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Sraffian Indeterminacy in General Equilibrium Revisited

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Abstract

In contrast to Mandler's (1999a; Theorem 6) impossibility result about the Sraffian indeterminacy of the steady-state equilibrium, we first show that any regular Sraffian steady-state equilibrium is *indeterminate* in terms of Sraffa (1960) under the simple overlapping generation economy. Moreover, we also check that this indeterminacy is *generic*. These results are obtained by explicitly defining a simple model of overlapping generation economies with Leontief production techniques, in which we also explain the main source of the difference between our results and Mandler (1999a; section 6).

JEL Classification Code: B51, D33, D50.

Keywords: Sraffian indeterminacy; factor income distribution; general equilibrium framework

1 Introduction

It is well-known that Sraffa's (1960) system of equilibrium price equations contains one more unknown than equation, which leads to the indeterminacy of the steady-state equilibrium. This Sraffian indeterminacy has been regarded

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as a basis to argue that some non-market-competitive force is indispensable to determine the factor income distribution between capital and labor, which also sets equilibrium prices of commodities. Mandler (1999a) critically examined Sraffian indeterminacy by embedding the Sraffian system of price equations in a general equilibrium framework. In section 3 of his study, Mandler confirmed the generic indeterminacy of equilibria under the condition of fixed production coefficients, time varying prices and the price inelastic supply of endowments. However, unlike the claim of Sraffa (1960), in section 6 of Mandler (1999a), he argued that the steady-state equilibria are generically determinate if only the labor endowment is fixed and physical commodities are supplied elastically.[1] [2]

The steady-state model in section 6 of Mandler (1999a) presumes a structure of overlapping generations of agents with 2-periods lives. In this paper we reproduce the underlying narrative of this overlapping economic structure defined in section 6 of Mandler (1999a). A simple overlapping generation model is constructed, in which each generation consists of a single representative agent, who lives for two periods. The agent works only in his youth. In his old age, the agent is retired and purchases consumption goods from the wealth due to his past savings.

In such a model, given the same definition of steady-state equilibrium as Mandler (1999; section 6, p. 705), we show that a steady-state equilibrium is generically indeterminate, unlike the result of Mandler (1999a; section 6). This general possibility is due to the fact that the system of equations characterizing the steady-state equilibrium still preserves one degree of freedom as it contains one more unknown than equations. This is because, unlike Mandler (1999a; section 6; p. 705), the (reduced form of) Walras' law can make one equilibrium equation redundant under the standard assumption of strongly monotonic utility functions. The equilibrium price equations, the equations of commodity market clearing, and the reduced form of the Walras' law together imply the equation of the labor market equilibrium.

In the rest of this paper, section 2 provides a review of the literature on indeterminacy issues in Walrasian general equilibrium theory and Sraffian economics. Section 3 introduces a simple model of overlapping generation economies and defines the steady-state equilibrium, following Mandler's own definition (Mandler, 1999a; section 6, p. 705). Then, section 4 argues the generic indeterminacy of such an equilibrium and explains under what alternative conditions Mandler's (1999a; section 6) claim of the opposite conclusion may be verified. Finally, section 5 provides concluding remarks, where the distinctive feature of the Sraffian indeterminacy, in comparison with some of the neoclassical indeterminacy, is discussed. The general existence theorem of a steady-state

[1] As Mandler (1999a, p. 699) himself points out, the Walrasian system of general equilibrium has an inherent problem of overdetermination: when the endowment of reproducible means of production is arbitrarily given, the system of equations is overdetermined under the uniform rate of profit. For further details of the implications of this issue, see Eatwell (1999), Garegnani (1990), and Petri (2004).

[2] The generic determinacy theorem applied in Mandler (1999a) is found in Mas-Colell (1985), Theorem 8.7.3, where technology is described by linear activities and where a subset of commodity excess demands can be (locally) inelastic with respect to price.

equilibrium is provided in the Appendix.

2 A brief literature review

In the historic work by Debreu (1970),[3] it was proven that exchange economies have only a finite number of Walrasian equilibria. This means that the Walrasian equilibrium prices and allocations change smoothly as a function of the parameters representing economic environments, so that agents in large economies can have only a negligible effect on equilibrium prices; i.e. there is no longer an incentive for market manipulation. In Debreu (1972), under the assumption of smooth preference ordering, the existence of a generic set of regular economies is proven. For production economies, Mas-Colell (1975) and Kehoe (1980, 1982) established generic determinacy for constant returns to scale technologies and for linear activity analysis. This implies that determinacy is now generic with almost any type of technology regardless of inelasticity in factor supply. For the model of an incomplete market with a nominal asset, generic determinacy is established in Geanakoplos and Plemarchakis (1987) and Balasko and Cass (1989).

In Mandler (1995), the genericity of sequential indeterminacy was established. Using Radner's (1972) method to decompose an intertemporal equilibrium into a sequential one, the second period production activity vector can be fixed in the second period continuation equilibrium by the vector of factors endowed and produced in the first period. Therefore, the continuation equilibrium condition consists of the second-period equilibrium price equations and the equations of the second-period excess demand condition for consumption goods, where the only unknown variables are the second-period prices of consumption goods and factors. Under this structure, it is shown that if an intertemporal equilibrium has fewer activities using positively priced second period factors than the number of those factors (implicitly degenerated), then there is a generic set of economies such that the continuation equilibrium of almost every induced second-period economy is indeterminate; and if it is not implicitly degenerated, then the continuation equilibrium of almost every induced second-period economy is regular. In Mandler (1997), the determinacy of both the intertemporal equilibria and the endogenously generated second period equilibria is verified under differentiable production technology.

This conclusion of generic sequential indeterminacy results from the assumptions of linear activities, the production of a fixed quantity, and the investment of part of the first period products into the second period production. These features are also observed in Sraffa's (1960) model. Given the same features, Mandler (1999a) investigated Sraffa's indeterminacy claim for an equilibrium with a non-stationary price vector. This equilibrium is defined by the zero-profit condition for a non-stationary price vector and the excess demand conditions for commodities and factors by reflecting Hahn's (1982) criticism of stationary prices and the lack of the demand side in the original model of Sraffa (1960). It

[3]For further details, see Mandler (1999b)

has a similar structure to the above-mentioned sequential second-period equilibrium, in that the equilibrium production activity vector is exogenously fixed by the endowment vector of factors given at the beginning of a production period. The only unknown variables in the system of equilibrium equations for the zero-profit and excess demand conditions are the prices of commodities, a wage rate, and an interest rate. In such an equilibrium, the Sraffian indeterminacy is observed whenever the total number of commodity inputs, labor, and financial capital with positive prices is greater than the number of activities used in production. In particular, the former is $n + 1$ while the latter is n in a simple Leontief production model without any alternative production technique or joint production, so that one-dimensional indeterminacy is generically observed, as shown by Mandler (1999a, section 3).

Mandler (1999a, section 6) also examined the possibility of Sraffian indeterminacy for steady-state equilibria in an overlapping generation economy with a simple Leontief production model. In this case, no analogical reasoning developed in the above argument of sequential equilibrium can be applied, as the equilibrium production activity vector should be endogenously determined while the price vector of commodity inputs is equal to that of commodity outputs. Mandler (1999a, section 6) argued that in this case, generic determinacy is observed, as the number of equilibrium equations and unknown variables is identical because none of the market-clearing equations are redundant in a long-run OLG setting.

In contrast to the last argument by Mandler (1999a, section 6), in the following sections, we will show that one-dimensional Sraffian indeterminacy is generically observed even for the steady-state equilibria in the same OLG setting. The main reason is that the standard Walras' law can still work to make one market-clearing equation redundant.

3 An overlapping generation economy in section 6 of Mandler (1999a)

A simple overlapping generation model is constructed, in which each generation $t = 1, 2, \ldots,$ is a single individual who lives for two periods. The individual works only in his youth and in his old age is retired and so purchases consumption goods from the wealth due to his past saving. Let ω_l be the labor endowment of one generation. There are $n \geqq 2$ commodities which are produced in this economy and used as consumption goods or capital goods, respectively. Let (A, L) be a Leontief production technique prevailing in this economy, where A is a $n \times n$ non-negative square, productive and indecomposable matrix of reproducible input coefficients and L is a $1 \times n$ positive row vector of direct labor coefficients. Finally, let $u : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}$ be a welfare function of lifetime consumption activities, which is common to all generations. As usual, u is assumed to be continuous and strongly monotonic. Thus, an *overlapping generation economy* is given by a profile $\langle (A, L); \omega_l; u \rangle$.

For each period t, let $p_t \in \mathbb{R}^n_+$ represent a vector of *prices* of n commodities prevailing at the end of this period; $w_t \in \mathbb{R}_+$ represent a *wage rate* prevailing at the end of this period; and $r_t \in \mathbb{R}_+$ represent an *interest rate* prevailing at the end of this period. Assume also, for each generation t, that $l^t \in \mathbb{R}_+$ represents t's labor supplied at the beginning of their youth; $\omega^{t+1} \in \mathbb{R}^n_+$ represents a commodity bundle for the purpose of saving monetary value $p_t\omega^{t+1}$, which will be chosen by generation t at the end of their youth and will be used in their old age; $\delta^{t+1} \in \mathbb{R}^n_+$ represents a commodity bundle purchased for the purpose of speculative activities by generation t at the beginning of their old age; $y^{t+1} \in \mathbb{R}^n_+$ represents a production activity vector decided by generation t at the beginning of their old age; z^t_b is the consumption bundle consumed by the generation t in their youth; and z^t_a is the consumption bundle consumed by generation t in their old age.

Each generation t in their youth is faced with the following optimization program MP^t: for a given sequence of price vectors $\{(p_t, w_t, r_t), (p_{t+1}, w_{t+1}, r_{t+1})\}$,

$$\max_{l^t,\omega^{t+1},\delta^{t+1},y^{t+1},z^t_b,z^t_a} u\left(z^t_b, z^t_a\right)$$

subject to

$$\begin{aligned}
p_t z^t_b + p_t\omega^{t+1} &\leqq w_t l^t, \\
l^t &\leqq \omega^t_l, \\
p_t\delta^{t+1} + p_t A y^{t+1} &= p_t\omega^{t+1}, \text{ and} \\
p_{t+1} z^t_a &\leqq p_{t+1}\delta^{t+1} + p_{t+1} y^{t+1} - w_{t+1} L y^{t+1}.
\end{aligned}$$

That is, each generation t can supply l^t amount of labor in her youth as a worker employed by generation $t-1$. From the wage income $w_t l^t$ earned at the end of her youth, she can save $p_t\omega^{t+1}$ amount of money and can purchase a consumption bundle z^t_b. By using the saved money $p_t\omega^{t+1}$, generation t at the beginning of her old age can purchase δ^{t+1} for speculative purposes and can purchase a vector of capital goods Ay^{t+1} as a productive investment. As an industrial capitalist, she can employ Ly^{t+1} amount of generation $t+1$'s labor. Then, at the end of her old age, she can earn $p_{t+1}\delta^{t+1}$ as the revenue of the speculative investment and can earn $p_{t+1}y^{t+1} - w_{t+1}Ly^{t+1}$ as the return on the productive investment. From these revenues, she can purchase a consumption bundle z^t_a.

Let $\left(l^t, \omega^{t+1}, \delta^{t+1}, y^{t+1}, z^t_b, z^t_a\right)$ be a solution to the optimization program MP^t for each generation t. At the optimum, all of the weak inequalities in the above constraints should hold with equality, given the assumption of u. That is,

$$\begin{aligned}
p_t z^t_b + p_t\omega^{t+1} &= w_t l^t, \\
l^t &= \omega^t_l, \text{ and} \\
p_{t+1} z^t_a &= p_{t+1}\delta^{t+1} + p_{t+1} y^{t+1} - w_{t+1} L y^{t+1}.
\end{aligned}$$

Note that the production activity vector y^{t+1}, planned by generation t at the beginning of old age, should satisfy the profit maximization condition. As market prices should satisfy the zero-profit condition in equilibrium, the following condition holds for every period $t+1$, where $t \geq 0$:

$$p_{t+1} \leqq (1 + r_{t+1})\, p_t A + w_{t+1} L.$$

Therefore, the profit maximization condition in equilibrium for every period $t+1$ is represented by:

$$p_{t+1} y^{t+1} = (1 + r_{t+1})\, p_t A y^{t+1} + w_{t+1} L y^{t+1}.$$

Thus, the revenue constraint $p_{t+1} z_a^t = p_{t+1} \delta^{t+1} + p_{t+1} y^{t+1} - w_{t+1} L y^{t+1}$ of generation t at the end of the old age can be reduced to

$$p_{t+1} z_a^t = p_{t+1} \delta^{t+1} + (1 + r_{t+1})\, p_t A y^{t+1}.$$

Given a pair of sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$, let $\left(z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})\right)$ be a solution of the generations $t = 1, 2, \ldots,$ to the problem MP^t of utility maximization under the budget constraint. Then, a competitive equilibrium can be formulated as follows.

Definition 1: A *competitive equilibrium* under the overlapping generation economy $\langle (A, L); \omega_l; u \rangle$ is a pair of sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$ and sequence of each generation's optimal actions $\left\{\left(\omega^{t+1}, y^{t+1}, \delta^{t+1}, z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})\right)\right\}_{t \geq 1}$ satisfying the following conditions:

$$\begin{aligned}
p_t &\leqq (1 + r_t)\, p_{t-1} A + w_t L \ \ (\forall t); \ \ (1.1) \\
\delta^t + y^t &\geqq z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + \omega^{t+1} \ \ (\forall t); \ \ (1.2) \\
\text{where } z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) &\equiv z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \ \text{ is the aggregate consumption demands at each } t; \\
\delta^t + A y^t &\leqq \omega^t \ \ (\forall t); \ \ (1.3) \\
\text{and } L y^t &\leqq \omega_l^t \ \ (\forall t). \ \ (1.4)
\end{aligned}$$

In the above definition, the excess demand condition in commodity markets is given by (1.2). In each period t, the aggregate consumption demand vector is given by $z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$. It may contain some zero components. For commodity i such that $z_i^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = 0$, it follows that in equilibrium, $\delta_i^t + y_i^t \geqq \omega_i^{t+1}$. In the inequality of excess demand condition (1.2) above, y^t is the gross output vector which is planned by generation $t-1$ at the beginning of period t and is harvested at the end of this period, while δ^t is the commodity bundle purchased by generation $t-1$ at the beginning of period t and is sold by generation $t-1$ at the end of period t.

In each period t, the capital market equilibrium condition is given by (1.3) of Definition 1. Note that the choice between the speculative investment δ^t and the productive investment $A y^t$ is made by generation $t-1$ at the beginning of old age. Moreover, the bundle of saving commodities ω^t is chosen by generation $t-1$ at the end of the young age.

In each period t, the labor market equilibrium condition is given by (1.4) of Definition 1. Note that the aggregate labor demand Ly^t is chosen by generation $t-1$ in their old age, while the aggregate labor supply ω_l^t is given by generation t at the young age.

Mandler (1999a; section 6) is interested in examining the robustness of the Sraffian indeterminacy for a specific long run feature of competitive equilibrium where all of the investment activities are simply replacements. Such a long run feature is given as a steady-state equilibrium, where (p_t, w_t, r_t), y^t, and δ^t are constant throughout the whole periods. In such a case, given a pair of sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$ such that $(p_t, w_t, r_t) = (p, w, r)$ for every period t, the solution $(z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}))$ of generations $t = 1, 2, \ldots,$ to the optimization problem MP^t can be represented by $z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b(p, w, r)$ and $z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_a(p, w, r)$ for every generation t. Correspondingly, the aggregate demand function $z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) + z_a^{t-1}(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ for every period t can be represented by $z^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z(p, w, r)$. Such an equilibrium is given by Mandler (1999a; section 6) as follows.

Definition 2 [Mandler (1999a, section 6; Definition D6.2)]: A *steady-state equilibrium* under the overlapping economy $\langle (A, L); \omega_l; u \rangle$ is a competitive equilibrium $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ associated with

$$\left\{\left(\omega^{t+1}, y^{t+1}, \delta^{t+1}, z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})\right)\right\}_{t \geq 1},$$

such that there exists a profile of a stationary price vector (p, w, r), a gross output vector $y \geq \mathbf{0}$, and a speculative activity vector $\delta \geqq \mathbf{0}$, satisfying $(p_t, w_t, r_t) = (p, w, r)$, $y^{t+1} = y$, $\delta^{t+1} = \delta$, $\omega^{t+1} = Ay + \delta$, $z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_b(p, w, r)$, and $z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) = z_a(p, w, r)$ for every t, and the following conditions hold:

$$\begin{aligned} p &\leqq (1+r)\, pA + wL; \text{ (a)} \\ y &\geqq z(p, w, r) + Ay, \text{ (b)} \\ \text{where } z(p, w, r) &= z_b(p, w, r) + z_a(p, w, r); \text{ and} \\ Ly &\leqq \omega_l. \text{ (c)} \end{aligned}$$

Let us introduce a specific case of a steady-state equilibrium, which is defined as follows.

Definition 3 [Mandler (1999a, section 6; Definition D6.2)]: A *steady-state equilibrium* $((p, w, r), y)$ under the overlapping economy $\langle (A, L); \omega_l; u \rangle$ is called *Sraffian* if and only if all of the conditions (a), (b), and (c) in Definition 2 hold in equality.

In the above definitions of these three types of equilibria, we simply assume that the individual choice between speculative investment δ^{t+1} and productive investment Ay^{t+1} is the consequence of each generation's optimal action in the program MP^t. We impose no restriction on that choice. Therefore, even for

Definitions 2 and 3, $\delta > \mathbf{0}$ may be compatible with those equilibrium notions. However, we will see that $\delta = \mathbf{0}$ should hold under the steady-state equilibrium whenever the equilibrium interest rate r is positive.

To see this last point, let us consider under what conditions in general the market equilibrium holds with no speculative activity, $\delta^{t+1} = \mathbf{0}$ $(\forall t)$. Note that if the whole monetary wealth $p_t \omega^{t+1}$ of generation t is used for productive investment, she would earn $(1 + r_{t+1})\, p_t \omega^{t+1}$, while if it is used for speculative investment, she would earn $p_{t+1}\omega^{t+1}$. Therefore, allocating her whole monetary wealth to productive investment is an optimal action for generation t at the beginning of her old age if and only if $(1 + r_{t+1})\, p_t \omega^{t+1} \geqq p_{t+1}\omega^{t+1}$. In general, if

$$(1 + r_{t+1})\, p_t \geqq p_{t+1}$$

holds for every period $t \geq 0$, then $\delta^{t+1} = \mathbf{0}$ is an optimal action for every generation t at the beginning of the old age. Thus, under the steady-state equilibrium, this inequality condition holds automatically, as $(1 + r)\, p \geqq p$ holds whenever $r \geqq 0$. However, if $r = 0$, then the generation is indifferent between speculative investment and productive investment, and so $\delta \geq \mathbf{0}$ may constitute a steady-state equilibrium associated with $r = 0$. In contrast, if $r > 0$, then the productive investment is strictly preferred to the speculative investment for every generation t under the steady-state equilibrium. Thus, $\delta = \mathbf{0}$ should hold under the steady-state equilibrium whenever $r > 0$.

4 Indeterminacy of the steady-state equilibrium

In this section, we show that a Sraffian steady-state equilibrium is generically indeterminate, given Definition 3. Firstly, again following Mandler (1999a), let us formulate the notion of indeterminacy in this model.

Definition 4 (Mandler (1999a)): Let $\langle (A, L)\,; \omega_l; u\rangle$ be an overlapping generation economy as specified above. Then, a Sraffian steady-state equilibrium $((p, w, r)\,, y)$ under this economy is *indeterminate* if for any $\varepsilon > 0$, there is a Sraffian steady-state equilibrium $((p', w', r')\,, y')$ such that $(p', w', r') \neq (p, w, r)$ and $\|(p', w', r') - (p, w, r)\| < \varepsilon$.

Let the profile $((p, w, r)\,, y)$ be a Sraffian steady-state equilibrium. It can be shown that it is indeterminate. To see this point, let us examine the system of equations that characterizes the Sraffian steady-state equilibrium, which is given as follows:

$$\begin{aligned} p &= (1 + r)\, pA + wL; \text{ (1)} \\ y &= z\,(p, w, r) + Ay; \text{ (2) and} \\ Ly &= \omega_l. \text{ (3)} \end{aligned}$$

Note that (1) has n equations, (2) has n equations, and (3) has one equation. In contrast, there are n unknown variables regarding the vector y and there are $(n-1)+2$ unknown variables regarding (p, w, r), assuming hereafter that commodity n is selected as the *numeraire*. Therefore, there are $2n+1$ unknown variables in the system of $2n+1$ equations. However, we can decrease the number of equations using Walras' law. Based on this fact, we can show the indeterminacy of the Sraffian steady-state equilibrium in terms of Definition 4.

Given a Sraffian steady-state equilibrium $((p, w, r), y)$, define $\bar{p} \equiv (\frac{p_1}{p_n}, ..., \frac{p_{n-1}}{p_n}, 1)$ and the associated system of equilibrium equations as follows:

$$F(\bar{p}, w, r, y) \equiv \begin{bmatrix} z(p, w, r) - [I - A]\, y \\ (\bar{p} - (1+r)\bar{p}A - wL)^T \end{bmatrix}.$$

By the definition of Sraffian steady state-equilibrium, $F(\bar{p}, w, r, y) = 0$ holds. Note that the mapping F does not contain the counterpart of equation (3). This is because the equation (3) is shown to be redundant, as discussed below in the proof of Theorem 1. Therefore, let us introduce the notion of regular equilibria by means of this F.

Definition 5 (Mandler (1999a)): Let $\langle (A, L); \omega_l; u \rangle$ be an overlapping generation economy as specified above. Then, a Sraffian steady-state equilibrium $((p, w, r), y)$ under this economy is *regular* if the Jacobian of $F(\bar{p}, w, r, y) = 0$ has full row rank.

Now, we are ready to argue the indeterminacy of Sraffian steady-state equilibria, which is summarized as follows:

Theorem 1: Let $\langle (A, L); \omega_l; u \rangle$ be an overlapping generation economy as specified above, and let $((p, w, r), y)$ be a Sraffian steady-state equilibrium under this economy. Then, it is *indeterminate* whenever it is regular.

Proof. First, let us show that the equation (3) is redundant by means of Walras' law. In the overlapping generation economy, Walras' law is generally given by the following equation:

$$\left[p_t \left(z_b^t + z_a^{t-1}\right) + p_t \omega^{t+1}\right] - \left[p_t \delta^t + (1+r_t)\, p_{t-1} A y_t + w_t \omega_l^t\right] = 0, \quad (4)$$

which is derived from the aggregation of $p_t z_b^t + p_t \omega^{t+1} - w_t \omega_l^t = 0$ and $p_t z_a^{t-1} - p_t \delta^t - (1+r_t)\, p_{t-1} A y^t = 0$. Moreover, (4) can be reduced to the following form under stationary prices:

$$\left[p \left(z_b^t + z_a^{t-1}\right) + p \omega^{t+1}\right] - \left[p \delta^t + (1+r)\, p A y_t + w \omega_l^t\right] = 0. \quad \text{(4a)}$$

Note that (4a) can be rewritten to the following form:

$$\left[p \left(z_b^t + z_a^{t-1}\right) + p A y_{t+1} + p \delta^{t+1}\right] - \left[p \delta^t + (1+r)\, p A y_t + w \omega_l^t\right] = 0. \quad \text{(4b)}$$

As $z_b^t = z_b$, $z_a^{t-1} = z_a$, and $y_{t+1} = y_t = y$ hold for every t under the steady-state, (4b) can be reduced to

$$\left[p \left(z_b + z_a\right) + p \delta^{t+1}\right] - \left[p \delta^t + r p A y + w \omega_l\right] = 0. \quad \text{(4b*)}$$

Furthermore, $\delta_{t+1} = \delta_t = \delta$ also holds for every t under the steady-state. Indeed, $\omega^{t+1} = \omega^t = \omega$ holds in the steady-state. Thus, as $\delta_t + Ay_t = \omega$ holds for every t whenever $p > \mathbf{0}$, $y_{t+1} = y_t = y$ implies $\delta_{t+1} = \delta_t = \delta$. Finally, $p > \mathbf{0}$ follows from the definition of Sraffian steady-state equilibrium prices (1), given the assumption of productive and indecomposable A and the positivity of L. Thus, (4b*) can be reduced to

$$p\left(z_b + z_a\right) - \left[rpAy + w\omega_l\right] = 0. \quad (4c)$$

Let us take a profile $\left(\left(p, w, r\right), y\right)$ satisfying the system of equations (1) and (2). From (2), we have

$$\begin{aligned} py &= pz\left(p, w, r\right) + pAy \quad (5) \\ \text{where } z\left(p, w, r\right) &= z_b\left(p, w, r\right) + z_a\left(p, w, r\right). \end{aligned}$$

By combining (1), (5) can be written as:

$$pz\left(p, w, r\right) = p\left(I - A\right)y = rpAy + wLy. \quad (5a)$$

Note that the profile $\left(\left(p, w, r\right), y\right)$ meets Walras' law (4c), which implies that

$$pz\left(p, w, r\right) = rpAy + w\omega_l. \quad (6)$$

From (5a) and (6), we obtain the equation (3):

$$Ly = \omega_l.$$

Thus, the system of $2n+1$ equations (1), (2), and (3) characterizing the Sraffian steady-state equilibrium $\left(\left(p, w, r\right), y\right)$ can be reduced to the system of $2n$ equations (1) and (2), given the reduced form of Walras' law (4c). Then, since the system of $2n$ equations has $2n + 1$ unknown variables, it has freedom of degree one.

If the equilibrium $\left(\left(p, w, r\right), y\right)$ is regular, then the Jacobian matrix of the system of equations (1) and (2) at $\left(\left(\overline{p}, w, r\right), y\right)$ has rank $2n$. Therefore, we can show the indeterminacy of the Sraffian steady-state equilibrium by applying the implicit function theorem (A detailed proof is given in Theorem A2 of Appendix). ■

Remember that, given the same definition of steady-state equilibrium as Definition 2, Mandler (1999a; section 6) argues that such an equilibrium is determinate, which is incompatible with Theorem 1. He seems to reach this conclusion by the following reasoning: "Due to the way in which $1 + r$ appears in Walras' law, the standard argument that one of the equilibrium conditions is redundant is not valid in the present model" (Mandler, 1999a; section 6; p. 705). However, Mandler's argument is not valid, at least for Sraffian steady-state equilibria, in economies where Walras' law (4) holds, as it overlooks the point that one equation (3) can be eliminated by the reduced form of Walras' law (4c).

Note that an alternative scenario may follow if Walras' law (4) is not ensured. For instance, if the utility function is satiated, then it would be possible for the budget constraint at period t of the old generation $t-1$ to not hold with equality: $pz_a^{t-1} - [p\delta^t + (1+r)\,pAy_t] < 0$. In such a case, the reduced form (4c) of Walras' law cannot be obtained. Therefore, we could not derive equation (3) from (1) and (2) under Walras' law, and so the indeterminacy of the Sraffian steady-state equilibrium may not be verified. This may suggest that Mandler's (1999a) claim of generic determinacy for the steady state equilibrium could be verified if Walras' law does not hold.

4.1 Openness and genericity

Next, we examine the openness and genericity of parameter set of economies in which every steady-state equilibrium is regular. The openness and genericity are related to the stability and coverage of indeterminacy in the perturbation of parameters characterizing the set of economies.

For the demand function of two generations z^a, z^b, labor endowment ω_ℓ and for $h = (h_1, h_2, ..., h_n, h^o) \in \mathbb{R}^{n+1}$, define a perturbed demand function with similar form to Mandler (1999a) as

$$z_i(h) \ \equiv z_i^b(h) \ + z_i^a(h)$$

where

$$z_i^b(h) \ \equiv z_i^b(p, w, r) + \frac{w}{p_i} h_i, \ \ z_i^a(h) \ \equiv z_i^a(p, w, r) + \frac{w}{p_i} h^o$$

for each $i = 1, 2, ..., n$.

In order to preserve Walras' law and homogeneity, the perturbation of labor endowment is given as $\omega_l(h) \ \equiv \omega_\ell + \sum_{i=1}^n h_i + \frac{nh^o}{1+r}$.

Now define a function F on the space of $n+1$ price variables $(\bar{p}, w, r)$ where $\bar{p} \equiv (p_1, ...p_{n-1}, 1)$, n quantity variables $(y_1, y_2, ...y_n)$, and adding the parameter set (A, L, h) to $\mathbb{R}^{2n}$, *i.e.*

$$F : \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}^{n}_{++} \times \mathbb{R}^{n^2}_{+} \times \mathbb{R}^{n}_{++} \times \mathbb{R}^{n+1} \to \mathbb{R}^{2n}$$

such that

$$F(\bar{p}, w, r, y, A, L, h) = \begin{bmatrix} z(h) - [I - A]\,y \\ (\bar{p} - (1+r)\bar{p}A - wL)^T \end{bmatrix}.$$

Definition 6: An *economy* is a profile of (A, L, h) where (A, L) is a Leontief production technique, in which A is $n \times n$ non-negative square, productive and indecomposable matrix of reproducible input coefficients, L is $1 \times n$ positive row vector of direct labor coefficients, and $h = (h_1, h_2, ..., h_n, h^o) \in \mathbb{R}^{n+1}$ is for perturbation.

An economy (A, L, h) is *regular* if every Sraffian steady-state equilibrium $((p, w, r)\,, y)$ is regular, that is, the Jacobian $\boldsymbol{DF}$ has full-rank at $(\bar{p}, w, r, y)$. Denote the set of economies as P and the set of regular economies as P_R.

Theorem 2: P_R is open and has full measure in P.

Proof. Before examining whether P_R has full measure, let's first check whether the Jacobian $\boldsymbol{D}F$ has full rank with respect to $p_1, ...p_{n-1}, w, r, y_1, ..., y_n$ in order to check the regularity of an equilibrium whenever the economy (A, L, h) has the property that L cannot be the Frobenius eigenvector of A. The system of equations above has $2n$ equations and $n+1$ price variables $(p_1, ..., p_{n-1}, w, r)$. Hence, the quantity variables $(y_1, ..., y_n)$ are to be determined simultaneously in the Jacobian. Including perturbed parameters, for any (A, L, h), $\boldsymbol{D}_{(y,\bar{p},w,r)}(F_{A,L,h}(\bar{p}, w, r, y))$ is given by:

$$\begin{bmatrix} [A-I] & \boldsymbol{D}_{\bar{p}}z(h) & \boldsymbol{D}_w z(h) & \boldsymbol{D}_r z(p,w,r) \\ \\ \mathbf{0} & I^*_{n-1} - (1+r)A^T_{-n} & -L^T & -(\bar{p}A)^T \end{bmatrix}$$

where

$$\boldsymbol{D}_{\bar{p}}z(h) = \boldsymbol{D}_{\bar{p}}z(\bar{p}, w, r) - \begin{bmatrix} \frac{w}{p_1^2}(h_1+h^o) & \mathbf{0} & & \cdots & \\ \mathbf{0} & \frac{w}{p_2^2}(h_2+h^o) & \mathbf{0} & \cdots & \\ & & \ddots & & \\ \mathbf{0} & \cdots & & \mathbf{0} & \frac{w}{p_{n-1}^2}(h_{n-1}+h^o) \\ \mathbf{0} & \cdots & & & \mathbf{0} \end{bmatrix},$$

$$\boldsymbol{D}_w z(h) = \boldsymbol{D}_w z(\bar{p}, w, r) + [\frac{1}{p_1}(h_1+h^o), \frac{1}{p_2}(h_2+h^o), ..., \frac{1}{p_{n-1}}(h_{n-1}+h^o), (h_n+h^o)]^T,$$

A^T is the transpose of A and A^T_{-n} is the $n \times (n-1)$ matrix obtained by deleting the n-th column of A^T, and

$$I^*_{n-1} = \begin{bmatrix} I_{n-1} \\ \mathbf{0} \end{bmatrix}.$$

Here, note that the last row of $\boldsymbol{D}_{\bar{p}}z(h)$ is nonzero because the last row of $\boldsymbol{D}_{\bar{p}}z(p, w, r)$ is non-zero. As we observed in the calculation result above, the Jacobian has full rank of $2n$ unless the vectors $\begin{bmatrix} \mathbf{0} & I^*_{n-1} - (1+r)A^T_{-n} & -L^T & -(\bar{p}A)^T \end{bmatrix}$ are linearly dependent. Note that the linear dependence of the vectors is observed only in the exceptional case that L becomes the Frobenius eigenvector of A.

The full-measure claim of P_R is proven by the transversality theorem. Let's consider the perturbation of parameters (A, L, h) in $\mathbb{R}^{n^2}_+ \times \mathbb{R}^n_{++} \times \mathbb{R}^{n+1}$. If 0 is a regular value of F at $(\bar{p}, w, r, y)$ and $\boldsymbol{D}F$ has full rank $2n$ with respect to (A, L, h) in $\mathbb{R}^{n^2}_+ \times \mathbb{R}^n_{++} \times \mathbb{R}^{n+1}$, then except for a set of $(A', L', h') \in \mathbb{R}^{n^2}_+ \times \mathbb{R}^n_{++} \times \mathbb{R}^n$ of measure zero, $F_{A,L,h}(\bar{p}, w, r, y) : \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{2n}$ has 0 as a regular value.

Define the Jacobian $\boldsymbol{D}F$ with respect to (A, L, h), which is denoted by $\boldsymbol{D}_{A,L,h}F$, as below:

$$
\boldsymbol{D}_{A,L,h}F = \begin{bmatrix}
\frac{w}{p_1} & & & 0 & \frac{w}{p_1} & y^T & & & \\
& \ddots & & & & & & & \mathbf{0} \\
& & \frac{w}{p_{n-1}} & 0 & \frac{w}{p_{n-1}} & & \ddots & & \\
& 0 & & w & w & & & y^T & \\
& & & & & & & & \\
& & & & \mathbf{0} & & (*) & & -wI_n
\end{bmatrix}
$$

where the row vector y^T is the transpose of y, I_n is the $n \times n$ identity matrix, $(*) = -(1+r)[p_1 I_n \ldots p_{n-1} I_n \; I_n]$ is $n \times n^2$ matrix. Here, each $p_i I_n$ is an $n \times n$ matrix:

$$
p_i I_n = \begin{bmatrix} p_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & p_i \end{bmatrix}.
$$

The first $n+1$ columns are for $(h_1, ..., h_n, h^o)$, the next n^2 columns are for the components of A and the last n columns are for the components of L. We can see that the above matrix has full-rank.

As for openness, consider the contrary case. Suppose P_R is not open. Then there exists a sequence $\{(A, L, h)_k\}$ of non-regular economies converging to a regular economy $(A, L, h)_* \in P_R$. Correspondingly, there exists a sequence of non-regular equilibria $\{(\bar{p}, r, w, y)_k\}$ which converges to a regular equilibrium $(\bar{p}, r, w, y)_*$ at $(A, L, h)_*$. Then the corresponding Jacobian matrices $\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y)_k$ of $2n$ rows and $2n+1$ columns exist, which have less than full rank. For a Jacobian matrix, we can pick $2n+1$ separate square submatrices of order $2n$. The determinants of square submatrices of order $2n$ are all zero. Now we can define a continuous function, say c, from the set of Jacobian matrices to the set of $2n+1$-dimensional vectors whose components are determinants of square submatrices derived from the Jacobian $\boldsymbol{D}F_{A,L,h}$. Since $c(\boldsymbol{D}F_{A,L,h}) = (0, ..., 0) \in \mathbb{R}^{2n+1}$ for any $\boldsymbol{D}F_{A,L,h}$ of less than full rank, $c(\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k) = (0, ..., 0)_k \to (0, ..., 0) \in \mathbb{R}^{2n+1}$ as $k \to \infty$.

Since $\{(0, ..., 0)_k\}$ converging to $(0, ..., 0)$ is closed in $\mathbb{R}^{2n+1}$ and c is continuous, the inverse image $c^{-1}(\{(0, ..., 0)_k\}) = \left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k\right\}$ is closed. Its elements are Jacobian matrices from $P \backslash P_R$ of less than full rank. Since $\left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k\right\}$ is closed, $\boldsymbol{D}F_{(A,L,h)_*}(\bar{p}, w, r, y, g)_*$ is contained in $\left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k\right\}$.

Note that $c(\boldsymbol{D}F_{(A,L,h)_*}(\bar{p}, w, r, y, g)_*) = (0, ..., 0) \in \mathbb{R}^{2n+1}$. This implies that the converging point of the sequence $\left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k\right\}$, each element of which is correspondingly defined from $(A, L, h)_k \in P \backslash P_R$, must also have less than full rank. In other words, the convergent point of the sequence of non-regular economies must also be non-regular. This contradicts our initial assumption. Therefore, the set of regular economies P_R is open. ■

5 Concluding Remarks

In the above argument, we have shown that under the same overlapping generation economy as Mandler (1999a; section 6), Sraffian indeterminacy generically occurs in the Sraffian steady-state equilibrium, unlike Mandler's (1999a; section 6) claim. This possibility theorem may be due to the strong monotonicity assumption of individual utility functions in this paper. As mentioned in section 3, if the utility function is allowed to be satiated, the generic determinacy for the steady-state equilibrium may be observed.

Remember that indeterminacy arises in many places in neoclassical economics, such as the overlapping-generations indeterminacy and factor-price indeterminacy summarized by Mandler (2002). However, the Sraffian indeterminacy observed in this paper has a distinctive feature in comparison with such neoclassical types.

Firstly, the overlapping-generations indeterminacy, such as Calvo (1978) and Kehoe and Levine (1990), summarized by Mandler (2002) is characterized as the continuum set of equilibrium price sequences that results from the arbitrariness of initial commodity prices. However, all of the equilibrium price sequences converge uniquely to the common steady-state price. In contrast, here we focus on the case where an equilibrium price sequence constantly consists of a steady-state equilibrium price vector through the whole periods. However, a continuum set of steady-state equilibrium prices is observed due to the continuum of factor income distributions. This suggests that the Sraffian indeterminacy and the overlapping-generations indeterminacy are quite different.

Secondly, regarding factor-price indeterminacy, the mechanism to derive one dimension of indeterminacy in the model of three factors and two outputs discussed by Mandler (2002) is essentially the same as that discussed by Mandler (1999a, section 3). That is, all three factors can be interpreted so that two of them are reproducible commodities, the same types as output commodities, and the other is labor, but the equilibrium prices are not the steady-state ones. Another typical interpretation would be that all of the factors are primary ones. In contrast, here we focus on the steady-state equilibrium of the economy where labor is the unique primary factor, capital is a bundle of multiple reproducible commodities, and the equilibrium prices are stationary. Moreover, the main source of factor-price indeterminacy is the price-inelastic supply of all productive factors under the fixed production coefficients, which can fix the equilibrium outputs. The remaining unknowns are only the prices associated with a smaller number of price equations. In contrast, the generic indeterminacy discussed here is observed under the conditions that the only price-inelastically supplied factor is labor, and the capital endowments are endogenously determined simultaneously with the determination of equilibrium outputs. Again, the Sraffian indeterminacy discussed here and the factor-price indeterminacy are quite different.

Given the generic indeterminacy of steady-state equilibria in the simple Leontief production model, the natural next question would be whether this indeterminacy is robust in more general models. There may be at least two interesting more general models: a production model with alternative Leontief techniques to represent economies with the possibility of technical changes; and the von Neumann production model of economies with joint production. Note that the discussion developed in section 5 of Mandler (1999a), referring to both of these models, is irrelevant to this robustness question, as it refers only to the sequential equilibria with non-stationary prices, as in section 3 of Mandler (1999a).

For the model with alternative Leontief techniques, it can be verified that the generic feature of one-dimensional indeterminacy of steady-state equilibria is still observed. Moreover, it is still true even if the number of alternative Leontief production techniques is infinite or uncountable. Therefore, unlike the case of sequential equilibria in Mandler (1997), the differentiability of overall production techniques cannot affect the generic feature of the indeterminacy for the case of steady-state equilibria.

For the von Neumann model, unlike the Sraffian equilibrium with a non-stationary price vector discussed in Mandler (1999a, section 5), but similar to the standard Walrasian equilibrium discussed in Mas-Collel (1975) and Kehoe (1980, 1982), the comparison of the number of activities in use with the number of factors that have positive prices is irrelevant. The former can vary across different equilibrium price vectors even in the case of steady-state equilibria. Given such features, we conjecture that the generic one-dimensional indeterminacy of steady-state equilibria is still observed in economies with joint production, but leave this for future research.

Finally, as Mandler's (2002) reference to Morishima (1961) indicates, it would also be interesting to investigate and characterize equilibrium paths in infinite-horison intertemporal economies as argued in the turnpike theorems, given that the continuum of Sraffian steady state equilibria exists. Related works are found in the literature such as Benhabib and Farmer (1994) and Behhabib and Nishimura (1998) which show local indeterminacy of equilibrium paths converging to the steady state.[4] In these works, however, the steady state equilibrium is assumed or shown to be unique, and the local indeterminacy implies infinitely many equilibrium paths from a given initial condition toward the steady state, which is shown to exist under economies with some degree of market imperfections.[5]

[4]Other than these two, there are many works regarding the local indeterminacy of equilibrium paths. See Nishimura and Venditti (2006) for a useful survey of these works.

[5]Note that Nishimura and Shimomura (2002, 2006) show the existence of a continuum of steady-state equilibria in dynamic Heckshar-Ohlin international economies. However, the generation of this continuum is due to the infinitely many allocations across two countries of a uniquely determined aggregate capital stock associated with a unique steady-state equilibrium price vector, which corresponds to the unique steady-state equilibrium in our terminology.

6 References

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7 Appendix: The Existence of Sraffian Steady-State Equilibrium

In this Appendix, we show that, given an economy $\langle (A, L)\,; \omega_l; u\rangle$, there exists an open subset of available non-negative interest rates such that for every interest rate in this subset, an associated steady-state equilibrium exists. By such an existence theorem, it is ensured that the generic indeterminacy discussed in Theorems 1 and 2 is not an empty claim.

Note that if speculative investment were allowed to be non-zero and non-negative under a steady-state equilibrium, then the commodity market clearing condition (b) in Definition 2 would be given by the following form:

$$y + \delta \geqq z\,(p, w, r) + Ay + \delta,$$

which is also the reduced form of condition (1.2) in Definition 1.

Finally, given that the utility function is strongly monotonic, $\delta \geq \mathbf{0}$ would appear under the steady-state equilibrium only when the equilibrium interest rate is zero. However, even when the equilibrium interest rate is zero, $\delta = \mathbf{0}$ is still an optimal action. Therefore, without loss of generality, we may focus on the case of no speculative investment when we discuss the indeterminacy of the Sraffian steady-state equilibrium.

With Definition 2, we can obtain the following existence theorem of the Sraffian steady-state equilibrium in this overlapping economy.

Theorem A1: Let $\langle (A, L); \omega_l; u \rangle$ be an economy as specified above. Then, there exists a *Sraffian steady-state equilibrium* $((p, w, r), y(p, w, r))$ under this economy.

Proof. Let us define $\triangle \equiv \left\{ (p, w) \in \mathbb{R}_+^{n+1} \mid \sum_{i=1}^n p_i + w = 1 \right\}$ and $\overset{\circ}{\triangle} \equiv \{(p, w) \in \triangle \mid (p, w) > \mathbf{0}\}$. For each $(p, w) \in \triangle$, consider the following opmizaition problem:

$$\max_{(z_b, z_a, y)} u(z_b, z_a)$$

subject to

$$\begin{aligned} pz_b + W &\leqq w\omega_l, \\ pAy &= W, \text{ and} \\ pz_a &\leqq \max\{py - wLy, W\}. \end{aligned}$$

Denote the set of solutions to this optimization problem by $\mathcal{O}(p, w)$.

Take $(z_b(p, w), z_a(p, w), y(p, w)) \in \mathcal{O}(p, w)$. Then,

$$y(p, w) \in \arg\max \left\{ \max_{y \geq \mathbf{0};\ pAy = W} py - pAy - wLy, 0 \right\}$$

holds. It is also shown that the correspondence $\mathcal{O} : \overset{\circ}{\triangle} \twoheadrightarrow \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n$ is non-empty, compact and convex-valued, and upper hemicontinuous.

Let us define the excess demand correspondence $\mathcal{D} : \overset{\circ}{\triangle} \twoheadrightarrow \mathbb{R}^n$ by

$$\begin{aligned} &\mathcal{D}(p, w) \\ \equiv\ &\{(z(p, w) - (I - A)y(p, w), Ly(p, w) - \omega_l) \mid (z_b(p, w), z_a(p, w), y(p, w)) \in \mathcal{O}(p, w)\}. \end{aligned}$$

It can be shown that this correspondence is non-empty, compact and convex-valued, and upper hemicontinuous. By the strong monotonicity of u, the following form of Walras' law holds: for any $(p, w) \in \overset{\circ}{\triangle}$ and any $d(p, w) \in \mathcal{D}(p, w)$, $(p, w) \cdot d(p, w) = 0$.

Let us take any price sequence $\left\{ \left(p^k, w^k\right) \right\} \subset \overset{\circ}{\triangle}$ such that $\left(p^k, w^k\right) \to (\overline{p}, \overline{w}) \in \triangle \backslash \overset{\circ}{\triangle}$. Take $d\left(p^k, w^k\right) \in \mathcal{D}\left(p^k, w^k\right)$ for each $\left(p^k, w^k\right)$.

Suppose that $(\overline{p}, \overline{w}) \in \triangle \backslash \mathring{\triangle}$ with $\overline{w} > 0$. Then, there exists a commodity i such that $\overline{p}_i = 0$. Then, for sufficiently large k, p_i^k is sufficiently close to zero. Then, $z_i\left(p^k, w^k\right)$ is sufficiently large by the strong monotonicity of u. In contrast, $y\left(p^k, w^k\right)$ is bounded by the condition $p^k A y\left(p^k, w^k\right) < w^k \omega_l$. Therefore, for sufficiently large k, $z_i\left(p^k, w^k\right) - y_i\left(p^k, w^k\right) + A_i y\left(p^k, w^k\right) > 0$ should hold, where A_i is the i-th row vector of A. Now, let us define $(p', w') \in \mathring{\triangle}$ such that $(p', w') \equiv \frac{1}{\lambda}\left(p^k, w^k\right) - \frac{1-\lambda}{\lambda}(\overline{p}, \overline{w})$ for some sufficiently small $\lambda \in (0, 1)$. Then, $(p', w') \cdot d\left(p^k, w^k\right) > 0$ holds as $p_i'\left[z_i\left(p^k, w^k\right) - y_i\left(p^k, w^k\right) + A_i y\left(p^k, w^k\right)\right] > 0$ is sufficiently greater.

Suppose that $(\overline{p}, \overline{w}) \in \triangle \backslash \mathring{\triangle}$ with $\overline{w} = 0$. Then, for sufficiently large k, w^k is sufficiently close to zero. Then, $y\left(p^k, w^k\right)$ must be sufficiently close to zero vector as $p^k A y\left(p^k, w^k\right) < w^k \omega_l$. Thus, for sufficiently large k, $Ly\left(p^k, w^k\right) < \omega_l$ should hold. Now, let us define $(p', w') \in \mathring{\triangle}$ such that $(p', w') \equiv \left(p^k\left(1 + \frac{\varepsilon}{1-w^k}\right), w^k - \varepsilon\right)$ for some sufficiently small $\varepsilon > 0$. Then,

$$
\begin{aligned}
& (p', w') \cdot d\left(p^k, w^k\right) \\
= & \left(p^k\left(1 + \frac{\varepsilon}{1-w^k}\right), w^k - \varepsilon\right) \cdot \left(z\left(p^k, w^k\right) - (I-A)\, y\left(p^k, w^k\right), Ly\left(p^k, w^k\right) - \omega_l\right) \\
= & \frac{\varepsilon}{1-w^k} p^k \cdot \left[z\left(p^k, w^k\right) - (I-A)\, y\left(p^k, w^k\right)\right] - \varepsilon\left(Ly\left(p^k, w^k\right) - \omega_l\right) \\
= & \frac{w^k}{1-w^k} \varepsilon\left(\omega_l - Ly\left(p^k, w^k\right)\right) - \varepsilon\left(Ly\left(p^k, w^k\right) - \omega_l\right) > 0.
\end{aligned}
$$

In summary, we have shown that for any price sequence $\left\{\left(p^k, w^k\right)\right\} \subset \mathring{\triangle}$ such that $\left(p^k, w^k\right) \to (\overline{p}, \overline{w}) \in \triangle \backslash \mathring{\triangle}$, and for any $d\left(p^k, w^k\right) \in \mathcal{D}\left(p^k, w^k\right)$, there exists $(p', w') \in \mathring{\triangle}$ such that $(p', w') \cdot d\left(p^k, w^k\right) > 0$ for infinitely many k. Then, by Grandmont (1977, Lemma 1), there exists $(p^*, w^*) \in \mathring{\triangle}$ such that $z(p^*, w^*) - (I-A)\, y(p^*, w^*) = \mathbf{0}$ and $Ly(p^*, w^*) - \omega_l = 0$. Thus, $y(p^*, w^*) = (I-A)^{-1} z(p^*, w^*)$, and so $y(p^*, w^*) > \mathbf{0}$ by the indecomposability of A, unless $z(p^*, w^*) = \mathbf{0}$. Since $p^* > \mathbf{0}$ and $w^* > 0$, $z(p^*, w^*) \geq \mathbf{0}$ follows from the strong monotonicity of u. Thus, $y(p^*, w^*) > \mathbf{0}$. Then, for $r^* \equiv \frac{p^* y(p^*, w^*) - Ly(p^*, w^*)}{p^* A y(p^*, w^*)} - 1$, $r^* \geqq 0$ holds from $y(p^*, w^*) \in \arg\max\left\{\max_{y \geq \mathbf{0};\ p^* A y = W} p^* y - p^* A y - w^* L y, 0\right\}$. Moreover, it should follow from the optimal behavior and $y(p^*, w^*) > \mathbf{0}$ that

$$
p^* = (1 + r^*)\, p^* A + w^* L.
$$

Thus, there exists a Sraffian steady-state equilibrium $((p^*, w^*, r^*), y(p^*, w^*, r^*))$ with $y(p^*, w^*, r^*) = y(p^*, w^*)$. ■

Denote the Frobenius eigenvalue of the matrix A by $(1+R)^{-1} \in (0, 1)$. Then, by Theorem A1 and Theorem 1, we have the following existence theorem.

Theorem A2: Let $\langle (A, L); \omega_l; u \rangle$ be an economy as specified above. Let $((p^*, w^*, r^*), y(p^*, w^*, r^*))$ be a Sraffian steady-state equilibrium, which is regular. Then, there exists an open neighborhood $\mathcal{N}(r^*) \subseteq [0, R)$ of r^* such that there exists a Sraffian steady-state equilibrium

$$((p(r), w(r), r), y(p(r), w(r), r))$$

for every $r \in \mathcal{N}(r^*)$.

Proof. Let us define a continuously differentiable function $F : \mathbb{R}_+^{n-1} \times \mathbb{R}_+ \times [0, R) \times \mathbb{R}_+^n \to \mathbb{R}^{2n}$ as:

$$F(\overline{p}, w, r, y) = \begin{bmatrix} z(p, w, r) - [I - A] y \\ \overline{p}_{-n} - (1 + r)\overline{p} A_{-n} - w L_{-n} \\ Ly - \omega_l \end{bmatrix}.$$

Let (p^*, w^*, r^*, y^*) be a Sraffian steady-state equilibrium, whose existence is ensured by Theorem A1. Assume it is a regular equilibrium. Then, the Jacobian of F at (p^*, w^*, r^*, y^*) is given by $\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}^*, w^*, r^*, y^*))$ is given by:

$$\begin{aligned} &\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}^*, w^*, r^*, y^*)) \\ = &\begin{bmatrix} [A - I] & \boldsymbol{D}_{\overline{p}} z(\overline{p}^*, w^*, r^*) & \boldsymbol{D}_w z(\overline{p}^*, w^*, r^*) & \boldsymbol{D}_r z(\overline{p}^*, w^*, r^*) \\ \mathbf{0} & I_{n-1} - (1 + r) A_{-n}^T & -L_{-n}^T & -(\bar{p} A_{-n})^T \\ L & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{aligned}$$

As (p^*, w^*, r^*, y^*) is regular, it follows that $rank\left[\boldsymbol{D}_{(y,\overline{p},w,r)}(F(\overline{p}^*, w^*, r^*, y^*))\right] = 2n$.[6]

Then, by the implicit function theorem, there exist an open neighborhood $\mathcal{N}(r^*) \subset [0, R)$ of r^* and also an open neighborhood $\mathcal{M}(\overline{p}^*, w^*, y^*) \subset \mathbb{R}_+^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+^n$ of $(\overline{p}^*, w^*, y^*)$ such that there exists a continuous single-valued mapping $\eta : \mathcal{N}(r^*) \to \mathcal{M}(\overline{p}^*, w^*, y^*)$ such that for any $r' \in \mathcal{N}(r^*)$, there exists $(\overline{p}', w', y') = \eta(r')$ with $F(\overline{p}', w', r', y') = \mathbf{0}$. By the definition of the mapping F, $F(\overline{p}', w', r', y') = \mathbf{0}$ implies that $\overline{p}' \cdot (z(\overline{p}', w', r') - [I - A] y') + w' \cdot (Ly' - \omega_l) = 0$. As $\overline{p}'_{-n} = (1 + r')\overline{p}' A_{-n} + w' L_{-n}$, it also follows that $1 = (1 + r')\overline{p}' A_n + w' L_n$. Thus, $\overline{p}' = (1 + r')\overline{p}' A + w' L$ holds, which implies that $(\overline{p}', w', r', y')$ is a Sraffian steady-state equilibrium associated with $r' \in \mathcal{N}(r^*)$. ■

In this way, we can show that for each non-negative interest rate within a subset of $[0, R)$, there exists a Sraffian steady-state equilibrium associated with this interest rate.

[6] As shown in the proof of Theorem 2, the regularity of the equilibrium (p^*, w^*, r^*, y^*) is indeed verified except for a non-generic case that the vector L becomes the Frobenius eigenvector of A.

8 Addendum: Indeterminacy of the stationary growth equilibrium

The definition of steady-state equilibrium, Definition 2, presumes that the aggregate net investment of capital is exogenously given to be zero. In this section, we will introduce an alternative equilibrium notion.

Note that in the case of steady-state equilibrium, gross investment is only for the replacement, and so no net investment appears. Alternatively, consider a case where positive net investment may be observed and its ratio to gross investment is invariant throughout the whole periods. Let g denote the ratio of net investment to replacement investment and call it an *investment growth rate*. Consider an economy of overlapping generations with an endogenous rate, g, of investment growth. Then, the corresponding long run feature of a competitive equilibrium with no speculative investment is given as follows.

Definition 7: A *stationary growth equilibrium* under the overlapping economy $\langle (A, L); \omega_l; u \rangle$ is a profile of a stationary price vector (p, w, r), a gross output vector $y \geq \mathbf{0}$, and a common ratio of new to replacement investment $g > -1$, such that the following conditions are satisfied:

$$\begin{aligned} p &\leqq (1+r)\, pA + wL; \text{ (a*)} \\ y &\geqq z\,(p, w, r; g) + (1+g)\, Ay, \text{ (b*)} \\ \text{where } z\,(p, w, r; g) &\equiv z_b\,(p, w, r) + \frac{z_a\,(p, w, r)}{1+g}; \\ Ly &\leqq \omega_l, \text{ (c*)} \\ \text{and } \frac{pz_a\,(p, w, r)}{1+r} &= (1+g)pAy. \text{ (d*)} \end{aligned}$$

Moreover, a *stationary growth equilibrium* $((p, w, r), y, g)$ under the overlapping economy $\langle (A, L); \omega_l; u \rangle$ is called *Sraffian* if and only if all of the conditions (a*), (b*), and (c*) hold in equality.

In the above definition, the equation (d*) for the present value of consumption expenditure in old age and gross productive investment is equivalent to the condition of no speculative investment. This additional requirement is consistent with individual optimization in the equilibria, because it is an optimal action for every generation to make no speculative investment whenever the equilibrium price vector of commodities is stationary. Some of the literature on overlapping generation models treats this condition as a part of the definition of competitive equilibrium, like Tvede (2010; p. 118; Definition 7.1).

Note that if speculative investment were allowed to be non-zero and non-negative under the stationary growth equilibrium, then the commodity market clearing condition in Definition 7 would be given by the following form:

$$y + \delta \geqq z\,(p, w, r; g) + (1+g)\, Ay + (1+g)\, \delta,$$

which is the reduced form of the condition (1.2) in Definition 1. If this inequality logically implied (b*) of Definition 7, then the non-zero and non-negative speculative investment under the stationary growth equilibrium would be compatible with the standard commodity market clearing condition for the definition of the stationary growth equilibrium. However, it does not imply (b*): for instance, in the case of $g < 0$.

Finally, given that the utility function is strongly monotonic, $\delta \geq \mathbf{0}$ would appear under the stationary growth equilibrium only when the equilibrium interest rate is zero. Therefore, without loss of generality, we may focus on the case of no speculative investment when we discuss the indeterminacy of the Sraffian stationary growth equilibrium.

Here, the aggregate consumption demand is given by $z\left(p, w, r; g\right) = z_b\left(p, w, r\right) + \frac{z_a(p,w,r)}{1+g}$, where $\frac{z_a(p,w,r)}{1+g}$ represents the old generation's consumption demand in the present period. The appearance of the denominator in the second component of this equation implies that g is also equal to the growth rate of labor endowments (population), in that the ratio of the old generation's population (labor endowment) to the young generation's is $\frac{1}{1+g}$. The endogenous determination of population growth is fixed outside of the market mechanism, and so it is not specified in the economic model. In the section of concluding remarks, we will discuss how the endogenous population growth rate would be matched with the equilibrium investment growth rate under the stationary growth equilibrium.

8.1 Indeterminacy of the stationary growth equilibrium

Given the Leontief production technique (A, L), let $y^* > \mathbf{0}$ be the Frobenius eigenvector associated with the Frobenius eigenvalue $(1+R)^{-1}$ such that it is normalized to satisfy $Ly^* = 1$. This commodity bundle is called the *standard commodity* by Sraffa (1960). In this section, we assume that the standard commodity is adopted as the *numeraire* of the price system: for any market price vector $p \in \mathbb{R}^n_+$, $py^* = 1$ is satisfied.

With Definition 7, we can obtain the following existence theorem of the Sraffian stationary growth equilibrium in this overlapping economy.

Theorem 3: Let $\langle (A, L); \omega_l; u \rangle$ be an overlapping generation economy as specified above. Then, for *each profit rate* $r \in [0, R)$, there exists a *Sraffian stationary growth equilibrium* $((p, w, r), y, g)$ under this economy.

Proof. Let $R > 0$ be the maximal profit rate under the technique (A, L). As is well-known, $\frac{1}{1+R}$ is the Frobenius eigen value of the productive and indecomposable matrix A such that there exists a unique Frobenius eigen vector $p^* > \mathbf{0}$ satisfying $p^* = (1+R)\, p^* A$.

Take any $r \in [0, R)$. Then, due to the Sraffian linear distribution function, we can specify $w > 0$ as satisfying

$$r = R\left(1 - w\right).$$

Given (w, r), let

$$p \equiv wL \left(1 - (1 + r) A\right)^{-1}.$$

Then, it is well-known that in this case, $p > \mathbf{0}$ and $p = (1 + r) pA + wL$ hold. That is, we obtain a stationary price vector (p, w, r), which prevails at each and every period.

Given this price information (p, w, r), consider the program MP^t of generation t. Let $\left(z_b^t (p, w, r), z_a^t (p, w, r)\right)$ be a solution of generation t to the program MP^t under the stationary prices (p, w, r). But the same solution is also optimal for generation $t-1$, as all generations have the same utility function. Therefore, without loss of generality, we get rid of the superscript "t" in the solution to each MP^t.

Thus, now without loss of generality, let $z (p, w, r; g) \equiv z_b (p, w, r) + \frac{z_a(p,w,r)}{1+g}$ be the aggregate consumption demand vector, where $g > -1$ denotes a common growth rate of outputs. In addition, let

$$y (p, w, r; g) \equiv [I - (1 + g) A]^{-1} z (p, w, r; g).$$

Note that since A is productive and indecomposable, we have $[I - (1 + g) A]^{-1} > \mathbf{0}$ for any $g \in (-1, R)$, and so $y > \mathbf{0}$ holds. Moreover, it follows that

$$\begin{aligned} py (p, w, r; g) &= (1 + r) pAy (p, w, r; g) + wLy (p, w, r; g) \\ &\Leftrightarrow pz (p, w, r) + (1 + g) pAy (p, w, r; g) = (1 + r) pAy (p, w, r; g) + wLy (p, w, r; g). \end{aligned}$$

Since the budget constraint of the program MP^t implies that

$$pz_b (p, w, r) + \frac{pz_a (p, w, r)}{1 + r} = w\omega_l,$$

we can establish $Ly (p, w, r; g) = \omega_l$ whenever

$$\frac{pz_a (p, w, r)}{1 + r} = (1 + g) pAy (p, w, r; g)$$

is satisfied. Therefore, let us show that for any $r \in [0, R)$, there exists a unique $g (r)$ such that

$$\frac{pz_a (p, w, r)}{1 + r} = (1 + g (r)) pAy (p, w, r; g (r))$$

holds.

Let $\Psi (r, g) \equiv (1 + g) pAy (p, w, r; g) - \frac{pz_a(p,w,r)}{1+r}$. Note that if g is sufficiently close to -1, then $\Psi (r, g) \approx \Psi (r, -1) < 0$. In contrast,

$$\lim_{g \to R} \Psi (r, g) = +\infty$$

holds, as the matrix $[I - (1 + g) A]^{-1}$ approaches the singular matrix. Since $\Psi (r, g)$ is continuous at every g, there exists $g (r)$ such that

$$\frac{pz_a (p, w, r)}{1 + r} = (1 + g (r)) pAy (p, w, r; g (r))$$

holds. Moreover, as

$$\begin{aligned}
\frac{\partial \Psi (r,g)}{\partial g} &= pAy\left(p,w,r;g\left(r\right)\right) + (1+g)\, pA\frac{\partial \left[I-(1+g)\,A\right]^{-1} z_b\left(p,w,r\right)}{\partial g} \\
&\quad + pA\frac{\partial \left[I-(1+g)\,A\right]^{-1} z_a\left(p,w,r\right)}{\partial g} - (1+g)\, pA\left[I-(1+g)\,A\right]^{-1}\frac{z_a\left(p,w,r\right)}{\left(1+g\right)^2} \\
&= pA\left[I-(1+g)\,A\right]^{-1} z_b\left(p,w,r\right) + (1+g)\, pA\frac{\partial \left[I-(1+g)\,A\right]^{-1} z\left(p,w,r;g\right)}{\partial g} \\
&> 0,
\end{aligned}$$

it follows that $g(r)$ is unique. Thus, for each $r \in [0, R)$, we obtain a stationary price vector $(p(r), w(r), r)$ such that

$$\begin{aligned}
p(r) &= (1+r)\, p(r)\, A + w(r)\, L; \\
y\left(p(r), w(r), r\right) &= z_b\left(p(r), w(r), r\right) + \frac{z_a\left(p(r), w(r), r\right)}{1+g(r)} + (1+g(r))\, Ay\left(p(r), w(r), r\right), \\
\frac{p(r)\, z_a\left(p(r), w(r), r\right)}{1+r} &= (1+g(r))\, p(r)\, Ay\left(p(r), w(r), r\right); \text{ and} \\
Ly\left(p(r), w(r), r\right) &= \omega_l.
\end{aligned}$$

In summary, the above-specified profile $((p(r), w(r), r), y(p(r), w(r), r), g(r))$ satisfies all of the conditions for a stationary growth equilibrium. ■

Now, we are ready to discuss the indeterminacy of the Sraffian stationary growth equilibrium. The definition of indeterminacy can be given analogical to Definition 4. Let the profile $((p, w, r), y, g)$ be a Sraffian stationary growth equilibrium. Let us take any $\varepsilon > 0$. Take $r' (\neq r)$, which is sufficiently close to r such that $\|(p', w', r'), (p, w, r)\| < \varepsilon$ holds, where $w' \equiv 1 - \frac{r'}{R}$ and $p' \equiv w'L\left(I - (1+r')\, A\right)^{-1}$. Then, by Theorem 3, there exists a stationary growth equilibrium $((p', w', r'), y', g')$. As $r' \neq r$, it is obvious that $(p', w', r') \neq (p, w, r)$. This implies that $((p, w, r), y, g)$ is indeterminate. Thus, we can summarize:

Theorem 4: Let $\langle (A, L); \omega_l; u\rangle$ be an overlapping generation economy as specified above. Then, the corresponding Sraffian stationary growth equilibrium $((p, w, r), y, g)$ under this economy is *indeterminate*.

Note that Theorem 4 can also be established by a standard proof similar to that developed for Theorem 1. Developing this proof parallel to the case of Theorem 1, we obtain the following system of $2n+1$ equations:

$$\begin{aligned}
\bar{p} &= (1+r)\,\bar{p}A + wL; \ (1^*) \\
\left(I-(1+g)\,A\right) y &= z\left(\bar{p}, w, r; g\right), \ (2^*); \\
(1+g)\bar{p}Ay &= \frac{\bar{p}z_a\left(\bar{p}, w, r\right)}{1+r}. \ (3^*)
\end{aligned}$$

Then, since the system of $2n+1$ equations has $2n+2$ unknown variables, it has freedom of degree one.

It is not difficult to see that the Jacobian matrix of the system of equations (1*), (2*), and (3*) has rank $2n+1$. Therefore, we can show the indeterminacy of the Sraffian stationary growth equilibrium by applying the implicit function theorem.

Note that the above result is due to the fact that the growth rate g is endogenously determined and so is an unknown variable. If the growth rate g is exogenously given as a parameter, then the system of equations (1*), (2*) with a fixed number $g > -1$, and (3*) has only $2n+1$ unknown variables. Again, it is not difficult to see that the Jacobian matrix of the system of equations (1*), (2*) with a fixed number $g > -1$, and (3*) has rank $2n+1$. Therefore, we can show the determinacy of such an equilibrium by applying the implicit function theorem:

Corollary 1: Let $\langle (A, L)\, ; \omega_l ; u \rangle$ be an overlapping generation economy as specified above. Then, the Sraffian stationary growth equilibrium with an exogenous growth rate $g \neq 0$, $((p, w, r)\, , y)$, under this economy is *determinate.*

Note that, as in the case of steady-state equilibrium, the Sraffian stationary growth equilibrium with an exogenous growth rate is not necessarily guaranteed to exist. This is because each agent's optimization program is solved independently of the exogenous growth rate $g \neq 0$.

8.2 Openness and genericity

Now let's investigate the openness and genericity of a parameter set of economies in which every Sraffian stationary growth equilibrium is regular.

For the demand functions of two generations z^a, z^b, labor endowment ω_ℓ and for $h = (h_1, h_2, ..., h_n, h^o) \in \mathbb{R}^{n+1}$, define a perturbed demand function as

$$z_i^b(h) \ \equiv z_i^b(p, w, r) + \frac{w}{p_i} h_i, \quad z_i^a(h) \ \equiv z_i^a(p, w, r) + \frac{w}{p_i} h^o,$$

$$z_i(p, w, r) \ \equiv z_i^b(p, w, r) \ + \frac{z_i^a(p, w, r)}{1+g}$$

for each $i = 1, 2, ..., n$, and $\omega_l(h) \ \equiv \omega_\ell + \sum_{i=1}^n h_i + \frac{nh^o/1+g}{1+r}$. Hence the perturbed demand function is represented as,

$$z_i(h) \ \equiv z_i^b(h) \ + \frac{z_i^a(h)}{1+g}$$

for each $i = 1, 2, ..., n$. The perturbed functions satisfy Walras' law and homogeneity.

Define a function F on the space of $n+1$ price variables $(\bar{p}, w, r)$ where $\bar{p} \equiv (p_1, ...p_{n-1}, 1)$ is the normalized price, there are n quantity variables $(y_1, y_2, ...y_n)$, the growth rate of investments is g and add the parameter set (A, L, h) toward $\mathbb{R}^{2n+1}$, *i.e.*

$$F : \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}^{n}_{++} \times \mathbb{R} \times \mathbb{R}^{n^2}_{+} \times \mathbb{R}^{n}_{++} \times \mathbb{R}^{n+1} \to \mathbb{R}^{2n+1}$$

such that

$$F(\bar{p}, w, r, y, g, A, L, h) = \begin{bmatrix} z(h) - [I - (1+g)A]\, y \\ \bar{p} - (1+r)\bar{p}A - wL \\ (1+g)\, \bar{p}Ay - \frac{\bar{p}z^a(h)}{1+r} \end{bmatrix}$$

where $\bar{p}$ and L are row vectors, y is a column vector, and R is the maximum rate of profit.

A *regular stationary growth equilibrium* is a normalized equilibrium vector $(\bar{p}, w, r, y, g)$ such that zero is a *regular value* of F, i.e. Jacobian $\boldsymbol{D}F$ has full-rank at $(\bar{p}, w, r, y, g)$.[7] An economy (A, L, h) is *regular* if every normalized

[7] The system of equation ; (1*), ; (2*) and ; (3*) has $2n+1$ equations and $n+1$ price variables $(p_1, ..., p_{n-1}, w, r)$. Hence the growth rate g and quantity variables $(y_1, ..., y_n)$ are to be determined simultaneously in the Jacobian. Including perturbed parameters, for any (A, L, h), $\boldsymbol{D}_{(g,y,\bar{p},w,r)}(F_{A,L,h}(\bar{p}, w, r, y, g))$ is given by:

$$\begin{bmatrix} \text{(i)} & [(1+g)A - I] & \boldsymbol{D}_{\bar{p}}z(h) & \boldsymbol{D}_w z(h) & \boldsymbol{D}_r z(p,w,r) \\ \mathbf{0} & \mathbf{0} & I^*_{n-1} - (1+r)A^T_{-n} & -L^T & -\bar{p}A \\ \bar{p}Ay & (1+g)\bar{p}A & \text{(ii)} & \text{(iii)} & \text{(iv)} \end{bmatrix}$$

where

$$I^*_{n-1} = \begin{bmatrix} I_{n-1} \\ \mathbf{0} \end{bmatrix}$$

and

$$(i) = \boldsymbol{D}_g z(p,w,r) - \frac{-wh^o}{(1+g)^2}[p_1^{-1}, ..., p_{n-1}^{-1}, 1\,]^T + Ay$$

$$(ii) = [(1+g)A_{-n}y]^T - \frac{1}{1+r}[z_1^a(p,w,r) + \sum_{i=1}^{n} p_i \frac{\partial z_i^a(p,w,r)}{\partial p_1}, ..., z_{n-1}^a(p,w,r) + \sum_{i=1}^{n} p_i \frac{\partial z_i^a(p,w,r)}{\partial p_{n-1}}],$$

$$(iii) = -\sum_{1}^{n}[\frac{p_i}{1+r}\frac{\partial z_i^a(p,w,r)}{\partial w}] - \frac{nh^o}{1+r}.$$

$$(iv) = \sum_{1}^{n} \frac{p_i}{1+r}[\frac{z_i^a(p,w,r)}{1+r} - \frac{\partial z_i^a(p,w,r)}{\partial r}] + \frac{nwh^o}{(1+r)^2}.$$

In addition, $\boldsymbol{D}_{\bar{p}}z(h)$ and $\boldsymbol{D}_w z(h)$ are calculated as:

stationary growth equilibrium vector $(\bar{p}, w, r, y, g)$ is regular.[8] Denote the set of economies as P and the set of regular economies as P_R.

Theorem 5: P_R is open and has full measure in P.

Proof. The full measure claim of P_R is proven by the transversality theorem. Trivially, the function F defined above is smooth on the domain including all $(\bar{p}, w, r, y, g)$ and parameter (A, L, h) in $\mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}^{n}_{++} \times \mathbb{R} \times \mathbb{R}^{n^2}_{+} \times \mathbb{R}^{n}_{++} \times \mathbb{R}^{n+1}$. If zero is a regular value of F at $(\bar{p}, w, r, y; g)$ and the Jacobian $\boldsymbol{D}F$ with respect to A, L, h has full rank $2n+1$, then except a set of $(A', L', h') \in \mathbb{R}^{n^2}_{+} \times \mathbb{R}^{n}_{++} \times \mathbb{R}^{n+1}$ of measure zero, $F_{A,L,h}(\bar{p}, w, r, y, g) : \mathbb{R}^{n-1}_{++} \times \mathbb{R}_{++} \times \mathbb{R}_{+} \times \mathbb{R}^{n}_{++} \times \mathbb{R} \to \mathbb{R}^{2n+1}$ has 0 as a regular value. Define the Jacobian $\boldsymbol{D}F$ with respect to (A, L, h), which is denoted by $\boldsymbol{D}_{A,L,h}F$, as below:

$$
\boldsymbol{D}_{A,L,h}F = \left[\begin{array}{cccccccc}
\frac{w}{p_1} & & & \frac{w}{(1+g)p_1} & (1+g)y^T & & & \\
& \ddots & & & & & & \mathbf{0} \\
& & \frac{w}{p_{n-1}} & \frac{w}{(1+g)p_{n-1}} & & \ddots & & \\
& & w & \frac{w}{(1+g)} & & & (1+g)y^T & \\
& \mathbf{0} & & & & \text{(v)} & & -wI_n \\
& \mathbf{0} & & -\frac{nw}{1+r} & & \text{(vi)} & & \mathbf{0}
\end{array}\right]
$$

where the row vector y^T is the transpose of y, I_n is the $n \times n$ identity matrix, (v)$= -(1+r)[p_1 I_n \ldots p_{n-1} I_n, I_n]$ is $n \times n^2$ and (vi) $= (1+g)[p_1 y^T \ldots p_{n-1} y^T, y^T]$

$$
\boldsymbol{D}_{\bar{p}} z(h) = \boldsymbol{D}_{\bar{p}} z(\bar{p}, w, r) - \begin{bmatrix}
\frac{w}{p_1^2}(h_1 + \frac{h^o}{1+g}) & \mathbf{0} & & \ldots & \mathbf{0} \\
\mathbf{0} & \frac{w}{p_2^2}(h_2 + \frac{h^o}{1+g}) & \mathbf{0} & \ldots & \mathbf{0} \\
& & \ddots & & \\
\mathbf{0} & \ldots & & \mathbf{0} & \frac{w}{p_{n-1}^2}(h_{n-1} + \frac{h^o}{1+g}) \\
\mathbf{0} & \ldots & & \mathbf{0} & \mathbf{0}
\end{bmatrix},
$$

$$
\boldsymbol{D}_w z(h) = \boldsymbol{D}_w z(p, w, r) + [\frac{1}{p_1}(h_1 + \frac{h^o}{1+g}), \frac{1}{p_2}(h_2 + \frac{h^o}{1+g}), \ldots, \frac{1}{p_{n-1}}(h_{n-1} + \frac{h^0}{1+g}), h_n + \frac{h^0}{1+g}\ ]^T.
$$

[8]Likewise, we can define an economy (A, L, h) is as *regular* if every normalized steady-state equilibrium vector $(\bar{p}, w, r, y)$ is regular. Then, as similar to the proof of Theorem 5 discussed below, it can be shown that such a regular economy is open and has full measure.

is $1 \times n^2$ matrix.[9] The first $n+1$ columns are for $(h_1, ..., h_n, h^o)$, the next n^2 columns are for the components of A and the last n columns are for the components of L.

To see the full-measureness of regular economy P_R, it is sufficient to check that $\boldsymbol{D}_{A,L,h}F$ has full rank. Observe that the first $n \times (n+1)$ submatrix of the upper-left is nonsingular. Next, the shape of (v) and $-wI_n$ guarantees n nonzero rows which are linearly independent. The bottom row will also be nonzero in elementary column operation. Therefore, $\boldsymbol{D}_{A,L,h}F$ has full rank. Since we have shown that $\boldsymbol{D}_{A,L,h}F$ has full row rank at all $(\bar{p}, w, 1+r, y, g, A, L, h)$ with $F = 0$, then by the transversality theorem, $F_{A,L,h}(\bar{p}, w, r, y, g)$ has 0 as a regular value almost everywhere in P. In other words, P_R has full measure.

As for the openness, consider the contrary case. Suppose P_R is not open. Then there would be a sequence $\{(A, L, h)_k\}$ of non-regular economies converging to a regular economy $(A, L, h)_* \in P_R$. Correspondingly, there exists a sequence of non-regular equilibria $\{(\bar{p}, r, w, y, g)_k\}$ which converges to a regular equilibrium $(\bar{p}, r, w, y, g)_*$ at $(A, L, h)_*$. Then the corresponding Jacobian matrices $\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k$ of $2n+1$ rows and $2n+2$ columns exist, as seen in footnote 4, which have less than full rank. For a Jacobian matrix, we can pick $2n+2$ separate square submatrices of order $2n+1$. The determinants of square submatrices of order $2n+1$ are all zero. Now we can define a continuous function, say c, from the set of Jacobian matrices to the set of $2n+2$-dimensional vectors whose components are determinants of square submatrices derived from the Jacobian $\boldsymbol{D}F_{A,L,h}$. Since $c(\boldsymbol{D}F_{A,L,h}) = (0, ..., 0) \in \mathbb{R}^{2n+2}$ for any $\boldsymbol{D}F_{A,L,h}$ of less than full rank, $c(\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k) = (0, ..., 0)_k \to (0, ..., 0) \in \mathbb{R}^{2n+2}$ as $k \to \infty$.

Since $\{(0, ..., 0)_k\}$ converging to $(0, ..., 0)$ is closed in $\mathbb{R}^{2n+2}$ and c is continuous, the inverse image $c^{-1}(\{(0, ..., 0)_k\}) = \left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k\right\}$ is closed. Its elements are Jacobian matrices from $P \backslash P_R$ of less than full rank. Since $\left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k\right\}$ is closed, $\boldsymbol{D}F_{(A,L,h)_*}(\bar{p}, w, r, y, g)_*$ is contained in $\left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k\right\}$.

Note that $c(\boldsymbol{D}F_{(A,L,h)_*}(\bar{p}, w, r, y, g)_*) = (0, ..., 0) \in \mathbb{R}^{2n+2}$. This implies that the converging point of the sequence $\left\{\boldsymbol{D}F_{(A,L,h)_k}(\bar{p}, w, r, y, g)_k\right\}$, each element of which is correspondingly defined from $(A, L, h)_k \in P \backslash P_R$, must also have less than full rank. In other words, the convergent point of the sequence of non-regular economies must be also non-regular. This contradicts to our initial assumption. Therefore the set of regular economies P_R is open. ■

[9] Here, each $p_i I_n$ is $n \times n$ matrix:

$$p_i I_n = \begin{bmatrix} p_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & p_i \end{bmatrix}.$$