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A Simple Example of Sraffian Indeterminacy in Walrasian General Equilibrium Framework

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A Simple Example of Sraffian Indeterminacy in Walrasian General Equilibrium Framework*

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Abstract

In contrast to Mandler's (1999; Theorem 6.1) impossibility result about the Sraffian indeterminacy of the steady-state equilibrium, we have proposed a simple example of overlapping generation economy in which generic indeterminacy occurs in the Sraffian steady-state equilibrium.

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1 Introduction

It is well-known that Sraffa's (1960) system of equilibrium price equations contains one more unknown than equation, which leads to the indeterminacy of the steady-state equilibrium, that is, so-called *Sraffian indeterminacy.* Mandler (1999) critically examined Sraffian indeterminacy by embedding the Sraffian system of price equations in a general equilibrium framework. In section 6 of Mandler (1999), generic determinacy of the steady-state equilibria is argued in overlapping generation economies where Walras' law holds. This conclusion follows from his claim that "Due to the way in which $1+r$ appears in Walras' law,

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the standard argument that one of the equilibrium conditions is redundant is not valid in the present model" [Mandler (1999; section 6; p. 705)], though the claim is not seriously verified in his paper.

In contrast to Mandler (1999; section 6), this paper provides a concrete example of overlapping generation economy with a locally nonsatiated utility function, in which a steady-state equilibrium is generically indeterminate. To reach to this result, it explicitly shows that, unlike the claim of Mandler (1999, p.705), one of the equilibrium conditions becomes redundant due to Walras' law even if $1 + r$ appears (the interest rate r is positive).

In the rest of this paper, section 2 introduces a simple model of overlapping generation economies and defines the steady-state equilibrium. Section 3 argues the generic indeterminacy of such an equilibrium. Section 4 concludes.

2 A simple example of overlapping generation economy

A simple overlapping generation model is constructed, in which each generation $t = 1, 2, \ldots,$ is a single individual who lives for two periods. The individual works only in her youth, retires in her old age and purchases her old-aged consumption goods with her past saving. Let $\omega_l > 0$ be the labor endowment of each and every generation. There are two goods produced in this economy and used as consumption goods as well as capital goods, respectively. Let (A, L) be a Leontief production technique, where A is a 2×2 positive square, productive matrix of reproducible input coefficients and L is a 1×2 positive row vector of direct labor coefficients. Finally, every generation has the following specialized form of utility function of lifetime consumption activities:

$$u(z_b, z_a) \equiv \left[z_{b1}^{0.5} \cdot (|z_{b2} - \varepsilon|)^{0.5} \right] \cdot \left[z_{a1}^{0.5} \cdot (|z_{a2} + \varepsilon|)^{0.5} \right],$$

where z_b (*resp.* z_a) is the consumption bundle consumed by the generation in her youth (*resp.* old age), and $\varepsilon > 0$ is sufficiently small.

For each period t, let $p_t \in \mathbb{R}_+^2$ represent a *price vector* of 2 commodities prevailing at the end of this period; $w_t \in \mathbb{R}_+$ represent a *wage rate* prevailing at the end of this period; and $r_t \in \mathbb{R}_+$ represent an *interest rate* prevailing at the end of this period. Then, given a sequence of price vectors $\{(p_t, w_t, r_t), (p_{t+1}, w_{t+1}, r_{t+1})\}$, each generation t in the youth would like to maximize her lifetime utility by the following economic activities: she can supply l^t amount of labor in her youth as a worker employed by generation $t-1$. From the wage income $w_t l^t$ earned at the end of her youth, she can save $p_t \omega^{t+1}$ amount of money and can purchase a consumption bundle z_b^t. By using the saving $p_t \omega^{t+1}$, generation t at the beginning of her old age can purchase δ^{t+1} for speculative purposes and a bundle of capital goods Ay^{t+1} as a productive investment. As an industrial capitalist, she can employ Ly^{t+1} amount of generation $t+1$'s labor. Then, at the end of her old age, she can earn $p_{t+1}\delta^{t+1}$ as the revenue of the speculative and $p_{t+1}y^{t+1} - w_{t+1}Ly^{t+1}$ as the return on the productive investment. From these revenues, she can purchase a consumption bundle z_a^t.

Formally, for a given sequence of price vectors $\{(p_t, w_t, r_t), (p_{t+1}, w_{t+1}, r_{t+1})\}$, each generation t in her youth is faced with the following optimization program MP^t:

$$\max_{l^t, \omega^{t+1}, \delta^{t+1}, y^{t+1}, z_b^t, z_a^t} u\left(z_b^t, z_a^t\right)$$

subject to

$$p_t z_b^t + p_t \omega^{t+1} \leqq w_t l^t,\ l^t \leqq \omega_l,\ p_t \delta^{t+1} + p_t A y^{t+1} = p_t \omega^{t+1}, \text{ and } p_{t+1} z_a^t \leqq p_{t+1} \delta^{t+1} + p_{t+1} y^{t+1} - w_{t+1} L y^{t+1}.$$

At the optimum, all of the weak inequalities in these constraints should hold with equality, on the basis of the assumption of u. Given a sequence of price vectors $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}) \equiv \{(p_t, w_t, r_t)\}_{t \geq 0}$, let $\left(l^t, \omega^{t+1}, \delta^{t+1}, y^{t+1}, z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})\right)$ be a solution of the generations $t = 1, 2, \ldots$, to the problem MP^t.

As our concern is of indeterminacy of equilibria with stationary prices, consider that a *sequence of price vectors* $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ given in the economy is *stationary*: $(p, w, r) = (p_t, w_t, r_t)$ for each and every $t \geq 0$. Then, a solution to MP^t has the following form for every $t \geq 0$: $l^t = \omega_l$, $\delta^{t+1} = \mathbf{0}$, and there exists $y > \mathbf{0}$ such that $y = y^{t+1}$ and $\omega^{t+1} = Ay$ for every $t \geq 0$. Moreover, the demand functions for each $t \geq 0$ is reduced to a pair of *stationary demand functions*: $(z_b^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r}), z_a^t(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})) = (z_b(p, w, r), z_a(p, w, r))$ for every $t \geq 0$.

With these notions, let us introduce a steady-state equilibrium in the following way:

Definition 1 [Mandler (1999, section 6; Definition D6.2)]: A *steady-state equilibrium* under the overlapping economy $\langle (A, L); \omega_l; u \rangle$ is a pair of a stationary price vector $(p, w, r) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R}_+$ and a gross output vector $y \in \mathbb{R}_{++}^2$, such that the following conditions hold:

$$\begin{aligned} p &\leqq (1+r) pA + wL; \text{ (a)} \\ y &\geqq z(p, w, r) + Ay, \text{ (b)} \\ \text{where } z(p, w, r) &= z_b(p, w, r) + z_a(p, w, r); \text{ and} \\ Ly &\leqq \omega_l. \text{ (c)} \end{aligned}$$

In particular, a steady-state equilibrium $((p, w, r), y)$ is called *Sraffian* if all of the above (a), (b), and (c) hold with equality.

3 Indeterminacy of the Sraffian steady-state equilibrium

Given the economy specified above, we will see that there exists an indeterminate Sraffian steady-state equilibrium.

Definition 2 (Mandler (1999)): A Sraffian steady-state equilibrium $((p, w, r), y)$ under the economy $\langle (A, L); \omega_l; u \rangle$ is *indeterminate* if for any $\epsilon > 0$, there is a Sraffian steady-state equilibrium $((p', w', r'), y')$ such that $((p', w', r'), y') \neq ((p, w, r), y)$ and $\|(p', w', r') - (p, w, r)\| < \epsilon$.

Let (p, w, r) be a stationary price vector such that $w > 0$, $r > 0$, and the condition (a) of Definition 1 holds with equality. In the following discussion, let us take commodity 1 as the numéraire, so that the commodity price vector is given by $p = (1, p_2)$ with $p_2 > 0$. Given this price vector, as $\varepsilon > 0$ is sufficiently small, the Marshallian demand vector is the following:

$$\begin{aligned} z_{b1} &= \frac{1}{2}(w\omega_l - Y - p_2\varepsilon) > 0;\ z_{b2} = \frac{1}{2}\left(\frac{w\omega_l - Y}{p_2} + \varepsilon\right) > 0; \\ z_{a1} &= \frac{1}{2}((1+r)Y + p_2\varepsilon) > 0;\ z_{a2} = \frac{1}{2}\left(\frac{(1+r)Y}{p_2} - \varepsilon\right) > 0, \end{aligned}$$

where $Y \equiv pAy$ is the monetary amount of productive investment.

From this information, we will comupute an equilibirum production activity vector $y > \mathbf{0}$ corresponfing to (p, w, r). As the aggregate demand vector is given by $z_b + z_a = \left(\frac{1}{2}(w\omega_l + rY), \frac{1}{2}\frac{(w\omega_l + rY)}{p_2}\right)$ and the condition (b) of Definition 1 holds with equality, we have

$$y = \left(\frac{1}{2}\left(b_1 + \frac{b_2}{p_2}\right)(w\omega_l + rY), \frac{1}{2}\left(b_3 + \frac{b_4}{p_2}\right)(w\omega_l + rY)\right), \quad (1)$$

where $b_1 > 0, b_2 > 0, b_3 > 0, b_4 > 0$ are derived from

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \equiv [I - A]^{-1}.$$

Now, by substituting $Y = pAy = (a_{11} + p_2 a_{21}) y_1 + (a_{12} + p_2 a_{22}) y_2$ into the right hand side of (1) and then rearranging (1), we finally obtain the following system of equations:

$$\begin{bmatrix} 2 - \left(b_1 + \frac{b_2}{p_2}\right) rpA_1 & -\left(b_1 + \frac{b_2}{p_2}\right) rpA_2 \\ -\left(b_3 + \frac{b_4}{p_2}\right) rpA_1 & 2 - \left(b_3 + \frac{b_4}{p_2}\right) rpA_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \left(b_1 + \frac{b_2}{p_2}\right) w\omega_l \\ \left(b_3 + \frac{b_4}{p_2}\right) w\omega_l \end{bmatrix}, \quad (2)$$

where $pA_1 \equiv a_{11} + p_2 a_{21}$ and $pA_2 \equiv a_{12} + p_2 a_{22}$. We can see that the above matrix of (2) satisfies the Hawkins-Simon condition for at least sufficiently small $r > 0$. Therefore, for at least sufficiently small $r > 0$, we can solve the equilibrium production activity vector as

$$y(p, w, r) \equiv \frac{1}{2 - \left(b_3 + \frac{b_4}{p_2}\right) rpA_2 - \left(b_1 + \frac{b_2}{p_2}\right) rpA_1}\left(\left(b_1 + \frac{b_2}{p_2}\right) w\omega_l, \left(b_3 + \frac{b_4}{p_2}\right) w\omega_l\right) > \mathbf{0}. \quad (3)$$

Next, we will show that for any stationary price vector $(p, w, r) > \mathbf{0}$ with any sufficiently small $r > 0$, the pair $((p, w, r), y(p, w, r))$ constitutes a Sraffian steady-state equilibrium. To do so, it is sufficient to show that this pair satisfies Definition 1-(c) with equality. First, as (p, w, r) satisfies Definition 1-(a) with equality, we have the following condition:

$$p\,[I - (1 + r)\,A]\,y\,(p, w, r) = wLy\,(p, w, r)\,. \qquad (4)$$

Second, as Definition 1-(b) holds with equality for the aggregate demand vector $z_b + z_a = \left(\frac{1}{2}\,(w\omega_l + rpAy)\,, \frac{1}{2}\frac{(w\omega_l + rpAy)}{p_2}\right)$, we have:

$$p\,[I - A]\,y = p \cdot (z_b + z_a) = w\omega_l + rpAy$$

for $y = y\,(p, w, r)$, where the equality $p \cdot (z_b + z_a) = w\omega_l + rpAy$ at the right hand sides of the above equation implies that *Walras' law* holds in this economy. Therefore, we have:

$$p\,[I - (1 + r)\,A]\,y\,(p, w, r) = w\omega_l. \qquad (5)$$

From (4) and (5), we have $Ly\,(p, w, r) = \omega_l$. Thus, Definition 1-(c) holds with equality, which also verifies that *Definition 1-(c) with equality is redundant.*

Finally, given the stationary price vector (p, w, r), we can see that $p = p\,(r)$ and $w = w\,(r)$, as

$$w\,(r) = \frac{1}{L\left(\left[I - (1 + r)\,A\right]^{-1}\right)_1} \text{ and } p_2\,(r) = \frac{L\left(\left[I - (1 + r)\,A\right]^{-1}\right)_2}{L\left(\left[I - (1 + r)\,A\right]^{-1}\right)_1}$$

hold, where $\left(\left[I - (1 + r)\,A\right]^{-1}\right)_j$ is the j-th column vector of the matrix $[I - (1 + r)\,A]^{-1}$. Therefore, the equilibirum production activity vector (3) is also given by a continuous function $y\,(p\,(r)\,, w\,(r)\,, r)$. As $y\,(p\,(r)\,, w\,(r)\,, r) > \mathbf{0}$ for at least sufficiently small $r > 0$, the profile $(p\,(r)\,, w\,(r)\,, r, y\,(p\,(r)\,, w\,(r)\,, r))$ is a Sraffian steady-state equilibrium for at least sufficiently small $r > 0$. As all of $p\,(r)\,, w\,(r)\,, r$, and $y\,(p\,(r)\,, w\,(r)\,, r)$ are continuous functions of $r \geqq 0$, there exists a non-empty and open subset of interest rates over which every component of the profile $(p\,(r)\,, w\,(r)\,, r, y\,(p\,(r)\,, w\,(r)\,, r))$ represents a continuous function of interest rates. Thus, by Definition 2, the Sraffian steady-state equilibrium $(p\,(r)\,, w\,(r)\,, r, y\,(p\,(r)\,, w\,(r)\,, r))$ is one-dimensionally *indeterminate.*

3.1 Genericity

In this subsection, we will argue the genericity of the indeterminacy of Sraffian steady-state equilibrium. For the demand functions $z_b\,(p, w, r)$, $z_a\,(p, w, r)$, labor endowment ω_l and for $h = (h_1, h_2, h^o) \in \mathbb{R}^3$, define a perturbed demand functions with similar form to Mandler (1999) as $z_i(h) \equiv z_i^b(h) + z_i^a(h)$ where $z_i^b(h) \equiv z_i^b(p, w, r) + \frac{w}{p_i}h_i$ and $z_i^a(h) \equiv z_i^a(p, w, r) + \frac{w}{p_i}h^o$ for each $i = 1, 2$. Then, to preserve Walras' law and homogeneity, the perturbation of labor endowment is given as $\omega_l(h) \equiv \omega_l + \sum_{i=1}^{2} h_i + \frac{2h^o}{1+r}$.

Now define a function $F : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, R) \times \mathbb{R}_+^2 \times \mathbb{R}_+^{2^2} \times \mathbb{R}_+^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}^4$ as follows, where $(1 + R)^{-1}$ is the Frobenius eigenvalue of the matrix A:

$$F(p, w, r, y, A, L, h) \equiv \begin{bmatrix} z_1 (p, w, r) + w (h_1 + h^o) - (1 - a_{11})y_1 + a_{12}y_2 \\ z_2 (p, w, r) + \frac{w}{p_2} (h_2 + h^o) + a_{21}y_1 - (1 - a_{22}) y_2 \\ 1 - (1 + r) (a_{11} + p_2 a_{21}) - wL_1 \\ p_2 - (1 + r) (a_{12} + p_2 a_{22}) - wL_2 \end{bmatrix}. \quad (6)$$

Definition 3: An *economy* is a profile of (A, L, h) where (A, L) is a Leontief production technique, in which A is 2×2 positive square and productive matrix of reproducible input coefficients, L is 1×2 positive row vector of direct labor coefficients, and $h = (h_1, h_2, h^o) \in \mathbb{R}^3$ is for perturbation.

An economy (A, L, h) is *regular* if every Sraffian steady-state equilibrium $((p, w, r), y)$ is *regular*, that is, the Jacobian $\boldsymbol{D}F$ with respect to $((p, w, r), y)$ has full-rank.

For the system of equations (6), we can compute the corresponding Jacobian $\boldsymbol{D}F$. Then, it can be easily shown that for any fixed (A, L, h), every Sraffian steady-state equilibrium is regular. Moreover, by calculating the Jacobian $\boldsymbol{D}F$ with respect to (h, A, L), which is denoted by $\boldsymbol{D}_{h,A,L}F$, it can be easily shown that the matrix $\boldsymbol{D}_{h,A,L}F$ has full-rank. Therefore, by applying the Transversality Theorem, we can see that the set of regular economies has *full measure*. Finally, it can be shown in a usual way that the set of regular economies is open.

4 Conclusion

We have proposed a simple example of overlapping generation economy with a locally nonsatiated utility function, in which generic indeterminacy occurs in the Sraffian steady-state equilibrium. In particular, we have shown that in that economy, even if the interest rate is positive, the labor market equilibrium condition follows from the condition for stationary equilibrium prices, the excess demand condition for commodities, and Walras' law. This proof implies that Mandler's (1999; section 6, p. 705) claim is invalid, which is a crucial point to verify the generic Sraffian indeterminacy. Indeed, the system of equilibrium equations (6) must contain 5 independent equations with 5 unknown if his claim holds. Therefore, generic determinacy of Sraffian steady-state equilibria would occur only if the utility function is allowed to be locally satiated, and so Walras' law does not hold.

5 References

Mandler, M. (1999) Sraffian indeterminacy in general equilibrium, *Review of Economic Studies* 66, 693–711.

Sraffa, P. (1960) *Production of Commodities by Means of Commodities*: *Prelude to a Critique of Economic Theory*, Cambridge University Press, Cambridge.

A Simple Example of Sraffian Indeterminacy in Walrasian General Equilibrium Framework: Addendum (Not for Publication)

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Abstract

This is the addendum of the main text of "A Simple Example of Sraffian Indeterminacy in Walrasian General Equilibrium Framework," which is not for the purpose of publication, but for helping the referee's review process.

1 The indeterminacy of Sraffian steady-state equilibria

In this section of this Addendum, the detailed process to compute the equilibrium production activity vector $y(p, w, r) > \mathbf{0}$ for a Sraffian steady-state equilibrium price vector (p, w, r) is provided. First, we explain the process of how to derive the Marshallian demand vectors z_b and z_a from the optimization program MP^t. Second, with these Marshallian demand vectors z_b and z_a, we provide a detailed process to compute the equilibrium production activity vector $y(p, w, r) > \mathbf{0}$. Finally, we discuss Walras' law in the final subsection.

1.1 Derivation of Marshallian demand vectors

Remember that the optimization program is given as follows. For a given sequence of price vectors $\{(p_t, w_t, r_t), (p_{t+1}, w_{t+1}, r_{t+1})\}$, each generation t in her

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youth is faced with the following optimization program MP^t:

$$\max_{l^t,\omega^{t+1},\delta^{t+1},y^{t+1},z_b^t,z_a^t} u\left(z_b^t, z_a^t\right)$$

subject to

$$\begin{aligned} p_t z_b^t + p_t\omega^{t+1} &\leqq w_t l^t, \\ l^t &\leqq \omega_l, \\ p_t\delta^{t+1} + p_t A y^{t+1} &= p_t\omega^{t+1}, \\ \text{and } p_{t+1} z_a^t &\leqq p_{t+1}\delta^{t+1} + p_{t+1} y^{t+1} - w_{t+1} L y^{t+1}, \end{aligned}$$

where the locally nonsatiated utility function $u\left(z_b^t, z_a^t\right)$ of lifetime consumption activities is specified by the following form:

$$u\left(z_b, z_a\right) = \left[z_{b1}^{0.5} \cdot \left(\left|z_{b2} - \varepsilon\right|\right)^{0.5}\right] \cdot \left[z_{a1}^{0.5} \cdot \left(\left|z_{a2} + \varepsilon\right|\right)^{0.5}\right].$$

As our concern is of indeterminacy of equilibria with stationary prices, consider that a *sequence of price vectors* $(\boldsymbol{p}, \boldsymbol{w}, \boldsymbol{r})$ given in the economy is *stationary*: $(p, w, r) = (p_t, w_t, r_t)$ for each and every $t \geqq 0$. Then, MP^t for each and every generation $t \geqq 0$ has the following reduced form: given a market price vector (p, w, r) with $p = (1, p_2)$, $p_2 > 0$, $w > 0$, and $r \geqq 0$,

$$\max_{l^t,\omega^{t+1},\delta^{t+1},y^{t+1},z_b^t,z_a^t} u\left(z_b^t, z_a^t\right)$$

subject to

$$\begin{aligned} p z_b^t + p\omega^{t+1} &\leqq w l^t, \\ l^t &\leqq \omega_l, \\ p\delta^{t+1} + p A y^{t+1} &= p\omega^{t+1}, \\ \text{and } p z_a^t &\leqq p\delta^{t+1} + p y^{t+1} - w L y^{t+1}. \end{aligned}$$

In the above reduced form of MP^t, the second inequality constraint holds with equality: $l^t = \omega_l$, as the utility function is independent of the labor supply of this generation. Then, the first inequality constraint is reduced to the following form:

$$p z_b^t + p\omega^{t+1} = w\omega_l,$$

where the constraint holds with equality at the optimum, on the assumption of the locally nonsatiated utility function. By the way, as the third constraint represents the allocation of the saving $p\omega^{t+1}$ to the productive investment pAy^{t+1} and the speculative investment $p\delta^{t+1}$, the first constraint is rewritten as follows:

$$p z_b^t + p\delta^{t+1} + p A y^{t+1} = w\omega_l.$$

Likewise, the fourth constraint holds with equality at the optimum, $p z_a^t = p\delta^{t+1} + p y^{t+1} - w L y^{t+1}$. Then, given that the stationary price vector (p, w, r)

satisfies the condition (a) of Definition 1 with equality, the fourth constraint is reduced to the following form:

$$pz_a^t = p\delta^{t+1} + (1+r)\,pAy^{t+1}.$$

Thus, the first and the fourth constraints of the program MP^t at the optimum can be reduced to the following forms:

$$\begin{aligned} pz_b^t &= w\omega_l - p\delta^{t+1} - pAy^{t+1}, \\ pz_a^t &= p\delta^{t+1} + (1+r)\,pAy^{t+1}. \end{aligned}$$

Under a Sraffian steady-state equilibrium, all of the time subscript "t" and "$t+1$" can be removed, so that the program MP^t under the Sraffian steady-state equilibrium is reduced to the following form:

$$\max_{\delta,y,z_b,z_a} \left[z_{b1}^{0.5} \cdot (|z_{b2} - \varepsilon|)^{0.5}\right] \cdot \left[z_{a1}^{0.5} \cdot (|z_{a2} + \varepsilon|)^{0.5}\right]$$

subject to

$$\begin{aligned} pz_b &= w\omega_l - p\delta - pAy, \\ pz_a &= p\delta + (1+r)\,pAy. \end{aligned}$$

Let (p, w, r) be a stationary price vector such that $w > 0$, $r > 0$, and the condition (a) of Definition 1 holds with equality. In this case, the optimal investment plan satisfies $pAy = p\omega$ and $p\delta = 0$, as one unit of the productive investment is more profitable than one unit of the speculative investment, because of $r > 0$. Given such a price vector, as $\varepsilon > 0$ is sufficiently small, the Marshallian demand vectors z_b and z_a are derived as the following form:

$$\begin{aligned} z_{b1} &= \frac{1}{2}(w\omega_l - Y - p_2\varepsilon) > 0;\ z_{b2} = \frac{1}{2}\left(\frac{w\omega_l - Y}{p_2} + \varepsilon\right) > 0; \\ z_{a1} &= \frac{1}{2}((1+r)\,Y + p_2\varepsilon) > 0;\ z_{a2} = \frac{1}{2}\left(\frac{(1+r)\,Y}{p_2} - \varepsilon\right) > 0, \end{aligned}$$

where $Y \equiv p\omega = pAy$.[1]

1.2 Derivation of equilibrium production activities

Now consider the equilibrium production activity vector y. By the condition (b) of Definition 1, an equilibrium production activity vector $y > \mathbf{0}$ in the Sraffian steady-state equilibrium must satisfy:

$$[I - A]\,y = \begin{pmatrix} z_{b1} + z_{a1} \\ z_{b2} + z_{a2} \end{pmatrix}, \quad (*)$$

[1] Note that even in a stationary price (p, w, r) with $r = 0$, the Marshallian demand vectors can be represented by the same forms with $r > 0$, as the property $Y \equiv p\omega$ still holds in this case.

where $z_{b1}+z_{a1}=\frac{1}{2}\left(w\omega_l+rY\right)$ and $z_{b2}+z_{a2}=\frac{1}{2}\frac{(w\omega_l+rY)}{p_2}$. As the matrix A is productive, the above system of equations (*) can be solved in the following way:

$$\begin{aligned} y &= [I-A]^{-1}\begin{pmatrix} z_{b1}+z_{a1} \\ z_{b2}+z_{a2} \end{pmatrix} = [I-A]^{-1}\begin{pmatrix} \frac{1}{2}\left(w\omega_l+rY\right) \\ \frac{1}{2}\frac{(w\omega_l+rY)}{p_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\left(b_1+\frac{b_2}{p_2}\right)\left(w\omega_l+rY\right) \\ \frac{1}{2}\left(b_3+\frac{b_4}{p_2}\right)\left(w\omega_l+rY\right) \end{pmatrix} \qquad (**) \end{aligned}$$

where each b_i for every $i=1,2,3,4$ comes from

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \equiv [I-A]^{-1}.$$

As A is productive and indecomposable, $b_i>0$ holds for every $i=1,2,3,4$.

Now, as $Y=pAy=\left(a_{11}+p_2a_{21}\right)y_1+\left(a_{12}+p_2a_{22}\right)y_2$, the right hand side of the above system of equations (**) can be written as:

$$[I-A]^{-1}\begin{pmatrix} z_{b1}+z_{a1} \\ z_{b2}+z_{a2} \end{pmatrix} = \begin{bmatrix} \frac{1}{2}\left(b_1+\frac{b_2}{p_2}\right)\left(w\omega_l+r\left[\left(a_{11}+p_2a_{21}\right)y_1+\left(a_{12}+p_2a_{22}\right)y_2\right]\right) \\ \frac{1}{2}\left(b_3+\frac{b_4}{p_2}\right)\left(w\omega_l+r\left[\left(a_{11}+p_2a_{21}\right)y_1+\left(a_{12}+p_2a_{22}\right)y_2\right]\right) \end{bmatrix},$$

where y_1 and y_2 are the components of the optimal solution to MP^t. Therefore, we have the following reduced form of the system of equations (**):

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(b_1+\frac{b_2}{p_2}\right)\left(w\omega_l+r\left[\left(a_{11}+p_2a_{21}\right)y_1+\left(a_{12}+p_2a_{22}\right)y_2\right]\right) \\ \frac{1}{2}\left(b_3+\frac{b_4}{p_2}\right)\left(w\omega_l+r\left[\left(a_{11}+p_2a_{21}\right)y_1+\left(a_{12}+p_2a_{22}\right)y_2\right]\right) \end{bmatrix}. \qquad (***)$$

Under the Sraffian steady-state equilibrium, the *ex-ante* production plan $y=(y_1,y_2)$ presented in the right hand side of the above system of equations (***) must be identical to the *ex-post* production activity vector $y=(y_1,y_2)$ presented in the left hand side. Therefore, the above (***) can be reduced to the following system of equations:

$$\begin{aligned} &\begin{bmatrix} 1-\left(\frac{1}{2}\left(b_1+\frac{b_2}{p_2}\right)\right)r\left(a_{11}+p_2a_{21}\right) & -\left(\frac{1}{2}\left(b_1+\frac{b_2}{p_2}\right)\right)r\left(a_{12}+p_2a_{22}\right) \\ -\left(\frac{1}{2}\left(b_3+\frac{b_4}{p_2}\right)\right)r\left(a_{11}+p_2a_{21}\right) & 1-\left(\frac{1}{2}\left(b_3+\frac{b_4}{p_2}\right)\right)r\left(a_{12}+p_2a_{22}\right) \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ = &\begin{bmatrix} \left(\frac{1}{2}\left(b_1+\frac{b_2}{p_2}\right)\right)w\omega_l \\ \left(\frac{1}{2}\left(b_3+\frac{b_4}{p_2}\right)\right)w\omega_l \end{bmatrix}, \end{aligned}$$

which is also equivalent to the following form:

$$\begin{bmatrix} 2-\left(b_1+\frac{b_2}{p_2}\right)r\left(a_{11}+p_2a_{21}\right) & -\left(b_1+\frac{b_2}{p_2}\right)r\left(a_{12}+p_2a_{22}\right) \\ -\left(b_3+\frac{b_4}{p_2}\right)r\left(a_{11}+p_2a_{21}\right) & 2-\left(b_3+\frac{b_4}{p_2}\right)r\left(a_{12}+p_2a_{22}\right) \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \left(b_1+\frac{b_2}{p_2}\right)w\omega_l \\ \left(b_3+\frac{b_4}{p_2}\right)w\omega_l \end{bmatrix}.$$

Let us denote $D \equiv \left(b_1 + \frac{b_2}{p_2}\right)$, $E \equiv \left(b_3 + \frac{b_4}{p_2}\right)$, $rpA_1 \equiv r\left(a_{11} + p_2 a_{21}\right)$, and $rpA_2 \equiv r\left(a_{12} + p_2 a_{22}\right)$. Then, the above system of equations can be represented by:

$$\begin{bmatrix} 2 - DrpA_1 & -DrpA_2 \\ -ErpA_1 & 2 - ErpA_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Dw\omega_l \\ Ew\omega_l \end{bmatrix}.$$

Note that the matrix $\begin{bmatrix} 2 - DrpA_1 & -DrpA_2 \\ -ErpA_1 & 2 - ErpA_2 \end{bmatrix}$ satisfies the Hawkins-Simon condition for at least sufficiently small $r > 0$. Therefore, for at least sufficiently small $r > 0$, we can solve the vector $y = (y_1, y_2)$ as follows:

$$\begin{aligned}
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \frac{1}{\left(2 - DrpA_1\right)\left(2 - ErpA_2\right) - ErpA_1 DrpA_2} \begin{bmatrix} 2 - ErpA_2 & DrpA_2 \\ ErpA_1 & 2 - DrpA_1 \end{bmatrix} \begin{bmatrix} Dw\omega_l \\ Ew\omega_l \end{bmatrix} \\
&= \frac{1}{\left(2 - DrpA_1\right)\left(2 - ErpA_2\right) - ErpA_1 DrpA_2} \begin{bmatrix} 2Dw\omega_l \\ 2Ew\omega_l \end{bmatrix} \\
&= \begin{bmatrix} \frac{2Dw\omega_l}{(2-DrpA_1)(2-ErpA_2)-ErpA_1DrpA_2} \\ \frac{2Ew\omega_l}{(2-DrpA_1)(2-ErpA_2)-ErpA_1DrpA_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\left(b_1+\frac{b_2}{p_2}\right)}{\left(2-\left(b_1+\frac{b_2}{p_2}\right)rpA_1\right)\left(2-\left(b_3+\frac{b_4}{p_2}\right)rpA_2\right)-\left(b_1+\frac{b_2}{p_2}\right)\left(b_3+\frac{b_4}{p_2}\right)rpA_1rpA_2} w\omega_l \\ \frac{2\left(b_3+\frac{b_4}{p_2}\right)}{\left(2-\left(b_1+\frac{b_2}{p_2}\right)rpA_1\right)\left(2-\left(b_3+\frac{b_4}{p_2}\right)rpA_2\right)-\left(b_1+\frac{b_2}{p_2}\right)\left(b_3+\frac{b_4}{p_2}\right)rpA_1rpA_2} w\omega_l \end{bmatrix} \\
&= \begin{bmatrix} \frac{2\left(b_1+\frac{b_2}{p_2}\right)}{4-2\left(b_3+\frac{b_4}{p_2}\right)rpA_2-2\left(b_1+\frac{b_2}{p_2}\right)rpA_1} w\omega_l \\ \frac{2\left(b_3+\frac{b_4}{p_2}\right)}{4-2\left(b_3+\frac{b_4}{p_2}\right)rpA_2-2\left(b_1+\frac{b_2}{p_2}\right)rpA_1} w\omega_l \end{bmatrix}.
\end{aligned}$$

Thus, the *equilibrium production activity vector* is specified as:

$$y\left(p, w, r\right) = \begin{bmatrix} y_1\left(p, w, r\right) \\ y_2\left(p, w, r\right) \end{bmatrix} = \begin{bmatrix} \frac{\left(b_1+\frac{b_2}{p_2}\right)}{2-\left(b_3+\frac{b_4}{p_2}\right)rpA_2-\left(b_1+\frac{b_2}{p_2}\right)rpA_1} w\omega_l \\ \frac{\left(b_3+\frac{b_4}{p_2}\right)}{2-\left(b_3+\frac{b_4}{p_2}\right)rpA_2-\left(b_1+\frac{b_2}{p_2}\right)rpA_1} w\omega_l \end{bmatrix} > \mathbf{0}.$$

1.3 Walras' Law

Let (p, w, r) be a stationary price vector such that $w > 0$, $r > 0$, and the condition (a) of Definition 1 holds with equality. Remember that in this case, the budget constraints of the lifetime utility maximization are reduced to

$$\begin{aligned}
pz_b &= w\omega_l - p\delta - pAy, \\
pz_a &= p\delta + (1 + r)\, pAy.
\end{aligned}$$

These two equations respectively represent the young generation's and the old generation's consumption expenditures. Therefore, the aggregate consumption expenditure is given by

$$pz_b + pz_a = w\omega_l + rpAy. \quad (1.3.1)$$

In this equation, the right hand side presents the aggregate net revenue of this economy, and so the equation implies that the aggregate consumption expenditure is equal to the aggregate net revenue. This implies that *Walras' law* holds.

Remember that Mandler (1999, section 6, p. 704) introduces Walras' law by the following form:[2]

$$(1+r)(pz_b - w\omega_l) + pz_a = 0. \quad (1.3.2)$$

It is easy to see that (1.3.1) and (1.3.2) are equivalent. Indeed, the first component $(1+r)(pz_b - w\omega_l)$ of the left hand side of (1.3.2) implies that all of the residual of the wage revenue after purchasing the young generation's consumption bundle, $w\omega_l - pz_b$, is invested in production activity. Therefore, the revenue of the old generation $(1+r)(w\omega_l - pz_b)$ represents the gross return of the productive investment $w\omega_l - pz_b$ with the return rate $r > 0$, that is expended to the consumption of the old generation, pz_a. This implies that there exists a production activity $y \geq \mathbf{0}$ such that $pAy = w\omega_l - pz_b$. Therefore, as the equation (1.3.2) can be rewritten as follows:

$$pz_b + pz_a - w\omega_l + r(pz_b - w\omega_l) = 0,$$

it is equivalent to

$$pz_b + pz_a - w\omega_l - rpAy = 0,$$

that is (1.3.1). Therefore, the equation (1.3.1) represents Walras' law even according to the primitive definition given by Mandler (1999, p. 704).

2 Regularity and genericity of the Sraffian steady-state Equilibrium

In this section, we will provide a detailed process to examine the regularity and the genericity of the Sraffian steady-state equilibrium, where the existence of an indeterminate Sraffian steady-state equilibrium is shown in the main text of the paper. First, remember that the perturbed demand functions are given by: for the demand functions $z_b(p,w,r)$, $z_a(p,w,r)$ and the parameter $h = (h_1, h_2, h^o) \in \mathbb{R}^3$, $z_i^b(h) \equiv z_i^b(p,w,r) + \frac{w}{p_i}h_i$ and $z_i^a(h) \equiv z_i^a(p,w,r) + \frac{w}{p_i}h^o$ for

[2] More precisely speaking, Walras' law in a stationary price vector $(p,w,r) > \mathbf{0}$ is given by Mandler (1999, section 6, p. 704) as the following form:

$$(1+r)(pz_b - w\omega_l) + pz_a - w\omega_l^a = 0,$$

where the last component $-w\omega_l^a$ in the left hand side of the above equation represents the labor endowment of the old generation. Though Mandler (1999, section 6) presumes that $\omega_l^a \geqq 0$, our model in our main text considers the simplest case that $\omega_l^a = 0$.

each commodity $i = 1, 2$. Moreover, the perturbed aggregate demand functions are given by $z_i(h) \equiv z_i^b(h) + z_i^a(h)$ for each commodity $i = 1, 2$, and the perturbation of labor endowment is given as $\omega_l(h) \equiv \omega_l + \sum_{i=1}^{2} h_i + \frac{2h^o}{1+r}$. Second, the system of Sraffian steady-state equilibrium equations is given by:

$$\begin{cases} z(h) - [I - A]\, y \\ p - (1+r)pA - wL \end{cases}.$$

Note that the equation of the labor market equilibrium, $Ly - \omega_l(h) = 0$, is not included, as it is shown, in the main text of our paper, to be redundant.

Given this, we can see that in our example of a simple economy, there are 4 independent equations and 5 unknown, y_1, y_2, p_2, w, and r. Remember that $R > 0$ is used to argue that $(1+R)^{-1}$ is the Frobenius eigenvalue of the matrix A. Then, a continuously differentiable function $F : \mathbb{R}_+ \times \mathbb{R}_+ \times [0, R) \times \mathbb{R}_+^2 \times \mathbb{R}_+^3 \times \mathbb{R}_+^{2^2} \times \mathbb{R}_+^2 \to \mathbb{R}^4$ is given by:

$$F(p, w, r, y, h, A, L) \equiv \begin{bmatrix} z_1(p, w, r) + w(h_1 + h^o) - (1 - a_{11})y_1 + a_{12}y_2 \\ z_2(p, w, r) + \frac{w}{p_2}(h_2 + h^o) + a_{21}y_1 - (1 - a_{22})\, y_2 \\ 1 - (1+r)(a_{11} + p_2 a_{21}) - wL_1 \\ p_2 - (1+r)(a_{12} + p_2 a_{22}) - wL_2 \end{bmatrix}.$$

2.1 Regularity of the Sraffian steady-state Equilibrium

Let $z_1(p_2, w, r, h) \equiv z_1(p, w, r) + w(h_1 + h^o)$ and $z_2(p_2, w, r, h) \equiv z_2(p, w, r) + \frac{w}{p_2}(h_2 + h^o)$. Therefore, for any fixed (h, A, L), the regularity of a Sraffian steady-state equilibrium in the economy (A, L, h) can be checked by examining the following Jacobian matrix:

$$\boldsymbol{D}_{(p,w,r,y)} F_{h,A,L}(p, w, r, y)$$
$$\equiv \begin{bmatrix} -(1 - a_{11}) & a_{12} & \boldsymbol{D}_{p_2} z_1(p, w, r, h) & \boldsymbol{D}_w z_1(p, w, r, h) & \boldsymbol{D}_r z_1(p, w, r, h) \\ a_{21} & -(1 - a_{22}) & \boldsymbol{D}_{p_2} z_2(p, w, r, h) & \boldsymbol{D}_w z_2(p, w, r, h) & \boldsymbol{D}_r z_2(p, w, r, h) \\ 0 & 0 & -(1+r)\, a_{21} & -L_1 & -(a_{11} + p_2 a_{21}) \\ 0 & 0 & 1 - (1+r)\, a_{22} & -L_2 & -(a_{12} + p_2 a_{22}) \end{bmatrix},$$

where $\boldsymbol{D}_{p_2} z_1(p, w, r, h)$ represents the partial derivative of the demand function $z_1(p_2, w, r, h)$ at the Sraffian steady-state equilibrium (p, w, r, y) with respect to p_2, and a similar argument is applied to each of the other notations, $\boldsymbol{D}_w z_1(p, w, r, h)$, $\boldsymbol{D}_r z_1(p, w, r, h)$, $\boldsymbol{D}_{p_2} z_2(p, w, r, h)$, $\boldsymbol{D}_w z_2(p, w, r, h)$, and $\boldsymbol{D}_r z_2(p, w, r, h)$.

To show the regularity, it is sufficient to show the following condition:

$$\begin{vmatrix} -(1 - a_{11}) & a_{12} & \boldsymbol{D}_{p_2} z_1(p, w, r, h) & \boldsymbol{D}_w z_1(p, w, r, h) \\ a_{21} & -(1 - a_{22}) & \boldsymbol{D}_{p_2} z_2(p, w, r, h) & \boldsymbol{D}_w z_2(p, w, r, h) \\ 0 & 0 & -(1+r)\, a_{21} & -L_1 \\ 0 & 0 & 1 - (1+r)\, a_{22} & -L_2 \end{vmatrix} \neq 0.$$

Note that the following property holds:

$$\begin{vmatrix} -(1-a_{11}) & a_{12} & \boldsymbol{D}_{p_2}z_1(p,w,r,h) & \boldsymbol{D}_w z_1(p,w,r,h) \\ a_{21} & -(1-a_{22}) & \boldsymbol{D}_{p_2}z_2(p,w,r,h) & \boldsymbol{D}_w z_2(p,w,r,h) \\ 0 & 0 & -(1+r)\,a_{21} & -L_1 \\ 0 & 0 & 1-(1+r)\,a_{22} & -L_2 \end{vmatrix}$$
$$= \begin{vmatrix} -(1-a_{11}) & a_{12} \\ a_{21} & -(1-a_{22}) \end{vmatrix} \times \begin{vmatrix} -(1+r)\,a_{21} & -L_1 \\ 1-(1+r)\,a_{22} & -L_2 \end{vmatrix}.$$

In the above equation, the first component of the right hand side is non-zero, because the Hawkins-Simon condition holds. Moreover, the second component is also positive as $1-(1+r)\,a_{22} > 0$. Therefore, the above determinant is non-zero, which implies that the Jacobian matrix $\boldsymbol{D}_{(p,w,r,y)}F_{h,A,L}(p,w,r,y)$ has full-rank.

Note that the above argument can be applied to any Sraffian steady-state equilibrium in the economy (A,L,h). Thus, we can see that *every Sraffian steady-state equilibrium is regular* in the economy (A,L,h), which implies that (A,L,h) is regular.

2.2 Genericity of the Sraffian steady-state Equilibrium

To check the *genericity* stated in section 3.1 of the main text of the paper, now define the Jacobian $\boldsymbol{D}F$ with respect to (h,A,L), which is denoted by $\boldsymbol{D}_{h,A,L}F$, as below:

$$\boldsymbol{D}_{h,A,L}F = \begin{bmatrix} w & 0 & w & \boldsymbol{D}_{a_{11}}F^1 & \boldsymbol{D}_{a_{12}}F^1 & 0 & 0 & 0 & 0 \\ 0 & \frac{w}{p_2} & \frac{w}{p_2} & 0 & 0 & \boldsymbol{D}_{a_{21}}F^2 & \boldsymbol{D}_{a_{22}}F^2 & 0 & 0 \\ 0 & 0 & 0 & -(1+r) & 0 & -(1+r)p_2 & 0 & -w & 0 \\ 0 & 0 & 0 & 0 & -(1+r) & 0 & -(1+r)p_2 & 0 & -w \end{bmatrix}$$

where $\boldsymbol{D}_{a_{ij}}F^i$ is the partial derivative of the function $F^i(p,w,r,y,h,A,L) \equiv z_i(p,w,r,h) - (1-a_{ii})y_i + a_{ik}y_k$ (for $i,k = 1,2$ with $i \neq k$) with respect to a_{ij} (for $i,j = 1,2$). The first 3 columns of $\boldsymbol{D}_{h,A,L}F$ are for (h_1,h_2,h^o), the next 4 columns are for the components of A and the last 2 columns are for the components of L. It is easy to see that the matrix $\boldsymbol{D}_{h,A,L}F$ has full-rank, as the 1st and the 2nd column vectors and the 8th and the 9th column vectors constitute a 4×4 nonsingular square matrix:

$$\begin{bmatrix} w & 0 & 0 & 0 \\ 0 & \frac{w}{p_2} & 0 & 0 \\ 0 & 0 & -w & 0 \\ 0 & 0 & 0 & -w \end{bmatrix}.$$

Therefore, by applying the Transversality Theorem, we can see that the set of regular economies has *full measure*.

Now let us examine the openness of the set of regular economies. First, denote the set of economies as P and the set of regular economies as P_R. Suppose P_R is not open. Then there exists a sequence $\{(A, L, h)_k\}$ of non-regular economies converging to a regular economy $(A, L, h)_* \in P_R$. Correspondingly, there exists a sequence of non-regular equilibria $\{(p, w, r, y)_k\}$ which converges to a regular equilibrium $(p, w, r, y)_*$ at $(A, L, h)_*$. Then the corresponding Jacobian matrices $\boldsymbol{D}F_{(h,A,L)_k}(p, w, r, y)_k$ of 4 rows and 5 columns exist, which have less than full rank. For a Jacobian matrix, we can pick 5 separate square submatrices of order 4. The determinants of square submatrices of order 4 are all zero. Now we can define a continuous function, say c, from the set of Jacobian matrices to the set of 5-dimensional vectors whose components are determinants of square submatrices derived from the Jacobian $\boldsymbol{D}F_{h,A,L}$. Since $c(\boldsymbol{D}F_{h,A,L}) = (0, ..., 0) \in \mathbb{R}^5$ for any $\boldsymbol{D}F_{h,A,L}$ of less than full rank, $c(\boldsymbol{D}F_{(h,A,L)_k}(p, w, r, y)_k) = (0, ..., 0)_k \to (0, ..., 0) \in \mathbb{R}^5$ as $k \to \infty$.

Since $\{(0, ..., 0)_k\}$ converging to $(0, ..., 0)$ is closed in $\mathbb{R}^5$ and c is continuous, the inverse image $c^{-1}(\{(0, ..., 0)_k\}) = \left\{\boldsymbol{D}F_{(h,A,L)_k}(p, w, r, y)_k\right\}$ is closed. Its elements are Jacobian matrices from $P \backslash P_R$ of less than full rank. Since $\left\{\boldsymbol{D}F_{(h,A,L)_k}(p, w, r, y)_k\right\}$ is closed, $\boldsymbol{D}F_{(h,A,L)_*}(p, w, r, y)_*$ is contained in $\left\{\boldsymbol{D}F_{(h,A,L)_k}(p, w, r, y)_k\right\}$.

Note that $c(\boldsymbol{D}F_{(h,A,L)_*}(p, w, r, y)_*) = (0, ..., 0) \in \mathbb{R}^5$. This implies that the converging point of the sequence $\left\{\boldsymbol{D}F_{(h,A,L)_k}(p, w, r, y)_k\right\}$, each element of which is correspondingly defined from $(A, L, h)_k \in P \backslash P_R$, must also have less than full rank. In other words, the convergent point of the sequence of non-regular economies must also be non-regular. This contradicts the supposition of $(A, L, h)_* \in P_R$. Therefore, the set of regular economies P_R is *open*. ■