{i=1}^{N} \left(\ln(p_{ik})\left(\phi_i^{jk} - \phi_i^{kl}\right) - \phi_i^{jk} \ln(p_{ij}) + \phi_i^{kl} \ln(p_{il}) - \phi_i^{jl} \ln(p_{il}) + \phi_i^{jl} \ln(p_{ij})\right) \\
&= \sum_{i=1}^{N} \left(\ln(p_{ik})\left(\phi_i^{jk} - \phi_i^{kl}\right) - \left(\phi_i^{jl} - \phi_i^{jk}\right) \ln(p_{ij}) + \left(\phi_i^{kl} - \phi_i^{jl}\right) \ln(p_{il})\right) \\
&= 0.
\end{aligned}
$$

Since the above equation must hold for all j, l, and k and for all the observed prices, the necessary and sufficient conditions for transitive to hold are that, for all j, k, l, and i,

$$\begin{aligned} \phi_i^{jk} - \phi_i^{kl} &= 0, \\ \phi_i^{jl} - \phi_i^{jk} &= 0, \\ \phi_i^{kl} - \phi_i^{jl} &= 0. \end{aligned} \tag{A3}$$

From the second equation in $(A3)$, we can infer that for given i, the matrix, $\phi_i^{j,k}$ (j, k) should have an identical value within the same rows. Next, from the third equation in $(A3)$, we can show that the matrix has an identical value within the same columns. Therefore, for given i, all the elements of the matrix $\phi_i^{j,k}$ (j, k) should be identical, that is,

$$\begin{pmatrix} \phi_i^{1,1} & \phi_i^{1,2} & \phi_i^{1,3} & ... & \phi_i^{1,N-1} & \phi_i^{1,N} \\ \phi_i^{2,1} & \phi_i^{2,2} & \phi_i^{2,3} & ... & \phi_i^{2,N-1} & \phi_i^{2,N} \\ \phi_i^{3,1} & \phi_i^{3,2} & \phi 3 & ... & \phi_i^{3,N-1} & \phi_i^{3,N} \\ ... & ... & ... & ... & ... & ... \\ \phi_i^{N-1,1} & \phi_i^{N-1,2} & \phi_i^{N-1,3} & ... & \phi_i^{N-1,N-1} & \phi_i^{N-1,N} \\ \phi_i^{N,1} & \phi_i^{N,2} & \phi_i^{N,3} & ... & \phi_i^{N,N-1} & \phi_i^{N,N} \end{pmatrix}$$
$$= \begin{pmatrix} \phi_i & \phi_i & \phi_i & ... & \phi_i & \phi_i \\ \phi_i & \phi_i & \phi_i & ... & \phi_i & \phi_i \\ \phi_i & \phi_i & \phi_i & ... & \phi_i & \phi_i \\ ... & ... & ... & ... & ... & ... \\ \phi_i & \phi_i & \phi_i & ... & \phi_i & \phi_i \\ \phi_i & \phi_i & \phi_i & ... & \phi_i & \phi_i \end{pmatrix}$$

Therefore, we can state that the necessary and sufficient condition for a log change index number to satisfy transitivity is that the weights used in the index for each commodity i should be identical for all different countries (or over time).

3 Proof of Result 2

Consider the following minimization problem,

$$\min \sum_{k=1}^{M} \sum_{j=1}^{M} \sum_{i=1}^{N} \frac{1}{2} \left(\phi_i^{j,k} - \phi_i \right)^2, \text{s t}, \sum_{i=1}^{N} \phi_i = 1 \text{ for all } i$$

The minimization problem can be formulated using Lagrangian,

$$\min_{,\phi_i} \sum_{k=1}^{M} \sum_{j=1}^{M} \sum_{i=1}^{N} \frac{1}{2} \left(\phi_i^{j,k} - \phi_i \right)^2 + \lambda \left(\sum_{i=1}^{N} \phi_i - 1 \right).$$

The first order conditions are

$$\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\left(\phi_i^{j,k}-\phi_i\right)\right]=\lambda \text{ for all } i$$

$$\sum_{i=1}^{N}\phi_i=1.$$

After simple manipulation we have

$$M^2\phi_i=\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)-\lambda$$

$$M^2\sum_{i=1}^{N}\phi_i=\sum_{i=1}^{N}\left[\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)-\lambda\right]$$

$$M^2=\sum_{i=1}^{N}\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)-N\lambda$$

Therefore, the multiplier can be written as,

$$\lambda=\frac{\sum_{i=1}^{N}\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)-M^2}{N}.$$

That is, the optimal weights, $\widehat{\phi}_i$, become

$$\begin{aligned}\widehat{\phi}_i&=\frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)-\frac{\sum_{i=1}^{N}\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)-M^2}{N}}{M^2}\\&=\frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)}{M^2}-\frac{1}{N}\frac{\sum_{i=1}^{N}\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)}{M^2}+\frac{1}{N}.\end{aligned}\tag{A4}$$

Define the mean of the weights as,

$$\overline{\phi}_i=\frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)}{M^2},$$

then, the average share, $\overline{\phi}$, becomes

$$\overline{\phi} = \frac{1}{N}\sum_{i=1}^{N}\overline{\phi}_i = \frac{\sum_{i=1}^{N}\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)}{M^2 N}.$$

Substituting $\overline{\phi}_i$ and $\overline{\phi}$ to $(A4)$ gives us

$$\widehat{\phi}_i = \frac{1}{N} - \overline{\phi} + \overline{\phi}_i.$$

Due to the adding up condition, we have

$$\begin{aligned}
&\frac{1}{N}\frac{\sum_{i=1}^{N}\sum_{k=1}^{M}\sum_{j=1}^{M}\left(\phi_i^{j,k}\right)}{M^2} \\
&= \frac{1}{N}\frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\sum_{i=1}^{N}\left(\phi_i^{j,k}\right)}{M^2} \\
&= \frac{1}{N}\frac{\sum_{k=1}^{M}\sum_{j=1}^{M}1}{M^2} \\
&= \frac{1}{N}.
\end{aligned}$$

Therefore, we get the simple formula for the optimal weight,

$$\widehat{\phi}_i = \overline{\phi}_i.$$

4 Proof of Proposition 4

Consider the following Lagrangian function for the minimization problem with weights,

$$\min_{\phi_i}\sum_{k=1}^{M}\sum_{j=1}^{M}\sum_{i=1}^{N}\frac{1}{2}\alpha_{ijk}\left(\phi_i^{j,k} - \phi_i\right)^2 + \lambda\left(\sum_{k=1}^{N}\phi_i - 1\right)$$

The first order conditions become

$$\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k} - \phi_i\right)\right] = \lambda, \text{ for all } i,$$

$$\sum_{i=1}^{N}\phi_i = 1.$$

The first order conditions can be transformed as,

$$
\begin{aligned}
\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right] - \sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i\right)\right] = \lambda \\
\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i\right)\right] &= \sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right] - \lambda. \\
\phi_i\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\right] &= \sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right] - \lambda \\
\phi_i &= \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right] - \lambda}{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\right]}.
\end{aligned}
$$

Substituting the last equation to the adding up condition, we can solve for λ

$$1 = \sum_{i=1}^{N} \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right] - \lambda}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]}$$

$$1 = \sum_{i=1}^{N} \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right]}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]} - \lambda \sum_{i=1}^{N} \frac{1}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]}$$

$$1 = \sum_{i=1}^{N} \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right]}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]} - \sum_{i=1}^{N} \frac{\lambda}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]}$$

$$\lambda = \left(\sum_{i=1}^{N}\left[\frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right]}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]}\right] - 1\right)\left(\frac{1}{\sum_{i=1}^{N}\frac{1}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]}}\right).$$

Therefore, the optimal weight can be written as

$$\widehat{\phi}_i = \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right] - \lambda}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]} \tag{A5}$$

$$= \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right]}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]} - \frac{1}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]}\left(\sum_{i=1}^{N}\left[\frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right]}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]}\right] - 1\right)\left(\frac{1}{\sum_{i=1}^{N}\frac{1}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{ijk}]}}\right)$$

Now, if the following conditions,

$$\sum_{k=1}^{M}\alpha_{ijk} = 1, \sum_{j=1}^{M}\alpha_{ijk} = 1.$$

are satisfied, we get

$$\sum_{k=1}^{M}\sum_{j=1}^{M}\alpha_{ijk} = \sum_{k=1}^{M}1 = M.$$

Define the weighted average of $\phi_i^{j,k}$ given i as

$$\overline{\phi}_i^{\alpha} = \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\alpha_{ijk}\left(\phi_i^{j,k}\right)}{M},$$

and the weighted average for all the observations as

$$\overline{\phi}^{\alpha} = \frac{1}{N}\sum_{i=1}^{N}\overline{\phi}_i^{\alpha}.$$

Then, some algebraic manipulation gives us

$$\begin{aligned}\widehat{\phi}_i &= \overline{\phi}_i^{\alpha} - \frac{1}{M}\left(N\overline{\phi}_i^{\alpha} - 1\right)\frac{M}{N} \\ &= \frac{1}{N} + \overline{\phi}_i^{\alpha} - \overline{\phi}^{\alpha}.\end{aligned}$$

5 Proof of Corollary

Suppose the following relations hold,

$$\alpha_{1jk} = \alpha_{2jk} = ... = \alpha_{Njk} = \alpha_{jk}.$$

Consider a variable υ_{jk} which is defined as:

$$\upsilon_{jk} \equiv \frac{\alpha_{jk}}{\sum_{k=1}^{M}\sum_{j=1}^{M}[\alpha_{jk}]}.$$

Then, after some calculations using $\alpha_{1jk} = \alpha_{2jk} = ... = \alpha_{Njk} = \alpha_{jk}$, we can obtain

$$\sum_{i=1}^{N}\left[\frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right]}{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\right]}\right] = \sum_{i=1}^{N}\sum_{k=1}^{M}\sum_{j=1}^{M}\upsilon_{jk}\phi_i^{j,k}$$

$$= \sum_{k=1}^{M}\sum_{j=1}^{M}\upsilon_{jk}\left(\sum_{i=1}^{N}\phi_i^{j,k}\right)$$

$$= \sum_{k=1}^{M}\sum_{j=1}^{M}\upsilon_{jk}$$

$$= 1.$$

Therefore,

$$\sum_{i=1}^{N}\left[\frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right]}{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\right]}\right] - 1 = 0.$$

Substituting this to $(A5)$

$$\widehat{\phi}_i = \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right]}{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\right]}$$

$$= \frac{\sum_{k=1}^{M}\sum_{j=1}^{M}\left[\alpha_{ijk}\left(\phi_i^{j,k}\right)\right]}{M}$$

$$= \overline{\phi}_i^{\alpha}$$

6 Data

For our empirical analyses, we obtained Basic Heading PPPs from the OECD for the year 2011 from the ICP unit at the OECD. The data cover 47 OECD countries with 202 basic headings. In this paper, we restrict the expenditure categories to household consumptions, which leads to 139 basic headings covering various type of consumption of goods and services.[1]

[1] For the detail of the basic heading and price data, see World Bank (2014).

7 Weighting Formulae

In this paper, we use three weighting systems.

(1) Expenditure Share Differences: Weights here are inversely related to the extent to which expenditure shares for a commodity differ between countries k and j. We take absolute differences.

$$\alpha_{ijk} = \frac{\frac{1}{|w_{ijk} - w_{ij}|}}{\sum_{j} \frac{1}{|w_{ijk} - w_{ij}|}}$$

(2) Country Level Expenditure Shares: In this specification the weights reflect the importance of the item in these two countries which is measured by the average of expenditure shares.

$$\alpha_{jk} = \frac{\sum_{i} \left(\frac{\frac{w_{ijk} + w_{ij}}{2}}{\sum_{j} \frac{w_{ijk} + w_{ij}}{2}} \right)}{N}$$

(3) Laspeyres-Paasche Gap: In the index number literature reliability of a bilateral comparison between two countries is measured through the Laspeyres-Paasche spread, higher the spread less reliable is the index. We take the absolute difference between Laspeyres and Paasche index expressed relative to the Laspeyres index to define our weights. It is usually expected that Laspeyres index is greater than Paasche index.

$$\alpha_{jk} = \frac{PL_{jk}}{N\,|PL_{jk} - PP_{jk}|}$$

PL_{jk} : Laspeyres price index when j is the base country

PP_{jk} : Paasche price index when k is the comparison country

The list of countries is given in Appendix Table 1, and the list of the basic headings as well as the optimal weights for our four transitive Sato-Vartia indices are shown in Appendix Table 2. The transitive price indices are reported in Appendix Table 3.

8 References

Sato K., (1976) "The ideal log-change index number," *The Review of Economics and Statistics*, 58, n. 2, 223-228.

World Bank (2015) *Purchasing Power Parities and the Real Size of World Economies: A Comprehensive Report of the 2011 International Comparison Program*. Available on line: https://elibrary.worldbank.org/doi/10.1596/978-1-4648-0329-1