Discussion Paper Series A No.716

Computational Methods and Classical-Marxian Economics

Jonathan F. Cogliano,
(Economics Department, University of Massachusetts Boston),
Roberto Veneziani,
(School of Economics and Finance, Queen Mary University of London)
and
Naoki Yoshihara
(Department of Economics, University of Massachusetts Amherst,
Institute of Economic Research, Hitotsubashi University and
School of Management, Kochi University of Technology)

October 2020

Institute of Economic Research
Hitotsubashi University
Kunitachi, Tokyo, 186-8603 Japan

Computational Methods and Classical-Marxian Economics*

Jonathan F. Cogliano†, Roberto Veneziani‡, Naoki Yoshihara§

September 15, 2020

Abstract

This article surveys computational approaches to classical-Marxian economics. These approaches include a range of techniques – such as numerical simulations, agent-based models, and Monte Carlo methods – and cover many areas within the classical-Marxian tradition. We focus on three major themes in classical-Marxian economics, namely price and value theory; inequality, exploitation, and classes; and technical change, profitability, growth and cycles. We show that computational methods are particularly well-suited to capture certain key elements of the vision of the classical-Marxian approach and can be fruitfully used to make significant progress in the study of classical-Marxian topics.

Keywords: Computational Methods; Agent-Based Models; Classical Economists; Marx.

JEL Classification Codes: C63 (Computational Techniques, Simulation Modeling); B51 (Socialist, Marxian, Sraffian); B41 (Economic Methodology).

*We would like to thank Peter Flaschel, Duncan Foley, Heinz Kurz, David Laibman, Peter Matthews, Bertram Schefold, Mark Setterfield, and Lefteris Tsoulfidis for helpful comments and suggestions. The usual disclaimer applies.

†(Corresponding author) Economics Department, University of Massachusetts Boston, Wheatley Hall, 5th floor, 100 William T. Morrissey Blvd., Boston, MA 02125, U.S. (jonathan.cogliano@umb.edu)

‡School of Economics and Finance, Queen Mary University of London, Mile End Road, London E1 4NS, U.K. (r.veneziani@qmul.ac.uk)

§Department of Economics, University of Massachusetts Amherst, Crotty Hall, 412 North Pleasant Street, Amherst, MA 01002, U.S.; The Institute of Economic Research, Hitotsubashi University, Naka 2-1, Kunitachi, Tokyo 186-8603, Japan; and School of Management, Kochi University of Technology, 2-22, Eikokuji-Chou, Kochi, Kochi, 780-8515, Japan. (nyoshihara@econs.umass.edu)

1 Introduction

Computational economics focuses on mathematical models analysed using a computer. Traditionally, economics has relied on computation for "empirical analysis of data and computing equilibria of conventional models" (Judd 2006, p.883). Computational economics builds on this tradition to study both analytical and statistical economic questions that may not be easily addressed with standard tools of analysis.

Computational models can be stylised and simple or highly detailed and complex. When they incorporate a microeconomic structure, simulations often begin by specifying a set of heterogeneous agents and behavioural rules, which can range from simple actions taken by agents in isolation, to complicated decisions that require interaction with other agents and a complex informational basis. In the latter case, the simulation exercise takes on the properties of full *agent-based models* (ABMs).

In the last few decades computational economics has seen tremendous advances. Yet, simulation-based approaches and ABMs are not at the forefront of mainstream economics. This is perhaps unsurprising. The core of mainstream economics typically focuses on the equilibrium states of models with forward-looking optimising agents in which macroeconomic outcomes can be inferred simply by aggregating individual behaviour (e.g. thanks to the representative agent assumption).[1] In these models, markets appear as a neutral institutional mechanism that coordinates economic activity and tends to yield efficient allocations, while distributive outcomes are driven by exogenous determinants. This particular focus of the 'mainstream core' is based on a vision of the economy that, by construction, eschews the need for simulations, resorting to such approaches only when all else fails.

Simulations and ABM are not at the forefront of classical-Marxian approaches either. More precisely, while a number of contributions in the classical-Marxian tradition have adopted computational techniques to analyse macroeconomic models, and in particular macrodynamics, the computational literature exploring the core of classical-Marxian economics – in particular, price and value theory, and the theory of distribution – is relatively recent and the full potential of computational techniques is yet to be realised.

The relatively late blooming is explained by the dominance of two different methodological stances in the recent classical-Marxian literature. Following the publication of Piero Sraffa's *Production of Commodities by Means of Commodities*, and the revival of mathematical Marxian economics in the 1970s, a large number of authors have adopted standard analytical tools to explore classical theories of price, value, and distribution in order to develop an alternative theoretical framework at the same level of generality as the mainstream core.[2]

The mathematical turn in classical-Marxian theory in the 1970s has not gone unchallenged. According to critics, it shifted the focus from broader conceptual issues toward relatively minor technical details. Furthermore, some authors have argued that classical theories, particularly Marx's own approach, are not easily amenable to mathematical formalisation (Fine and Harris 1979; Fine and Saad-Filho 2010), while others have raised doubts on the use of formal methods in economics (Lawson 2003).

[1] See Kirman (1992) for a discussion of problems with the representative agent.

[2] The literature is too vast for a comprehensive list of contributions. An illustrative but far from exhaustive selection includes Okishio (1963), Morishima (1973), Roemer (1981, 1982), Kurz and Salvadori (1995).

The two broad methodological stances summarily sketched here are very different—indeed, contradictory—and yet have jointly contributed to relegating computational methods to the margins of classical-Marxian economics, either by advocating the adoption of standard, closed-form mathematical models or by insisting on the shortcomings of *all* formal models. This is surprising since—as this survey shows—computational simulations are particularly well-suited to capture certain key elements of the vision of the classical authors and Marx, and conversely, the classical-Marxian approach offers robust theoretical foundations for computational methods.

In the classical-Marxian vision, macroeconomic phenomena and individual behaviour are linked in complex and bidirectional ways. While individual decisions matter, individual preferences and beliefs are influenced by social structures and social positions and class is a relevant unit of analysis. Individual actors rely on aggregate information when making their decisions but this information is itself influenced by their decisions. Thus, the aggregate behaviour of the economy is not conceived of as the sum of individual actions, and macroeconomic outcomes are neither driven by natural laws, nor are they presumed to be efficient. The distribution of income and wealth is shaped by institutional structures, social norms, and power relations. Persistent unemployment and recurrent crises are inherent in capitalist economies, which are characterised by incessant technological innovations, yielding structural change and disequilibrium that interact with classes and distribution. Computational methods are particularly well-suited to analyse these phenomena, which often do not lend themselves to closed-form solutions, and to incorporate classical-Marxian assumptions.

The classical-Marxian approach takes a long-run evolutionary view of capitalist dynamics in which the long-run tendencies of capitalist economies are the outcome of a series of complex local and short-run dynamics. This overarching vision of classical-Marxian economics bears a strong resemblance to modern views on complex adaptive systems, or complexity science (Colander (2000), Matthews (2000), Arthur (2006), Foley (2003), Miller and Page (2007)). Complexity theory analyses "highly organized but decentralized systems composed of very large numbers of individual components" (Foley 2003, p.1). These systems conform to the following criteria:

> [P]otential to configure their component parts in an astronomically large number of ways (they are *complex*), constant change in response to environmental stimulus and their own development (they are *adaptive*), a strong tendency to achieve recognizable, stable patterns in their configuration (they are *self-organizing*), and an avoidance of stable, self-reproducing states (they are *non-equilibrium systems*) (Foley 2003, p.2).

The above description maps, in a general sense, onto the vision at the heart of classical-Marxian economics. For instance, the deep heterogeneity of agents implicit in the classical-Marxian vision treats the economy as complex, the feedback between the aggregate behaviour of the economy and individual decisions is an adaptive mechanism, and the identification of long-run tendencies as the outcome of a series of protracted local decisions and actions treats the economy as self-organising and inherently a non-equilibrium system (at least at the local level over short periods of time).

To be sure, we are *not* suggesting that computational methods are uniquely suited to model classical-Marxian economics, nor are we claiming that standard, closed-form models are fundamentally inadequate. Rather, our survey aims to show the insights gained specifically thanks to computational methods, and the role that computational models can play in theory development by potentially allowing one to derive novel, unexpected results.

The classical economists themselves used what may be considered as rudimentary computational methods. For instance, François Quesnay's *Tableau*, Marx's schemes of reproduction in volume two of *Capital*, and even his attempts to use numerical examples to solve the transformation problem in volume three of *Capital*, can be seen as early experiments in the use of numerical demonstrations and ways of framing economic questions as computational problems. These examples, obviously, do not involve the use of computers, but they can be seen as the intellectual ancestors of the literature surveyed in this paper.

This article surveys recent contributions using computational methods to model different aspects of the classical-Marxian vision. The focus is on classical-Marxian contributions in microeconomics and in particular on the core topics of price and value theory, inequality and exploitation, and technical change and profitability. While a large number of contributions contain numerical examples, we focus on papers that use computational methods as a fundamental tool of empirical or theoretical analysis.

The rest of the paper is structured as follows. Section 2 lays out the basic notation and baseline formal framework. Section 3 analyses classical-Marxian price and value theory. Section 4 deals with inequality, exploitation, and class. Section 5 discusses work on technical change and profitability. Section 6 concludes.

2 The baseline framework and notation

In this section, we specify the basic notation and baseline framework used throughout the paper, unless otherwise stated. Vectors and matrices are in boldface and for any matrix $\mathbf{X}$, $\mathbf{X}^T$ denotes its transpose. $\mathbf{1} = (1, \ldots, 1)$ is the summation vector and $\mathbf{0} = (0, \ldots, 0)$ is the null vector. For any vector $\mathbf{x}$, $\dot{\mathbf{x}}$ denotes a time derivative, g_{x_i} denotes the growth rate of the ith component, and $\widehat{\mathbf{x}}$ is the diagonal matrix formed from $\mathbf{x}$. We let $\mathbf{I} = \widehat{\mathbf{1}}$.

2.1 Technology

In the economy, there is a set $\mathcal{M} = \{1, \ldots, m\}$ of produced goods. At the beginning of each production period $t = 1, 2, \ldots$, there is a finite set, $\mathcal{P}_t$, of linear production techniques, or *activities*, $(\mathbf{B}_t, \mathbf{A}_t, \mathbf{L}_t)$, where $\mathbf{B}_t$ is the $m \times m$ matrix of produced outputs, $\mathbf{A}_t$ is the $m \times m$ matrix of produced inputs, and $\mathbf{L}_t$ is the $1 \times m$ vector of effective (or skill-adjusted) labour. A_{ijt} and L_{jt} denote, respectively, the amounts of physical input i and labour used in the jth production process. B_{ijt} denotes the amount of good i produced in production process j. The ith column of $\mathbf{A}_t$ is denoted by $\mathbf{A}_{\star it}$.

$\mathbf{x}_t$ is the $m \times 1$ vector denoting the aggregate level at which the various activities are activated. The vector of aggregate net output is $\mathbf{y}_t = (\mathbf{B}_t - \mathbf{A}_t)\mathbf{x}_t$. If every sector produces one good (no joint production), then $\mathbf{B}_t$ is diagonal and activities can be normalised such that $\mathbf{B}_t = \mathbf{I}$, and can be simply denoted as $(\mathbf{A}_t, \mathbf{L}_t)$.

The set $\mathcal{P}_t$ contains the blueprints that can be used at t to produce the m goods. We assume that all techniques in $\mathcal{P}_t$ are known to, and can be activated by all agents, and they are all such that the economy can produce a nonnegative net output vector. If technical change takes place between $t-1$ and t, then $\mathcal{P}_t \neq \mathcal{P}_{t-1}$. If $(\mathbf{B}^*_t, \mathbf{A}^*_t, \mathbf{L}^*_t) \in \mathcal{P}_t \backslash \mathcal{P}_{t-1}$, then $(\mathbf{B}^*_t, \mathbf{A}^*_t, \mathbf{L}^*_t)$ is an *innovation*.

2.2 Agents

In the economy, there are a set $\mathcal{F} = \{1, ..., N_f\}$ of firms and a set $\mathcal{N} = \{1, ..., N\}$ of agents. In every t, each agent $\nu \in \mathcal{N}$ owns an $m \times 1$ vector of capital endowments $\boldsymbol{\omega}^{\nu}_{t-1}$ and has an endowment of labour *time* normalised to one. Agent ν's skill level is $\sigma^{\nu} > 0$, and therefore her endowment of *effective labour* is $l^{\nu} = \sigma^{\nu}$. An agent $\nu \in \mathcal{N}$ endowed with $\left(l^{\nu}, \boldsymbol{\omega}^{\nu}_{t-1}\right)$ can sell a quantity z^{ν}_t of her labour power and/or she can hire others to operate a technique $(\mathbf{B}_t, \mathbf{A}_t, \mathbf{L}_t) \in \mathcal{P}_t$ at the level $\mathbf{x}^{\nu}_t$. Agent ν can allocate her income to consumption, denoted by the $m \times 1$ vector $\mathbf{c}^{\nu}_t$, and savings, given by the $m \times 1$ vector $\mathbf{s}^{\nu}_t = \boldsymbol{\omega}^{\nu}_t - \boldsymbol{\omega}^{\nu}_{t-1}$.

2.3 Prices and values

In every period t, let $\mathbf{p}_t$ be the $1 \times m$ vector of market prices and let $\mathbf{w}_t$ be the $m \times 1$ vector of nominal wage rates (per unit of effective labour) paid in the m production processes. Supposing wages to be paid at the end of t, if a production technique $(\mathbf{B}_t, \mathbf{A}_t, \mathbf{L}_t) \in \mathcal{P}_t$ is activated, then in sector j it yields a profit rate $r_{jt} = \frac{\mathbf{p}_t \mathbf{B}_{.jt} - \mathbf{p}_{t-1} \mathbf{A}_{.jt} - w_{jt} \mathbf{L}_{jt}}{\mathbf{p}_{t-1} A_{.jt}}$ if historical pricing is considered and $r_{jt} = \frac{\mathbf{p}_t \mathbf{B}_{.jt} - \mathbf{p}_t \mathbf{A}_{.jt} - w_{jt} \mathbf{L}_{jt}}{\mathbf{p}_t A_{.jt}}$ if inputs are valued at replacement cost. Letting $\mathbf{W}_t = \widehat{\mathbf{w}}_t$, $\mathbf{r}_t = (r_{1t}, \ldots, r_{mt})$, and $\mathbf{R}_t = \widehat{\mathbf{r}}_t$, we can write, respectively:

$$\mathbf{p}_t \mathbf{B}_t = \mathbf{p}_{t-1} \mathbf{A}_t (\mathbf{I} + \mathbf{R}_t) + \mathbf{L}_t \mathbf{W}_t \tag{1}$$

and

$$\mathbf{p}_t \mathbf{B}_t = \mathbf{p}_t \mathbf{A}_t (\mathbf{I} + \mathbf{R}_t) + \mathbf{L}_t \mathbf{W}_t. \tag{2}$$

If competitive forces prevail, and both wage and profit rates are equalised across sectors, then one obtains the standard classical production prices

$$\mathbf{p}^e_t \mathbf{B}_t = (1 + r^e_t) \mathbf{p}^e_t \mathbf{A}_t + w^e_t \mathbf{L}_t. \tag{3}$$

If the $m \times 1$ vector $\mathbf{n}$ is the numéraire, then $\mathbf{p}^e_t \mathbf{n} = 1$ and from equation (3) it is possible to obtain the *wage-profit curve* corresponding to technique $(\mathbf{B}_t, \mathbf{A}_t, \mathbf{L}_t) \in \mathcal{P}_t$

$$w(r^e) = \frac{1}{\mathbf{L}_t \left[\mathbf{B}_t - (1 + r^e) \mathbf{A}_t\right]^{-1} \mathbf{n}}. \tag{4}$$

Equation (4) derives an inverse relation between the equilibrium profit rate and the wage rate, and there is one wage-profit curve for any production technique in $\mathcal{P}_t$. The outer envelope of all such curves is the *wage profit frontier* associated with $\mathcal{P}_t$. The wage-profit frontier is an important analytical and theoretical construct as it allows one to analyse optimal choice of technique: at any given wage rate, capitalists will choose a technique $(\mathbf{B}_t, \mathbf{A}_t, \mathbf{L}_t) \in \mathcal{P}_t$ that yields the maximum profit rate, i.e. a technique on the frontier.

Finally, in single-product systems, the vector of labour values (employment multipliers) associated with $(\mathbf{A}_t, \mathbf{L}_t) \in \mathcal{P}_t$ is defined as follows

$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_t \mathbf{A}_t + \mathbf{L}_t \Rightarrow \boldsymbol{\lambda}_t = \mathbf{L}_t(\mathbf{I} - \mathbf{A}_t)^{-1}. \quad (5)$$

3 Price and value theory

Price and value theory is one of the key areas of difference between the classical economists, including Marx, and neoclassical economics. In neoclassical theory, equilibrium prices ensure market clearing and are ultimately determined by preferences, technology, and endowments. In the classical-Marxian approach, production prices are ultimately determined by technological conditions on the one hand, and by social and institutional factors which set wages, on the other hand. Equilibrium is reached when the sectoral allocation of capital is determined and capitalists have no incentive to transfer their capital. In the specific case of long-period positions, this corresponds to the equalisation of profit rates across sectors (Kurz and Salvadori 1995).

In other words, classical-Marxian theory adopts a different notion of equilibrium and focuses on a different concept of prices. Thus, while some of the questions that arise are similar to those investigated in the mainstream – such as existence, uniqueness, and stability of equilibrium prices – others are rather different. They regard, for example, the determination of the distributive variables, or the relation between production prices (3) and labour values (5), as in the debate on the famous "transformation problem".[3]

Some answers to these questions have been provided by standard, analytical means. Below, we review recent contributions that have tackled them using computational methods.

3.1 Capital theory

One of the key contributions of economists working in the classical-Marxian tradition is the analysis of a number of phenomena in capital theory which cast doubts on some central neoclassical propositions. Sraffa (1960) showed that a technique that is the most profitable at a given profit rate, r^e, may be abandoned when r^e decreases, only to be chosen again after r^e *decreases further*. This phenomenon, known as *reswitching of techniques* runs against the neoclassical postulate that techniques with lower intensities of capital become eligible at higher rates of profit. *Reverse capital deepening* may also occur: contrary to standard neoclassical theory, as the profit rate *rises* relative to the wage rate, the technique adopted by capitalists may become *more* capital-intensive.

These findings sparked the so-called *capital controversy* between the Sraffians and the neoclassicals.[4] The debate conclusively proved that there is no way of ruling out reswitching and reverse capital deepening in general, multisectoral production economies. But establishing the *possible* occurrence of such phenomena left the issue of their *likelihood* open. And ever since the capital controversy started, one standard rebuttal of the Sraffian criticisms relies precisely on considering the paradoxes empirically as flukes.

[3] For a comprehensive discussion, see Desai (1988) and Mohun and Veneziani (2017).

[4] For a survey focusing especially on more recent contributions, see Fratini (2019).

But are they? It is difficult to answer this question. Reswitching and reverse capital deepening occur when the wage-profit curves (4) of a given set of technologies $\mathcal{P}_t$, and the associated wage-profit frontier have certain properties. However, it is hard to derive general theoretical results, or even to define a proper probability measure, because wage-profit curves are rather complicated expressions depending on a large number of parameters. Empirically, the set $\mathcal{P}_t$, and therefore the wage-profit frontier, are not observable.

A small but growing literature has started to circumvent these problems and address the issue of the likelihood of reswitching and reverse capital deepening by means of computational methods.[5] In a pioneering contribution, D'Ippolito (1989) uses Monte Carlo simulations to explore the properties of wage-profit curves emerging from a randomly generated $\mathcal{P}_t$. He analyses economies with up to 30 sectors and finds that the percentage of such techniques displaying reverse capital deepening at various profit rates is positive but generally low (less than 1% for $r = .1$), and seems unrelated to the number of sectors.

Although D'Ippolito (1989) derives some interesting insights, it does not provide an explicit formula for the probability of reverse capital deepening and does not discuss reswitching. Mainwaring and Steedman (2000) tackle both issues. In order to derive an explicit formula, they analyse a two-sector economy with single-output techniques. They consider activities $(\mathbf{A}_t, \mathbf{L}_t)$,$(\mathbf{A}'_t, \mathbf{L}'_t)$ which differ in only one sector say, sector j,[6] and derive an analytical expression for the probability $\psi(r^*)$ that, if the wage profit curves of the two techniques intersect at r^*, there is another switch point at a different r. For the sake of illustration, suppose $j = 1$. This probability is the ratio of the 'size' of two sets: one is the set $\mathcal{S}$ of all vectors $(\mathbf{A}'_{\star 2t}, L'_{2t})$ such that $((\mathbf{A}_{\star 1t}, \mathbf{A}'_{\star 2t}), (L_{1t}, L'_{2t}))$,$((\mathbf{A}'_{\star 1t}, \mathbf{A}'_{\star 2t}), (L'_{1t}, L'_{2t}))$ yield the same vector $\mathbf{p}^e_t$ at $r_t = r^*$. The other is a subset $\mathcal{T} \subseteq \mathcal{S}$ which collects all vectors $(\mathbf{A}'_{\star jt}, L'_{jt}) \in \mathcal{S}$ such that $((\mathbf{A}_{\star 1t}, \mathbf{A}'_{\star 2t}), (L_{1t}, L'_{2t}))$,$((\mathbf{A}'_{\star 1t}, \mathbf{A}'_{\star 2t}), (L'_{1t}, L'_{2t}))$ yield again the same vector of production prices at $r_t > r^*$.

The probability of reswitching thus depends on $(\mathbf{A}_t, \mathbf{L}_t)$ and r^*. Hence, even in a two-sector economy, the expression for $\psi(r^*)$ is exceedingly complicated and no insights can be derived analytically. Therefore, starting from a baseline technique $(\mathbf{A}_t, \mathbf{L}_t)$, Mainwaring and Steedman (2000) change the coefficients of $\mathbf{A}_t$, and $\mathbf{L}_t$, in order to derive computationally the various functions $\psi(r)$ and see how their shape changes depending – among other things – on the concavity of the wage-profit curve. They find that in general $\psi(r)$ is first increasing and then decreasing in r, dropping to 0 when r reaches its maximum value. Further, for realistic values of a_{22}, "intermediate concavity $w(r)$ functions are associated with techniques for which $\psi(r)$ averages approximately four-five percent" (Mainwaring and Steedman 2000, p.344). Sufficiently high values of a_{22} yield significantly higher values of $\psi(r)$. "If, however, it is assumed that techniques are distributed uniformly over all coefficient combinations consistent with a given productivity level ... the probability of finding techniques generating very high $\psi(r)$ is, itself, very low" (Mainwaring and Steedman 2000, p.344). If the domain is restricted to values of r such that $w > 0$ then the probability of reswitching at any two rates of profit decreases further, dropping below one percent.[7]

[5]Earlier contributions used analytical tools and numerical examples to show that phenomena such as reswitching and reverse capital deepening are not a fluke, and may occur with positive – albeit low – probability; see, for example, Eltis (1973); Schefold (1976).

[6]Formally, for an activity $(\mathbf{A}, \mathbf{L}) \in \mathcal{P}_t$, sector j is represented by $(\mathbf{A}_{\star j}, L_j)$.

[7]Similar results are obtained by D'Ippolito (1987) in his computational analysis of the so-called Samuelson-

While this analysis is innovative, there are two problems. First, as Salvadori (2000) has shown, the probability of reswitching $\psi(r)$ depends on the description of the techniques and therefore the results are not robust to alternative specifications. Second, it provides an abstract measure of probability over the entire space of conceivable activities, regardless of their being actually implementable and thus it does not answer the question of the empirical likelihood of reswitching.

Han and Schefold (2006) adopt a different, empirically-oriented approach. They start off from thirty-two 36-sectoral input-output tables and the corresponding vectors of labour inputs from the OECD database (for nine countries from the period 1986 – 1990). Their main methodological innovation is to assume that a given process used in a country c at a certain date t – say, $(\mathbf{A}^c_{\star jt}, L^c_{jt})$ – is also available in principle in a different country c' at a different point in time t'. Therefore they implement pairwise comparisons of the input-output datasets - for example, they compare Germany 1990 and Canada 1990, - which gives two possible processes for each of the $m = 36$ industries and thus 2^{36} possible economy-wide techniques, or mixed input-output matrices, on the basis of which the envelopes of the wage-profit curves can be calculated.

This procedure yields a very large amount of data, because there are $\binom{32}{2} = 496$ pairs of mixed input-output matrices. From each of these pairs an envelope can be derived, and the switchpoints on those envelopes and the frequency of reswitching and reverse capital deepening can be analysed. Han and Schefold (2006, pp.758-761) find that most of the 2^{36} "wage curves . . . never appear on a relevant part of the envelope, yet substitution exists on all envelopes; on average, about ten wage curves constitute the envelope which we calculated. There is one case of reswitching and nearly 4% of cases with reverse capital deepening or reverse substitution of labour".

Formally, the approach pioneered by Han and Schefold (2006) has the advantage of rigour because the envelopes emerging from each pair of input-output tables are calculated by means of linear programming. Nonetheless, conceptually, if it is true that any technique $(\mathbf{A}^c_{\star jt}, L^c_{jt})$ should be considered to be available at any other time t' and location c', then it would be more appropriate to compute a single envelope for all the techniques given by 32 input-output tables taken together.[8] The problem, of course, is that "While the mathematical notion of an envelope is conceptually straightforward, the brute force algorithm associated with the computation of such an envelope that takes into account every single point is computationally infeasible" (Zambelli, Fredholm, and Venkatachalam 2017, p.39).

Zambelli et al. (2017) have developed an algorithm that, while theoretically less rigorous than linear programming, provides a satisfactory approximation in practice and drastically reduces computational intensity. The algorithm exploits two key properties of wage-profit frontiers: "the set of methods at the frontier . . . will not change as the numéraire changes, but will change as a function of the profit rate. Furthermore, it is also known that two adjacent

Hicks-Spaventa model. See, however, Petri (2011) for a critique of D'Ippolito (1987) and a computational reconsideration of his results leading to much higher probabilities of reverse capital deepening.

[8] The interpretation of the activities $(\mathbf{A}, \mathbf{L})$ arguably relies on the assumption that only circulating capital is used in production. If fixed capital is considered, then as Flaschel, Franke, and Veneziani (2012, p.455) have argued, it is questionable to combine the intermediate inputs of different sectors from different years "into one single technique, and its hypothetical wage-profit curve, without recognizing that these inputs may be specific to the capital that is tied up in their production process".

set of methods [on the frontier] will differ only in one method" (Zambelli 2018, p.390). As a result "for a given profit rate we have a unique determination of the combination of the frontier set of methods" (Zambelli 2018, p.390), and this property can be used to simplify the computation of the wage-profit frontier considerably.

Zambelli (2018) considers the 31-sectoral input-output systems of 30 OECD countries, during 1995-2011, and constructs 17 yearly wage-profit envelopes starting from 31^{30} wage curves, considering 32 different numéraires. He then computes the wage-profit frontier deriving from "the outer envelope of all the possible combinations of methods observed during the period 1995-2011 ... [computed] based on the 100 curves that dominate all the other $31^{30\times 17} (\approx 3.6 \times 10^{760})$ possible wage-profit curves" (Zambelli 2018, p.402). The findings are quite unambiguous: although there is virtually no evidence of reswitching, so-called 'Wicksell effects' are pervasive as the capital-labour ratio per unit of output varies *positively* with the rate of profit, both sectorally and in the aggregate, in a large number of instances.[9]

Overall, these studies confirm the main findings of the capital controversy: "observed cases of reswitching and reverse capital deepening are more than flukes, hence they seem to suffice to undermine the neoclassical production and distribution theory, both in a stochastical and falsificatorial sense" (Han and Schefold 2006, p.758). Furthermore, the existence of Wicksell effects raises doubts on the use of neoclassical aggregate production functions.

Nonetheless, these findings do suggest that the likelihood of reswitching and reverse capital deepening is low.[10] This result is relevant beyond the capital controversy because the shape of wage-profit curves, and of the wage-profit frontier, is linked to another central topic in classical-Marxian theory, namely the relation between labour values (5) and production prices (3). For, it is well-known that if wage-profit curves are linear, then labour values are good predictors for production prices.

Although the contributions reviewed above do not prove that empirical wage-profit curves are straight lines, they do suggest that their curvature is – loosely speaking – less pronounced than one may surmise theoretically. In this sense, they confirm a key finding of the empirical literature on the so-called "transformation problem". Theoretically, unless all industries adopt techniques with the same capital intensity relative to labour, labour values and production prices will differ significantly. Empirically, however, a strong correlation between production prices and labour values has been identified (see, among the many others, Flaschel (2010) and Cockshott and Cottrell (1997)). We discuss this in the next subsections.

3.2 Classical dynamics

In the previous subsection, we have analysed contributions focusing on some static, or equilibrium properties of classical price theory. Yet, one key area of classical-Marxian economics that has seen fruitful application of computational methods concerns the analysis of the dynamics of capitalist competition. Two dynamic mechanisms are central in classical-Marxian theorising, and are usually at work simultaneously. One postulates that market adjustment processes drive changes in prices caused by the 'law of excess demand'. According to the

[9] Zambelli et al. (2017) use the algorithm to derive some interesting measures of productivity and technical progress in a classical vein.

[10] For a theoretical explanation of these findings see Schefold (2016). For a counterpoint see Petri (2011) and Dvoskin and Petri (2017).

other, the allocation of capital, and thus output quantities, changes in response to differences in sectoral profit rates. Analysing the dynamics of high-dimensional systems, and the stability of long-period positions is in principle rather complex: closed-form solutions are not easily available and only partial insights can be gained analytically.

Computational methods are particularly suited to analyse the gravitational movement of prices around long-period positions because, at the broad conceptual level, this is the area of classical-Marxian economics that bears the closest resemblance to modern views on complex adaptive systems. In the classical-Marxian view, dynamic processes are characterised by both *local instability*, as market prices are constantly moving toward and away from long-period positions; and *global stability* since the whole system does not unravel.

In this perspective, production prices only emerge as centers of gravity for the ongoing movement of market prices over a sufficiently long timespan. This type of oscillatory behaviour and turbulent process of competition also characterises the equalisation of profit rates and wages, and is seen as the basis of the labour theory of value (LTV) originally derived by Smith (2000) in his famous beaver-deer example. As Foley(2003, p.4) aptly puts it, the classical vision

> does not insist that each and every component of the economy achieve its own equilibrium as part of a larger master equilibrium of the system as a whole. In fact, it is precisely from the *disequilibrium* behavior of individual households and firms that the classical vision of competition sees the orderliness of gravitation of market prices around natural prices as arising. In the language of complex systems theory, classical gravitation is a self-organized outcome of the competitive economic system.

This section examines contributions that analyse the processes of open-ended gravitation of prices around a long-period equilibrium. The central aim of these contributions is “to construct a decentralised model of competition without an auctioneer, based on the classical perspective” (Duménil and Lévy 1987, p.160). As shown below, computational methods are particularly well suited for the task.

3.2.1 Cross-dual dynamics

The first approach to explore the gravitation of market prices around long-period positions is the literature on “cross-dual” dynamics.[11] The central idea of this literature is that the time-paths of quantities, $\mathbf{x}_t$, and prices, $\mathbf{p}_t$, are functions of each other, of profit rates $\mathbf{R}_t$, of the input-output structure of the economy ($\mathbf{B}_t$, $\mathbf{A}_t$), and of workers' consumption $\mathbf{b}_t\mathbf{L}_t$ where $\mathbf{b}_t$ is a $m \times 1$ vector of subsistence requirements. In the most general form, assuming wages to be paid ex ante, the cross-dual dynamics can be formalised as follows:

$$\dot{\mathbf{x}}_t = f\left(\mathbf{p}_t, \mathbf{R}_t, \mathbf{B}_t, \tilde{\mathbf{A}}_t\right), \tag{6}$$

$$\dot{\mathbf{p}}_t = g\left(\mathbf{x}_t, \mathbf{R}_t, \mathbf{B}_t, \tilde{\mathbf{A}}_t\right), \tag{7}$$

[11] The literature is vast here too. A representative but far from comprehensive sample includes: Morishima (1976); Flaschel and Semmler (1986b, 1987); Duménil and Lévy (1990); Flaschel (1990); Semmler (1990); Duménil and Lévy (1991); Flaschel (2010); Cogliano, Flaschel, Franke, Fröhlich, and Veneziani (2018).

where $\tilde{\mathbf{A}}_t = \mathbf{A}_t + \mathbf{b}_t\mathbf{L}_t$, and f and g are general functions.

Equations (6)-(7) incorporate the following intuition (Flaschel and Semmler 1987). Capitalists respond to profitability when deciding how to allocate their capital: they move capital toward (away from) sectors with profits above (below) the equilibrium rate r^e. The decisions to shift capital alter the composition of $\mathbf{x}_t$, thus changing $\mathbf{p}_t$ and also affecting sectoral profit rates, which in turn prompt further capital movements. These capital movements across sectors cause feedback loops between $\mathbf{x}_t$ and $\mathbf{p}_t$, inducing cyclical behaviour. Depending on the properties of f and g, the economy may display explosive cycles; a limit cycle; or damped cycles with price-quantity dynamics eventually converging to a steady state.

In its simplest form, the cross-dual dynamics model can be expressed as follows

$$\dot{\mathbf{x}}_t = \mathbf{\Phi^1}\widehat{\mathbf{x}}_t \left(\mathbf{B} - (1+r^e)\tilde{\mathbf{A}}\right)^T \mathbf{p}_t^T, \tag{8}$$

$$\dot{\mathbf{p}}_t = -\mathbf{\Phi^2}\widehat{\mathbf{p}}_t \left(\mathbf{B} - (1+r^e)\tilde{\mathbf{A}}\right) \mathbf{x}_t, \tag{9}$$

where $\mathbf{\Phi^1}, \mathbf{\Phi^2}$ are diagonal matrices of reaction coefficients. Equation (8) states that capitalists move their capital to sectors which obtain profits in excess of the equilibrium profit rate, thus increasing output in those sectors. Equation (9) incorporates a standard market adjustment mechanism: if aggregate supply of a good is higher (lower) than aggregate demand, then its price decreases (increases).[12]

By means of Lyapunov functions, Flaschel and Semmler (1987, 1992) prove that the cross dual dynamics formulated in equations (8)-(9) is stable and gives rise to purely oscillatory behaviour of prices and quantities. Further, if one assumes in equation (8) that capitalists allocate capital also taking into account the *rate of change* of profit rates – one instance of a set of similar mechanisms known as *derivative controls* – then the system generates globally converging properties (Flaschel and Semmler 1987).

The cross-dual dynamics system can be extended to analyse product and process innovation (and extinction) by allowing the matrices $\mathbf{B}, \tilde{\mathbf{A}}$ to be rectangular and time-dependent (Flaschel and Semmler 1987, 1992; Cogliano et al. 2018), and to incorporate Keynesian output adjustment processes (Flaschel and Semmler 1988; Flaschel 2010), and various stability results can be derived analytically.

These results, however, have two important limitations. First, they make "use of the equilibrium rate of profit [r^e] instead of . . . the (perhaps) more convincing benchmark [$1+r = 1+r(\mathbf{x}, \mathbf{p}) = \frac{\mathbf{pBx}}{\mathbf{p}\tilde{\mathbf{A}}\mathbf{x}}$] - i.e., the average rate of profit - for the definition of extra profits (or losses) and the separation of expanding from shrinking sectors" (Flaschel and Semmler 1992, p.201). More generally, the results rely on the linearity of equations (8)-(9), and the assumption of a given profit rate which is equal to the growth rate of the economy, g. Second, stability often depends on restrictive assumptions, especially on the reaction parameters in $\mathbf{\Phi^1}, \mathbf{\Phi^2}$.

By using computer simulations, however, one can show that the stability results of the cross dual dynamic system remain valid even if r^e is replaced by $r_t = \frac{\mathbf{p}_t\mathbf{B}\mathbf{x}_t}{\mathbf{p}_t\mathbf{A}\mathbf{x}_t} - 1$ (Flaschel

[12] Aggregate demand is proxied by $(1+r^e)\tilde{\mathbf{A}}\mathbf{x}$, where $\tilde{\mathbf{A}}\mathbf{x}$ represents the workers' demand for consumption goods plus the capitalists' replacement investment. The implicit assumption is that aggregate demand grows uniformly at the equilibrium rate $g = r^e$.

and Semmler 1986a,b), and if one considers process and product innovation, provided output does not react too strongly to changes in the profit rate (Flaschel and Semmler 1992).

Let $1/(1+r^*)$ be the maximum eigenvalue of $\tilde{\mathbf{A}}$. Flaschel and Semmler (1988) and Semmler (1990) prove, analytically, the stability of mixed classical-Marxian and Keynesian dynamics only under relatively stringent assumptions on the adjustment speeds in the classical-Marxian dynamics. Yet, simulation results show that similar stability properties hold under "less stringent restrictions on adjustment speeds in the limit case $[r^e = g = r^*]$ and in the case of a time dependent rate of profit for the classical cross–dual, the Keynesian dual dynamics and the composite dynamics" (Flaschel 2010, p.412). Stability does not obtain throughout, however, as numerical counterexamples can be generated of unstable systems.

Semmler (1990) uses computational methods in order to explore the set of unstable mixed classical-Marxian and Keynesian dynamic systems. Focusing on randomly generated matrices, he shows that "for the case $[g = r^e < r^*]$, . . . unstable matrices are not easily found. There are, however, unstable matrices for $[g < r^e < r^*]$" (Semmler 1990, pp.206-7). Moreover, Semmler (1990) adopts eigenvalue computations and numerical simulations to prove that even the trajectories of the dynamics of unstable systems (stemming from either unstable matrices or high reaction coefficients) can be stabilised by means of plausible economic mechanisms acting as derivative controls, giving rise to stable or bounded fluctuations. A derivative control on the rate of change of the profit rate is, however, more effective than derivative controls on the excess demand functions.

As complex as the dynamic system (6)-(7) and its variants may be, they are reduced form models with a relatively terse description of the economy. In a series of contributions, Duménil and Lévy (1987, 1993) have laid out an ambitious general disequilibrium model of classical-Marxian cross-dual dynamics. For, "The classical competitive process is a decentralised dynamic procedure realised in disequilibrium (in particular, markets do not necessarily clear)" (Duménil and Lévy 1987, p.133).

Duménil and Lévy (1987) consider a linear economy with $m = 2$, one firm per sector, and one capitalist. In every period t exchange takes place after production, and aggregate supply is the sum of new outputs and inherited inventories. The real wage is fixed at the subsistence level, and so $w_t = \mathbf{p}_t\mathbf{b}$, and wages are paid ex ante. The capitalist receives a fraction a of the profits which is spent on consumption goods in fixed proportions given by a vector $\mathbf{d}^k$. A fraction $1 - a$ of profits is retained and finances capital accumulation.

Before exchange takes place, capital is allocated to different sectors according to their profitability, and all capital is used in production. After exchange, firms discover the new level of inventories – equal to aggregate supply minus aggregate demand – and make their pricing decisions for the following period $t + 1$ based on inventories: an increase in the inventory of a good leads to a reduction in its price, and vice versa.

In this economy, "An equilibrium exists, in which prices are equal to prices of production, and the proportions of output correspond to homothetical growth" (Duménil and Lévy 1987, p.148). The model is, however, too complex to obtain analytical results concerning stability. Using computational methods, Duménil and Lévy (1987) prove that stability obtains provided the initial values of prices, outputs, and inventories are not too far from the equilibrium values, and both the reaction of prices to stockpiling, and the sensitivity of capital movements to profit rate differentials across sectors are neither too low, nor too high. Similar results are obtained when the model is extended to include rationing (in production and consumption), output adjustment to variation of inventories, and a third sector. If two heterogeneous capitalists are considered, then again stability obtains but the capitalist with the lowest accumulation rate eventually disappears.

Similarly, if one allows for several firms producing the same commodity using different techniques and selling their output at different prices, then the economy converges to the long-period position associated with the dominating technology and firms adopting inferior technologies eventually disappear. In this case, Duménil and Lévy (1987) use computational methods to trace the evolution of the system during the disequilibrium phase: in an initial stage, prices are equalised in every sector, then when the law of one price is nearly established, competition drives the process of equalisation of profit rates across sectors, which in turn leads to the elimination of any dominated technique (and the firm using it). Only in the last stage does the economy reach (asymptotically) its long period position.

Duménil and Lévy (1993, Ch.8) significantly extend the general disequilibrium framework to include fixed capital, variable capacity utilisation, and a Keynesian consumption function in an economy with three sectors (producing a capital good, an intermediate good and a consumption good), three firms, and three capitalists which own different shares of the firms. Firms have (heterogeneous) target rates of capacity utilisation and of inventories. Their output decisions depend on the differences between the actual and target rates, and are aimed at closing any gaps between the two. Capitalists invest the same proportion of profits but have different reaction functions to profit rate differentials across sectors.

Duménil and Lévy (1993, pp.142-3) prove that a classical long-term equilibrium exists with profit rate equalisation, balanced growth, and given proportions of capital among firms. Further, the equilibrium rates of capacity utilisation and of inventories are equal to their target values.[13] However, the high dimensionality of the model makes it difficult to draw any conclusions on its dynamic properties analytically.[14] Using computational methods, Duménil and Lévy (1993) show that, for a large constellation of parameters, the classical equilibrium is stable and this conclusion is robust to a number of extensions, including different definitions of the profit rate (with or without circulating capital and inventories), nonlinear reaction functions, the inclusion of capacity utilisation in pricing decisions, alternative market pricing rules, the addition of bank loans to finance investment, a variable wage rate, some limitations to capital mobility, and the inclusion of expected profit rates (with adaptive expectations) in capitalists' evaluation of profitability differentials.[15] In general, "Stability can always be obtained, under specific conditions on reaction coefficients" (Duménil and Lévy 1993, p.141).

While the general disequilibrium economy is a model of convergence, Duménil and Lévy (1993, Sect. 8.3) extend it in order to analyse the classical-Marxian gravitation process conceived of as "a 'stationary' (but agitated) regime in which centrifugal forces are matched by centripetal convergence forces" (Duménil and Lévy 1993, p.147). Therefore they analyse computationally a version of the model in which demand is randomly shocked in the three

[13]Indeed, there is a *continuum* of equilibria since the shares of ownership can take any value when equilibrium is reached, depending on their initial values and of the exact trajectory to the equilibrium.

[14]Duménil and Lévy (1990) and Duménil and Lévy (1993, Chp.7) prove analytically the existence and stability of the classical long-period equilibrium in a simplified version of the model with two goods, two identical firms, and one capitalist.

[15]In other words, capitalists adjust their expectations of future rates of profit based on recent past experiences and events, including past values of the rates of profit themselves.

industries and show that the three profit rates persistently gravitate around their common equilibrium value (Duménil and Lévy 1993, p.147).

By explicitly modelling the competitive processes that characterise capitalist economies, the literature on cross-dual dynamics represents a milestone in classical-Marxian economics.[16] Nonetheless, the analysis of individual behaviour is at best rudimentary: it is based either on reduced form models or, when individual agents are explicitly modelled, on rather drastic simplifying assumptions limiting the number of agents or the degree of heterogeneity. Furthermore, the treatment of gravitation processes as purely driven by exogenous shocks in Duménil and Lévy (1993) is arguably unsatisfactory and far from the Classics' own theorisation. ABMs allow one to model the convergence and gravitation processes in economies with a large number of heterogeneous agents whose production, exchange, and consumption decisions can be explicitly considered. To these models we turn next.

3.2.2 Agent-Based approaches to the labour theory of value (LTV)

ABMs adopt a complexity-based approach that conceives of economies as "dynamic systems of interacting agents" (Tesfatsion 2006, p.835) where macro-level outcomes are emergent properties that cannot be directly inferred from micro-level behaviour. In its simplest form, an ABM is composed of an appropriate taxonomy of heterogeneous agents, a scale that fits the phenomena under examination, and a set of rules that govern the actions and interactions of agents (LeBaron and Tesfatsion 2008). Together, the set(s) of agents and the behavioral rules constitute what can be called the 'micro-specification' of the model, and the scale of the model sets the scope for the types of 'macro' phenomena examined. Among the benefits of ABMs, perhaps the most important is that they *generate* macroeconomic regularities from a microeconomic specification, and, given the recursive nature of ABMs, the micro and macro "co-evolve" (Epstein 2006, p.6). At each stage of the simulation, boundedly rational agents carry out some actions according to pre-specified decision rules, based on the limited information at their disposal. These actions collectively influence the macroeconomic behaviour of the system. Conversely, the resulting macrostructure accounts for much, and often all, of the information and conditions upon which agents base their subsequent decisions.

The agent-based approach is concerned with developing an account of how macroeconomic regularities – which unfold as *emergent properties* (Epstein 2006, pp.31-38) – are attained "by a decentralized system of heterogeneous autonomous agents" (Epstein 2006, p.8), and therein lies the explanatory power of ABMs. While ABMs can operate at a high level of abstraction and rest on assumptions that may not necessarily be realistic (Axtell 2010, p.36), they can capture a broad array of actual dynamic processes and empirical regularities.

The ABM approach to the LTV aims to model the gravitation of prices around a long-period equilibrium determined by labour values which emerges as a centre of gravity for the ongoing fluctuations in prices. To be specific, Wright's ABM begins from the "law of value", taken as "the process by which a simple commodity economy (i) reaches an equilibrium, in which (ii) prices correspond to labour values, and (iii) social labour is allocated to different branches of production according to social demand" (Wright 2008, p.369).

Wright considers an economy with N independent producers specialising in the production

[16] Indeed, several results can be extended to Walrasian models (Flaschel and Semmler 1987; Flaschel 1990).

of a single commodity at a time. Labour is homogeneous and the only productive input; hence, $\boldsymbol{\lambda} = \mathbf{L}$. At the beginning of the simulation, agents have zero endowments: $\boldsymbol{\omega}_0^\nu = \mathbf{0}$, all ν. Then, each agent ν generates a unit of commodity i every L_i time steps. The commodities each agent produces are stored in their endowment vector. There exists a common vector $\bar{\mathbf{c}}$ which sets the desired rate of consumption events for all individuals. Because agents rely on the production of others to fulfill their desired consumption, agent ν may experience a consumption deficit, denoted by $\mathbf{d}_t^\nu$. Setting $\mathbf{d}_0^\nu = \mathbf{0}$ all ν, each ν generates one unit of consumption deficit for each commodity i every $\frac{1}{\bar{c}_i}$ time steps. Consumption deficits create consumption desires that need to be fulfilled: during each time step, agents use their endowments to meet their consumption desires of each good both by consuming their holdings directly and to trade.

Agents own some amount of money $M_t^\nu \geq 0$ that they use to purchase goods and $\sum_{\nu \in \mathcal{N}} M_t^\nu = \bar{M}$ all t. Prior to exchanging with one another, agents randomly form subjective evaluations of commodity prices $p_{it}^\nu \in [0, M_t^\nu]$, then they meet in a marketplace. The market algorithm randomly selects an uncleared commodity i and identifies the set of agents who wish to sell i, $\mathcal{S}_t = \{\nu \in \mathcal{N} \mid \omega_{it}^\nu > d_{it}^\nu\}$ and the set of buyers $\mathcal{B}_t = \{\nu \in \mathcal{N} \mid \omega_{it}^\nu < d_{it}^\nu\}$. Individual sellers $\nu^s \in \mathcal{S}_t$ and buyers $\nu^b \in \mathcal{B}_t$ are randomly paired, then an exchange price is randomly selected from the interval $\left[p_{it}^{\nu^b}, p_{it}^{\nu^s}\right]$. If $M_t^{\nu^b}$ is sufficient, the exchange takes place. This procedure does not represent a typical Walrasian market: "transactions occur at disequilibrium prices, commodities may go unsold, and the same commodity type may exchange for many different prices in the same market period" (Wright 2008, p.373).

Agents decide whether or not to switch the sector in which they produce based on their consumption deficit. At the end of every t, each agent ν compares the Euclidean norm of $\mathbf{d}_t^\nu$, $\|\mathbf{d}_t^\nu\|$, to that of the previous period, $\|\mathbf{d}_{t-1}^\nu\|$. If $\|\mathbf{d}_t^\nu\| > \|\mathbf{d}_{t-1}^\nu\|$, then agent ν randomly selects a new sector to produce in.

The simulation is initialised by constructing $\mathbf{L}$ and $\bar{\mathbf{c}}$ such that the economy can achieve a "state of simple reproduction (where total production equals total consumption)" (Wright 2008, p.371), and setting $M_0^\nu = \frac{\bar{M}}{N}$ all ν. Over a sufficiently long period, the economy displays a dynamic consistent with the law of value. If $m = 4$, in each sector the allocation of agents is tightly distributed around the theoretical equilibrium where production equals social demand. When $m = 3$, Wright (2008) also shows that the Pearson correlation between $\boldsymbol{\lambda}$ and *average* market prices approaches one. Thus, over time, the LTV emerges from the interaction of dispersed, independent producers, and decentralised exchanges.[17]

To be sure, in equilibrium, labour values are proportional to *average* prices, but there is a whole distribution of individual prices around the market average, whose shape in general depends on the price offer rule (or rules) employed by agents.[18] Market prices regulate the social division of labour as different prices represent different transfers of social labour time between buyer and seller. "The role of the mismatch between labour embodied and labour commanded in regulating the division of labour is apparent 'on average' and is a property

[17]Perhaps surprisingly, however, competitive forces play no real role in determining market outcomes. Because offer prices are determined only by agent's availability of money, market imbalances have no (direct or indirect) effect on prices either in the long- or in the short-run. In this sense, the LTV does not emerge as the result of market competition.

[18]This is consistent with "the probabilistic approach to political economy initiated by Farjoun and Machover (1983)" (Wright 2008, p.385) discussed in section 3.3 below.

of the price distributions, not a property of individual transactions" (Wright 2008, p.385).

Cogliano's (2013) model draws on Wright (2008), but provides a more detailed description of behavioural rules and incorporates the exchange procedure developed by Albin and Foley (1992), which is built around geographically dispersed and boundedly rational agents who engage in bilateral exchanges at non-equilibrium prices given communication costs. To be specific, Cogliano (2013) models an economy with N direct producers and $m = 2$. During every period t, agents consume both commodities – in fixed proportions – but produce only one of them. Labour is the only input and the output of each agent during one time step is given by $x_i = 1/L_i$, with $L_i \in (0, 1]$ for $i = 1, 2$.

Unlike in Wright (2008), $\boldsymbol{\omega}_0^\nu > \mathbf{0}$ all ν, there is no money, and agents barter in order to obtain the good they do not produce. Each agent's willingness to trade is given by their marginal rate of substitution derived from the (common) Cobb-Douglas function $u(\mathbf{c}) = c_1 c_2$.[19] Formally, taking good 2 as the numéraire, initial offer prices for agent ν are given by:

$$p^\nu = \frac{\partial u(\boldsymbol{\omega}^\nu)/\partial c_1}{\partial u(\boldsymbol{\omega}^\nu)/\partial c_2} = \frac{\omega_2^\nu}{\omega_1^\nu}.$$

Agents in different sectors are randomly matched and exchange at the geometric mean of their offer prices. Formally, any pair of agents ν and ν' exchange a fixed quantity of the numéraire at the rate $\rho^{\nu\nu'} = (p^\nu \cdot p^{\nu'})^{1/2}$.

During each t, agents can engage in multiple exchanges and thus multiple exchange prices manifest, and exchanges continue to take place until the average offer prices across the sectors are close. Exchange prices and the allocation of commodities across agents depend on the path taken to reach equilibrium and vary over time. After exchange ends, agents consume and decide whether or not to reallocate their productive capacity across sectors in response to how they fared in exchange. Agent ν in sector i compares a moving average of their $\rho^{\nu\nu'}$ to the moving average of the exchange price for the agents in sector j. If the difference is to agent ν's disadvantage, then agent ν moves to sector j with a positive probability, and the bigger the difference the higher the probability.

Thus, market prices at t determine the allocation of producers and the supply of commodities across sectors at $t + 1$, which in turn determine the market prices at $t + 1$, and so on. Relative prices and the allocation of agents across sectors co-evolve and labour values emerge as the center of gravity of market prices. For the fleeting moments where relative price is equal to relative labour time the LTV holds. Otherwise, the movements of relative price and the allocation of producers occur within a fairly tight boundary of roughly 10% from the LTV equilibrium, even over a very long time period. As in Wright (2008), the classical-Marxian gravitation around the long-period equilibrium appears as an emergent property driven by the two-way relation between micro properties and the macro behaviour of the economic system. In this sense, ABMs aptly capture the kinds of open-ended processes of gravitation and chaotic cycles described by the Classics and Marx.

[19] Observe that the Cobb-Douglas function does *not* represent agents' preferences in a standard sense: it plays a role only in the determination of the agents' offer prices, and not in the derivation of their demand functions. In this model, agents do not use standard optimisation procedures: they adopt rules-of-thumb and consume the two commodities in fixed proportions.

3.3 Classical-Marxian econophysics

The first major contribution to classical-Marxian econophysics is Farjoun and Machover's (1983) *Laws of Chaos*, which lays the foundations of a *probabilistic* political economy. Farjoun and Machover (1983) adopt the classical-Marxian emphasis on labour as a fundamental category in economic analyses, and interpret labour magnitudes probabilistically as the best predictors of actual market monetary magnitudes (prices and profit rates), with labour values defined as in equation (5). However, they reject the standard approach to production prices based on the assumption of a uniform profit rate in equation (3). They argue that actual market variables should be analysed adopting "a probabilistic model, in which price, the rate of profit (and other economic parameters, such as capital intensity) are treated from the very beginning not as determinate numerical quantities, but as random variables, each having its own probability distribution" (Farjoun and Machover 1983, p.25).[20]

Consequently, Farjoun and Machover (1983) adopt a statistical equilibrium notion which differs from both the standard Walrasian concept and the classical-Marxian long-period approach, in that in equilibrium, a whole range of profit rates coexist and the rate of profit of each firm keeps changing all the time, and this "can only be a state of *equilibrium* in the sense that the *proportion* of capital (out of the total social capital) that yields any particular rate of profit remains approximately constant" (Farjoun and Machover 1983, p.36). Under competitive conditions, the system gravitates towards an equilibrium probability distribution of each random variable, whose general form (at least) is theoretically ascertainable and empirically verifiable. Under certain assumptions, for example, they show that the market prices of commodities are proportional to labour values on average.

Farjoun and Machover's (1983) objective is to build the fundations of a probabilistic approach to political economy. Such an approach is a natural fit for computational methods. For while some of the stochastic properties of the model can be studied using traditional mathematical techniques, the probabilistic model is too complex for a fine-grained analysis using closed form solutions, and the statistical equilibrium concept is inherently suited for a computational approach.

The probabilistic political economy pioneered by Farjoun and Machover (1983) has sparked a small but growing literature which studies the statistical properties of simulated economies.[21] Wright (2005) develops a computational simulation of the social relations of capitalism, – i.e. the distribution between workers and capitalists, – "grounded in Marx's distinction between the invariant social relations of production and the varying forces of production" (Wright 2005, p.617). At any time t each agent has an endowment of *money* and can be either a worker, or a capitalist, or unemployed. Firms consist of a single capitalist and a group of employees. Unemployed agents are randomly matched to employers and a wage rate is randomly chosen from an interval. If the capitalist's endowment allows her to pay the wage then the worker is hired. Agents spend a random amount of income on goods produced by firms. As they make purchases, their spending is added to a pool of consumer expenditure that firms compete for. If the revenue a firm receives cannot cover their wage costs, then some workers are dismissed. Workers employed by capitalists are then paid.

Considering $N = 1000$, Wright's (2005) relatively simple model replicates a number of

[20]For a thorough discussion of Farjoun and Machover's (1983) theory, see Mohun and Veneziani (2017).

[21]See, for example, the contributions in Cockshott, Cottrell, Michaelson, Wright, and Yakovenko (2009).

well-known empirical regularities of advanced capitalist economies. In particular, over a sufficiently long time horizon, the distribution of firm size (by employment) fits a power law, as in Axtell (2001), and their growth rate fits a Laplace distribution; the rate of firm disappearance and the duration of downturns in GDP are, respectively, lognormally and exponentially distributed; and the distribution of firm-level profit rates fits a gamma distribution, as proposed by Farjoun and Machover (1983). As for the income distribution, the lower tail fits a lognormal while the upper tail conforms to a Pareto power law.

While Wright (2005) focuses largely on distributive variables, Greenblatt (2014) explores a probabilistic relation between values and market prices in a model broadly similar to Wright's (2005), but with the important addition of the "separation between ownership of the means of production and the ownership of labor power" (Greenblatt 2014, p.519). He conceptualises the economy as a production/exchange network, including workers, capitalists, and firms. Each firm produces either a consumption good or a capital good, and the demand for capital goods varies across firms. Production functions vary across firms. Workers in a firm are paid a uniform wage per unit of time but wages may vary across firms. Capitalists receive profits from firms, which they spend on consumer goods. Simple reproduction holds across the economy: all income is spent on goods produced during the same period, and markets clear. Exchange takes place using money, which serves purely as a medium of exchange.

Equilibrium in Greenblatt's (2014, p.520) model balances "physical flows" of "production and distribution of consumer and producer goods" along with "labor flows". Formally, equilibrium in all markets requires

$$\mathbf{L}\boldsymbol{\Xi} = \mathbf{L}\left[\mathbf{C} + \boldsymbol{\Xi}\mathbf{A}\right] \tag{10}$$

where $\mathbf{C}$ is the square matrix of consumer demands with element $c_{ij}^{(W)} + c_{ij}^{(K)}$ up to the $m^{(c)}$-th firm, after which the elements of $\mathbf{C}$ are 0 up to the m-th firm and $\boldsymbol{\Xi}$ is a diagonal matrix of firm level productivities – defined as gross output per unit of labour – with element ξ_i. By equation (10), it follows that in equilibrium $\mathbf{L}$ must be the left eigenvector of $\left[\mathbf{C} + \boldsymbol{\Xi}\mathbf{A}\right]\boldsymbol{\Xi}^{-1}$.

As concerns prices, following Farjoun and Machover (1983), Greenblatt drops the assumption of uniform profit and wage rates and adopts a version of equation (2). The model is then closed assuming that expenditures equal income for all workers and capitalists and prices are given by $\mathbf{p} = \mathbf{p}\boldsymbol{\Xi}^{-1}\left[\mathbf{A}^T\boldsymbol{\Xi} + \mathbf{C}^T\right]$. This specification of $\mathbf{L}$ and $\mathbf{p}$ as transposes of one another establishes "a duality relation between labor and price" (Greenblatt 2014, p.525).

Greenblatt (2014, pp.526-527) uses Monte Carlo simulations to populate $(\boldsymbol{\Xi}, \mathbf{C}, \mathbf{A})$ randomly in an economy with $N_f = 64$ in order to study the relationship between price and labour values (5). The simulations show, first, that profit rates across firms vary widely, consistent with the probabilistic approach, and, second, that a strong relationship exists between price and (the monetary value of) labour content of commodities, with an R^2 "between unit price and labor content [of] 0.98 or better" (Greenblatt 2014, p.528). However, as "the economy subdivides into increasingly disconnected sub-sectors, [R^2] tends to decrease" (Greenblatt 2014, p.528).

Hahn and Rieu (2017) extend Greenblatt's model in order to explore the relationship between labour values and prices when labour is heterogeneous and its value-creating capacity varies. The types of heterogeneity Hahn and Rieu consider is not directly observed empirically and cannot be easily handled with standard analytical techniques, hence their use of simulations. Consistent with the probabilistic approach, Hahn and Rieu do not assume that wage and profit rates are equalised across sectors, and focus on equation (2). Hahn and Rieu (2017, p.600) define the vector of labour values $\boldsymbol{\lambda}^*$ as

$$\boldsymbol{\lambda}^* = \boldsymbol{\lambda}^* \mathbf{A} + \mathbf{L}\mathbf{V}, \tag{11}$$

where $\mathbf{V}$ is a diagonal matrix of coefficients "expressing the reduction of an hour of concrete labor to the corresponding unit of abstract labor" (Hahn and Rieu 2017, p.602) and is not directly observable but it can be retrieved from other observable variables, noting that the elements of $\mathbf{V}$ bear a relation with the rate(s) of surplus value obtained in each sector and the so-called "monetary expression of labour time" (MELT), a variable which allows one to convert labour magnitudes into monetary ones (Foley 1982).

Hahn and Rieu (2017) then plug their derivation of labour values into a version of Greenblatt's (2014) model to run a series of simulations exploring the relationship between $\boldsymbol{\lambda}^*$ and $\mathbf{p}$ under three scenarios. In Scenario 1, they assume a uniform rate of surplus value and uniform MELT, so that $\mathbf{V}$ is a scalar; in Scenario 2, they assume a uniform MELT but allow rates of surplus value to differ; in Scenario 3, both rates of surplus value and sectoral MELTs are allowed to differ. Over 10,000 simulations, Hahn and Rieu (2017) find that the R-squared for price-value regressions is distributed quite differently across these three scenarios.In the first two scenarios, the adjusted R-squared values are very closely clustered around 1, implying "an almost direct linear proportionality between price and value" (Hahn and Rieu 2017, p.608). In Scenario 3, however, when rates of surplus value and sectoral MELTs are allowed to vary, direct proportionality between prices and labour values "almost disappears" (Hahn and Rieu 2017, p.608).

Not all results in this literature are overly surprising – for example, both in Greenblatt (2014) and in Hahn and Rieu (2017) a relation between prices and labour values is somewhat built into the model, and as certain assumptions are relaxed and more randomness is introduced this relation weakens. Yet, these computational models are able to generate results consistent with well-known empirical findings. More generally, at the theoretical level, they suggest that the probabilistic approach proposed by Farjoun and Machover (1983), and the related notion of statistical equilibrium, may yield interesting and novel insights into the functioning of capitalist economies, and computational methods are fundamental for the development of a probabilistic political economy.

4 Systemic inequality

In classical-Marxian theory, inequality is a central concern: "The economists of the nineteenth century deserve immense credit for placing the distributional question at the heart of economic analysis" (Piketty 2014, pp.15-16). Of course, this concern is not exclusive to the classical-Marxian tradition: economic inequalities have been analysed by all of the main schools of thought and there exists a vast mainstream literature on income inequalities.

Yet the classical-Marxian approach to inequalities has some distinctive features. First, while the mainstream approach is essentially individualistic, classical-Marxian theory has a more systemic outlook. On the one hand, it places collective actors at centre stage, and social classes play a fundamental explanatory and normative role. On the other hand, it focuses on the distributive implications of the institutions, social relations and overall economic organisation of capitalist economies, while devising possible systemic alternatives.

Thus, and this is the second element of distinction, instead of concentrating primarily on personal income, the analytical focus is on the functional income distribution. In turn, this has led the classical economists and Marx to pay particular attention to property relations and *wealth* inequalities, a topic which, as Piketty (2014) has noted, has received only scant attention after the neoclassical revolution. What's more, at least in Marx's case, the analysis of wealth inequalities was associated also with a focus on exploitation, a concept that is at best of marginal relevance in neoclassical economics.

Capturing the systemic aspects of unequal social and economic relations in large economies with complex production and consumption networks; explaining the emergence and persistence of wealth inequalities, exploitation, and classes from the dispersed actions of heterogeneous agents; and analysing hypothetical policies or even entire systemic alternatives are precisely the kind of things that computational methods are well suited to handle.

4.1 Computable general equilibrium and the socialist calculation debate

Historically, the issue of using computational methods in the analysis of classical-Marxian themes emerged in the context of the so-called 'socialist calculation debate' of the 1920s and 1930s. The debate revolved around the possibility that a planned economy would have similar welfare and efficiency properties to those of a market-based economy. Among other things, this would require being able to calculate competitive prices that would direct the economy to an optimal allocation. von Mises (1935) argued that decentralised rational actions coordinated by markets were necessary for the efficient functioning of the economy. Lange (1936, 1937) adopted a standard neoclassical general equilibrium model to counter this criticism arguing that, at least in theory, a Central Planning Board would be able to set optimal equilibrium prices adopting a trial-and-error process responding to notional excess demand in the various markets, thus effectively acting as a Walrasian auctioneer.

While the main interlocutors in the debate did not use computational methods, nor were computational methods powerful enough to handle the calculation of ideal prices at the time they were writing, at the heart of the calculation debate is also a computational problem. This is the starting point of Cockshott and Cottrell's (1993a, p.73) reassessment of the original calculation debate, whose arguments must be reconsidered "in the light of the development of the theory and technology of computation since that time".

The computational vision of socialism put forth by Cockshott and Cottrell (1993a,b) is different from earlier conceptions. Rather than building their vision of socialism around a planning body that makes decisions to coordinate the economy, Cockshott and Cottrell envision a socialist economy that allows for feedbacks between prices and consumer choices, thereby tackling a key problem of centrally planned economies.

The central step of Cockshott and Cottrell's (1993a; 1993b) argument is their view—contra von Mises (1935)—that a strong case can be made for a Marxian approach to calculation in terms of labour time. Given the availability of sophisticated computational procedures, "labour-time calculation is defensible as a rational procedure" when using computational algorithms to account for consumer choice guiding "the allocation of ... economic resources" (Cockshott and Cottrell 1993a, p.74). The calculation of labour values, $\boldsymbol{\lambda}$, is not as straightforward as it may appear. Computationally, finding the Leontief inverse $(\mathbf{I} - \mathbf{A})^{-1}$ "is the hard nut to crack in socialist planning" (Cockshott and Cottrell 1993a, p.102). In the Soviet economy, for example, Cockshott and Cottrell estimate $m = 10^7$, and the inversion of a $10^7 \times 10^7$ matrix proves difficult even with modern computing capabilities. However, once $(\mathbf{I} - \mathbf{A})^{-1}$, and thus $\boldsymbol{\lambda}$, are computed, then one may proceed "to calculate the vector of gross outputs of all products which is required to support any given vector of desired final outputs, for consumption and accumulation" (Cockshott and Cottrell 1993a, p.102).

In the system proposed by Cockshott and Cottrell (1993a, pp.105-106), workers are paid and goods are purchased with "labour certificates". System-wide equilibrium requires that "planned supplies and consumer demands for the individual goods ... coincide when the goods are priced in accordance with their labour values" (Cockshott and Cottrell 1993a, p.106).[22] Of course, the economy is unlikely to find itself at this equilibrium. If the available supplies are inconsistent with labour values, then a "marketing authority" must adjust prices (measured in labour certificates) to achieve "[approximate] short-run balance" (Cockshott and Cottrell 1993a, p.106). Over time, the authority adjusts quantities produced to achieve equilibrium where market-clearing prices are equal to labour values.The equilibrium production plans that support a given level of output will be arrived at through an "iterative process ... performed in 'computer time'" (Cockshott and Cottrell 1993a, p.106). This does not presuppose that levels of consumer demand be known. Consumer demand is allowed to vary and adjustments to demand are "left to a 'trial and error' process which takes place in historical time" (Cockshott and Cottrell 1993a, p.106).[23]

Cockshott and Cottrell's (1993a; 1993b) focus on labour value pricing and their cursory treatment of informational and incentive problems – arguably one of the main problems faced by centrally planned economies – may be deemed unsatisfactory. Nonetheless, at the methodological level, their analysis forcefully shows that advances in computational methods may lead one to reconsider apparently incontrovertible conclusions – such as von Mises's (1935) proof of the impossibility of an efficient planned economy. More generally, computational methods can be fruitfully used to explore alternatives to capitalism that go beyond central planning, as discussed in the next two subsections.

4.2 Socialism and social democracy

In the last four decades, John Roemer (1982; 1994; 2008; 2019; 2020) has analysed the distributive, and more generally normative, implications of alternative forms of economic organisation. In his view, socialism is not defined by centralised decision making and the abolition of private ownership of productive assets. Rather, it is defined by a set of distributive, and more generally normative, goals to be reached and, in more recent work, by a cooperative ethos. From this perspective, he advocates the use of markets as efficient

[22]That is, labour values and prices in labour certificates coincide or are proportional. The moral and efficiency arguments behind this proposal are elaborated on in Cockshott and Cottrell (1997).

[23]Cockshott and Cottrell (1993b) also offer some methods for handling issues of skilled or complex labour in their labour-time-based socialism, as well as expanding on issues of international trade, ownership and property rights, and democracy.

coordination devices even in socialism. In Roemer (1994), he has designed a blueprint for a market socialist economy.

It is difficult to assess Roemer's proposals at a purely theoretical level. What are the efficiency implications of various models of socialism? What would income distribution look like in a socialist economy? What redistribution of assets would be required to achieve a fair society? What educational policy may attenuate inequalities arising from differential skills? These questions cannot be fully answered using analytical tools only.

Roemer (2008) studies the normative implications of socialism compared to the current redistributive tax system in the U.S. He considers a single good economy with a continuum of agents and a Cobb-Douglas production function, and therefore we need to adjust our notation slightly. Output is given by $y = \delta K^{1-\alpha}\ell^{\alpha}$, where K is the capital stock and ℓ is labour in efficiency units. Given a skill level σ, an agent working for L units of time contributes σL efficiency units of labour. Agents also own portions $\beta(\sigma)^{\mu}$ of K, where β and μ are given parameters. Thus, capital is distributed according to a power function, an assumption "meant to represent the empirical fact that capital ownership increases sharply with earned income" (Roemer 2008, p.16). Per capita endowments of capital and labour in efficiency units are, respectively, $\bar{K} = \beta \int (\sigma)^{\mu} dF(\sigma)$ and $\int \sigma dF(\sigma)$. Each agent has a utility function over consumption and leisure given by: $u(c, L) = c - \gamma\ L^{1+1/\eta}$, where η and γ are positive parameters.

The economic environment is specified by $(F, \alpha, \beta, \delta, \mu, \gamma, \eta)$. A socialist allocation is defined as follows (Roemer 2008, p.17).

Definition 1. *A socialist allocation for* $(F, \alpha, \beta, \delta, \mu, \gamma, \eta)$ *is a pair of functions* $(c(\sigma), L(\sigma))$ *and a number* ι *such that:*
(1) for all σ, $c(\sigma) = \iota\sigma L(\sigma)$;
(2) $\delta\bar{K}^{1-\alpha}(\int \sigma L(\sigma)dF(\sigma))^{\alpha} = \int c(\sigma)dF(\sigma)$;
(3) $(c(\cdot), L(\cdot))$ *is Pareto efficient.*

In other words, a socialist allocation is (2) a feasible allocation that (3) no agent would block, in which (1) "output received by each individual is proportional to the efficiency units of labor that she expends in production" (Roemer 2008, p.17).[24] Because output is distributed according to labour in efficiency units exploitation does not exist at the socialist allocation (more on this in section 4.3 below). This is contrasted to a capitalist economy in which taxes are collected at a constant marginal tax rate and redistributed in a lump-sum to every agent (Roemer 2008, p.19).

The two allocations are examined by comparing Gini coefficients of income, denoted by G^I. First, the model is calibrated to the U.S. using data from 2001-2002. Because skills determine income, Roemer (2008) fits a lognormal distribution to the empirical distribution of earnings and uses the estimated parameters to specify the distribution of skills, normalising the mean skill to one. He then takes an estimate of the Lorenz curve for financial wealth in the U.S. as a proxy for the distribution of capital ownership. This is used to "try to fit [capital ownership] with the Lorenz curve from a power distribution" (Roemer 2008, p.20). An agent's pre-tax income is $y^{\text{pre}}(\sigma) = w\sigma L(\sigma) + r\beta(\sigma)^{\mu}$. Roemer calculates that at a socialist allocation $G^I = 0.406$, reflecting the variance in earnings due to the skill distribution. Quite

[24]This concept is also known as the *proportional solution*. For a discussion, see Roemer (2019).

surprisingly, under the assumptions of the model, in the capitalist economy with moderate taxation $G^I = 0.382$, and so income is distributed *more* equally than under socialism.

In recent work, Roemer has further refined his definition of socialism and the analysis of different models of socialism. Building on his theory of Kantian behaviour, Roemer (2019; 2020) has proposed several variants of socialism, including social democracy, which are characterised by different kinds of property relations *and* a cooperative ethos. The difference between a cooperative ethos and the individualistic ethos typical of capitalism is modelled as a difference in the kind of optimising behaviour agents engage in: individualistic agents are Nash optimisers, while under a cooperative ethos, they adopt an optimisation protocol that Roemer (2019) associates with the theory of Immanuel Kant.

According to Roemer (2020, p.13), social democracy is "an economic mechanism in which firms remain privately owned, individuals contribute factor inputs to firms, but taxation redistributes incomes, perhaps substantially". Socialism is defined as a *family* of sharing economies: there are no shareholders and "the firm is conceptualized as owned by workers and investors, who share in conventional profits after rental payments are paid to investors and wages are paid to workers" (Roemer 2020, p.13). Different variants of socialism correspond to different rules for sharing profits – ranging from the standard Marxist view that workers should get all profits, to a rule attributing profits entirely to investors.

Roemer (2020) analyses economies with a standard strictly concave production function and heterogeneous agents with preferences defined over consumption, leisure, and capital holdings and with different endowments of capital and labour that generalise Roemer (2008). Considering various types of Kantian equilibria, Roemer (2020, p.22) proves that in both socialism and social democracy, "the equilibria are Pareto efficient. Resource allocation is decentralized by the existence of markets and competitive prices, and optimization by individuals and firms". Kantian optimisation can decentralise resource allocation in ways that allow one to separate distributive concerns and efficiency. In the case of social democracy, any income distribution can be obtained in equilibrium via a linear taxation scheme, without any deadweight loss.

How about income distribution? In order to answer this question Roemer (2020, Sec.8) simulates three models: "capitalism with a positive tax rate, social-democracy with various positive tax rates, and the sharing economy with various distributions of the capital endowments" (Roemer 2020, p.26).[25] The key findings are as follows.

First, the capitalist equilibrium has $G^I = 0.37414$ and is Pareto inefficient because of the deadweight loss of taxation, owing to Nash behaviour. Second, the social democratic equilibrium is Pareto efficient at any linear tax rate, and therefore any value of G^I can be obtained without deadweight loss. Interestingly, however, when the tax rate is 30%, G^I is slightly larger than at the capitalist equilibrium at that tax rate. Third, at the socialist equilibrium $G^I \approx 0.5$ and is quite insensitive to the specific sharing rule: the socialist income distribution is rather unequal even if all profits go to workers. Inequality decreases somewhat only if wealth is significantly redistributed.

This broadly confirms the results obtained in Roemer (2008): perhaps surprisingly, major changes in property relations would not improve the distribution of income–indeed, they may

[25]The baseline assumptions on the key functions and parameters of the model are broadly similar to Roemer (2008).

worsen it– even if one assumes all agents to act cooperatively. "To achieve acceptable [G^I's] in the sharing economy ..., we need either a significant redistribution of financial wealth, as I have simulated, or income taxation – and the latter, as far as I know, will be inefficient" (Roemer 2020, p.46). This is a negative result, and one may be led to conclude that the socialist blueprint should be discarded, but there are various reasons to hesitate.

First, Roemer's (2008; 2020) results crucially depend on the specific assumptions made to calibrate the model. This includes not only the relevant functional forms representing technology and preferences, but also – crucially – the distribution of skills. Second, Roemer's (2008; 2020) analysis is static: it analyses one-shot policies *and* takes skills as given. Yet, in a dynamic perspective, significant wealth redistribution can be implemented with rather modest tax rates. Moreover, a socialist society would presumably implement policies that would redistribute *all* productive assets including skills – for example, by promoting public education. Finally, before discarding the socialist blueprint, it may be worth reflecting on its normative foundations. An allocation of income proportional to contribution was advocated by Marx as a means to eliminate exploitation, and not (only) to reduce income inequalities.[26]

These three sets of issues are central in the literature surveyed next.

4.3 Computational approaches to exploitation and class

In the most general sense, an agent is exploited (an exploiter) if she contributes more (less) labour to the economy than she receives via her income. As simple as this idea may seem, there exists no widely accepted definition of exploitation. Outside of the simplest two-class economies with homogeneous labour, the notions of labour contributed and labour received have no straightforward interpretation, and a number of conceptual problems arise, which, in the received view, make the concept of exploitation logically flawed and metaphysical.[27]

Cogliano, Veneziani, and Yoshihara (2016; 2019) aim to question this view by proposing a definition of exploitation that is theoretically robust and empirically relevant, *and* to reconsider Roemer's (2008; 2020) negative conclusions on socialism within a dynamic framework with a large (and possibly variable) number of heterogeneous agents.

To be specific, Cogliano et al. (2016; 2019) analyse a dynamic extension of Roemer's (1982) accumulating economy with a labour market to examine the dynamics of exploitation and classes, and their relation with income and wealth inequality. One good is consumed and produced in the economy. In every t, an agent ν can engage in two types of productive activity: she can either sell her (effective) labour or hire others to operate her capital.[28] In Cogliano et al. (2016) labour is assumed to be homogeneous, and so $l^\nu = 1$ for all ν. Cogliano et al. (2019) take a more general approach and allow for different skills σ^ν. In every t, agents choose their productive activities to maximise their wealth subject to consuming b_t per unit of (effective) labour performed, and to being able to lay out in advance the operating costs of their production activities (and without exceeding their labour endowment).

Prices are given by equation (1). In equilibrium, at all t, agents optimise, the labour

[26]As Roemer (2020, p.46) himself has noted, one should also be mindful "of the importance of the cooperative ethos to socialism, and the possible dynamic interaction between that ethos and property relations".

[27]For a discussion see Yoshihara (2010, 2017) and Mohun and Veneziani (2017).

[28]In principle, agents can also operate her own capital. However, Cogliano et al. (2016, 2019) prove that self-employment can be ignored in equilibrium without loss of generaility.

market clears, and there is no excess demand in the markets for capital and outputs. During every t, the equilibrium profit rate r_t and real wage $\tilde{w}_t$ are determined by the relative scarcity of labour and capital. By the linearity of the technology, Cogliano et al. (2019) show that if the economy is capital constrained, then $\tilde{w}_t = b_t$; if the economy is labour constrained, then $r_t = 0$; and if a knife-edge condition obtains, and labour and capital are equally abundant, it is possible to have $r_t > 0$ and $\tilde{w}_t > b_t$.[29]

The exploitation status of an agent is identified comparing the amount of effective labour expended, z_t^ν, and the amount of labour received via her income. Formally, let c_t^ν denotes an agent's *potential* consumption.[30] The amount of labour received by agent ν via their income is given by $\lambda_t c_t^\nu$ and an *exploitation intensity index* can be defined for each agent:

$$\varepsilon_t^\nu = \frac{z_t^\nu}{\lambda_t c_t^\nu}. \tag{12}$$

Based on equation (12), agent ν is exploited (an exploiter) if and only if $\varepsilon_t^\nu > 1$ ($\varepsilon_t^\nu < 1$) and is neither exploited nor an exploiter if and only if $\varepsilon_t^\nu = 1$. The index contains more information, however, because it provides a quantitative measure of the *intensity* of exploitation. Thus, it is possible to use ε_t^ν to qualify differences in how two exploited agents experience exploitation, or how the experience of exploitation compares across countries, or over time.

Indeed, as Cogliano et al. (2016, 2019) show, the distribution of $(\varepsilon_t^\nu)_{\nu\in\mathcal{N}}$ can be analysed using standard tools of inequality measurement. Denoting the Gini of $(\varepsilon_t^\nu)_{\nu\in\mathcal{N}}$ by G_t^ε, it is also possible to compare G_t^ε to the Gini coefficients of wealth, G_t^W, and income, G_t^I. The effect of alternative societal structures on exploitative relations, the interaction of inequality in exploitation intensity, wealth, and income, or the effect of different policies on $G_t^\varepsilon, G_t^W, G_t^I$ are not easily studied with standard analytical tools. Thus, Cogliano et al. (2016, 2019) adopt a computational approach.[31]

Cogliano et al. (2016, 2019) calibrate their models to empirical observations for the U.S. The initial distribution of wealth maps closely that of the U.S., and in the general case with heterogeneous labour (Cogliano et al. 2019), skill factors σ^ν are increasing in ω_0^ν over the first 80% of agents and decreasing thereafter. This structure of $(\sigma^\nu)_{\nu\in\mathcal{N}}$ captures the classical-Marxian view that very wealthy "capitalists" derive nearly all of their income from capital ownership rather than labour performed – a view consistent with Piketty's (2014) findings. It also produces an initial distribution of income that is close to empirical observations for the U.S. Cogliano et al. (2016, 2019) consider initially capital constrained economies where exploitation exists as a result of the unequal ownership of wealth. The simulations specify initial conditions for (N_0, A_0, L_0, b_0).[32] They then run by first checking whether the economy is capital constrained, labour constrained, or on the knife-edge and determine $\tilde{w}_t$ and r_t accordingly. Agents then optimise and the simulation repeats for T periods.

[29] The knife-edge is defined as a situation in which aggregate capital endowments are exactly sufficient to employ the aggregate amount of (effective) labour in the economy.

[30] Formally, $p_t c_t^\nu = p_t \omega_t^\nu + p_t b z_t^\nu - p_{t-1}\omega_{t-1}^\nu$.

[31] The earliest instance of a computational approach to study class formation in accumulation economies is an unpublished paper by Takamasu (2008). Yet this paper does not analyse exploitation and it only considers a very basic scenario with constant technology, population, and consumption, and a rudimentary treatment of agents' decisions.

[32] If there is no technical change and both population and consumption norms are constant, $A_t = A_0, L_t = L_0, N_t = N_0, b_t = b_0$, all t.

In the baseline scenario, accumulation progresses and labour demand increases. While the economy is capital constrained $\tilde{w}_t = b$ and r_t and the growth rate are steady, as are $G_t^\varepsilon, G_t^W, G_t^I$. The structure of exploitation (number of agents who are exploited versus exploiters) is stable and the pattern of ε_t^ν across agents clearly shows that agents with low or zero wealth have $\varepsilon_t^\nu > 1$ and the wealthiest agents $\varepsilon_t^\nu < 1$.[33]

Absent any countervailing mechanisms, however, as capital accumulates, eventually, the economy becomes labour constrained, and exploitation disappears. Therefore Cogliano et al. (2016) explore exogenous labour-saving technical change and population growth as possible mechanisms to maintain the relative abundance of labour. The simulations show that even modest paces of technical change and population growth maintain the relative abundance of labour while capital is accumulated. As the economy grows, the proportion of exploited agents increases as the new agents added to the simulation are propertyless and wealth accumulates in the hands of an ever-decreasing proportion of the population. Thus, there is a clear pattern of polarisation in the economy even though the real wage, consumption and living standards steadily increase.

Is it possible to design policies to tackle wealth and income inequalities, as well as eliminate exploitative relations, possibly moving towards an alternative economic system as in Roemer (2008, 2020)? Because differential ownership of productive assets is a necessary condition for the existence of exploitation (Cogliano et al. 2019, Theorem 1), and capital income is very unequally distributed, it is interesting to examine what happens to G^I, G^W, G^ε as wealth is redistributed. Based on his static, one shot simulations Roemer (2020) argues that massive wealth transfers are required in order to reduce *both* exploitation *and* income inequalities. Focusing on the economy with heterogeneous labour, and starting from the initial distributions of wealth and skills described above, Cogliano et al. (2019) explore the *dynamic* effects of different taxation policies.

First, Cogliano et al. (2019) consider fixed percentage tax rates on large fortunes, as suggested by Piketty (2014). They show that even with fairly small (progressive) tax rates, the redistribution of wealth quickly reduces ε_t^ν for low skilled agents with $\omega_0^\nu = 0$, whereas ε_t^ν rises for agents with very high initial wealth, although they remain exploiters so long as the simulation is capital constrained. As wealth is redistributed the burden of exploitation falls upon a robust "middle class" of agents with moderate to high skills. This result holds even as the cumulative impact of wealth taxes is a dramatic reduction in G_t^W and G_t^I: G_t^ε remains almost constant while the economy is capital constrained.

If, instead, taxation is designed to eliminate exploitation by allowing agents to own wealth in proportion to their effective labour, the structure of exploitation is immediately altered and even with relatively low taxation rates G_t^ε approaches zero after only 15 time periods and the socialist allocation is achieved by $t = 20$. However, even under the socialist allocation G_t^W and G_t^I remain positive and nonnegigible due to skill heterogeneity and the fact that wealth is allocated in proportion to effective labour.

These results confirm Roemer's (2008) general insight: eliminating exploitative relations

[33]Cogliano et al. (2016, 2019) also derive the class structure of the economy, where, following Roemer (1982), classes are defined based on the way in which agents relate to the labour process. The simulations confirm the existence of a correspondence between class and exploitation status and show that class positions are determined by property holdings rather than merely conventional factors or social norms, as in Axtell, Epstein, and Young's (2001) celebrated computational model.

does not guarantee an egalitarian distribution of income, and vice versa. Nonetheless, first, the trade-off between the two objectives may be less severe than suggested by Roemer, depending on the skill distribution. Under Cogliano et al.'s (2019) assumptions, in the socialist allocation, $G_t^I \approx .25$. Second, and perhaps more important, a socialist economic policy does not consist of a single instrument – wealth or income taxation, – and aims at a more egalitarian distribution of *all* assets, including (acquired) skills. Cogliano et al. (2019) run a simulation of the model with socialist wealth taxation in which a fraction of the tax proceeds are used to fund education, which increases skills. The education system is aimed at mitigating inequalities in skills by providing more educational resources to low-skilled agents. Under reasonable assumptions on parameters values, they show that even with relatively modest taxation rates, the trade-off between socialist and egalitarian commitments becomes less severe over time and can be led to vanish in the long run.

5 Technical change and profits

The rate of profit is a key variable in classical-Marxian theory. As discussed in section 3, it is one of the determinants of capitalist choices in production and in the allocation of capital across sectors. In the classical worldview, growth is driven essentially by capitalist decisions, which are influenced by the long-run profit rate. As shown in section 4.3, the profit rate is a central determinant of income and wealth inequalities, and exploitation. It is also important to understand the long-run tendencies of capitalism and its inherently crisis-ridden nature.

Therefore the analysis of the determinants of the profit rate is a central topic in classical-Marxian economics. While analytical tools can be used to derive important insights in relatively stylised models, computational methods are naturally suited for a comprehensive analysis of the dynamics of profitability taking into account the complexity of the production network and accounting for the fundamental structural change produced by innovations. Simulations are crucial to explore intersectoral dynamics and accumulation "and their relation to the emergence—and etiology—of periodic crisis" (Laibman 1988, p.107).

5.1 The falling rate of profit

The Classics and Marx believed that, absent any countervailing forces, there was an inherent tendency in capitalist economies for the profit rate to fall. While classical economists emphasised the constraints deriving from fixed inputs – typically, land, — Marx focused on capital-using labour-saving innovations – the type of technical change that, in his view, characterised capitalist development. While ex ante profitable for individual capitalists, such innovations would lead to a decrease in the general rate of profit, when widely adopted, due to the increase in capital intensity.

This 'law of the tendency of the rate of profit to fall' has sparked a heated controversy and has been called into question by the celebrated Okishio theorem (Okishio 1963), which proves that ex-ante profitable technical change cannot lead to a decrease in the equilibrium profit rate, if the real wage rate is constant. While this is an important theoretical insight, it does not shed much light on the dynamics of the profit rate in capitalist economies. As Okishio (2001, p.493) himself has noted, the theorem "rests on two questionable assumptions: (1)

that the real wage is constant, and (2) that the new production prices are established".

In an extension of the general disequilibrium framework with adaptive expectations described in section 3.2, Duménil and Lévy (1993, Sect.8.9) retain assumption (1), while dropping (2). They adopt an 'evolutionary' model of technical progress in which technology changes randomly, and "innovations are represented by shocks that affect the three inputs (fixed capital, circulating capital, and labor) in any proportions, and in different manners in the three firms" (Duménil and Lévy 1993, p.161), and new techniques are adopted when they yield a higher profit rate at prevailing prices. They largely confirm the Okishio Theorem: the model displays a classical-Marxian gravitation mechanism with the profit rates of the three sectors oscillating around an increasing trend equilibrium rate.

In later work, Okishio (2001) has analysed a two sector dynamic model in which assumptions (1) and (2) are both dropped. Sector 1 produces an intermediate capital good by means of itself and labour. Sector 2 produces a consumption good by means of good 1 and labour. In each t, a uniform wage rate obtains and market prices determine profit rates, which may differ across sectors (see equation (2)). Market clearing conditions determine prices in every period t. At t, aggregate supply is given, determined from past choices. Aggregate demand for the capital good by capitalists depends on their state of confidence, which measures their desire to expand production, and on their expectations of prices in the next period (which are assumed to be adaptive). Because only workers are assumed to consume, aggregate demand for the consumption good depends on aggregate employment and the real wage. Finally, the money wage rate is assumed to be a (nonlinear) function of the expected price of consumption goods and aggregate employment.

The dynamic system comprises 13 variables and equations and not much insight can be gained by analytical means. Numerical simulations, however, show that the wage rate increases over time and tends to its upper bound, driving both profit rates to zero. As Okishio (2001, p.499) notes, "If there is no technical change, competition makes profits vanish". Furthermore, innovations act only as a partial countervailing factor. "If a new technique is introduced, though the capitalists who use it can obtain extra profit, this extra profit decreases owing to competition and eventually vanishes. The profit of the capitalists who adhere to the old technique becomes negative. In order for profit to be positive, *incessant* technical change is necessary" (Okishio 2001, pp.500-501).

In the computational literature on classical-Marxian economics, analogous results are derived by Duménil and Lévy (1993, Sect. 8.10) in a version of their general disequilibrium model with an endogenous wage rate and no technical change, and by Cogliano et al. (2016), as discussed in section 4.3 above. Absent any countervailing factors, accumulation drives the demand of labour up, which tends to increase the wage rate and lower the profit rate. In the long run, the profit rate is ultimately driven to zero – an outcome incompatible with the functioning of a capitalist economy. Therefore the question arises, what kind of (endogenous) mechanisms guarantee the persistence of a positive profit rate in a capitalist economy?

Perhaps the first instance of the explicit application of computational methods to classical-Marxian questions is Laibman's (1981) two-sector Marxian growth model with endogenous technical change. Sector 1 produces fixed capital goods, which can be scrapped but do not depreciate; sector 2 produces consumer goods. In each sector $i = 1, 2$, if the newest technique is used, then fixed capital K_{it} and labour L_{it} are necessary to produce output, x_{it}. Because capitalists are assumed to aim "to both enlarge the capital within [their] control and install the latest techniques" (Laibman 1981, p.51), in each sector a range of vintages of production techniques are operated. Let L_{it}^S and x_{it}^S denote, respectively, the total amount of labour employed and total output in sector i aggregated over all vintages and let K_{it}^S be the total amount of capital employed in sector i in terms of latest vintage equivalents.[34]

In each period, labour productivities and capital-labour ratios are determined in each sector $i = 1, 2$ both for the most up-to-date technology, $u_{it} = x_{it}/L_{it}$ and $k_{it} = K_{it}/L_{it}$, respectively, and similarly for all vintages, $u_{it}^S = x_{it}^S/L_{it}^S$ and $k_{it}^S = K_{it}^S/L_{it}^S$. In each sector i, the labour values $\lambda_{it} = 1/u_{it}^S$ are the inverses of productivity.

Prices are normalised by setting $p_{2t} = \lambda_{2t}$ and thus $\mathbf{p}_t = (\lambda_{1t} z_t m_t, \lambda_{2t})$, where z_t and m_t are multipliers such that z_t transforms λ_{1t} into a price of production, which m_t further transforms into a market price. w_t is the wage rate in terms of labour values, or the share of wages in value added. The rates of profit are $r_{1t} = \frac{\lambda_{1t} z_t m_t x_{1t}^S - w_t L_{1t}^S}{\lambda_{1t} z_t m_t K_{1t}^S}$ and $r_{2t} = \frac{\lambda_{2t} x_{2t}^S - w_t L_{2t}^S}{\lambda_{1t} z_t m_t K_{2t}^S}$.

Capitalists invest a fraction a of their profits, while workers consume all of their income. Given w_t, the general condition for goods market equilibrium can be derived by equating spending on consumption goods by workers and capitalists in sector 1, $w_t L_{1t}^S + (1 - a)(\lambda_{1t} z_t m_t x_{1t}^S - w_t L_{1t}^S)$, and spending on capital goods by capitalists in sector 2, $a(\lambda_{2t} x_{2t}^S - w_t L_{2t}^S)$. In equilibrium, $m_t = \frac{a}{(1-a) z_t} \frac{L_{2t}^S (1-w_t) - w_t L_{1t}^S}{L_{1t}^S}$.

Plugging the latter expression back into the equations of r_{1t}, r_{2t} gives the market profit rates. By definition, if $m_t = 1$ then market prices are equal to production prices. Therefore, setting $m_t = 1$ and $r_{1t} = r_{2t} = r_t$, one obtains $z_t = w_t + (1 - w_t) k_{1t}^S / k_{2t}^S$ and the profit rate $r_t = \frac{x_{1t}^S (1-w_t)}{k_{1t}^S + w_t (k_{2t}^S - k_{1t}^S)}$ which yields the long-run equilibrium in the capital market. If $m_t \neq 1$, then $r_{1t} \neq r_{2t}$ induces capital movements across sectors. To be specific, if sector j yields a higher return, then a fraction of sector i's profits is invested in sector j, and the bigger the profit rate differential the higher such fraction.[35]

Innovation is costless and therefore profitable innovations are always adopted. Capitalists constantly strive to find new techniques that maximise the profit rate on the most recently "installed vintage, and consequently to the entire capital stock" (Laibman 1981, p.56). Letting $g_{u_{it}} = u_{it+1}/u_{it}$ and $g_{k_{it}} = k_{it+1}/k_{it}$, $i = 1, 2$, the profit rates on the newly innovated technique are the "conjunctural profit rate[s]" (Laibman 1981, p.56):

$$\rho_{1t} = \frac{\lambda_{1t} z_t m_t g_{u_{1t}} u_{1t} - w_t}{\lambda_{1t} z_t m_t g_{k_{1t}} k_{1t}}, \tag{13}$$

$$\rho_{2t} = \frac{\lambda_{2t} g_{u_{2t}} u_{2t} - w_t}{\lambda_{1t} z_t m_t g_{k_{2t}} k_{2t}}. \tag{14}$$

The key behavioural assumption is that capitalists choose the *direction* of innovation – the growth rates of productivity and of capital intensity $g_{u_{it}}, g_{k_{it}}$ – subject to a feasibility constraint formalised by two mechanisation functions which embody existing technical knowledge: $g_{u_{it}} = \gamma (g_{k_{it}})^{\alpha_i}$, $i = 1, 2$, with $0 < \alpha_1, \alpha_2 < 1, \ldots \gamma \geq 1$ (Laibman 1981, p.58).[36]

[34] In other words, K_{it}^S is "the quantity of the ... latest-vintage machine that would be necessary to produce $[x_{it}^S]$" (Laibman 1981, p.50). This is necessary since capitals goods of different vintages are heterogeneous and cannot be simply summed.

[35] Old vintages of capital are assumed to be 'congealed' and cannot be moved across sectors.

[36] "In reality, capitalists never choose among statically preexisting techniques; they choose a *course of action*, i.e., a path of technical change" (Laibman 1981, p.70).

The capitalists' optimal choice of technique drives the evolution of labour productivities and capital-labour ratios, which in turn determine the time-paths for labour values, profit rates, and outputs. Over time, as new techniques are adopted, the oldest techniques become increasingly uneconomical and are eventually scrapped when they yield negative profits.

The simulations yield a number of interesting insights. First, in line with the literature surveyed in section 3.2, over time k_{1t}^S/k_{2t}^S approaches 1, leading market prices to converge to labour values (with some oscillations if $\alpha_1 \neq \alpha_2$). Second, in the long run the growth rate of productivity goes to zero. Of course, the result depends on the shape of the mechanisation functions, but it does highlight a potential "immanent barrier within the capitalist system" (Laibman 1981, p.70). Third, and perhaps more importantly for our discussion, "given reasonable values of the parameters [it is possible] for either sector or both to exhibit a rising composition of capital and ... a falling rate of profit over time" (Laibman 1981, p.64). This result is less counterintuitive than it may seem. For, capitalists adopt innovations based on the 'conjunctural' profit rates ρ_i and ignoring the general equilibrium effects of technical change on prices and wages. Moreover, given the assumption of a constant wage share w, the increase in productivity implies a rising real wage. This result is well-known. "What the present model adds is a demonstration that such path is consistent with capitalist ρ-maximizing rationality, and a statement of the specific conditions in which this rationality will result in a falling-r path" (Laibman 1981, p.65).

Cogliano et al. (2016) address the issue of the persistence of profits by incorporating two key ingredients of the classical-Marxian approach into the one-good model described in section 4.3 – namely, endogenous technical change and a more complex determination of the wage rate that takes into account social norms and structures of power in a bargaining framework. Bargaining becomes relevant when the linear economy is on the knife-edge. Cogliano et al. (2016) consider economies characterised by population growth and Hicks-neutral technical change driven by profit-squeeze dynamics, whereby cost-reducing innovations are adopted when the profit rate drops below a certain threshold. Growth is balanced and the economy remains on the knife-edge.

On the knife-edge, the distribution of income is undetermined in the equilibrium of the linear economy and Cogliano et al. (2016) close the model considering an N-agent Nash bargaining solution. Formally, they suppose that $\tilde{w}_t$ solves[37]

$$\arg \max_{\tilde{w}_t \in \left[b, \frac{1}{\lambda_t}\right]} \prod_{\nu \in \mathcal{N}_t} \left[(1+r_t)\,\omega_{t-1}^{\nu} + (\tilde{w}_t - b)\, l_t^{\nu}\right]^{\beta_t^{\nu}}.$$

where $(\beta_t^{\nu})_{\nu \in \mathcal{N}_t} \in \mathbb{R}_+^{N_t}$ with $\sum_{\nu \in \mathcal{N}_t} \beta_t^{\nu} = 1$ is a profile of weights capturing the agents' *bargaining power*, which Cogliano et al. (2016) specify as follows:

$$\beta_t^{\nu} \equiv (1-\epsilon)\frac{\omega_{t-1}^{\nu}}{\omega_{t-1}} + \epsilon \frac{N_t^{\nu}}{N_t}, \text{ some } \epsilon \in [0,1], \tag{15}$$

where $N_t^{\nu} \geq 1$ is the number of agents who possess the same wealth as ν. Equation (15) frames bargaining power as a combination of wealth and class solidarity: an agent's bargaining power is determined by their share of aggregate capital and by the number of agents who

[37]In other words, agents bargain over income net of subsistence consumption. Observe that $\tilde{w}_t \in \left[b, \frac{1}{\lambda_t}\right]$ guarantees $r_t \geq 0$. Observe also that population is not constant, hence the time subscript in $\mathcal{N}_t$.

share “the same objective condition as ν (in terms of capital ownership)” (Cogliano et al. 2016, p.270). The weight ϵ is adjusted by to explore a wide range of power dynamics.

If capitalists dominate ($\epsilon = 0$), there is a pattern of polarisation in exploitation status as the bargaining power afforded to wealth owners allows them to keep the wage at subsistence. An ever-growing number of propertyless workers are exploited by a relatively decreasing share of the population whose wealth increases dramatically over time. If $\epsilon = 1$, the bargaining strength of propertyless agents is such that r_t quickly falls to zero. Working class solidarity can effectively eliminate exploitation. If $\epsilon \in (0, 1)$, the model exhibits more complex dynamics. The bargaining power of propertyless agents drives profits initially to zero but, once technical change occurs, r_t starts to increase. However, as population also grows, the number of propertyless agents becomes large and their class solidarity begins to outweigh the power of the wealthy. This leads r_t to decrease and continued technical change only allows for short-lived recoveries of profitability until profits and exploitation disappear for good. In other words, setting aside robust population dynamics (which in the long-run is largely exogenous), profitability persists if sustained labour-saving technical change occurs *and* capitalists' control in the long-run is sufficiently strong.

The literature reviewed in this subsection focuses largely on long-run tendencies, but the profit rate is a fundamental variable regulating the functioning of capitalist economies also at business cycle frequency. Conversely, many of the factors analysed here – such as bargaining conditions, technical change, market conditions – do not change smoothly: they are all subject to shocks and short-run perturbations which affect the profit rate. The relation between profitability and the short run dynamics of capitalist economies is discussed next.

5.2 Cycles

Laibman (1988, 1992) studies fluctuations in profitability and business cycles in an extension of the two-sector model outlined in section 5.1. In particular, he incorporates heterogeneity in accumulation ratios in the two sectors (a_i) and explores the effect of labour market dynamics on the wage, while abstracting from technical progress. [38]

If L_s denotes the labour supply, then $v = L/L_s$ is the employment rate. In Laibman (1988) the growth rate of the wage share is $g_w = (\Delta/(1 - w + w\Delta)) - 1$, where $\Delta = (1 + \rho(v - \alpha))^2$; in Laibman (1992), it is $g_w = \phi(v - \alpha)$, where ρ, α and ϕ and positive parameters. Laibman (1992) endogenises the accumulation ratios by assuming that their growth rates depend on profitability: $g_{a_i} = (\theta_i/(1 - a_i + a_i\theta_i)) - 1$, where $\theta_i = 0.01\left(\frac{r_{it}^2}{r_{jt}r_{it-1}} - 1\right) + 1$. Thus, as r_i increases relative to r_j, g_{a_i} goes up, and vice versa. The model provides further details on intersectoral dynamics beyond shifts in a_i, where profit rate differentials induce shifts in capital and labour across sectors.

In both specifications of Laibman (1988, 1992), the relationship between w, v, and sectoral profitability drives cycles and recurrent crises: (w, v) display cyclical behaviour “with a constant period of about 43 periods ... High unemployment implies falling w, excess supply of (demand for) consumer goods (capital goods), and therefore a p_1 peak/ p_2 trough; and conversely” (Laibman 1988, pp.111-112). Interestingly, these cycles are unstable, illustrating the link between accumulation, profitability, and crises, even though the model does not show

[38] Under these assumptions, the growth rate of employment in each sector $i = 1, 2$ is $g_{L_i} = a_i r_i$.

any trend in profitability.

A different computational analysis of the dynamics of capitalist economies can be found in Foley (1987). The economy bears a close resemblance to models of Marx's circuit of capital (Foley 1983). Profit-seeking capitalist firms are the "fundamental actors" (Foley 1987, p.363). Liquidity plays an important role in firms' decisions: production takes time and firms necessarily have money tied up in the production and sales process at any given time in the form of "inventories of raw materials and partly finished commodities, fixed capital, and inventories of finished commodities awaiting sale" (Foley 1987, p.364).

Formally, the balance sheet of a firm $f \in \mathcal{F}$ consists of productive capital assets, K_f, money, M_f, financial assets, F_f, and financial liabilities D_f. Firms sell goods for a profit, and the percentage of sales that goes to profit for firm f is q_f. The key decision of each firm f is "the outlay of capital $[O_f]$ to initiate production, and borrowing $[Z_f]$ to finance production" (Foley 1987, p.364).[39] Letting Λ_f, S_f, denote, respectively, the loans and sales of firm f, and denoting the interest rate on loans by i, the differential equations that describe the evolution of the enterprises' state, in matrix form, are:[40]

$$\begin{aligned}
\dot{\mathbf{M}} &= \mathbf{S} + i(\mathbf{F} - \mathbf{D}) + \mathbf{Z} - \boldsymbol{\Lambda} - \mathbf{O}, \\
\dot{\mathbf{F}} &= \boldsymbol{\Lambda}, \\
\dot{\mathbf{K}} &= \mathbf{O} - (\mathbf{I} - \widehat{\mathbf{q}})\mathbf{S}, \\
\dot{\mathbf{D}} &= \mathbf{Z}.
\end{aligned}$$

Let $r_f = q_f S_f / K_f$. Firms' borrowing and lending decisions depend on the differential $\delta_f = r_f - i$, and, in the latter case, on their liquidity position $m_f = M_f/K_f$: $Z_f/K_f = \chi(\delta_f)$, $\chi_{\delta_f} > 0$ and $\Lambda_f/K_f = \pi(\delta_f, m_f)$, $\pi_{\delta_f} < 0$, $\pi_{m_f} > 0$. Firms adjust capital outlays "with an eye to their current money and financial assets" (Foley 1987, p.367): $\dot{O}_f/O_f = \psi(r_f, e_f)$, $\psi_r > 0$ and $\psi_e > 0$, where $e_f = m_f + F_f/K_f$.

Therefore, the condition for the market of loanable funds to clear is

$$\mathbf{1}\widehat{\boldsymbol{\pi}}(\boldsymbol{\delta}, \mathbf{m})\mathbf{K} = \mathbf{1}\widehat{\boldsymbol{\chi}}(\boldsymbol{\delta})\mathbf{K} + \mathbf{1}\dot{\mathbf{M}}.$$

Finally, assuming that a constant fraction of firms' capital outlays go to pay wages and the rest pays for non-labour inputs, and that a constant fraction of the j-th firm's outlays that create demand for the f-th firm are a given parameter, γ_{fj}, the sales of the f-th firm are $S_f = \sum_j \gamma_{fj} O_j$.

The dynamics of this model can quickly take on complex characteristics, particularly as the number of firms becomes large. Foley (1987, pp.372-373) explores its properties computationally and shows that the system exhibits a limit cycle in liquidity and profit rates. "[T]he cycle in (m, r) space moves counter-clockwise...from a low profit rate, liquidity increases because the profit rate is below the growth rate of money" (Foley 1987, p.372), this increases m, the interest rate falls and borrowing increases, which causes firms to become more liquid and expand their capital outlays, expanding aggregate demand (sales) and increasing the rate of profit. As r rises, eventually liquidity dries up and m begins to decline, the gap $r - i$

[39] In Foley (1987) F consists of holdings of other firms' debt, and in the aggregate: $M + F = D$. The state of a firm at any time period is described by its balance sheet, which includes K, M, F, D, and net worth E.

[40] Observe that while sales increase monetary assets, $S_f(1 - q_f)$ decreases firm f's productive capital.

becomes smaller and borrowing slows, which causes a decline in capital outlays and a fall in the profit rate. Thus, consistent with the classical-Marxian vision, these dynamics forcefully portray a "cyclical, not a smooth, process of accumulation" (Foley 1987, p.374).[41]

Jiang's (2015) ABM builds on Foley (1987) in order to explore the cyclical behaviour of the flow of capital and realisation of profit, while connecting Marxian concerns to post-Keynesian treatments of effective demand. The basic structure and notation is as in Foley (1987) but the model is set in discrete time, and the growth rate of capital outlays is assumed to be increasing in the liquidity ratio m_f because a higher "liquidity ratio will encourage a firm to invest more" (Jiang 2015, p.128), and increasing in $\delta_{f,t}$. The smaller $\delta_{f,t}$, the more inclined firms are to lend rather than invest in production (Jiang 2015, p.129).

For each firm a portion of capital outlays go to wages, which are consumed immediately. The portion of capital outlays for means of production "must be shared across all other firms as their sales" (Jiang 2015, p.129). It is assumed that capitalists invest all surplus and therefore aggregate effective demand always equals aggregate supply. This is not true for individual firms, however, whose situation is captured by the elements of the $N_f \times N_f$ transition matrix $\mathbf{\Pi}$, with $\mathbf{1} = \mathbf{1\Pi}$ and $\mathbf{\Pi} \cdot \mathbf{O} = \mathbf{S}$. Thus, the matrix $\mathbf{\Pi}$ then essentially distributes the capital outlays across firms as their sales (Jiang 2015, p.129).[42] The composition of $\mathbf{\Pi}$ makes this model an ABM because it renders the N_f firms heterogeneous and "organizes [them] into a network of buyers and sellers with various strengths of linkages between them" (Jiang 2015, p.131). Firm's profit rates, and the equations of motion of each firm's productive and capital are the same as in Foley (1987).[43] However, it is assumed that the central bank determines the interest rate in each t, based on the average liquidity ratio $\bar{m}$ across the economy: if liquidity increases, the interest rate is lowered, and vice versa.

Jiang (2015) simulates the ABM for $N_f = 200$. The growth rate of GDP exhibits turbulent upward and downward movement—close to a cycle of booms and busts—much like the growth rate of output for real economies. The ABM also displays liquidity-profit rate cycles as in Foley (1987), though with an unsteady orbit. Further, the system is globally stable as the set of steady states forms a trapping region, which ensures that the chaotic trajectories of the system always remain in the neighbourhood of the steady state. In economic terms, the steady states are essentially a set of classical-Marxian long-period positions which emerge as a result of a complex gravitation process.

6 Conclusions

In this paper, we have critically discussed the growing literature which applies computational methods in the analysis of classical-Marxian topics, focusing in particular on three major topics in classical-Marxian microeconomics – price and value theory, systemic inequalities,

[41]Foley (1992) develops a theory of liquidity cycles that is, at the broad conceptual level, related to Foley (1987), and it models endogenous cycles that arise "from the inherent instability of decentralized decision-making in capitalist economies", something that is "deeply rooted in the Marxian and Keynesian tradition" (Foley 1992, p.1071). Yet the paper has a more standard focus on macroeconomic aggregates and does not deal with the profit rate, and hence we shall not discuss it in detail here.

[42]In other words, problems of effective demand arise at the firm level and affect their decisions from one time period to the next.

[43]Assuming $\mathbf{B} = \mathbf{L} = \mathbf{0}$.

and technical change and profitability. We have argued that, although standard analytical tools remain fundamental to gain insight into the functioning of capitalist economies, computational methods – including simulations, Monte Carlo experiments, and ABMs, among others – are particularly well-suited to analyse certain key topics in classical-Marxian theory, such as the characteristics of wage-profit curves and wage-profit frontiers, which require high computational power, or systemic alternatives to capitalism as a social organisation, which necessarily involve the examination of complex counterfactual scenarios.

Perhaps more importantly, computational methods are well suited to capture certain key elements of the vision of the classical-Marxian approach, including the emphasis on the complex working of competitive processes; the turbulent, crisis-ridden path of development of capitalist economies; the complex, if not chaotic, gravitational movement of the economy around long-period positions; the long-run effects of innovations and structural changes; the complex interaction of micro behaviour and macro structures in complex production and exchange networks. For this reason, we believe that the classical-Marxian approach offers robust theoretical foundations for computational methods.

References

Albin, P. S. and D. K. Foley (1992). Decentralized, dispersed exchange without an auctioneer. *Journal of Economic Behavior and Organization 18*(1), pp. 27–51.

Arthur, W. B. (2006). Out-of-Equilibrium Economics and Agent-Based Modeling. In K. L. Judd and L. Tesfatsion (Eds.), *Handbook of Computational Economics, Volume 2*. Amsterdam, NL: Elsevier B.V.

Axtell, R. (2001). Zipf Distribution of US Firm Sizes. *Science 293*, pp. 1818–1820.

Axtell, R. (2010). Many Agent Firms. Working Paper, Center for Social Complexity, Krasnow Institute for Advanced Study, George Mason University. Version: July 2010.

Axtell, R. L., J. M. Epstein, and H. P. Young (2001). The Emergence of Classes in a Multi-Agent Bargaining Model. In S. N. Durlauf and H. P. Young (Eds.), *Social Dynamics*. Washington, DC: Brookings Institution Press.

Cockshott, W. P. and A. Cottrell (1993a). Calculation, complexity and planning: the socialist calculation debate once again. *Review of Political Economy 5*(1), pp. 73–112.

Cockshott, W. P. and A. Cottrell (1993b). *Towards A New Socialism*. Nottingham, UK: Spokesman, Bertrand Russell House.

Cockshott, W. P. and A. F. Cottrell (1997). Value, Markets and Socialism. *Science & Society 61*(3), pp. 330–357.

Cockshott, W. P., A. F. Cottrell, G. J. Michaelson, I. P. Wright, and V. M. Yakovenko (2009). *Classical Econophysics*. Abingdon, Oxon: Routledge.

Cogliano, J. F. (2013). *New Directions in Political Economy: Value Theory, Agent-Based Computational Modeling, and the Dynamics of Labor Mobility*. Ph. D. thesis, New School for Social Research, New York, NY.

Cogliano, J. F., P. Flaschel, R. Franke, N. Fröhlich, and R. Veneziani (2018). *Value, Competition, and Exploitation: Marx's Legacy Revisited*. Cheltenham: Edward Elgar.

Cogliano, J. F., R. Veneziani, and N. Yoshihara (2016). The dynamics of exploitation and class in accumulation economies. *Metroeconomica 67*(2), pp. 242–290.

Cogliano, J. F., R. Veneziani, and N. Yoshihara (2019). Exploitation, skills, and inequality. *Review of Social Economy 77*(2), pp. 208–249.

Colander, D. (2000). A thumbnail sketch of the history of thought from a complexity perspective. In D. Colander (Ed.), *Complexity and the History of Economic Thought*. London: Routledge.

Desai, M. (1988). The transformation problem. *Journal of Economic Surveys 2*(4), pp. 295–333.

D'Ippolito, G. (1987). Probabilità di perverso comportamento del capitale al variare del saggio di profitto. il modello embrionale a due settori. *Note Economiche 2*, pp. 5–37.

D'Ippolito, G. (1989). Delimitazione dell'area dei casi di comportamento perverso del capitale in un punto di mutamento della tecnica. In L. L. Pasinetti (Ed.), *Aspetti controversi della teoria del valore*. Bologna: Il Mulino.

Duménil, G. and D. Lévy (1987). The Dynamics of Competition: A Restoration of the Classical Analysis. *Cambridge Journal of Economics 11*(1), pp. 133–164.

Duménil, G. and D. Lévy (1990). Stability in Capitalism: Are Long-Term Positions the Problem. *Political Economy 6*(1-2), pp. 229–264.

Duménil, G. and D. Lévy (1991). Micro Adjustment toward Long-Term Equilibrium. *Journal of Economic Theory 53*(2), pp. 369–395.

Duménil, G. and D. Lévy (1993). *The Economics of the Profit Rate: Competition, Crises, and Historical Tendencies in Capitalism*. Aldershot: Edward Elgar.

Dvoskin, A. and F. Petri (2017). Again on the relevance of reverse capital deepening and reswitching. *Metroeconomica 68*(4), pp. 625–659.

Eltis, W. (1973). *Growth and Distribution*. London: MacMillan.

Epstein, J. M. (2006). Agent-Based Computational Models and Generative Social Science. In J. M. Epstein (Ed.), *Generative Social Science: Studies in Agent-Based Computational Modeling*. Princeton, NJ: Princeton University Press.

Farjoun, E. and M. Machover (1983). *Laws of Chaos*. London, UK: Verso.

Fine, B. and L. Harris (1979). *Rereading Capital*. New York, NY: Columbia University Press.

Fine, B. and A. Saad-Filho (2010). *'Marx's Capital'* (5th edition ed.). New York, NY: Pluto Press.

Flaschel, P. (1990). Cross-Dual Dynamics, Derivative Control and Global Stability: A Neoclassical Presentation of a Classical Theme. *Political Economy 6*(1-2), pp. 73–91.

Flaschel, P. (2010). *Topics in Classical Micro- and Macroeconomics*. New York, NY: Springer.

Flaschel, P., R. Franke, and R. Veneziani (2012). The measurement of prices of production: An alternative approach. *Review of Political Economy 24*(3), pp. 417–435.

Flaschel, P. and W. Semmler (1986a). Cross-Dual Dynamics for Sraffa/von Neumann Models with Nonsubstitution Properties. Mimeo. University of Bielefeld and the New School for Social Research, New York, NY.

Flaschel, P. and W. Semmler (1986b). The dynamic equalization of profit rates for input-output models with fixed capital. In W. Semmler (Ed.), *Competition, Instability and Nonlinear Cycles*. Berlin: Springer Verlag.

Flaschel, P. and W. Semmler (1987). Classical and neoclassical competitive adjustment processes. *The Manchester School 55*(1), pp. 13–37.

Flaschel, P. and W. Semmler (1988). On the integration of dual and cross-dual adjustment processes in leontief systems. *Ricerche Economiche 42*, pp. 403–432.

Flaschel, P. and W. Semmler (1992). Classical Competitive Dynamics and Technical Change. In J. Halevi, D. Laibman, and E. J. Nell (Eds.), *Beyond the Steady State*. New York, NY: St. Martin's Press.

Foley, D. K. (1982). The value of money the value of labor power and the Marxian transformation problem. *Review of Radical Political Economics 14*(2), pp. 37–47.

Foley, D. K. (1983). Realization and Accumulation in a Marxian Model of the Circuit of Capital. *Journal of Economic Theory 28*(2), pp. 300–319.

Foley, D. K. (1987). Liquidity-profit rate cycles in a capitalist economy. *Journal of Economic Behavior & Organization 8*(3), pp. 363–376.

Foley, D. K. (1992). A Contribution to the Theory of Business Cycles. *Quarterly Journal of Economics 107*(3), pp. 1071–1088.

Foley, D. K. (2003). *Unholy Trinity: Labor, capital, and land in the new economy*. New York, NY: Routledge.

Fratini, S. M. (2019). On the second stage of the cambridge capital controversy. *Journal of Economic Surveys 33*(4), pp. 1073–1093.

Greenblatt, R. E. (2014). A dual theory of price and value in a meso-scale economic model with stochastic profit rate. *Physica A 416*, pp. 518–531.

Hahn, S. B. and D.-M. Rieu (2017). Generalizing the New Interpretation of the Marxian Value Theory: A Simulation Analysis with Stochastic Profit Rate and Labor Heterogeneity. *Review of Political Economy 29*(4), pp. 597–612.

Han, Z. and B. Schefold (2006). An empirical investigation of paradoxes: reswitching and reverse capital deepening in capital theory. *Cambridge Journal of Economics 30*, pp. 737–765.

Jiang, X. (2015). Endogenous Cycles and Chaos in a Capitalist Economy: A Circuit of Capital Model. *Metroeconomica 66*(1), pp. 123–157.

Judd, K. L. (2006). Computationally Intensive Analyses in Economics. In L. Tesfatsion and K. L. Judd (Eds.), *Handbook of Computational Economics, Volume 2*. Amsterdam: Elsevier B.V.

Kirman, A. P. (1992). Whom or what does the representative individual represent? *Journal of Economic Perspectives 6*(2), pp. 117–136.

Kurz, H. D. and N. Salvadori (1995). *Theory of Production: A long-period analysis*. Cambridge, UK: Cambridge University Press.

Laibman, D. (1981). Two-Sector Growth with Endogenous Technical Change: A Marxian Simulation Model. *Quarterly Journal of Economics 96*(1), pp. 47–75.

Laibman, D. (1988). Cyclical Growth and Intersectoral Dynamics. *Review of Radical Political Economics 20*(2&3), pp. 107–113.

Laibman, D. (1992). Cyclical Growth and Intersectoral Dynamics: A Simulation Approach. In J. Halevi, D. Laibman, and E. J. Nell (Eds.), *Beyond the Steady State*. New York, NY: St. Martin's Press.

Lange, O. (1936). On the Economic Theory of Socialism: Part One. *Review of Economic Studies 4*(1), pp. 53–71.

Lange, O. (1937). On the Economic Theory of Socialism: Part Two. *Review of Economic Studies 4*(2), pp. 123–142.

Lawson, T. (2003). *Reorienting Economics*. London, UK: Routledge.

LeBaron, B. and L. Tesfatsion (2008). Modeling Macroeconomies as Open-Ended Dynamic Systems of Interacting Agents. *The American Economic Review 98*(2), pp. 246–250. Papers and Proceedings of the One Hundred Twentieth Annual Meeting of the American Economic Association.

Mainwaring, L. and I. Steedman (2000). On the probability of reswitching and capital reversing in a two-sector Sraffian model. In H. D. Kurz (Ed.), *Critical Essays on Piero Sraffa's Legacy in Economics*. Cambridge, UK: Cambridge University Press.

Marx, K. (1978). *Capital: Volume II.* New York, NY: Penguin Group.

Matthews, P. H. (2000). Did Marx know the way to Santa Fe? In D. Colander (Ed.), *Complexity and the History of Economic Thought*. London: Routledge.

Miller, J. H. and S. E. Page (2007). *Complex Adaptive Systems.* Princeton, NJ: Princeton University Press.

Mohun, S. and R. Veneziani (2017). Values, price and exploitation: the logic of the transformation problem. Mimeo. Queen Mary University of London.

Morishima, M. (1973). *Marx's Economics.* New York, NY: Cambridge University Press.

Morishima, M. (1976). *The Economic Theory of Modern Society.* Cambridge, UK: Cambridge University Press.

Okishio, N. (1963). A mathematical note on Marxian theorems. *Weltwirtschaftliches Archiv 91*, pp. 287–298.

Okishio, N. (2001). Competition and production prices. *Cambridge Journal of Economics 25*(4), pp. 493–501.

Petri, F. (2011). On the likelihood and relevance of reswitching and reverse capital deepening. In C. Gehrke and N. Salvadori (Eds.), *Keynes, Sraffa and the Criticism of Neoclassical Theory*. London: Routledge.

Piketty, T. (2014). *Capital in the Twenty-First Century.* Cambridge, MA: Harvard University Press.

Ricardo, D. (1951). *On the Principles of Political Economy and Taxation.* Cambridge, UK: Cambridge University Press. Reprinted in 2004 by Liberty Fund, Inc., Indianapolis, IN.

Roemer, J. E. (1981). *Analytical Foundations of Marxian Economic Theory.* Cambridge: Cambridge University Press.

Roemer, J. E. (1982). *A General Theory of Exploitation and Class.* Cambridge, MA: Harvard University Press.

Roemer, J. E. (1994). *A future for socialism.* Cambridge, MA: Harvard University Press.

Roemer, J. E. (2008). Socialism vs Social Democracy as Income-Equalizing Institutions. *Eastern Economic Journal 34*(1), pp. 14–26.

Roemer, J. E. (2019). *How we cooperate: A theory of Kantian optimization.* New Haven, CT: Yale University Press.

Roemer, J. E. (2020). What is socialism today? Conceptions of a cooperative economy. Cowles Foundation Discussion Paper No. 220.

Salvadori, N. (2000). Comment on mainwaring and steedman. In H. D. Kurz (Ed.), *Critical Essays on Piero Sraffa's Legacy in Economics*. Cambridge, UK: Cambridge University Press.

Schefold, B. (1976). Relative prices as a function of the rate of profit. *Zeitschrift für Nationalökonomie 36*, pp. 21–48.

Schefold, B. (2016). Marx, the Production Function and the Old Neoclassical Equilibrium: Workable under the Same Assumptions? Centro Sraffa Working Paper No. 19.

Semmler, W. (1990). On Composite Market Dynamics: Simultaneous Microeconomic Price and Quantity Adjustments. *Political Economy 6*(1-2), pp. 193–220.

Smith, A. ([1776] 2000). *The Wealth of Nations*. New York, NY: Random House, Inc. Modern Library Edition. Originally published in 1776 as: *An Inquiry into the Nature and Causes of the Wealth of Nations*.

Sraffa, P. (1960). *Production of Commodities by Means of Commodities*. Cambridge, UK: Cambridge University Press.

Takamasu, A. (2008). Classes in a Virtual Economy. Kansai University, Mimeo.

Tesfatsion, L. (2006). Agent-Based Computational Economics: A Constructive Approach to Economic Theory. In K. L. Judd and L. Tesfatsion (Eds.), *Handbook of Computational Economics, Volume 2*. Amsterdam: Elsevier B.V.

von Mises, L. (1935). Economic Calculation in the Socialist Commonwealth. In F. A. Hayek (Ed.), *Collectivist Economic Planning*. London: George Routledge & Sons.

Wright, I. (2005). The social architecture of capitalism. *Physica A 346*(1/2), pp. 589–620.

Wright, I. (2008). The emergence of the law of value in a dynamic simple commodity economy. *Review of Political Economy 20*(3), pp. 367–391.

Yoshihara, N. (2010). Class and exploitation in general convex cone economies. *Journal of Economic Behavior & Organization 75*, pp. 281–296.

Yoshihara, N. (2017). A progress report on Marxian economic theory: On the controversies in exploitation theory since Okishio. *Journal of Economic Surveys 31*(2), pp. 632–659.

Zambelli, S. (2018). The aggregate production function is not neoclassical. *Cambridge Journal of Economics 42*, pp. 383–426.

Zambelli, S., T. Fredholm, and R. Venkatachalam (2017). Robust measurement of national technological progress. *Structural Change and Economic Dynamics 42*, pp. 38–55.