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# A new test for common breaks in heterogeneous panel data models

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# A new test for common breaks in heterogeneous panel data models [1]

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**Abstract**

In this paper, we develop a new test to detect whether break points are common in heterogeneous panel data models where the time series dimension T could be large relative to cross-section dimension N. The error process is assumed to be cross-sectionally independent. The test is based on the cumulative sum (CUSUM) of ordinary least squares (OLS) residuals. We derive the asymptotic distribution of the detecting statistic under the null hypothesis, while proving the consistency of the test under the alternative. Monte Carlo simulations and an empirical example show good performance of the test.

*JEL classification:* C12, C23

*Key words:* CUSUM test, panel data, structural change, common breaks

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## 1. Introduction

In recent years, panel data models have become increasingly popular in theoretical and empirical analyses, since richer information from both the cross-section and time series dimension leads to more powerful inferences than with a single cross-section or a single time series. In particular, the modeling and inferences of structural changes in panel frameworks have attracted significant attention in the literature. Compared to applying the single detection method for structural changes separately to each series, using cross-sectional datasets improves break detection power. The detection procedures in panels are often designed to test for the null hypothesis that the regression parameters in each series are constant over time against the alternative that at least one series exhibits structural changes. See, for example, Horváth and Hušková (2012) in a mean-shift panel model, De Wachter and Tzavalis (2012) and Hidalgo and Schafgans (2017) in dynamic panels, Pauwels *et al.* (2012) in panel data models allowing for heterogeneous coefficients, Chen and Huang (2018) in a time-varying panel data model, and Antoch *et al.* (2019) in panels with fixed T and large N, to name a few. However, the rejection of the null hypothesis leaves the researcher with no information as to which cross-sectional unit exhibits structural changes. Furthermore, it naturally leads to the issue of change point estimation in panel data models.

Classical change point estimation methodologies in panel literature often assume that break point occurred in each series at the same location, referred to as the common break point. This assumption is particularly attractive, as the common break phenomenon occurs in many practical applications. The other major advantage of this assumption is the increased accuracy of the change point estimate, as noted by Bai (2010). It is well known that only the break fraction (i.e., the break date divided by the sample size) can be consistently estimated in a single time series. In panel frameworks, however, the failure of the consistency of the break point in time series models has been overcome under the common break assumption. This enhanced precision of the common break point estimate has been widely confirmed under various frameworks in panel data analyses. Kim (2011, 2014) focused on panel deterministic time trend models and considered a factor structure for the error component. Although the former study stated that the ordinary least squares break date estimator fails to achieve consistency by imposing the factor structure, the latter overcame this problem and developed a new estimation strategy, where the common break date is estimated jointly with the common factor to successfully sustain the precision advantage of the common break point estimate in panels. In addition, Qian and Su (2016) used a panel data model in which the parameters of interest are homogeneous and errors are assumed to be cross-sectionally independent, while Baltagi *et al.* (2016) considered a more general panel framework allowing for heterogeneous parameters across individuals and multifactor error structure. More related works, including Li *et al.* (2016), Baltagi *et al.* (2017), Horváth *et al.* (2017), Westerlund (2019), and others, have documented that the break date estimate obtains increased precision via imposing a common break assumption in panels.

In practice, however, the common break assumption is restrictive, and some evidence has verified that the break points are likely to vary significantly across individuals (see Claeys and Vašíček 2014; Adesanya 2020). To the best of our knowledge, no study has focused on the validity of the common break assumption in panels. In this paper, we contribute to the literature in three ways. First, we fill in this gap to introduce a test for the null hypothesis that the panels exhibit a common break against the alternative that break dates can vary across units. The closest related work is that of Oka and Perron (2018), who considered common break detection in maximum likelihood frameworks in multiple equation systems. We extend their model to a more general framework where both the number of series N and the number of observations T are sufficiently large, which makes it available using panel or macroeconomic data in applications.

The second major contribution of this paper is that we investigate the statistical properties of the estimated common break point when the common break assumption fails. It is verified that the common break estimate cannot be consistent for each series, but will be restricted to a specific region. Based on this property, our test has a non-degenerate distribution under the null hypothesis and achieves consistency under the alternative.

Third, our test delivers monotonic power as the magnitude of the breaks increases. The statistic is established by the squares of the cumulative sum of the residuals, and we use a normalization factor to replace the long-run variance estimator to avoid power loss when the shift increases under the alternative (the so-called nonmonotonic power problem). Monte Carlo simulations show good size performance for large T. Moreover, the test can successfully reject the null hypothesis of a common break against various types of alternatives and has nontrivial power for large breaks. An empirical example demonstrates that a common break exists in the mutual fund data during the sub-prime crisis.

From a different perspective, recent clustering literature suggested an estimation methodology as an alternative strategy to identify distinct breaks across units in panels. The panel data are modeled using a grouped pattern, in which the regression coefficients containing break dates are heterogeneous across groups but homogeneous within a group. In this framework, Okui and Wang (2020) and Lumsdaine *et al*. (2020) proposed iterative estimation approaches to jointly estimate the break point, group membership structure, and coefficients. The consistency of all estimates can be achieved simultaneously within the prior information on the number of groups and an appropriate choice of the initial values for iteration. Researchers can determine whether to conduct a testing procedure, apply an estimation methodology, or use a hybrid of two approaches depending on their empirical purpose.

The remainder of this paper is organized as follows. Section 2 introduces the model and necessary assumptions. Section 3 explains the testing strategy for the common break assumption. Section 4 establishes the asymptotic distribution of the statistic under the null hypothesis and the consistency of the test under the alternative hypothesis. Monte Carlo simulations are conducted in Section 5. Section 6 provides an empirical example, and Section 7 provides concluding remarks. The mathematical proofs are relegated to the Appendix.

## 2. Model and Assumptions

We consider a panel data model allowing for heterogeneous coefficients across units, defined by

$$y_{it} = x_{it}'\beta_i + x_{it}'\delta_i 1_{\{t>k_i^0\}} + u_{it}, \quad 1 \le i \le N \,\text{and}\, 1 \le t \le T, \tag{1}$$

where $x_{it} = [x_{it}(1), \cdots, x_{it}(p)]'$ is $p$-dimensional explanatory variables including a constant term; thus, the first element is unity for all $t$. The coefficients $\beta_i = [\beta_{i1}, \cdots, \beta_{ip}]'$, $\delta_i = [\delta_{i1}, \cdots, \delta_{ip}]'$ are $p \times 1$ vectors of fixed parameters, and $1_{\{t>k_i^0\}}$ is an indicator function that takes the value one if $t > k_i^0$, and zero otherwise. $u_{it}$ is an unobservable stochastic disturbance. We assume that the regression parameters in the $i$th panel change from $\beta_i$ to $\beta_i + \delta_i$ at unknown time $k_i^0$, and we are interested in testing whether the break point in each series is common against the alternative that the break point varies across individuals. The null hypothesis is defined as

$$H_0: \ k_i^0 = k^0, \quad \text{for all } \ i = 1, 2, \cdots, N.$$

Under the alternative of distinct breaks across individuals, we suppose that there exist $G$ groups, and the regression coefficients share the common break point in each group $g = 1, 2, \cdots, G$. Then, the alternative hypothesis is defined by

$$H_A: \ k_{g_1}^0 \neq k_{g_2}^0, \quad \text{for some } \ g_1, g_2 \in \{1, 2, \cdots, G\}.$$

In this paper, we impose the following assumptions.

**Assumption 1** *$k_i^0 = [T\tau_i^0]$, where $\tau_i^0 \in (0, 1)$ and $[\cdot]$ is the greatest integer function.*

The break point $k_i^0$, which is a positive fraction of the total sample size, is assumed to be bounded away from the end points. This is a conventional assumption in the change point literature, see Bai (1997).

**Assumption 2** *Define $\phi_N = \sum_{i=1}^N \delta_i^{0\prime} \delta_i^0$. Suppose that*
*(i) $\phi_N \to \infty$ as $N \to \infty$,*
*(ii) $\frac{\phi_N}{N}$ is bounded as $N \to \infty$,*
*(iii) $\frac{T}{N} \to \infty$, $\phi_N \frac{\sqrt{T}}{N} \to \infty$ as $(T, N) \to \infty$.*

Denote $\delta_i^0$ as the true shift for individual $i$. Assumptions 2(i)–(ii) are borrowed from Assumption A2 in Baltagi *et al.* (2016). The additional condition $T/N \to \infty$ requires that $T$ grows at a faster rate than $N$. This is a significant condition to ensure a non-degenerate distribution of the statistic under the null hypothesis and consistency of the test under the alternative.

**Assumption 3** *(i) For each series $i$, $u_{it}$ is independent of $x_{it}$ for all $i$ and $t$;*
*(ii) $u_{it} = \sum_{j=0}^{\infty} a_{ij} \epsilon_{i,t-j}$, $\epsilon_{it} \sim (0, \sigma_{i\epsilon}^2)$ are i.i.d over all $i$ and $t$; $\sum_j j|a_{ij}| \leq M$ for all $i$.*

The idiosyncratic errors form a stationary time series, and it is assumed that $u_{it}$ are cross-sectionally independent, similar to the assumption in Bai (2010). In practice, this assumption is relatively restrictive, as cross-sectional dependence commonly exists in many panel datasets. As explained in Sections 3 and 4, the statistic of our test can have a non-degenerate distribution under the null hypothesis of the common break, crucially depending on the consistency of the common change point estimate. However, Kim (2011) indicated that imposing a factor structure on the error component may impede the consistency property. Some additional techniques are needed if we relax Assumption 3 to allow for cross-sectional dependence.

**Assumption 4** *(i) For $i = 1, \cdots, N$, the matrices $(1/j)\sum_{t=1}^{j} x_{it}x_{it}'$, $(1/j)\sum_{t=T-j+1}^{T} x_{it}x_{it}'$ $(1/j)\sum_{t=k_i^0-j+1}^{k_i^0} x_{it}x_{it}'$, and $(1/j)\sum_{t=k_i^0+1}^{k_i^0+j} x_{it}x_{it}'$ are stochastically bounded and have minimum eigenvalues uniformly bounded away from zero in probability for all large $j$.*
*(ii) For each $i$, $(1/T)\sum_{t=1}^{T} x_{it}x_{it}'$ converges in probability to a nonrandom and positive definite $p \times p$ matrix $C_i$ as $T \to \infty$.*
*(iii) For each $i$, $(1/T)\sum_{t=1}^{T} x_{it}$ converges in probability to a $p \times 1$ vector $c_{i1}$ as $T \to \infty$.*

Denote the $j$th row of $C_i$ by $c_{ij}$ for $j = 1, \cdots, p$. That is, $C = [c_{i1}, \cdots, c_{ip}]'$. Note that the vector $c_{i1}'$ is the first row of $C_i$.

**Assumption 5** *(i) For any positive finite integer $s$, the matrices $(1/N)\sum_{i=1}^{N}\sum_{t=k_i^0-s+1}^{k_i^0} x_{it}x_{it}'$ and $(1/N)\sum_{i=1}^{N}\sum_{t=k_i^0+1}^{k_i^0+s} x_{it}x_{it}'$ are stochastically bounded and have minimum eigenvalues uniformly bounded away from zero in probability for all large $N$.*
*(ii) For each $t$, $(1/N)\sum_{i=1}^{N} x_{it}x_{it}'$ is stochastically bounded as $N \to \infty$.*

Assumption 4 is a conventional assumption in time series models, see, for example, Bai (1997), while Assumption 5 is borrowed from Assumption 5 in Baltagi *et al.* (2016).

## 3. Test Statistic

The null hypothesis assumes that the panels exhibit one break occurring at an unknown common location. We first use the least squares method, as proposed by Baltagi *et al.* (2016), to estimate the common break point. Let

$$Y_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{iT} \end{bmatrix}, X_i = \begin{bmatrix} x'_{i1} \\ x'_{i2} \\ \vdots \\ x'_{iT} \end{bmatrix}, Z_i(k_i) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x'_{i(k_i+1)} \\ \vdots \\ x'_{iT} \end{bmatrix}, \text{and} \quad u_i = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iT} \end{bmatrix}.$$

The model with an unknown break point $k_i$ can be rewritten in matrix form as

$$\begin{aligned} Y_i &= X_i\beta_i + Z_i(k_i)\delta_i + u_i \\ &= [X_i, Z_i(k_i)] \begin{bmatrix} \beta_i \\ \delta_i \end{bmatrix} + u_i \\ &= \bar{X}_i(k_i)b_i + u_i, \end{aligned} \tag{2}$$

where $\bar{X}_i(k_i) = [X_i, Z_i(k_i)]$, and $b_i = [\beta'_i, \delta'_i]'$. Given any $k^* = 1, 2, \cdots, T-1$, $b_i$ can be estimated by

$$\hat{b}_i(k^*) = \begin{bmatrix} \hat{\beta}_i(k^*) \\ \hat{\delta}_i(k^*) \end{bmatrix} = [\bar{X}_i(k^*)'\bar{X}_i(k^*)]^{-1}\bar{X}_i(k^*)'Y_i, \quad i = 1, \cdots, N.$$

The sum of squared residuals for $i$th equation is given by

$$SSR_i(k^*) = [Y_i - \bar{X}_i(k^*)\hat{b}_i(k^*)]'[Y_i - \bar{X}_i(k^*)\hat{b}_i(k^*)], \quad i = 1, \cdots, N.$$

The least squares estimator of $k^*$ is defined as

$$\hat{k} = \arg \min_{1 \leq k^* \leq T-1} \sum_{i=1}^{N} \pi_i SSR_i(k^*). \tag{3}$$

where weights $\pi_i \in (0, 1), i = 1, \cdots, N$, $\sum_{i=1}^{N} \pi_i = 1$.

Our statistic is composed of ordinary least squares residuals based on the estimated common break point $\hat{k}$. We decompose the panels into two regimes using $\hat{k}$ in the time series dimension. Then, the OLS residuals are calculated by

$$\hat{u}_i = \begin{bmatrix} \hat{u}_{i1} \\ \hat{u}_{i2} \\ \vdots \\ \hat{u}_{iT} \end{bmatrix} = Y_i - \bar{X}_i(\hat{k})\hat{b}_i(\hat{k}), \tag{4}$$

and the squares of the partial sum of the OLS residuals $\hat{u}_{it}$ are defined by

$$US_{NT}(k,\hat{k}) = \left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}\hat{u}_{it}\right)^2, \quad \text{where } k=[T\tau] \text{ with } \tau\in(0,1). \tag{5}$$

The statistic is a CUSUM-type of residuals, motivated by the consistency of the break point estimate if the common break assumption holds. Under the null hypothesis that all individuals are assumed to share a common change point $k^0 = [T\tau^0]$ with $\tau^0 \in (0,1)$, Baltagi *et al.* (2016) verified that the common break date is consistently estimated. Based on the consistency of $\hat{k} \xrightarrow{p} k^0$, the regression parameters corresponding to regimes $\{x_{i1},\cdots,x_{i\hat{k}}\}$, $\{x_{i(\hat{k}+1)},\cdots,x_{iT}\}$ are asymptotically constant over time. Consequently, the cumulative sums of the corresponding residuals will not diverge and can have a non-degenerate distribution, which is derived as follows:

$$US_{NT}(k,\hat{k}) \quad \Rightarrow \quad \begin{cases}\sigma^2[W(\tau)-\frac{\tau}{\tau^0}W(\tau^0)]^2 & \text{if } \tau\le\tau^0\\ \sigma^2[W(\tau)-W(\tau^0)-\frac{\tau-\tau^0}{1-\tau^0}(W(1)-W(\tau^0))]^2 & \text{if } \tau>\tau^0\end{cases},$$

where $W(\cdot)$ is a one-dimensional Brownian motion, and $\sigma^2$ is the long-run variance defined below. Under the alternative of distinct breaks, since the estimated common break point cannot coincide with the true break point for each series, partial residuals will significantly deviate from the one under the null hypothesis. Hence, $US_{NT}(k,\hat{k})$ will diverge to infinity as $N,T\to\infty$ such that we can successfully reject the null hypothesis.

A traditional approach is to use a consistent estimate to replace the unknown $\sigma^2$, while the kernel estimator is commonly applied. Typically, the selection of the bandwidth for the kernel estimator significantly affects the size and power performance of the test. In time series analyses, it has been extensively mentioned that the structural change tests suffer from the so-called non-monotonic power problem; that is, the tests may lose power as the magnitude of the break increases. See Vogelsang (1999), Deng and Perron (2008), Yamazaki and Kurozumi (2015), and Jiang and Kurozumi (2019), among others. The main reason is that the long-run variance estimated under the null hypothesis is consistent but may be severely biased under the alternative hypothesis. To maintain nontrivial detection power for large breaks, we extend the self-normalization method proposed by Shao and Zhang (2010) to construct a normalization factor instead of using the long-run variance estimate. This normalization factor $V_{NT}(k_1, \hat{k}, k_2)$ is required to be proportional to $\sigma^2$ such that the long-run variance can be canceled out as

$$\frac{US_{NT}(k, \hat{k})}{V_{NT}(k_1, \hat{k}, k_2)} \Rightarrow \frac{\sigma^2 \text{ functional of Brownian motions}}{\sigma^2 \text{ functional of Brownian motions}},$$

where the long-run variance $\sigma^2$ is $\lim_{(N,T)\to\infty} E\left(\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{T} u_{it}\right)^2$. Furthermore, the normalization process cannot grow at a faster rate relative to the process $US_{NT}(k, \hat{k})$ under the alternative to avoid loss of power. To this end, we separate the panels into four regimes by flexible points $k_1, k_2$ and the estimated break point $\hat{k}$, where $k_1$ and $k_2$ take values in the interval $1 \leq k_1 < \hat{k} < k_2 \leq T-1$. We estimate the model on the basis of four regimes $\{x_{i1}, \cdots, x_{ik_1}\}$, $\{x_{i(k_1+1)}, \cdots, x_{i\hat{k}}\}$, $\{x_{i(\hat{k}+1)}, \cdots, x_{ik_2}\}$, and $\{x_{i(k_2+1)}, \cdots, x_{iT}\}$ for the $i$th equation. Denote $T \times p$ matrices by

$$\begin{aligned} X_{ji}(a, b) &= [0, \cdots, 0, x_{i,a+1}, \cdots, x_{i,b}, 0, \cdots, 0]', \ j = 1, 2, \quad (6) \\ X_{3i}(a) &= [0, \cdots, 0, x_{i,a+1}, \cdots, x_T]', \quad (7) \end{aligned}$$

where the elements of the $(a+1)$th-$b$th rows of $X_{ji}(a, b)$ are the same as that of $X_i$ and zero otherwise, and the elements of the $(a+1)$th–$T$th rows of $X_{3i}(a)$ are the same as that of $X_i$ and zero otherwise. Then, the model can be represented by

$$\begin{aligned} Y_i &= [X_i, X_{1i}(k_1, \hat{k}), X_{2i}(\hat{k}, k_2), X_{3i}(k_2)] \begin{bmatrix} \beta_i \\ \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{bmatrix} + u_i \\ &= X_i\beta_i + X_{1i}(k_1, \hat{k})\delta_{1i} + X_{2i}(\hat{k}, k_2)\delta_{2i} + X_{3i}(k_2)\delta_{3i} + u_i \\ &= \tilde{X}_i(k_1, \hat{k}, k_2)b_i + u_i, \quad (8) \end{aligned}$$

where $\tilde{X}_i(k_1, \hat{k}, k_2) = [X_i, X_{1i}(k_1, \hat{k}), X_{2i}(\hat{k}, k_2), X_{3i}(k_2)]$. Using the coefficient estimators $\tilde{\beta}_i$, $\tilde{\delta}_{1i}$, $\tilde{\delta}_{2i}$, and $\tilde{\delta}_{3i}$, the corresponding residuals are calculated as

$$\tilde{u}_i = \begin{bmatrix} \tilde{u}_{i1} \\ \tilde{u}_{i2} \\ \vdots \\ \tilde{u}_{iT} \end{bmatrix} = Y_i - X_i\tilde{\beta}_i - X_{1i}(k_1, \hat{k})\tilde{\delta}_{1i} - X_{2i}(\hat{k}, k_2)\tilde{\delta}_{2i} - X_{3i}(k_2)\tilde{\delta}_{2i}. \quad (9)$$

Then, we define the process $V_{NT}(k_1, \hat{k}, k_2)$ based on the residuals $\tilde{u}_{it}$ as

$$\begin{aligned} & V_{NT}(k_1, \hat{k}, k_2) \\ = \quad & \frac{1}{T}\sum_{s=1}^{k_1}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{s}\tilde{u}_{it}\right)^2 + \frac{1}{T}\sum_{s=k_1+1}^{\hat{k}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{\hat{k}}\tilde{u}_{it}\right)^2 \\ + \quad & \frac{1}{T}\sum_{s=\hat{k}+1}^{k_2}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{s}\tilde{u}_{it}\right)^2 + \frac{1}{T}\sum_{s=k_2+1}^{T}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{T}\tilde{u}_{it}\right)^2. \end{aligned} \tag{10}$$

Thus, our test statistic is composed of the squared CUSUM of residuals (5) and the normalization factor (10), defined by

$$\begin{aligned} & S_{NT}(k, k_1, k_2) = \sup_{(k,k_1,k_2)\in\Omega(\epsilon)} \frac{US_{NT}(k, \hat{k})}{V_{NT}(k_1, \hat{k}, k_2)} \\ = \quad & \sup_{(k,k_1,k_2)\in\Omega(\epsilon)} \left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}\hat{u}_{it}\right)^2 \left\{\frac{1}{T}\sum_{s=1}^{k_1}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{s}\tilde{u}_{it}\right)^2\right. \\ + \quad & \frac{1}{T}\sum_{s=k_1+1}^{\hat{k}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{\hat{k}}\tilde{u}_{it}\right)^2 + \frac{1}{T}\sum_{s=\hat{k}+1}^{k_2}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{s}\tilde{u}_{it}\right)^2 \\ + \quad & \left.\frac{1}{T}\sum_{s=k_2+1}^{T}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{T}\tilde{u}_{it}\right)^2\right\}^{-1}, \end{aligned}$$

where $\Omega(\epsilon) = \{(k, k_1, k_2)\,\text{or}\,(\tau, \tau_1, \tau_2) : [T\epsilon] \leq k \leq [T(1-\epsilon)], [T\epsilon] \leq k_1 \leq \hat{k} - [T\epsilon], \hat{k} + [T\epsilon] \leq k_2 \leq [T(1-\epsilon)]\}$. $k = [T\tau]$, $k_1 = [T\tau_1]$ and $k_2 = [T\tau_2]$ with $\tau, \tau_1, \tau_2 \in (0, 1)$.

## 4. Asymptotic Theory

We next derive the limiting properties of the test statistic.

**Theorem 1** *Suppose that Assumptions 1–5 hold. Then, under $H_0$, we have, as $N, T \to \infty$,*

$$
\begin{aligned}
& S_{NT}(k, k_1, k_2) \\
\Rightarrow \; & \sup_{(\tau,\tau_1,\tau_2)\in\Omega(\epsilon)} \left\{ W(\tau) - \tau \frac{W(\tau^0)}{\tau^0} - (\tau - \tau^0)\left[ \frac{W(1) - W(\tau^0)}{1-\tau^0} - \frac{W(\tau^0)}{\tau^0} \right] 1_{\{\tau > \tau^0\}} \right\}^2 \\
& \left\{ \int_0^{\tau_1} \left[ W(s) - s\frac{W(\tau_1)}{\tau_1} \right]^2 ds + \int_{\tau_1}^{\tau_0} \left[ W(\tau^0) - W(s) - (\tau^0 - s)\frac{W(\tau^0) - W(\tau_1)}{\tau^0 - \tau_1} \right]^2 ds \right. \\
& + \int_{\tau^0}^{\tau_2} \left[ W(s) - W(\tau^0) - (s - \tau^0)\frac{W(\tau_2) - W(\tau^0)}{\tau_2 - \tau^0} \right]^2 ds \\
& \left. + \int_{\tau_2}^{1} \left[ W(1) - W(s) - (1-s)\frac{W(1) - W(\tau_2)}{1-\tau_2} \right]^2 ds \right\}^{-1},
\end{aligned}
$$

*where $W(\cdot)$ is a standard Brownian motion, $k = [T\tau], k_1 = [T\tau_1], k^0 = [T\tau^0]$, and $k_2 = [T\tau_2]$ with $\tau, \tau_0, \tau_1, \tau_2, \in (0,1)$.*

Under the null hypothesis, the proposed test has a non-standard limit distribution depending on the true break fraction, which is unknown in practice. We choose $\tau^0 = 0.1, 0.2, \cdots, 0.9$, and approximate Brownian motions using 2,000 independent normal random variables with 10,000 replications to obtain the critical values in Table 1. A researcher can calculate an appropriate critical value based on the value of the estimated break fraction. For example, if $\hat{\tau} \in [0.4, 0.5)$, we obtain the critical value by the interpolation,

$$c = c_{0.4} + 10(\hat{\tau} - 0.4)(c_{0.5} - c_{0.4}),$$

where $c_{\tau^0}$ for $\tau^0 = 0.1, \ldots, 0.9$ are the critical values given in Table 1. Next, we investigate the behavior of the proposed test statistic when the breaks vary across individuals. We focus on the case in which there are two groups, and individuals in the same group share a common break $k_j^0, j = 1, 2$.

$$H_{1A} : |k_1^0 - k_2^0| \geq \Delta T, \text{ for some } \Delta > 0.$$

**Assumption 6** *Let $N_j, j = 1, 2$, denote the number of units in group $j$ ($N = N_1 + N_2$). Suppose that $N_j/N \to \pi_j > 0$ for $j = 1, 2$.*

To characterize the limiting properties of the test statistic under the alternative, it is useful to first state some preliminary results regarding the statistical properties of the estimated common break point. Define $K(C) = \{k : 1 \leq k < k_1^0 - C_1, k_2^0 + C_2 < k \leq T-1\}$, where $C_1, C_2$ are finite numbers.

**Proposition 1** *Suppose that Assumptions 1–6 hold. Then, under $H_{1A}$, for any given $\epsilon > 0$, for both large $N$ and $T$,*

$$P(\hat{k} \in K(C)) < \epsilon.$$

Proposition 1 states the possible region of the location of the common break date estimator when the common break assumption fails. This implies that this estimator will be bounded away from both end points. In other words, the estimated common break point may lie between the two true break points or be stochastically bounded by either of the true break dates.

**Proposition 2** *Suppose that Assumptions 1–6 hold. Under $H_{1A}$, for both large $N$ and $T$,*
*(i) if $\hat{k} < k_1^0$,*

$$\sup_{k \in \Omega(\epsilon)} US_{NT}(k, \hat{k}) = O_p(NT),$$

*(ii) if $k_1^0 \leq \hat{k} \leq k_2^0$,*

$$\sup_{k \in \Omega(\epsilon)} US_{NT}(k, \hat{k}) = O_p(NT),$$

*(iii) if $k_2^0 < \hat{k}$,*

$$\sup_{k \in \Omega(\epsilon)} US_{NT}(k, \hat{k}) = O_p(NT),$$

Proposition 2 derives the divergence rate of the process $US_{NT}(k, \hat{k})$ under the alternative. In case (i), Proposition 1 implies that the common change point estimate is bounded by the true break point $k_1^0$; that is, $k_1^0 - \hat{k} = O_p(1)$. Since we assume that the two true breaks are separated by some positive fraction of the sample size, $\hat{k}$ will become distant from the other break date $k_2^0$. Therefore, for individuals in group 2, the regression parameters will be estimated based on an inconsistent break fraction estimate. Then, we find that the CUSUM of the corresponding residuals $\hat{u}_{it}$ in $US_{NT}(k, \hat{k})$ will diverge to infinity at a rate of $NT$. For the second and third cases, it is shown that the divergence rate of the process $US_{NT}(k, \hat{k})$ is the same as that in case (i).

**Proposition 3** *Suppose that Assumptions 1–6 hold. Under $H_{1A}$, for any given $\epsilon > 0$, there exists a finite $M > 0$ such that, for both large $N$ and $T$,*

*(i)*

$$P\Big(\inf_{(k_1,k_2)\in\Omega(\epsilon)} V_{NT}(k_1,\hat{k},k_2) > M \Big| k_1^0 - C_1 < \hat{k} \leq k_1^0\Big) < \epsilon,$$

*(ii)*

$$P\Big(\inf_{(k_1,k_2)\in\Omega(\epsilon)} V_{NT}(k_1,\hat{k},k_2) > M \Big| k_1^0 < \hat{k} < k_2^0\Big) < \epsilon,$$

*(iii)*

$$P\Big(\inf_{(k_1,k_2)\in\Omega(\epsilon)} V_{NT}(k_1,\hat{k},k_2) > M \Big| k_2^0 \leq \hat{k} < k_2^0 + C_2\Big) < \epsilon.$$

Proposition 3 investigates the limiting properties of the normalization process under the alternative. The results indicate that $\inf_{(k_1,k_2)\in\Omega(\epsilon)} V_{NT}(k_1,\hat{k},k_2)$ is $O_p(1)$. Since the model is estimated based on four subsamples for the normalization factor, we can eventually find appropriate $k_1$ and $k_2$ such that the minimization will not diverge. The numerator of the statistic diverges at a rate of $NT$, and the denominator has a finite limit. Then, we derive the consistency of the test under the alternative in the following theorem:

**Theorem 2** *Suppose that Assumptions 1–6 hold. Then, under $H_{1A}$, we have, as $N, T \to \infty$,*

$$S_{NT}(k,k_1,k_2) \to \infty.$$

The consistency of this test is achieved under a particular and specified alternative $H_{1A}$. Nevertheless, our simulations confirm that this test is valid and powerful against a variety of alternatives.

## 5. Finite Sample Properties

In this section, we investigate the finite sample performance of the test considered in the previous sections. The data-generating process (DGP.1) under the null hypothesis of a common break is given by

$$y_{it} = x_{it}'\beta_i + x_{it}'\delta_i 1_{\{t>k^0\}} + u_{it},\ i = 1,\cdots,N,\ t = 1,\cdots,T.$$

where $x_{it} = [1, z_{it}]'$ includes a constant, each $z_{it}$ has a normal distribution $N(1, 1)$, and is independent of the errors $u_{it}$, $1 \leq t \leq T, 1 \leq i \leq N$. We assume that a common break $k^0 = [0.5T]$ exists in the slopes. The coefficients $\beta_i \sim i.i.d.U(0, 0.8)$ and $\delta_i$ are the jumps for each series with $\delta_i \sim i.i.d.U(0, 0.5)$. We allow for serial correlation in the errors $u_{it} = \rho u_{i(t-1)} + e_{it}$ with $e_{it} \sim i.i.d.N(0, (1-\rho)^2)$. The trimming parameter $\epsilon$ is 0.1, the number of replications is 2,000, and all computations are conducted using the GAUSS matrix language.

Table 2 summarizes the empirical sizes of the test for the different pairs of $(N, T)$. In the case of *i.i.d.* errors, the nominal rejection rate is close to the corresponding significance level of the test. When the errors are allowed to be serially correlated with $\rho = 0.4$, for small $N$ and $T$, the size distortion is quite noticeable. The size improves for large T and appears to be quite close to the nominal level at $T = 200$.[2]

In practice, no prior information is available on the form of structural changes for researchers. Therefore, we conduct extensive simulations to explore the empirical power of the test for various group patterns of structural change and different magnitudes of the break. We first impose a benchmark case as Assumption 6. There are two groups in panels, and the break points for individuals are common in the same group but distinct across groups. We next consider more general circumstances in which there are more than two groups in panels or the break dates can be distinct across individuals. Moreover, we are interested in the validity of the test when the break dates are common, but multiple common breaks occurred in the panels. Then, four types of alternative hypotheses are considered as follows:

- $H_{1A}$: There are two groups and the series in each group share common break $k_j^0$, $j = 1, 2$. Let $N_j$ denote the number of units in group $j$ and $N = N_1 + N_2$.
- $H_{2A}$: There are three groups and the series in each group share common break $k_j^0$, $j = 1, 2, 3$. Note that $N = N_1 + N_2 + N_3$.
- $H_{3A}$: Suppose that there is no group pattern. The break point for the $j$th series is given by $k_j^0$, $j = 1, 2, \cdots, N$.

[2]The size is distorted corresponding to strong serial correlation ($\rho = 0.8$) but appears to be controlled when $T$ increases. The results are similar and are thus omitted.

- $H_{4A}$: The panel data exhibit multiple common break dates.

The data generating process (DGP.2) under $H_{1A}$ is given by

$$\begin{cases} y_{it} = x_{it}'\beta_i + x_{it}'\delta_{1i}1_{\{t>k_1^0\}} + u_{it} & t = 1, \cdots, T, \quad \text{for } i \text{ in group 1,} \\ y_{it} = x_{it}'\beta_i + x_{it}'\delta_{2i}1_{\{t>k_2^0\}} + u_{it} & t = 1, \cdots, T, \quad \text{for } i \text{ in group 2,} \end{cases}$$

which is the same as DGP.1, except that the change point varies across groups. The first group exhibits one common break at $k_1^0 = [0.25T]$, and we set the time of change $k_2^0$ equal to $[0.75T]$ in the second group. We assume $\beta_i \sim i.i.d.U(0, 0.8)$ and the jumps $\delta_{1i}, \delta_{2i} \sim i.i.d.U(0, 0.5)$. The ratio of units among the groups is set to $N_1 : N_2 = 5 : 5$. Table 3 shows that the test is powerful (almost with a rejection probability of more than 80%), except for small N or T. Table 4 reports the effect of the magnitude of change on power. As expected, the proposed test delivers monotonic power. When the magnitude of the changes is larger than 0.4, our test almost perfectly rejects the null hypothesis (power tends to one). We can see that the test shows good performance in the case of two well-separated groups. We further investigate the sensitivity of the test when the group characteristics (distance between two change points or number of units in each group) change. In Table 5, we fix one common break at $[0.2T]$, and the other break changes from $[0.25T]$ to $[0.8T]$. If the distance between two breaks exceeds [0.3T], the rejection probability reaches at least 90% at the 10% significance level. When the two break dates become quite close (the distance is less than [0.1T]), the power of the test decreases to 0.325 at the 10% significance level. On the other hand, the power of the test is sensitive to the number of individuals in each group. Table 6 shows that the test rejects the null hypothesis with probability over 80% when the number of observations in each group is sufficiently large (the ratio of units between two groups is larger than 3/7). If the number of individuals in one group is much less than that in the second group, the heterogeneity between the two groups cannot be identified. Eventually, it is not easy to reject the null hypothesis, even if the two break dates are distinct.

We next investigate the power properties of the test under $H_{2A}$. The results for distinct change point locations and ratios of units among groups are reported in Table 7. The test can successfully reject the null of one common break for large N. The close break points among the three groups will reduce the power.

The alternative hypothesis $H_{3A}$ considers heterogeneous change points without a group pattern. The change point for individual $j$ is set to $k_j^0 = [T\tau_j^0]$, $j = 1, 2, \cdots, N$, while the break fraction $\tau_j^0$ is drawn from $U(0.15, 0.75)$. Table 8 shows that the test is still powerful for large N.

The data generating process (DGP.3) under $H_{4A}$ is given by

$$y_{it} = x_{it}'\beta_i + x_{it}'\delta_{1i}1_{\{k_1^0 < t \leq k_2^0\}} + x_{it}'\delta_{2i}1_{\{t > k_2^0\}} + u_{it},\ t = 1, \cdots, T,\ \ i = 1, \cdots, N,$$

where the coefficients change from $\beta_i$ to $\beta_i + \delta_{1i}$ in the second regime and change from $\beta_i + \delta_{1i}$ to $\beta_i + \delta_{2i}$ in the third regime. The change points are set to $k_1^0 = [0.25T]$ and $k_2^0 = [0.75T]$, while the coefficients $\beta_i \sim i.i.d.U(0, 0.8)$, $\delta_{1i} \sim i.i.d.U(0, 0.5)$, and $\delta_{2i} \sim i.i.d.U(0, 0.2)$. Table 9 shows the good performance of the test when there exist two common breaks in the panels.

In summary, the size of our test is controlled for large N and T. The test exhibits monotonic power as the magnitude of the break increases and is powerful against various alternatives.

## 6. Empirical Example

In this section, we apply our approach to detect common breaks in the capital asset pricing model (CAPM). We use the Fama-French three-factor model augmented with the Carhart (1997) momentum factor considered in Antoch *et al.* (2019), which is given by

$$R_{it} - R_t^f = \alpha_{it} + \left(R_t^M - R_t^f\right)\beta_{it}^M + R_t^{HML}\beta_{it}^{HML} + R_t^{SMB}\beta_{it}^{SMB} + R_t^{MOM}\beta_{it}^{MOM} + u_{it},$$

for $1 \leq t \leq T$ and $1 \leq i \leq N$, where $R_{it} - R_t^f$ denotes the excess return on the mutual fund; the three factors include market risk premium, returns on a high minus low (HML) portfolio, and returns on a small minus big (SMB) portfolio; the momentum factor $R_t^{MOM}$ describes the tendency of securities that have outperformed (or underperformed) the market over the past period to continue to outperform (or underperform) the market.

We test for common breaks in the coefficients for the mutual fund return data around the sub-prime crisis. Our sample period is from February 2005 to December 2011. Four factors can be downloaded from Ken French's data library.[3] The monthly return data of mutual funds

[3] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

are taken from Yahoo Finance. Mutual funds are classified according to their size, growth, value characteristics, and investment strategies. Using the Yahoo Finance classification, we focus on the characteristics of blend and growth to select ten categories of mutual funds. These are Foreign Large Blend, Foreign Small/Mid Blend, Foreign Large Growth, Foreign Small/Mid Growth, Large Blend, Mid-Cap Blend, Small Blend, Large Growth, Mid-Cap Growth, and Small Growth.

First, we apply the test to detect common breaks in the whole sample period of 2005M02–2011M12. The results in panel (a) of Table 10 show that the test rejects the one common break assumption at the 1% significance level for the nine categories. In this case, there are several possibilities such that there is no common break and each series (or some groups) has a distinct one or several breaks, or there are multiple common breaks. To see the overall tendency, we tentatively apply the Bai-Perron sequential test (Bai and Perron, 1998) to estimate the number and locations of the breaks for several mutual funds in each group. We find multiple break points, which are centralized at similar locations (early 2006, early 2008, and early 2009), even if the mutual funds are from different categories. These results suggest the possibility of multiple common breaks in mutual fund data.

Based on the above result and because Anotch *et al.* (2019) indicated that there exist structural changes in US mutual fund data during the sub-prime crisis period (mid-2008 to early 2009), we split the whole sample period into (b) the period before the sub-prime crisis (2005M02–2008M05), and (c) the period during the sub-prime crisis (2008M06–2011M12). In panel (b) of Table 10, the test rejects the null hypothesis for all categories in the period before the sub-prime crisis (before the middle of 2008); there still exists the possibility of distinct breaks in each series or multiple common breaks in this sub-period. On the other hand, as in panel (c) of Table 10, our test cannot reject the null hypothesis for the period 2008M06–2011M12. This result implies that the mutual fund data exhibit one common break during the sub-prime crisis.

## 7. Conclusion

In this study, we developed a new test based on the OLS residuals to detect whether structural breaks across individuals occurred at the common location in panel data models. The asymptotic properties of the test were investigated under the null and alternative hypotheses. The simulation results indicated that the test is powerful against various alternatives. In application, we found evidence of the common break phenomenon in mutual fund data during the sub-prime crisis. Although we assumed cross-sectional independence throughout the paper, it may be interesting for the cross-sectional dependence in the error component to be generally taken into account. We leave such an extension for our future research.

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## Appendix A. Proof of Theorem 1

Supposing that the structural change occurred at a common location, Baltagi *et al*. (2016) showed the consistency of the common break estimator,

$$\lim_{(N,T)\to\infty} P(\hat{k}=k^0)=1, \quad \text{which implies} \quad |\hat{k}-k^0|=o_p(1). \tag{A.1}$$

In this Appendix, we derive the asymptotic distribution of the test statistic under the null hypothesis using this consistency property. We first focus on the limiting properties of the numerator of the statistic. Model (2) with the true common break $k^0$ is expressed as follows:

$$\begin{aligned} Y_i &= \bar{X}_i(k^0)b_i^0 + u_i \\ &= \bar{X}_i(\hat{k})b_i^0 + u_i + [\bar{X}_i(k^0)-\bar{X}_i(\hat{k})]b_i^0 \\ &= \bar{X}_i(\hat{k})b_i^0 + u_i + [Z_i(k^0)-Z_i(\hat{k})]\delta_i^0, \end{aligned} \tag{A.2}$$

where $b_i^0=[\beta_i^{0\prime},\delta_i^{0\prime}]'$. Replacing $Y_i$ with (A.2), the residuals in (4) can be rewritten as

$$\begin{aligned} \hat{u}_i &= \bar{X}_i(\hat{k})b_i^0 + u_i + [Z_i(k^0)-Z_i(\hat{k})]\delta_i^0 - \bar{X}_i(\hat{k})\hat{b}_i(\hat{k}) \\ &= u_i - \bar{X}_i(\hat{k})[\hat{b}_i(\hat{k})-b_i^0] + [Z_i(k^0)-Z_i(\hat{k})]\delta_i^0 \\ &= u_i - X_i[\hat{\beta}_i(\hat{k})-\beta_i^0] - Z_i(\hat{k})[\hat{\delta}_i(\hat{k})-\delta_i^0] + [Z_i(k^0)-Z_i(\hat{k})]\delta_i^0, \end{aligned} \tag{A.3}$$

whose vector form is represented by

$$\begin{bmatrix} \hat{u}_{i1} \\ \hat{u}_{i2} \\ \vdots \\ \hat{u}_{iT} \end{bmatrix} = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iT} \end{bmatrix} - \begin{bmatrix} x_{i1} \\ x'_{i2} \\ \vdots \\ x'_{iT} \end{bmatrix} (\hat{\beta}_i(\hat{k})-\beta_i^0) - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x'_{i(\hat{k}+1)} \\ \vdots \\ x'_{iT} \end{bmatrix} (\hat{\delta}_i(\hat{k})-\delta_i^0) + \left( \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x'_{i(k^0+1)} \\ \vdots \\ x'_{iT} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x'_{i(\hat{k}+1)} \\ \vdots \\ x'_{iT} \end{bmatrix} \right) \delta_i^0$$

For the sake of simplicity, $\hat{k}$ is suppressed in $\hat{\beta}_i(\hat{k})$ and $\hat{\delta}_i(\hat{k})$. Then, the cumulative sum of the residuals is

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}\hat{u}_{it} \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}u_{it} - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}x_{it}'(\hat{\beta}_i-\beta_i^0) - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{k}x_{it}'(\hat{\delta}_i-\delta_i^0)1_{\{k>\hat{k}\}} \\
&+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=k^0+1}^{k}x_{it}'\delta_i^0 1_{\{k^0<k\le\hat{k}\}} + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=k^0+1}^{\hat{k}}x_{it}'\delta_i^0 1_{\{k^0<\hat{k}<k\}} \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{k}x_{it}'\delta_i^0 1_{\{\hat{k}<k\le k^0\}} - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{k^0}x_{it}'\delta_i^0 1_{\{\hat{k}<k^0<k\}} \\
&= U_1 - U_2 - U_3 + U_4 + U_5 - U_6 - U_7. \qquad \text{(A.4)}
\end{aligned}
$$

We can show that the terms $U_4, U_5, U_6$, and $U_7$ are negligible as $N, T \to \infty$. Since $k$ is bounded by $k^0$ and $\hat{k}$ in $U_4$, using the convergence property (A.1),

$$
U_4 = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=k^0+1}^{k}x_{it}'\delta_i^0 1_{\{k^0<k\le\hat{k}\}} = \sqrt{\frac{N}{T}}o_p(1) = o_p\left(\sqrt{\frac{N}{T}}\right). \qquad \text{(A.5)}
$$

Similarly, it is shown that the orders of terms $U_5, U_6$, and $U_7$ are $o_p\left(\sqrt{\frac{N}{T}}\right)$, which will vanish since $N/T \to 0$ in Assumption 2(iii). The asymptotic distributions of the dominating terms $U_1, U_2$, and $U_3$ are derived from Lemma A.1.

**Lemma A.1** *Suppose that Assumptions 1–5 hold. We have, uniformly in* $\tau \in (0,1)$,

*(i)* $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}u_{it} \Rightarrow \sigma W(\tau)$,

*(ii)* $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}x_{it}'(\hat{\beta}_i-\beta_i^0) \Rightarrow \sigma\tau\frac{W(\tau^0)}{\tau^0}$,

*(iii)* $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{k}x_{it}'(\hat{\delta}_i-\delta_i^0)1_{\{k>\hat{k}\}} \Rightarrow \sigma(\tau-\tau^0)\left[\frac{W(1)-W(\tau^0)}{1-\tau^0} - \frac{W(\tau^0)}{\tau^0}\right]1_{\{\tau>\tau^0\}}$,

*where* $k = [T\tau]$, $k^0 = [T\tau^0]$, $W(\cdot)$ *is a standard Brownian motion, and the long-run variance* $\sigma^2$ *is* $\lim_{(N,T)\to\infty} E\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}u_{it}\right)^2$.

**Proof of Lemma A.1**. (i) Denote the process

$$X_{N,T}\left(\frac{k}{T}\right) = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{k} u_{it}.$$

It is shown that, for a particular $\tau$,

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{[T\tau]} u_{it} \xrightarrow{d} \sigma W(\tau),$$

as $N, T \to \infty$. It remains to be shown that weak convergence holds uniformly in $\tau \in (0, 1)$. To this end, by Billingsley's (1968) Theorem 12.1, we next show that the moment condition (A.8) is satisfied such that the process $X_{N,T}(\tau)$ is tight. Applying Rosenthal's inequality, we have,

$$\begin{aligned}
& E\left|X_{N,T}\left(\frac{l}{T}\right) - X_{N,T}\left(\frac{k}{T}\right)\right|^{2\gamma} = E\left|\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=k+1}^{l} u_{it}\right|^{2\gamma} \\
& \leq c_1 \sum_{i=1}^{N} E\left|\frac{1}{\sqrt{NT}} \sum_{t=k+1}^{l} u_{it}\right|^{2\gamma} + c_2 \left[\frac{1}{N} \sum_{i=1}^{N} E\left(\frac{1}{\sqrt{T}} \sum_{t=k+1}^{l} u_{it}\right)^2\right]^{\gamma} \\
& \leq c_1 \sum_{i=1}^{N} E\left|\frac{1}{\sqrt{NT}} \sum_{t=k+1}^{l} u_{it}\right|^{2\gamma} + c_3 \left(\frac{l-k}{T}\right)^{\gamma},
\end{aligned} \tag{A.6}$$

with some constants $c_1, c_2$, and $c_3$. According to Phillips and Solo (1992) and p.637 of Horváth and Hušková (2012), the partial sum of $u_{it}$ is composed of two parts,

$$\sum_{t=1}^{k} u_{it} = a_i \sum_{t=1}^{k} \epsilon_{it} + \eta_{ik},$$

where $\eta_{ik} = e_{i0}^* - e_{ik}^*$, $e_{it}^* = \sum_{l=1}^{\infty} c_{il}^* \epsilon_{i(t-l)}$, and $c_{il}^* = \sum_{k=l+1}^{\infty} c_{ik}$. For the term $\eta_{ik}$, Horváth and Hušková (2012, p.640) indicated that $E|\eta_{ik}|^{\gamma} \leq cE|\epsilon_{i0}|^{\gamma}$. Then, using Minkowski's inequality and Rothenthal's inequality, we show that for $\gamma > 1$,

$$
\begin{aligned}
& E\left|\sum_{t=k+1}^{l} u_{it}\right|^{2\gamma} = E\left|a_i \sum_{t=k+1}^{l} \epsilon_{it} + \eta_{il} - \eta_{ik}\right|^{2\gamma} \\
& \leq \left[\left(E\left|a_i \sum_{t=k+1}^{l} \epsilon_{it}\right|^{2\gamma}\right)^{\frac{1}{2\gamma}} + \left(E\left|\eta_{il} - \eta_{ik}\right|^{2\gamma}\right)^{\frac{1}{2\gamma}}\right]^{2\gamma} \\
& \leq \left\{\left[c_4 \sum_{t=k+1}^{l} E\left|\epsilon_{it}\right|^{2\gamma} + c_5\left(\sum_{t=k+1}^{l} E(\epsilon_{it})^2\right)^{\gamma}\right]^{\frac{1}{2\gamma}} + \left(E\left|\eta_{il} - \eta_{ik}\right|^{2\gamma}\right)^{\frac{1}{2\gamma}}\right\}^{2\gamma} \\
& \leq \left\{\left[c_4(l-k) E\left|\epsilon_{i0}\right|^{2\gamma} + c_5(l-k)^{\gamma}\left(E(\epsilon_{i0})^2\right)^{\gamma}\right]^{\frac{1}{2\gamma}} + \left(E\left|\eta_{il} - \eta_{ik}\right|^{2\gamma}\right)^{\frac{1}{2\gamma}}\right\}^{2\gamma} \\
& \leq \left\{\left[c_6(l-k)^{\gamma} E\left|\epsilon_{i0}\right|^{2\gamma}\right]^{\frac{1}{2\gamma}} + \left(E\left|\epsilon_{i0}\right|^{2\gamma}\right)^{\frac{1}{2\gamma}}\right\}^{2\gamma} \\
& \leq c_7(l-k)^{\gamma} E\left|\epsilon_{i0}\right|^{2\gamma},
\end{aligned}
$$

with some constants $c_4$–$c_7$. Then, we have, for $\gamma > 1$ ,

$$
\begin{aligned}
& \sum_{i=1}^{N} E\left|\frac{1}{\sqrt{NT}} \sum_{t=k+1}^{l} u_{it}\right|^{2\gamma} = \frac{1}{(NT)^{\gamma}} \sum_{i=1}^{N} E\left|\sum_{t=k+1}^{l} u_{it}\right|^{2\gamma} \\
& \leq \frac{1}{(NT)^{\gamma}} \sum_{i=1}^{N} c_7(l-k)^{\gamma} E\left|\epsilon_{i0}\right|^{2\gamma} \\
& \leq c_7\left(\frac{l-k}{T}\right)^{\gamma} \frac{1}{N} \sum_{i=1}^{N} E\left|\epsilon_{i0}\right|^{2\gamma} \\
& \leq c_8\left(\frac{l-k}{T}\right)^{\gamma},
\end{aligned} \tag{A.7}
$$

with a constant $c_8$. Combining (A.6) and (A.7), we can show that there exists constants $\gamma > 1$ and $c_9$ such that

$$
E\left|X_{N,T}\left(\frac{l}{T}\right) - X_{N,T}\left(\frac{k}{T}\right)\right|^{2\gamma} \leq c_9\left(\frac{l-k}{T}\right)^{\gamma}. \tag{A.8}
$$

(ii) By regressing $Y_i$ on $\bar{X}_i(\hat{k})$, the coefficient $\beta_i$ is estimated as, if $\hat{k} \leq k^0$,

$$
\hat{\beta}_i = \left(\sum_{t=1}^{\hat{k}} x_{it} x_{it}'\right)^{-1} \sum_{t=1}^{\hat{k}} x_{it} y_{it} = \left(\sum_{t=1}^{\hat{k}} x_{it} x_{it}'\right)^{-1} \sum_{t=1}^{\hat{k}} x_{it}(x_{it}' \beta_i^0 + u_{it}) = \beta_i^0 + \left(\sum_{t=1}^{\hat{k}} x_{it} x_{it}'\right)^{-1} \sum_{t=1}^{\hat{k}} x_{it} u_{it},
$$

and if $\hat{k} > k^0$,

$$\begin{aligned}\hat{\beta}_i &= \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1}\left[\sum_{t=1}^{k^0} x_{it}(x_{it}'\beta_i^0 + u_{it}) + \sum_{t=k^0+1}^{\hat{k}} x_{it}(x_{it}'\beta_i^0 + x_{it}'\delta_i^0 + u_{it})\right] \\ &= \beta_i^0 + \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{\hat{k}} x_{it}u_{it} + \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1}\sum_{t=k^0+1}^{\hat{k}} x_{it}x_{it}'\delta_i^0. \qquad \text{(A.9)}\end{aligned}$$

Then, we can see that,

$$\sqrt{T}(\hat{\beta}_i - \beta_i^0) = \left(\frac{1}{T}\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{\hat{k}} x_{it}u_{it} + \left(\frac{1}{T}\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=k^0+1}^{\hat{k}} x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}>k^0\}} \qquad \text{(A.10)}$$

$$\begin{aligned}&= \left(\frac{1}{T}\sum_{t=1}^{k^0} x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0} x_{it}u_{it} + \left[\left(\frac{1}{T}\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} - \left(\frac{1}{T}\sum_{t=1}^{k^0} x_{it}x_{it}'\right)^{-1}\right]\frac{1}{\sqrt{T}}\sum_{t=1}^{\hat{k}} x_{it}u_{it} \\ &\quad + \left(\frac{1}{T}\sum_{t=1}^{k^0} x_{it}x_{it}'\right)^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{\hat{k}} x_{it}u_{it} - \frac{1}{\sqrt{T}}\sum_{t=1}^{k^0} x_{it}u_{it}\right] + \left(\frac{1}{T}\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=k^0+1}^{\hat{k}} x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}>k^0\}} \\ &= \left(\frac{1}{T}\sum_{t=1}^{k^0} x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0} x_{it}u_{it} + o_p\left(\frac{1}{T}\right)O_p(1) + O_p(1)o_p\left(\frac{1}{\sqrt{T}}\right) + o_p\left(\frac{1}{\sqrt{T}}\right)1_{\{\hat{k}>k^0\}} \\ &= \left(\frac{1}{T}\sum_{t=1}^{k^0} x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0} x_{it}u_{it} + o_p\left(\frac{1}{T}\right) + o_p\left(\frac{1}{\sqrt{T}}\right), \qquad \text{(A.11)}\end{aligned}$$

where we replace $\hat{k}$ with $k^0$ using the consistency property (A.1) and the following orders:

$$\begin{aligned}&\left(\frac{1}{T}\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} - \left(\frac{1}{T}\sum_{t=1}^{k^0} x_{it}x_{it}'\right)^{-1} \\ &= \left(\frac{1}{T}\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{k^0} x_{it}x_{it}' - \frac{1}{T}\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)\left(\frac{1}{T}\sum_{t=1}^{k^0} x_{it}x_{it}'\right)^{-1} \\ &= O_p(1)o_p\left(\frac{1}{T}\right)O_p(1) = o_p\left(\frac{1}{T}\right),\end{aligned}$$

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\hat{k}} x_{it}u_{it} - \frac{1}{\sqrt{T}}\sum_{t=1}^{k^0} x_{it}u_{it} = o_p\left(\frac{1}{\sqrt{T}}\right),$$

and

$$\left(\frac{1}{T}\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1}\frac{1}{T}\sum_{t=k^0+1}^{\hat{k}} x_{it}x_{it}'\delta_i^0 = O_p(1)o_p\left(\frac{1}{T}\right) = o_p\left(\frac{1}{T}\right).$$

Substituting (A.11) into the term $U_2$ in (A.4), we have

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}x_{it}'(\hat{\beta}_i-\beta_i^0)=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{k}x_{it}'\sqrt{T}(\hat{\beta}_i-\beta_i^0)\\
&=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{k}x_{it}'\left(\frac{1}{T}\sum_{t=1}^{k^0}x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0}x_{it}u_{it}+\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{k}x_{it}'\left(o_p\left(\frac{1}{T}\right)+o_p\left(\frac{1}{\sqrt{T}}\right)\right)\\
&=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{k}x_{it}'\left(\frac{1}{T}\sum_{t=1}^{k^0}x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0}x_{it}u_{it}+o_p\left(\frac{\sqrt{N}}{T}\right)+o_p\left(\sqrt{\frac{N}{T}}\right), \qquad \text{(A.12)}
\end{aligned}$$

where the second and third terms in the last equality vanish since $N/T\to 0$ by Assumption 2(iii). From Assumptions 4(ii)–(iii), we can see that,

$$\left\|\frac{1}{k}\sum_{t=1}^{k}x_{it}'-c_{i1}'\right\|=o_p(1),\quad\text{and}\quad\left\|\left(\frac{1}{k}\sum_{t=1}^{k}x_{it}x_{it}'\right)^{-1}-C_i^{-1}\right\|=o_p(1). \qquad \text{(A.13)}$$

Using orders in (A.13) and equality $c_{i1}'C_i^{-1}=[1,0,\cdots,0]$, we have,

$$\begin{aligned}
&\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{k}x_{it}'\left(\frac{1}{T}\sum_{t=1}^{k^0}x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0}x_{it}u_{it}-\frac{k}{k^0}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k^0}u_{it}\right|\\
=&\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{k}x_{it}'\left(\frac{1}{T}\sum_{t=1}^{k^0}x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0}x_{it}u_{it}-\frac{k}{k^0}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}c_{i1}'C_i^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0}x_{it}u_{it}\right|\\
\leq&\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{k}x_{it}'\left(\frac{1}{T}\sum_{t=1}^{k^0}x_{it}x_{it}'\right)^{-1}-\frac{k}{k^0}c_{i1}'C_i^{-1}\right\|\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0}x_{it}u_{it}\right\|\\
=&\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k^0}x_{it}u_{it}\right\|o_p(1)=o_p(1). \qquad \text{(A.14)}
\end{aligned}$$

Applying the functional central limit theorem (FCLT), we can see that

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k^0}u_{it}\Rightarrow\sigma W(\tau^0). \qquad \text{(A.15)}$$

Hence, we have, uniformly in $\tau$,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=1}^{k}x_{it}'\left(\frac{1}{T}\sum_{t=1}^{k^0}x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0}x_{it}u_{it}\Rightarrow\sigma\tau\frac{W(\tau^0)}{\tau^0}, \qquad \text{(A.16)}$$

(iii) The coefficient $\delta_i$ is estimated as, if $\hat{k} < k^0$,

$$\begin{aligned}
\hat{\delta}_i &= \left(\sum_{t=\hat{k}+1}^{T} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{T} x_{it}y_{it} - \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{\hat{k}} x_{it}y_{it} \\
&= \left(\sum_{t=\hat{k}+1}^{T} x_{it}x_{it}'\right)^{-1} \left[\sum_{t=\hat{k}+1}^{k^0} x_{it}(x_{it}'\beta_i^0 + u_{it}) + \sum_{t=k^0+1}^{T} x_{it}(x_{it}'\beta_i^0 + x_{it}'\delta_i^0 + u_{it})\right] \\
&\quad - \left[\beta_i^0 + \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{\hat{k}} x_{it}u_{it}\right] \\
&= \left(\sum_{t=\hat{k}+1}^{T} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{T} x_{it}u_{it} - \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{\hat{k}} x_{it}u_{it} + \left(\sum_{t=\hat{k}+1}^{T} x_{it}x_{it}'\right)^{-1} \sum_{t=k^0+1}^{T} x_{it}x_{it}'\delta_i^0 \\
&= \left(\sum_{t=\hat{k}+1}^{T} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{T} x_{it}u_{it} - \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{\hat{k}} x_{it}u_{it} + \delta_i^0 - \left(\sum_{t=\hat{k}+1}^{T} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{k^0} x_{it}x_{it}'\delta_i^0,
\end{aligned}$$

and if $\hat{k} \geq k^0$,

$$\begin{aligned}
\hat{\delta}_i &= \left(\sum_{t=\hat{k}+1}^{T} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{T} x_{it}y_{it} - \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{\hat{k}} x_{it}y_{it} \\
&= \left(\sum_{t=\hat{k}+1}^{T} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{T} x_{it}(x_{it}'\beta_i^0 + x_{it}'\delta_i^0 + u_{it}) \\
&\quad - \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} \left[\sum_{t=1}^{k^0} x_{it}(x_{it}'\beta_i^0 + u_{it}) + \sum_{t=k^0+1}^{\hat{k}} x_{it}(x_{it}'\beta_i^0 + x_{it}'\delta_i^0 + u_{it})\right] \\
&= \delta_i^0 + \left(\sum_{t=\hat{k}+1}^{T} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{T} x_{it}u_{it} - \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{\hat{k}} x_{it}u_{it} - \left(\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} \sum_{t=k^0+1}^{\hat{k}} x_{it}x_{it}'\delta_i^0.
\end{aligned}$$

Using the consistency property (A.1), the term $\sum_{t=\hat{k}+1}^{k^0} x_{it}x_{it}'\delta_i^0 = o_p(1)$ is negligible, and we can see that,

$$\sqrt{T}(\hat{\delta}_i - \delta_i^0) = \left(\frac{1}{T}\sum_{t=\hat{k}+1}^{T} x_{it}x_{it}'\right)^{-1} \frac{1}{\sqrt{T}} \sum_{t=\hat{k}+1}^{T} x_{it}u_{it} - \left(\frac{1}{T}\sum_{t=1}^{\hat{k}} x_{it}x_{it}'\right)^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{\hat{k}} x_{it}u_{it} + o_p\left(\frac{1}{\sqrt{T}}\right). \tag{A.17}$$

Similar to the proof of (ii), $\hat{k}$ in (A.17) can be replaced by $k^0$ due to the consistency of

$\hat{k} \xrightarrow{p} k^0$. Then, (A.17) is transformed into

$$\sqrt{T}(\hat{\delta}_i - \delta_i^0) = \left(\frac{1}{T}\sum_{t=k^0+1}^{T} x_{it}x_{it}'\right)^{-1} \frac{1}{\sqrt{T}}\sum_{t=k^0+1}^{T} x_{it}u_{it} - \left(\frac{1}{T}\sum_{t=1}^{k^0} x_{it}x_{it}'\right)^{-1} \frac{1}{\sqrt{T}}\sum_{t=1}^{k^0} x_{it}u_{it} + o_p\left(\frac{1}{T}\right) + o_p\left(\frac{1}{\sqrt{T}}\right) \tag{A.18}$$

Thus, we have,

$$\begin{aligned}
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{k} x_{it}'(\hat{\delta}_i - \delta_i^0) &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=\hat{k}+1}^{k} x_{it}'\sqrt{T}(\hat{\delta}_i - \delta_i^0) \\
&= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=k^0+1}^{k} x_{it}' 1_{\{k>k^0\}} + o_p\left(\frac{1}{T}\right)\right)\sqrt{T}(\hat{\delta}_i - \delta_i^0) \\
&= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=k^0+1}^{k} x_{it}' 1_{\{k>k^0\}}\right)\sqrt{T}(\hat{\delta}_i - \delta_i^0) + o_p\left(\frac{\sqrt{N}}{T}\right)O_p(1) \\
&= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=k^0+1}^{k} x_{it}' 1_{\{k>k^0\}}\right)\left[\left(\frac{1}{T}\sum_{t=k^0+1}^{T} x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=k^0+1}^{T} x_{it}u_{it}\right. \\
&\quad \left. - \left(\frac{1}{T}\sum_{t=1}^{k^0} x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k^0} x_{it}u_{it}\right] + o_p\left(\frac{\sqrt{N}}{T}\right) + o_p\left(\sqrt{\frac{N}{T}}\right).
\end{aligned} \tag{A.19}$$

The terms in the brackets of (A.19) dominate the others. Similar to (A.14)–(A.16), we can see that, uniformly in $\tau \in (0,1)$,

$$U_3 \Rightarrow \sigma(\tau - \tau^0)\left[\frac{W(1) - W(\tau^0)}{1-\tau^0} - \frac{W(\tau^0)}{\tau^0}\right] 1_{\{\tau>\tau^0\}}.$$

Thus, we complete the proof of Lemma A.1. ■

Using (A.4), (A.5), and Lemma A.1, we can show that,

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{[T\tau]} \hat{u}_{it} \Rightarrow \sigma W(\tau) - \sigma\tau\frac{W(\tau^0)}{\tau^0} - \sigma(\tau-\tau^0)\left[\frac{W(1) - W(\tau^0)}{1-\tau^0} - \frac{W(\tau^0)}{\tau^0}\right] 1_{\{\tau>\tau^0\}},$$

uniformly in $\tau$. Applying the continuous mapping theorem, we obtain,

$$\sup_{\tau\in\Omega(\epsilon)}\left|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{[T\tau]} \hat{u}_{it}\right|^2 \Rightarrow \sup_{\tau\in\Omega(\epsilon)} \sigma^2\left|W(\tau) - \tau\frac{W(\tau^0)}{\tau^0} - (\tau-\tau^0)\left[\frac{W(1) - W(\tau^0)}{1-\tau^0} - \frac{W(\tau^0)}{\tau^0}\right] 1_{\{\tau>\tau^0\}}\right|^2. \tag{A.20}$$

Next, we derive the asymptotic distribution of the normalization process under the null hypothesis. By definition (10), the normalization factor is based on the residuals $\tilde{u}_{it}$, which are calculated by regressing $Y_i$ on $\tilde{X}_{it}(k_1, \hat{k}, k_2)$ in (8). We assume that $[T\epsilon] \leq k_1 \leq \hat{k} - [T\epsilon]$, and $\hat{k} + [T\epsilon] \leq k_2 \leq [T(1-\epsilon)]$, where $k_1, k_2$ are bounded away from endpoints and the common break estimate $\hat{k}$. Since $\hat{k}$ converges in probability to $k^0$, we have $|k_j - \hat{k}| > |k^0 - \hat{k}|$ for $j = 1, 2$. Thus, we only consider the case in which $k_1$ and $k_2$ take values in $k_1 < k^0 < k_2$. In this case, the true model with a common break $k^0$ is written as

$$\begin{aligned} Y_i &= [X_i, X_{1i}(k_1, k^0), X_{2i}(k^0, k_2), X_{3i}(k_2)] \begin{bmatrix} \beta_i^0 \\ 0 \\ \delta_i^0 \\ \delta_i^0 \end{bmatrix} + u_i \\ &= \tilde{X}_i(k_1, k^0, k_2) b_{1i}^0 + u_i \\ &= \tilde{X}_i(k_1, \hat{k}, k_2) b_{1i}^0 + u_i + [\tilde{X}_i(k_1, k^0, k_2)) - \tilde{X}_i(k_1, \hat{k}, k_2)] b_{1i}^0. \end{aligned} \tag{A.21}$$

The residuals are calculated by

$$\begin{aligned} \tilde{u}_i &= \tilde{X}_i(k_1, \hat{k}, k_2) b_{1i}^0 + u_i + [\tilde{X}_i(k_1, k^0, k_2) - \tilde{X}_i(k_1, \hat{k}, k_2)] b_{1i}^0 - \tilde{X}_i(k_1, \hat{k}, k_2) \tilde{b}_{1i}(\hat{k}) \\ &= u_i + \tilde{X}_i(k_1, \hat{k}, k_2) b_{1i}^0 - \tilde{X}_i(k_1, \hat{k}, k_2) \tilde{b}_{1i}(\hat{k}) \\ &\quad + [0, X_{1i}(k_1, k^0) - X_{1i}(k_1, \hat{k}), X_{2i}(k^0, k_2) - X_{2i}(\hat{k}, k_2), 0] \begin{bmatrix} \beta_i^0 \\ 0 \\ \delta_i^0 \\ \delta_i^0 \end{bmatrix} \\ &= u_i - \tilde{X}_i(k_1, \hat{k}, k_2)(\tilde{b}_{1i}(\hat{k}) - b_{1i}^0) + [X_{2i}(k^0, k_2) - X_{2i}(\hat{k}, k_2)] \delta_i^0, \end{aligned}$$

whose vector form is

$$\begin{bmatrix} \tilde{u}_{i1} \\ \tilde{u}_{i2} \\ \vdots \\ \tilde{u}_{iT} \end{bmatrix} = \begin{bmatrix} u_{i1} \\ u_{i2} \\ \vdots \\ u_{iT} \end{bmatrix} - \begin{bmatrix} x'_{i1} \\ x'_{i2} \\ \vdots \\ x'_{iT} \end{bmatrix} (\tilde{\beta}_i(\hat{k}) - \beta_i^0) - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x'_{i(k_1+1)} \\ \vdots \\ x'_{i\hat{k}} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \tilde{\delta}_{1i}(\hat{k}) - \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ x'_{i(\hat{k}+1)} \\ \vdots \\ x'_{ik_2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} (\tilde{\delta}_{2i}(\hat{k}) - \delta_i^0)$$

$$- \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ x'_{i(k_2+1)} \\ \vdots \\ x'_{iT} \end{bmatrix} (\tilde{\delta}_{3i}(\hat{k}) - \delta_i^0) + \left( \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ x'_{i(k^0+1)} \\ \vdots \\ x'_{ik_2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ x'_{i(\hat{k}+1)} \\ \vdots \\ x'_{ik_2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \delta_i^0. \qquad \text{(A.22)}$$

For simplicity, $\hat{k}$ is suppressed in $\tilde{\beta}_i(\hat{k})$, $\tilde{\delta}_{1i}(\hat{k})$, $\tilde{\delta}_{2i}(\hat{k})$, and $\tilde{\delta}_{3i}(\hat{k})$. The normalization factor is constructed by four terms $V_1, V_2, V_3$, and $V_4$, which are defined by

$$\begin{aligned} V_1 &= \frac{1}{T} \sum_{s=1}^{k_1} \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{s} \tilde{u}_{it} \right)^2, \\ V_2 &= \frac{1}{T} \sum_{s=k_1+1}^{\hat{k}} \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=s}^{\hat{k}} \tilde{u}_{it} \right)^2, \\ V_3 &= \frac{1}{T} \sum_{s=\hat{k}+1}^{k_2} \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=\hat{k}+1}^{s} \tilde{u}_{it} \right)^2, \\ V_4 &= \frac{1}{T} \sum_{s=k_2+1}^{T} \left( \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=s}^{T} \tilde{u}_{it} \right)^2. \end{aligned}$$

Lemma A.2 derives the asymptotic distributions of the four terms under the null hypothesis.

**Lemma A.2** *Suppose that Assumptions 1–5 hold. We have, as $N, T \to \infty$,*

*(i)* $V_1 \Rightarrow \sigma^2 \int_0^{\tau_1} \left( W(r) - \frac{r}{\tau_1} W(\tau_1) \right)^2 dr,$

*(ii)* $V_2 \Rightarrow \sigma^2 \int_{\tau_1}^{\tau^0} \left[ W(\tau^0) - W(r) - \frac{\tau^0 - r}{\tau^0 - \tau_1} (W(\tau^0) - W(\tau_1)) \right]^2 dr,$

*(iii)* $V_3 \Rightarrow \sigma^2 \int_{\tau^0}^{\tau_2} \left[ W(r) - W(\tau^0) - \frac{r - \tau^0}{\tau_2 - \tau^0} (W(\tau_2) - W(\tau^0)) \right]^2 dr,$

*(iv)* $V_4 \Rightarrow \sigma^2 \int_{\tau_2}^{1} \left[ W(1) - W(r) - \frac{1 - r}{1 - \tau_2} (W(1) - W(\tau_2)) \right]^2 dr.$

**Proof of Lemma A.2**. (i) Using (A.22), the first term $V_1$ can be rewritten as

$$\begin{aligned} V_1 &= \frac{1}{T} \sum_{s=1}^{k_1} \left\{ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \left[ \sum_{t=1}^{s} u_{it} - \sum_{t=1}^{s} x'_{it} (\tilde{\beta}_i - \beta_i^0) \right] \right\}^2 \\ &= \frac{1}{T} \sum_{s=1}^{k_1} (V_{11} - V_{12})^2 . \end{aligned}$$

Using the FCLT, it is shown that

$$V_{11} \Rightarrow \sigma W(r). \tag{A.23}$$

By the definition of $\tilde{\beta}_i$, we can see that,

$$\tilde{\beta}_i = \left( \sum_{t=1}^{k_1} x_{it} x'_{it} \right)^{-1} \sum_{t=1}^{k_1} x_{it} y_{it} = \left( \sum_{t=1}^{k_1} x_{it} x'_{it} \right)^{-1} \sum_{t=1}^{k_1} x_{it} (x'_{it} \beta_i^0 + u_{it}) = \beta_i^0 + \left( \sum_{t=1}^{k_1} x_{it} x'_{it} \right)^{-1} \sum_{t=1}^{k_1} x_{it} u_{it}.$$

Thus, we have,

$$\begin{aligned} V_{12} &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{T} \sum_{t=1}^{s} x'_{it} \left( \frac{1}{T} \sum_{t=1}^{k_1} x_{it} x'_{it} \right)^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{k_1} x_{it} u_{it} \\ &\Rightarrow \sigma \frac{r}{\tau_1} W(\tau_1). \end{aligned} \tag{A.24}$$

Combining the results (A.23) and (A.24) and using the continuous mapping theorem, we can derive the asymptotic distribution of the first term $V_1$ as follows:

$$V_1 \Rightarrow \sigma^2 \int_0^{\tau_1} \left( W(r) - \frac{r}{\tau_1} W(\tau_1) \right)^2 dr.$$

(ii) The second term $V_2$ can be rewritten as

$$\begin{aligned}
V_2 &= \frac{1}{T}\sum_{s=k_1+1}^{\hat{k}}\left\{\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\sum_{t=s}^{\hat{k}}u_{it}-\sum_{t=s}^{\hat{k}}x_{it}'(\tilde{\beta}_i-\beta_i^0)-\sum_{t=s}^{\hat{k}}x_{it}'\tilde{\delta}_{1i}+\sum_{t=s}^{\hat{k}}x_{it}'\delta_i^0 1_{\{k^0<s<\hat{k}\}}\right]\right\}^2 \\
&= \frac{1}{T}\sum_{s=k_1+1}^{\hat{k}}\left\{\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{\hat{k}}u_{it}-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{\hat{k}}x_{it}'(\tilde{\beta}_i-\beta_i^0)-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{\hat{k}}x_{it}'\tilde{\delta}_{1i}+o_p\left(\sqrt{\frac{N}{T}}\right)\right\}^2 \\
&= \frac{1}{T}\sum_{s=k_1+1}^{\hat{k}}\left(V_{21}-V_{22}-V_{23}+o_p\left(\sqrt{\frac{N}{T}}\right)\right)^2.
\end{aligned}$$

Since $\hat{k}$ coincides asymptotically with the true break date from (A.1), $\hat{k}$ in $V_{21}, V_{22}, V_{23}$ can be replaced by $k^0$. Then, we can show that

$$V_{21} = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{\hat{k}}u_{it}-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{s-1}u_{it} \Rightarrow \sigma(W(\tau^0)-W(r)). \tag{A.25}$$

$$\begin{aligned}
V_{22} &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=s}^{\hat{k}}x_{it}'\sqrt{T}(\tilde{\beta}_i-\beta_i^0) \\
&= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=s}^{k^0}x_{it}'+o_p\left(\frac{1}{T}\right)\right)\left(\frac{1}{T}\sum_{t=1}^{k_1}x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k_1}x_{it}u_{it} \\
&\Rightarrow \sigma(\tau^0-r)\frac{W(\tau_1)}{\tau_1}.
\end{aligned} \tag{A.26}$$

The coefficient estimator $\tilde{\delta}_{1i}$ in $V_{23}$ can be calculated as

$$\begin{aligned}
\tilde{\delta}_{1i} &= \left(\sum_{t=k_1+1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=k_1+1}^{\hat{k}}x_{it}y_{it}-\left(\sum_{t=1}^{k_1}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1}x_{it}y_{it} \\
&= \left(\sum_{t=k_1+1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\left\{\sum_{t=k_1+1}^{\hat{k}}x_{it}(x_{it}'\beta_i^0+u_{it})1_{\{\hat{k}\le k^0\}}+\left[\sum_{t=k_1+1}^{k^0}x_{it}(x_{it}'\beta_i^0+u_{it})\right.\right. \\
&\quad \left.\left.+\sum_{t=k^0+1}^{\hat{k}}x_{it}(x_{it}'\beta_i^0+x_{it}'\delta_i^0+u_{it})\right]1_{\{\hat{k}>k^0\}}\right\}-\left[\beta_i^0+\left(\sum_{t=1}^{k_1}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1}x_{it}u_{it}\right] \\
&= \left(\sum_{t=k_1+1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=k_1+1}^{\hat{k}}x_{it}u_{it}-\left(\sum_{t=1}^{k_1}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1}x_{it}u_{it}+\left(\sum_{t=k_1+1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=k^0+1}^{\hat{k}}x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}>k^0\}} \\
&= \left(\sum_{t=k_1+1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=k_1+1}^{\hat{k}}x_{it}u_{it}-\left(\sum_{t=1}^{k_1}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1}x_{it}u_{it}+o_p\left(\frac{1}{T}\right).
\end{aligned}$$

Then, the third term $V_{23}$ becomes, as $N, T \to \infty$,

$$\begin{aligned}
V_{23} &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{T} \sum_{t=s}^{\hat{k}} x_{it}' \sqrt{T} \tilde{\delta}_{1i} \\
&= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( \frac{1}{T} \sum_{t=s}^{k^0} x_{it}' + o_p\left(\frac{1}{T}\right) \right) \left[ \sqrt{T} \left( \sum_{t=k_1+1}^{k^0} x_{it} x_{it}' \right)^{-1} \sum_{t=k_1+1}^{k^0} x_{it} u_{it} - \sqrt{T} \left( \sum_{t=1}^{k_1} x_{it} x_{it}' \right)^{-1} \sum_{t=1}^{k_1} x_{it} u_{it} \right. \\
&\quad \left. + o_p\left(\frac{1}{T}\right) + o_p\left(\frac{1}{\sqrt{T}}\right) + o_p\left(\frac{1}{\sqrt{T}}\right) \right] \\
&= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{T} \sum_{t=s}^{k^0} x_{it}' \left[ \sqrt{T} \left( \sum_{t=k_1+1}^{k^0} x_{it} x_{it}' \right)^{-1} \sum_{t=k_1+1}^{k^0} x_{it} u_{it} - \sqrt{T} \left( \sum_{t=1}^{k_1} x_{it} x_{it}' \right)^{-1} \sum_{t=1}^{k_1} x_{it} u_{it} \right] \\
&\quad + o_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) + o_p\left(\frac{\sqrt{N}}{T}\right) + o_p\left(\frac{\sqrt{N}}{T^{3/2}}\right) + o_p\left(\frac{\sqrt{N}}{T^2}\right) \\
&\Rightarrow \sigma(\tau^0 - r) \left( \frac{W(\tau^0) - W(\tau_1)}{\tau^0 - \tau_1} - \frac{W(\tau_1)}{\tau_1} \right), \qquad \text{(A.27)}
\end{aligned}$$

since $N/T \to 0$. Combining results (A.25), (A.26), and (A.27), we have

$$\begin{aligned}
V_2 &\Rightarrow \int_{\tau_1}^{\tau^0} \left[ \sigma(W(\tau^0) - W(r)) - \sigma(\tau^0 - r) \frac{W(\tau_1)}{\tau_1} - \sigma(\tau^0 - r) \left( \frac{W(\tau^0) - W(\tau_1)}{\tau^0 - \tau_1} - \frac{W(\tau_1)}{\tau_1} \right) \right]^2 dr \\
&= \sigma^2 \int_{\tau_1}^{\tau^0} \left( W(\tau^0) - W(r) - (\tau^0 - r) \frac{W(\tau^0) - W(\tau_1)}{\tau^0 - \tau_1} \right)^2 dr.
\end{aligned}$$

(iii) The third term $V_3$ can be rewritten as

$$\begin{aligned}
V_3 &= \frac{1}{T} \sum_{s=\hat{k}+1}^{k_2} \left\{ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \left[ \sum_{t=\hat{k}+1}^{s} u_{it} - \sum_{t=\hat{k}+1}^{s} x_{it}' (\tilde{\beta}_i - \beta_i^0) - \sum_{t=\hat{k}+1}^{s} x_{it}' (\tilde{\delta}_{2i} - \delta_i^0) \right. \right. \\
&\quad \left. \left. - \sum_{t=\hat{k}+1}^{k^0} x_{it}' \delta_i^0 1_{\{\hat{k} < k^0 < s\}} - \sum_{t=\hat{k}+1}^{s} x_{it}' \delta_i^0 1_{\{s \le k^0\}} \right] \right\}^2 \\
&= \frac{1}{T} \sum_{s=\hat{k}+1}^{k_2} \left[ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=\hat{k}+1}^{s} u_{it} - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=\hat{k}+1}^{s} x_{it}' (\tilde{\beta}_i - \beta_i^0) - \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=\hat{k}+1}^{s} x_{it}' (\tilde{\delta}_{2i} - \delta_i^0) \right. \\
&\quad \left. - o_p\left( \sqrt{\frac{N}{T}} \right) \right]^2 \\
&= \frac{1}{T} \sum_{s=\hat{k}+1}^{k_2} \left( V_{31} - V_{32} - V_{33} - o_p\left( \sqrt{\frac{N}{T}} \right) \right)^2.
\end{aligned}$$

Similar to (A.25) and (A.27), we can find that

$$V_{31} \Rightarrow \sigma(W(r) - W(\tau^0)), \tag{A.28}$$

$$\begin{aligned} V_{32} &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=k^0+1}^{s} x_{it}' + o_p\left(\frac{1}{T}\right)\right)\left(\frac{1}{T}\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k_1} x_{it}u_{it} \\ &\Rightarrow \sigma(r-\tau^0)\frac{W(\tau_1)}{\tau_1}. \end{aligned} \tag{A.29}$$

The coefficient $\delta_{2i}$ is estimated as

$$\begin{aligned} \tilde{\delta}_{2i} &= \left(\sum_{t=\hat{k}+1}^{k_2} x_{it}x_{it}'\right)^{-1}\sum_{t=\hat{k}+1}^{k_2} x_{it}y_{it} - \left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1} x_{it}y_{it} \\ &= \left(\sum_{t=\hat{k}+1}^{k_2} x_{it}x_{it}'\right)^{-1}\left\{\sum_{t=\hat{k}+1}^{k_2} x_{it}(x_{it}'\beta_i^0 + x_{it}'\delta_i^0 + u_{it})1_{\{k^0\le\hat{k}\}} + \left[\sum_{t=\hat{k}+1}^{k^0} x_{it}(x_{it}'\beta_i^0 + u_{it})\right.\right. \\ &\quad \left.\left. + \sum_{t=k^0+1}^{k_2} x_{it}(x_{it}'\beta_i^0 + x_{it}'\delta_i^0 + u_{it})\right]1_{\{k^0<\hat{k}\}}\right\} - \left[\beta_i^0 + \left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1} x_{it}u_{it}\right] \\ &= \delta_i^0 + \left(\sum_{t=\hat{k}+1}^{k_2} x_{it}x_{it}'\right)^{-1}\sum_{t=\hat{k}+1}^{k_2} x_{it}u_{it} - \left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1} x_{it}u_{it} - o_p\left(\frac{1}{T}\right). \end{aligned}$$

Then, similar to $V_{23}$, the term $V_{33}$ becomes, as $N, T \to \infty$,

$$\begin{aligned} &\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=\hat{k}+1}^{s} x_{it}'\sqrt{T}(\tilde{\delta}_{2i} - \delta_i^0) \\ =\ & \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=k^0+1}^{s} x_{it}' + o_p\left(\frac{1}{T}\right)\right)\left[\sqrt{T}\left(\sum_{t=k^0+1}^{k_2} x_{it}x_{it}'\right)^{-1}\sum_{t=k^0+1}^{k_2} x_{it}u_{it} - \sqrt{T}\left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1} x_{it}u_{it}\right. \\ & \left. + o_p\left(\frac{1}{T}\right) + o_p\left(\frac{1}{\sqrt{T}}\right) + o_p\left(\frac{1}{\sqrt{T}}\right)\right] \\ =\ & \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=k^0+1}^{s} x_{it}'\left[\sqrt{T}\left(\sum_{t=k^0+1}^{k_2} x_{it}x_{it}'\right)^{-1}\sum_{t=k^0+1}^{k_2} x_{it}u_{it} - \sqrt{T}\left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1} x_{it}u_{it}\right] \\ & + o_p\left(\frac{\sqrt{N}}{\sqrt{T}}\right) + o_p\left(\frac{\sqrt{N}}{T}\right) + o_p\left(\frac{\sqrt{N}}{T^{3/2}}\right) + o_p\left(\frac{\sqrt{N}}{T^2}\right) \\ \Rightarrow\ & \sigma(r-\tau^0)\left(\frac{W(\tau_2) - W(\tau^0)}{\tau_2 - \tau^0} - \frac{W(\tau_1)}{\tau_1}\right), \end{aligned} \tag{A.30}$$

since $N/T \to 0$. Combining results (A.28), (A.29), and (A.30), we have

$$V_3 \Rightarrow \sigma^2 \int_{\tau^0}^{\tau_2} \left( W(r) - W(\tau^0) - (r-\tau^0)\frac{W(\tau_2)-W(\tau^0)}{\tau_2-\tau^0} \right)^2 dr.$$

(iv) The fourth term $V_4$ can be rewritten as

$$\begin{aligned} V_4 &= \frac{1}{T}\sum_{s=k_2+1}^{T}\left\{\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\sum_{t=s}^{T}u_{it} - \sum_{t=s}^{T}x_{it}'(\tilde{\beta}_i-\beta_i^0) - \sum_{t=s}^{T}x_{it}'(\tilde{\delta}_{3i}-\delta_i^0)\right]\right\}^2 \\ &= \frac{1}{T}\sum_{s=k_2+1}^{T}\left[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{T}u_{it} - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{T}u_{it}x_{it}'(\tilde{\beta}_i-\beta_i^0) - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{T}u_{it}x_{it}'(\tilde{\delta}_{3i}-\delta_i^0)\right]^2 \\ &= \frac{1}{T}\sum_{s=k_2+1}^{T}\left(V_{41}-V_{42}-V_{43}\right)^2. \end{aligned}$$

It is easily seen that

$$V_{41} \Rightarrow \sigma(W(1)-W(r)), \tag{A.31}$$

$$\begin{aligned} V_{42} &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=s}^{T}x_{it}'\right)\left(\frac{1}{T}\sum_{t=1}^{k_1}x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=1}^{k_1}x_{it}u_{it} \\ &\Rightarrow \sigma(1-r)\frac{W(\tau_1)}{\tau_1}. \end{aligned} \tag{A.32}$$

The coefficient estimator $\tilde{\delta}_{3i}$ can be written as

$$\begin{aligned} \tilde{\delta}_{3i} &= \left(\sum_{t=k_2+1}^{T}x_{it}x_{it}'\right)^{-1}\sum_{t=k_2+1}^{T}x_{it}y_{it} - \left(\sum_{t=1}^{k_1}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1}x_{it}y_{it} \\ &= \left(\sum_{t=k_2+1}^{T}x_{it}x_{it}'\right)^{-1}\left[\sum_{t=k_2+1}^{T}x_{it}(x_{it}'\beta_i^0 + x_{it}'\delta_i^0 + u_{it})\right] - \left[\beta_i^0 + \left(\sum_{t=1}^{k_1}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1}x_{it}u_{it}\right] \\ &= \delta_i^0 + \left(\sum_{t=k_2+1}^{T}x_{it}x_{it}'\right)^{-1}\sum_{t=k_2+1}^{T}x_{it}u_{it} - \left(\sum_{t=1}^{k_1}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1}x_{it}u_{it}. \end{aligned}$$

Then, it is shown that, as $N, T \to \infty$,

$$\begin{aligned} V_{43} &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{1}{T}\sum_{t=s}^{T}x_{it}'\sqrt{T}(\tilde{\delta}_{3i}-\delta_i^0) \\ &= \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=s}^{T}x_{it}'\right)\sqrt{T}\left[\left(\sum_{t=k_2+1}^{T}x_{it}x_{it}'\right)^{-1}\sum_{t=k_2+1}^{T}x_{it}u_{it} - \left(\sum_{t=1}^{k_1}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{k_1}x_{it}u_{it}\right] \\ &\Rightarrow \sigma(1-r)\left(\frac{W(1)-W(\tau_2)}{1-\tau_2} - \frac{W(\tau_1)}{\tau_1}\right), \end{aligned} \tag{A.33}$$

since $N/T \to 0$. From (A.31), (A.32), and (A.33), we have

$$V_4 \Rightarrow \sigma^2 \int_{\tau_2}^{1} \left[ W(1) - W(r) - \frac{1-r}{1-\tau_2} \left( W(1) - W(\tau_2) \right) \right]^2 dr.$$

The proof of Lemma A.2 is complete. ■

**Proof of Theorem 1**. Combining the asymptotic distributions in (A.20) and Lemma A.2, we can complete the proof.■

## Appendix B. Proof of Theorem 2

We first investigate the statistical properties of the common break estimator $\hat{k}$ under the alternative hypothesis in Proposition 1. The proof follows the proof of Lemmas 1–2 and Theorem 1 in Baltagi *et al.* (2016). Suppose that there are two groups and the individuals in the same group share a common break date. These groups are denoted by $G_1 = \{i :$ individuals in group 1 with a common break $k_1^0\}$ and $G_2 = \{i :$ individuals in group 2 with common break $k_2^0\}$. The model under the alternative can be specified as

$$\begin{cases} y_{it} = x_{it}'\beta_i^0 + x_{it}'\delta_i^0 1_{(t>k_1^0)} + u_{it} & t = 1, \cdots, T, \text{ for } i \in G_1, \\ y_{it} = x_{it}'\beta_i^0 + x_{it}'\delta_i^0 1_{(t>k_2^0)} + u_{it} & t = 1, \cdots, T, \text{ for } i \in G_2. \end{cases}$$

The vector form can be rewritten as

$$Y_i = [X_i, Z_i(k_j^0)] b_i^0 + u_i, \text{ for } i \in G_j, j = 1, 2.$$

The common break point is estimated in (3) by minimizing the total sum of the squared OLS residuals. Let $SSR_i$ denote the sum of the squared residuals of regression $Y_i$ on $X_i$ (no break case). Using the equality on page 185 of Baltagi *et al.* (2016),

$$SSR_i - SSR_i(k^*) = \hat{\delta}_i(k^*)'[Z_i(k^*)'M_i Z_i(k^*)]\hat{\delta}_i(k^*),$$

estimation (3) can be transformed into

$$
\begin{aligned}
\hat{k} &= \arg\min_{1\le k^*\le T-1}\sum_{i=1}^{N} SSR_i(k^*)\\
&= \arg\max_{1\le k^*\le T-1}\sum_{i=1}^{N}(SSR_i - SSR_i(k^*))\\
&= \arg\max_{1\le k^*\le T-1}\sum_{i=1}^{N} SV_i(k^*)\\
&= \arg\max_{1\le k^*\le T-1}\left[\sum_{i\in G_1}(SV_i(k^*)-SV_i(k_1^0))+\sum_{i\in G_2}(SV_i(k^*)-SV_i(k_1^0))\right], \qquad \text{(B.1)}
\end{aligned}
$$

where

$$
\begin{aligned}
M_i &= I - X_i(X_i'X_i)^{-1}X_i',\\
SV_i(k^*) &= \hat{\delta}_i(k^*)'[Z_i(k^*)'M_iZ_i(k^*)]\hat{\delta}_i(k^*),\\
SV_i(k_j^0) &= \hat{\delta}_i(k_j^0)'[Z_i(k_j^0)'M_iZ_i(k_j^0)]\hat{\delta}_i(k_j^0),\ \text{for},\ j=1,2.
\end{aligned}
$$

For individuals in group 1, we can see that the coefficient estimators are given by

$$
\begin{aligned}
\hat{\delta}_i(k^*) &= [Z_i(k^*)'M_iZ_i(k^*)]^{-1}Z_i(k^*)M_iY_i,\\
\hat{\delta}_i(k_1^0) &= [Z_i(k_1^0)'M_iZ_i(k_1^0)]^{-1}Z_i(k_1^0)M_iY_i.
\end{aligned}
$$

Replacing $Y_i$ with

$$Y_i = X_i\beta_i^0 + Z_i(k_1^0)\delta_i^0 + u_i,$$

we have

$$
\begin{aligned}
\hat{\delta}_i(k^*) &= [Z_i(k^*)'M_iZ_i(k^*)]^{-1}Z_i(k^*)'M_i[X_i\beta_i^0 + Z_i(k_1^0)\delta_i^0 + u_i]\\
&= [Z_i(k^*)'M_iZ_i(k^*)]^{-1}Z_i(k^*)'M_iZ_i(k_1^0)\delta_i^0 + [Z_i(k^*)'M_iZ_i(k^*)]^{-1}Z_i(k^*)'M_iu_i,\\
\hat{\delta}_i(k_1^0) &= [Z_i(k_1^0)'M_iZ_i(k_1^0)]^{-1}Z_i(k_1^0)'M_i[X_i\beta_i^0 + Z_i(k_1^0)\delta_i^0 + u_i]\\
&= \delta_i^0 + [Z_i(k_1^0)'M_iZ_i(k_1^0)]^{-1}Z_i(k_1^0)'M_iu_i.
\end{aligned}
$$

Similarly, for individuals in group 2, by replacing $Y_i$ with

$$Y_i = X_i\beta_i^0 + Z_i(k_2^0)\delta_i^0 + u_i,$$

the coefficients estimators are rewritten as

$$\begin{aligned}\hat{\delta}_i(k^*) &= [Z_i(k^*)'M_iZ_i(k^*)]^{-1}Z_i(k^*)'M_i[X_i\beta_i^0 + Z_i(k_2^0)\delta_i^0 + u_i] \\ &= [Z_i(k^*)'M_iZ_i(k^*)]^{-1}Z_i(k^*)'M_iZ_i(k_2^0)\delta_i^0 + [Z_i(k^*)'M_iZ_i(k^*)]^{-1}Z_i(k^*)'M_iu_i, \\ \hat{\delta}_i(k_1^0) &= [Z_i(k_1^0)'M_iZ_i(k_1^0)]^{-1}Z_i(k_1^0)'M_i[X_i\beta_i^0 + Z_i(k_2^0)\delta_i^0 + u_i] \\ &= [Z_i(k_1^0)'M_iZ_i(k_1^0)]^{-1}Z_i(k_1^0)'M_iZ_i(k_2^0)\delta_i^0 + [Z_i(k_1^0)'M_iZ_i(k_1^0)]^{-1}Z_i(k_1^0)'M_iu_i.\end{aligned}$$

To simplify the notations, we use $Z_i, Z_{1i}^0, Z_{2i}^0$ to replace $Z_i(k^*), Z_i(k_1^0), Z_i(k_2^0)$. For the individuals in group 1, we have

$$\begin{aligned}SV_i(k^*) &= \delta_i^{0\prime}Z_{1i}^{0\prime}M_iZ_i(Z_i'M_iZ_i)^{-1}Z_i'M_iZ_{1i}^0\delta_i^0 + 2\delta_i^{0\prime}Z_{1i}^{0\prime}M_iZ_i(Z_i'M_iZ_i)^{-1}Z_i'M_iu_i \\ &\quad +u_i'M_iZ_i'(Z_i'M_iZ_i)^{-1}Z_i'M_iu_i, &\text{(B.2)} \\ SV_i(k_1^0) &= \delta_i^{0\prime}Z_{1i}^{0\prime}M_iZ_{1i}^0\delta_i^0 + 2\delta_i^{0\prime}Z_{1i}^{0\prime}M_iu_i + u_i'M_iZ_{1i}^0(Z_{1i}^{0\prime}M_iZ_{1i}^0)^{-1}Z_{1i}^{0\prime}M_iu_i. &\text{(B.3)}\end{aligned}$$

Using (B.2) and (B.3), $SV_i(k^*) - SV_i(k_1^0)$ becomes

$$\begin{aligned}SV_i(k^*) - SV_i(k_1^0) &= -\delta_i^{0\prime}\left[Z_{1i}^{0\prime}M_iZ_{1i}^0 - Z_{1i}^{0\prime}M_iZ_i(Z_i'M_iZ_i)^{-1}Z_i'M_iZ_{1i}^0\right]\delta_i^0 \\ &\quad +2\delta_i^{0\prime}Z_{1i}^{0\prime}M_iZ_i(Z_i'M_iZ_i)^{-1}Z_i'M_iu_i - 2\delta_i^{0\prime}Z_{1i}^{0\prime}M_iu_i \\ &\quad +u_i'M_iZ_i'(Z_i'M_iZ_i)^{-1}Z_i'M_iu_i - u_i'M_iZ_{1i}^0(Z_{1i}^{0\prime}M_iZ_{1i}^0)^{-1}Z_{1i}^{0\prime}M_iu_i,\end{aligned}$$

and can be decomposed into the term defined by

$$J_{1i}(k^*) = \delta_i^{0\prime}\left[Z_{1i}^{0\prime}M_iZ_{1i}^0 - Z_{1i}^{0\prime}M_iZ_i(Z_i'M_iZ_i)^{-1}Z_i'M_iZ_{1i}^0\right]\delta_i^0 \qquad \text{(B.4)}$$

and the term related to disturbance $u_i$ defined by

$$\begin{aligned}H_{1i}(k^*) &= 2\delta_i^{0\prime}Z_{1i}^{0\prime}M_iZ_i(Z_i'M_iZ_i)^{-1}Z_i'M_iu_i - 2\delta_i^{0\prime}Z_{1i}^{0\prime}M_iu_i \\ &\quad +u_i'M_iZ_i'(Z_i'M_iZ_i)^{-1}Z_i'M_iu_i - u_i'M_iZ_{1i}^0(Z_{1i}^{0\prime}M_iZ_{1i}^0)^{-1}Z_{1i}^{0\prime}M_iu_i. &\text{(B.5)}\end{aligned}$$

Then, we have $SV_i(k^*) - SV_i(k_1^0) = -J_{1i}(k^*) + H_{1i}(k^*)$ for $i \in G_1$. A similar transformation for individuals in group 2 shows that

$$\begin{aligned}SV_i(k^*) &= \delta_i^{0\prime}Z_{2i}^{0\prime}M_iZ_i(Z_i'M_iZ_i)^{-1}Z_i'M_iZ_{2i}^0\delta_i^0 + 2\delta_i^{0\prime}Z_{2i}^{0\prime}M_iZ_i(Z_i'M_iZ_i)^{-1}Z_i'M_iu_i \\ &\quad +u_i'M_iZ_i'(Z_i'M_iZ_i)^{-1}Z_i'M_iu_i, &\text{(B.6)} \\ SV_i(k_1^0) &= \delta_i^{0\prime}Z_{2i}^{0\prime}M_iZ_{1i}^0(Z_{1i}^{0\prime}M_iZ_{1i}^0)^{-1}Z_{1i}^0M_iZ_{2i}^0\delta_i^0 + 2\delta_i^{0\prime}Z_{2i}^{0\prime}M_iZ_{1i}^0(Z_{1i}^{0\prime}M_iZ_{1i}^0)^{-1}Z_{1i}^0M_iu_i \\ &\quad +u_i'M_iZ_{1i}^0(Z_{1i}^{0\prime}M_iZ_{1i}^0)^{-1}Z_{1i}^{0\prime}M_iu_i. &\text{(B.7)}\end{aligned}$$

Using (B.6) and (B.7), we can see that, for $i \in G_2$,

$$SV_i(k^*) - SV_i(k_1^0) = -J_{2i}(k^*) + H_{2i}(k^*),$$

where the term $J_{2i}(k^*)$ is denoted by

$$J_{2i}(k^*) = \delta_i^{0\prime} \left[ Z_{2i}^{0\prime} M_i Z_{1i}^0 (Z_{1i}^{0\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^0 M_i Z_{2i}^0 - Z_{2i}^{0\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i Z_{2i}^0 \right] \delta_i^0 \quad \text{(B.8)}$$

and the term $H_{2i}(k^*)$ related to disturbance is denoted by

$$\begin{aligned} H_{2i}(k^*) &= 2\delta_i^{0\prime} Z_{2i}^{0\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i - 2\delta_i^{0\prime} Z_{2i}^{0\prime} M_i Z_{1i}^0 (Z_{1i}^{0\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^0 M_i u_i \\ &\quad + u_i' M_i Z_i' (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i - u_i' M_i Z_{1i}^0 (Z_{1i}^{0\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^{0\prime} M_i u_i. \end{aligned} \quad \text{(B.9)}$$

Thus, (B.1) can be rewritten as

$$\hat{k} = \arg \max_{1 \le k^* \le T-1} \left[ \sum_{i \in G_1} (-J_{1i}(k^*) + H_{1i}(k^*)) + \sum_{i \in G_2} (-J_{2i}(k^*) + H_{2i}(k^*)) \right].$$

Define the sets $K(C_1) = \{k : 1 \le k < k_1^0 - C_1\}$, $K(C_2) = \{k : k_2^0 + C_2 < k \le T\}$, and $K(C) = K(C_1) \cup K(C_2) = \{k : 1 \le k < k_1^0 - C_1 \text{ or } k_2^0 + C_2 < k \le T\}$ with positive constants $C_1, C_2$. Next, we show that the common break estimator cannot appear in the set $K(C_1)$ by Lemmas B.1–B.2. A similar result can be obtained for the set $K(C_2)$ by symmetry; thus, the details are omitted. Define

$$Z_{1i}^{\Delta} = \begin{cases} Z_i(k^*) - Z_i(k_1^0) & \text{if } k^* < k_1^0 \\ -(Z_i(k^*) - Z_i(k_1^0)) & \text{if } k^* \ge k_1^0 \end{cases} \text{ and } Z_{2i}^{\Delta} = \begin{cases} Z_i(k^*) - Z_i(k_2^0) & \text{if } k^* < k_2^0 \\ -(Z_i(k^*) - Z_i(k_2^0)) & \text{if } k^* \ge k_2^0 \end{cases}.$$

**Lemma B.1** *Under Assumptions 1–6, for all large $N$ and $T$, with probability tending to 1,*

$$\inf_{k^* \in K(C_1)} \frac{1}{k_1^0 - k^*} \left( \sum_{i \in G_1} J_{1i}(k^*) + \sum_{i \in G_2} J_{2i}(k^*) \right) \ge \lambda \phi_N.$$

**Proof of Lemma B.1**. We first show that the summation of part $J_{1i}(k^*)$ has a lower bound in the case of $k^* \in K(C_1)$. From Lemma A.2 in Bai(1997), if $k^* < k_1^0$,

$$\begin{aligned} J_{1i}(k^*) &= \delta_i^{0\prime} \left[ Z_{1i}^{0\prime} M_i Z_{1i}^0 - Z_{1i}^{0\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i Z_{1i}^0 \right] \delta_i^0 \\ &= \delta_i^{0\prime} Z_{1i}^{\Delta\prime} Z_{1i}^{\Delta} \left( Z_i' Z_i \right)^{-1} Z_{1i}^{0\prime} Z_{1i}^0 \delta_i^0. \end{aligned} \quad \text{(B.10)}$$

Since the matrix

$$\frac{Z_{1i}^{\Delta\prime} Z_{1i}^{\Delta}}{k_1^0 - k^*} \left( \frac{Z_i' Z_i}{T} \right)^{-1} \frac{Z_{1i}^{0\,\prime} Z_{1i}^0}{T} \tag{B.11}$$

is symmetric and positive definite from Assumption 4, we have

$$\frac{1}{k_1^0 - k^*} \sum_{i \in G_1} J_{1i}(k^*) = \sum_{i \in G_1} \delta_i^{0\prime} S_i' \Lambda_i S_i \delta_i^0 = \sum_{i \in G_1} \tilde{\delta}_i^{0\prime} \Lambda_i \tilde{\delta}_i^0 \geq \sum_{i \in G_1} \lambda_i \tilde{\delta}_i^{0\prime} \tilde{\delta}_i^0, \tag{B.12}$$

where $\Lambda_i$ is a diagonal matrix comprising of the eigenvalues of matrix (B.11), $\tilde{\delta}_i^0 = S_i \delta_i^0$, and $\lambda_i$ is the minimum eigenvalue of (B.11). Since $\tilde{\delta}_i^{0\prime} \tilde{\delta}_i^0 = \delta_i^{0\prime} S_i' S_i \delta_i^0 = \delta_i^{0\prime} \delta_i^0$, with probability tending to one for large N and T, we show that

$$\frac{1}{k_1^0 - k^*} \sum_{i \in G_1} J_{1i}(k^*) \geq \lambda_1 \sum_{i \in G_1} \delta_i^{0\prime} \delta_i^0 = \lambda_1 \phi_{N_1}, \tag{B.13}$$

where $\lambda_1 = min_{i \in G_1} \{\lambda_i\}$. Next, we investigate the lower bound of $J_{2i}(k^*)$ for individuals in group 2. Denote

$$V_i(a,b) = \begin{bmatrix} x_{i(a+1)}' \\ x_{i(a+2)}' \\ \vdots \\ x_{ib}' \end{bmatrix}, \quad V_i^0(a,b,c) = \begin{bmatrix} 0_{(b-a) \times p} \\ x_{i(b+1)}' \\ x_{i(b+2)}' \\ \vdots \\ x_{ic}' \end{bmatrix}, \quad \text{and} \quad S = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix},$$

where $V_i(a,b)$ is a $(b-a) \times p$ matrix whose $j$th row is the same as the $(a+j)$th row of $X_i$, $V_i^0(a,b,c)$ is a $(c-a) \times p$ matrix whose first $b-a$ rows are zeros and the $j$th row is the same as the $(a+j)$th row of $X_i$ for $j > b-a$, and $S$ is a $2p \times 2p$ matrix constructed by $p \times p$ identity matrix $I$. The second term $J_{2i}(k)$ can be transformed into

$$\begin{aligned} J_{2i}(k^*) &= \delta_i^{0\prime} \left[ Z_{2i}^{0\,\prime} M_i Z_{1i}^0 (Z_{1i}^{0\,\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^0 M_i Z_{2i}^0 - Z_{2i}^{0\,\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i Z_{2i}^0 \right] \delta_i^0 \\ &= \delta_i^{0\prime} Z_{2i}^{0\,\prime} \left[ M_i Z_{1i}^0 (Z_{1i}^{0\,\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^0 M_i - M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i \right] Z_{2i}^0 \delta_i^0 \\ &= \delta_i^{0\prime} Z_{2i}^{0\,\prime} \left[ M_i - M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i - \left( M_i - M_i Z_{1i}^0 (Z_{1i}^{0\,\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^0 M_i \right) \right] Z_{2i}^0 \delta_i^0 \\ &= \delta_i^{0\prime} Z_{2i}^{0\,\prime} \left( M_W - M_{W_1^0} \right) Z_{2i}^0 \delta_i^0 \\ &= \delta_i^{0\prime} Z_{2i}^{0\,\prime} \left( M_{\bar{W}} - M_{\bar{W}_1^0} \right) Z_{2i}^0 \delta_i^0, \end{aligned} \tag{B.14}$$

where

$$W=[X_i,Z_i(k^*)]=\left[\begin{array}{cc} V_i(0,k^*) & 0 \\ V_i(k^*,T) & V_i(k^*,T)\end{array}\right],\quad W_1^0=[X_i,Z_i(k_1^0)]=\left[\begin{array}{cc} V_i(0,k_1^0) & 0 \\ V_i(k_1^0,T_1^0) & V_i(k_1^0,T)\end{array}\right],$$
$$\bar{W}=\left[\begin{array}{cc} V_i(0,k^*) & 0 \\ 0 & V_i(k^*,T)\end{array}\right],\quad \bar{W}_1^0=\left[\begin{array}{cc} V_i(0,k_1^0) & 0 \\ 0 & V_i(k_1^0,T)\end{array}\right],$$
$$M_X=I-X(X'X)^{-1}X', \text{ for matrix } X.$$

The final equality (B.14) holds because

$$M_W=I-W(W'W)^{-1}W'=I-WS(S'W'WS)^{-1}S'W'=I-\bar{W}\left(\bar{W}'\bar{W}\right)^{-1}\bar{W}=M_{\bar{W}},$$
$$M_{W_1^0}=I-W_1^0(W_1^{0\prime}W_1^0)^{-1}W_1^{0\prime}=I-W_1^0S(S'W_1^{0\prime}W_1^0S)^{-1}S'W_1^{0\prime}=I-\bar{W}_1^0\left(\bar{W}_1^{0\prime}\bar{W}_1^0\right)^{-1}\bar{W}_1^{0\prime}=M_{\bar{W}_1^0}.$$

Since $\bar{W}$ and $\bar{W}_1^0$ are block matrices, it follows that

$$\begin{aligned}
& Z_{2i}^{0\,\prime}M_{\bar{W}}Z_{2i}^0 \\
= \;& Z_{2i}^{0\,\prime}\left[I-\bar{W}\left(\bar{W}'\bar{W}\right)^{-1}\bar{W}\right]Z_{2i}^0 \\
= \;& [0,V_i^0(k^*,k_2^0,T)]'\left[I-\bar{W}\left(\bar{W}'\bar{W}\right)^{-1}\bar{W}\right]\left[\begin{array}{c} 0 \\ V_i^0(k^*,k_2^0,T)\end{array}\right] \\
= \;& V_i^0(k^*,k_2^0,T)'V_i^0(k^*,k_2^0,T)-V_i^0(k^*,k_2^0,T)'V_i(k^*,T)\left(V_i(k^*,T)'V_i(k^*,T)\right)^{-1}V_i(k^*,T)V_i^0(k^*,k_2^0,T) \\
= \;& V_i(k_2^0,T)'V_i(k_2^0,T)-V_i(k_2^0,T)'V_i(k_2^0,T)\left(V_i(k^*,T)'V_i(k^*,T)\right)^{-1}V_i(k_2^0,T)'V_i(k_2^0,T) \\
= \;& V_i(k_2^0,T)'V_i(k_2^0,T)\left[\left(V_i(k_2^0,T)'V_i(k_2^0,T)\right)^{-1}-\left(V_i(k^*,T)'V_i(k^*,T)\right)^{-1}\right]V_i(k_2^0,T)'V_i(k_2^0,T), \quad \text{(B.15)}
\end{aligned}$$

$$\begin{aligned}
& Z_{2i}^{0\,\prime}M_{\bar{W}_1^0}Z_{2i}^0 \\
= \;& Z_{2i}^{0\,\prime}\left[I-\bar{W}_1^0\left(\bar{W}_1^{0\prime}\bar{W}_1^0\right)^{-1}\bar{W}_1^{0\prime}\right]Z_{2i}^0 \\
= \;& [0,V_i^0(k_1^0,k_2^0,T)]'\left[I-\bar{W}_1^0\left(\bar{W}_1^{0\prime}\bar{W}_1^0\right)^{-1}\bar{W}_1^{0\prime}\right]\left[\begin{array}{c} 0 \\ V_i^0(k_1^0,k_2^0,T)\end{array}\right] \\
= \;& V_i^0(k_1^0,k_2^0,T)'V_i^0(k_1^0,k_2^0,T)-V_i^0(k_1^0,k_2^0,T)'V_i(k_1^0,T)\left(V_i(k_1^0,T)'V_i(k_1^0,T)\right)^{-1}V_i(k_1^0,T)V_i^0(k_1^0,k_2^0,T) \\
= \;& V_i(k_2^0,T)'V_i(k_2^0,T)\left[\left(V_i(k_2^0,T)'V_i(k_2^0,T)\right)^{-1}-\left(V_i(k_1^0,T)'V_i(k_1^0,T)\right)^{-1}\right]V_i(k_2^0,T)'V_i(k_2^0,T). \quad \text{(B.16)}
\end{aligned}$$

Substituting (B.15) and (B.16) into (B.14), we have

$$
\begin{aligned}
J_{2i}(k^*) &= \delta_i^{0\prime} V_i(k_2^0,T)' V_i(k_2^0,T) \left[ \left( V_i(k_1^0,T)' V_i(k_1^0,T) \right)^{-1} - \left( V_i(k^*,T)' V_i(k^*,T) \right)^{-1} \right] V_i(k_2^0,T)' V_i(k_2^0,T) \delta_i^0 \\
&= \delta_i^{0\prime} Z_{2i}^{0\,\prime} Z_{2i}^0 \left[ \left( Z_{1i}^{0\,\prime} Z_{1i}^0 \right)^{-1} - \left( Z_i' Z_i \right)^{-1} \right] Z_{2i}^{0\,\prime} Z_{2i}^0 \delta_i^0 \\
&= \delta_i^{0\prime} Z_{2i}^{0\,\prime} Z_{2i}^0 \left( Z_i' Z_i \right)^{-1} \left( Z_i' Z_i - Z_{1i}^{0\,\prime} Z_{1i}^0 \right) \left( Z_{1i}^{0\,\prime} Z_{1i}^0 \right)^{-1} Z_{2i}^{0\,\prime} Z_{2i}^0 \delta_i^0 \\
&= \delta_i^{0\prime} Z_{2i}^{0\,\prime} Z_{2i}^0 \left( Z_i' Z_i \right)^{-1} Z_{1i}^{\Delta\prime} Z_{1i}^{\Delta} \left( Z_{1i}^{0\,\prime} Z_{1i}^0 \right)^{-1} Z_{2i}^{0\,\prime} Z_{2i}^0 \delta_i^0, \qquad \text{(B.17)}
\end{aligned}
$$

which is symmetric from the first equality. Hence, under Assumptions 4 and 5,

$$
\frac{Z_{2i}^{0\,\prime} Z_{2i}^0}{T} \left( \frac{Z_i' Z_i}{T} \right)^{-1} \frac{Z_{1i}^{\Delta\prime} Z_{1i}^{\Delta}}{k_1^0 - k^*} \left( \frac{Z_{1i}^{0\,\prime} Z_{1i}^0}{T} \right)^{-1} \frac{Z_{2i}^{0\,\prime} Z_{2i}^0}{T} \qquad \text{(B.18)}
$$

is positive definite. Then, we have

$$
\frac{1}{k_1^0 - k^*} \sum_{i \in G_2} J_{2i}(k^*) \geq \lambda_2 \sum_{i \in G_2} \delta_i^{0\prime} \delta_i^0 = \lambda_2 \phi_{N_2}, \qquad \text{(B.19)}
$$

where $\lambda_2 = min_{i \in G_2}\{\lambda_i\}$, and $\lambda_i$ is the minimum eigenvalue of matrix (B.18). From inequalities (B.13) and (B.19), the proof of Lemma B.1 is complete.■

**Lemma B.2** *Under Assumptions 1–6, uniformly on $k^* \in K(C_1)$,*

*(i)* $\sum_{i \in G_1} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{1i}^{\Delta\prime} u_i = O_p(\sqrt{\phi_{N_1}})$,

*(ii)* $\sum_{i \in G_1} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{1i}^{\Delta\prime} X_i \left(X_i' X_i\right)^{-1} X_i' u_i = O_p\left(\sqrt{\frac{\phi_{N_1}}{T}}\right)$,

*(iii)* $\sum_{i \in G_1} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{1i}^{\Delta\prime} M_i Z_i \left(Z_i' M_i Z_i\right)^{-1} Z_i' M_i u_i = O_p\left(\sqrt{\frac{\phi_{N_1}}{T}}\right)$,

$\sum_{i \in G_2} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{2i}^{0\prime} M_i Z_{1i}^{\Delta} \left(Z_i' M_i Z_i\right)^{-1} Z_i' M_i u_i = O_p\left(\sqrt{\frac{\phi_{N_2}}{T}}\right)$,

$\sum_{i \in G_2} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{2i}^{0\prime} M_i Z_{1i}^{0} \left(Z_i' M_i Z_i\right)^{-1} Z_{1i}^{\Delta\prime} M_i u_i = O_p\left(\sqrt{\phi_{N_2}}\right) + O_p\left(\sqrt{\frac{\phi_{N_2}}{T}}\right)$,

$\sum_{i \in G_2} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{2i}^{0\prime} M_i Z_{1i}^{0} \left[\left(Z_i{}' M_i Z_i\right)^{-1} - \left(Z_{1i}^{0\prime} M_i Z_{1i}^{0}\right)^{-1}\right] Z_{1i}^{0\prime} M_i u_i = O_p\left(\sqrt{\frac{\phi_{N_2}}{T}}\right)$,

*(iv)* $\sum_{i=1}^{N} \frac{1}{k_1^0 - k^*} u_i' M_i Z_{1i}^{\Delta} \left(Z_i' M_i Z_i\right)^{-1} Z_{1i}^{\Delta\prime} M_i u_i = O_p\left(\frac{N}{T}\right)$,

*(v)* $\sum_{i=1}^{N} \frac{1}{k_1^0 - k^*} u_i' M_i Z_{1i}^{\Delta} \left(Z_i' M_i Z_i\right)^{-1} Z_{1i}^{0\prime} M_i u_i = O_p\left(\frac{N}{T}\right) + O_p\left(\frac{N}{\sqrt{T}}\right)$,

*(vi)* $\sum_{i=1}^{N} \frac{1}{k_1^0 - k^*} u_i' M_i Z_{1i}^{0} \left[\left(Z_i' M_i Z_i\right)^{-1} - \left(Z_{1i}^{0\prime} M_i Z_{1i}^{0}\right)^{-1}\right] Z_{1i}^{0\prime} M_i u_i = O_p\left(\frac{N}{T}\right)$.

**Proof of Lemma B.2**. (i) It is shown that

$$\frac{1}{\sqrt{k_1^0 - k^*}} Z_{1i}^{\Delta\prime} u_i = O_p(1), \quad \text{since} \quad Var\left(\frac{1}{\sqrt{k_1^0 - k^*}} Z_{1i}^{\Delta\prime} u_i\right) < \infty.$$

Then, we have

$$\sum_{i \in G_1} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{1i}^{\Delta\prime} u_i = O_p(\sqrt{\phi_{N_1}}).$$

(ii) We can show that

$$\sum_{i \in G_1} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{1i}^{\Delta\prime} X_i \left(X_i' X_i\right)^{-1} X_i' u_i = \frac{1}{\sqrt{T}} \sum_{i \in G_1} \delta_i^{0\prime} \frac{Z_{1i}^{\Delta\prime} X_i}{k_1^0 - k^*} \left(\frac{X_i' X_i}{T}\right)^{-1} \frac{1}{\sqrt{T}} X_i' u_i = O_p\left(\sqrt{\frac{\phi_{N_1}}{T}}\right),$$

since for large T

$$\frac{1}{\sqrt{T}} X_i{}' u_i = O_p(1).$$

(iii) By expanding $M_i$, we can show that

$$\begin{aligned}
&\sum_{i\in G_1}\frac{1}{k_1^0-k^*}\delta_i^{0\prime}Z_{1i}^{\Delta\prime}M_iZ_i\left(Z_i'M_iZ_i\right)^{-1}Z_i'M_iu_i \\
=\ &\frac{1}{\sqrt{T}}\sum_{i\in G_1}\delta_i^{0\prime}\frac{Z_{1i}^{\Delta\prime}Z_i}{k_1^0-k^*}\left(\frac{Z_i'M_iZ_i}{T}\right)^{-1}\frac{1}{\sqrt{T}}Z_i'M_iu_i \\
&-\frac{1}{\sqrt{T}}\sum_{i\in G_1}\delta_i^{0\prime}\frac{Z_{1i}^{\Delta\prime}X_i}{k_1^0-k^*}\left(\frac{X_i'X_i}{T}\right)^{-1}\frac{X_i'Z_i}{T}\left(\frac{Z_i'M_iZ_i}{T}\right)^{-1}\frac{1}{\sqrt{T}}Z_i'M_iu_i \\
=\ &O_p\left(\sqrt{\frac{\phi_{N_1}}{T}}\right).
\end{aligned}$$

To prove the second order, since $Z_{2i}^{0\prime}Z_{1i}^{\Delta}=0$, we have

$$\begin{aligned}
&\sum_{i\in G_2}\frac{1}{k_1^0-k^*}\delta_i^{0\prime}Z_{2i}^{0\prime}M_iZ_{1i}^{\Delta}\left(Z_i'M_iZ_i\right)^{-1}Z_i'M_iu_i \\
=\ &\sum_{i\in G_2}\frac{1}{k_1^0-k^*}\delta_i^{0\prime}Z_{2i}^{0\prime}Z_{1i}^{\Delta}\left(Z_i'M_iZ_i\right)^{-1}Z_i'M_iu_i \\
&-\frac{1}{\sqrt{T}}\sum_{i\in G_2}\delta_i^{0\prime}\frac{Z_{2i}^{0\prime}X_i}{T}\left(\frac{X_i'X_i}{T}\right)^{-1}\frac{X_i'Z_{1i}^{\Delta}}{k_1^0-k^*}\left(\frac{Z_i'M_iZ_i}{T}\right)^{-1}\frac{1}{\sqrt{T}}Z_i'M_iu_i \\
=\ &O_p\left(\sqrt{\frac{\phi_{N_2}}{T}}\right).
\end{aligned}$$

Considering the third order, we can show that

$$\begin{aligned}
&\sum_{i\in G_2}\frac{1}{k_1^0-k^*}\delta_i^{0\prime}Z_{2i}^{0\prime}M_iZ_{1i}^{0}\left(Z_i'M_iZ_i\right)^{-1}Z_{1i}^{\Delta\prime}M_iu_i \\
=\ &\sum_{i\in G_2}\frac{1}{k_1^0-k^*}\delta_i^{0\prime}Z_{2i}^{0\prime}M_iZ_{1i}^{0}\left(Z_i'M_iZ_i\right)^{-1}Z_{1i}^{\Delta\prime}u_i \\
&-\sum_{i\in G_2}\frac{1}{k_1^0-k^*}\delta_i^{0\prime}Z_{2i}^{0\prime}M_iZ_{1i}^{0}\left(Z_i'M_iZ_i\right)^{-1}Z_{1i}^{\Delta\prime}X_i(X_i'X_i)^{-1}X_i'u_i \\
=\ &\frac{1}{\sqrt{k_1^0-k^*}}O_p\left(\sqrt{\phi_{N_2}}\right)+O_p\left(\sqrt{\frac{\phi_{N_2}}{T}}\right) \\
=\ &O_p\left(\sqrt{\phi_{N_2}}\right)+O_p\left(\sqrt{\frac{\phi_{N_2}}{T}}\right).
\end{aligned}$$

The last term can be transformed into

$$
\begin{aligned}
& \sum_{i \in G_2} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{2i}^{0\prime} M_i Z_{1i}^0 \left( Z_i' M_i Z_i \right)^{-1} \left( Z_{1i}^{0\prime} M_i Z_{1i}^0 - Z_i' M_i Z_i \right) \left( Z_{1i}^{0\prime} M_i Z_{1i}^0 \right)^{-1} Z_{1i}^{0\prime} M_i u_i \\
= & \sum_{i \in G_2} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{2i}^{0\prime} M_i Z_{1i}^0 \left( Z_i' M_i Z_i \right)^{-1} \left( -Z_{1i}^{\Delta\prime} Z_{1i}^{\Delta} - Z_{1i}^{\Delta\prime} M_i Z_{1i}^0 - Z_{1i}^{0\prime} M_i Z_{1i}^{\Delta} \right) \left( Z_{1i}^{0\prime} M_i Z_{1i}^0 \right)^{-1} Z_{1i}^{0\prime} M_i u_i \\
& + \sum_{i \in G_2} \frac{1}{k_1^0 - k^*} \delta_i^{0\prime} Z_{2i}^{0\prime} M_i Z_{1i}^0 \left( Z_i' M_i Z_i \right)^{-1} \left( Z_{1i}^{\Delta\prime} X_i (X_i' X_i)^{-1} X_i' Z_{1i}^{\Delta} \right) \left( Z_{1i}^{0\prime} M_i Z_{1i}^0 \right)^{-1} Z_{1i}^{0\prime} M_i u_i \\
= & O_p \left( \sqrt{\frac{\phi_{N_2}}{T}} \right) + O_p \left( \sqrt{\frac{\phi_{N_2}}{T}} \right).
\end{aligned}
$$

(iv) The term $\sum_{i=1}^{N} \frac{1}{T(k_1^0 - k^*)} u_i' M_i Z_{1i}^{\Delta} \left( \frac{Z_i' M_i Z_i}{T} \right)^{-1} Z_{1i}^{\Delta\prime} M_i u_i$ has the same order as that of $\sum_{i=1}^{N} \frac{1}{T(k_1^0 - k^*)} u_i' M_i Z_{1i}^{\Delta} Z_{1i}^{\Delta\prime} M_i u_i$ since the matrix $\frac{Z_i' M_i Z_i}{T} = O_p(1)$ for large T. Expanding matrix $M_i$, we have

$$
\begin{aligned}
& \sum_{i=1}^{N} \frac{1}{T(k_1^0 - k^*)} u_i' M_i Z_{1i}^{\Delta} Z_{1i}^{\Delta\prime} M_i u_i \\
= & \frac{1}{T(k_1^0 - k^*)} \sum_{i=1}^{N} u_i' Z_{1i}^{\Delta} Z_{1i}^{\Delta\prime} u_i - \frac{2}{T(k_1^0 - k^*)} \sum_{i=1}^{N} u_i' X_i (X_i' X_i)^{-1} X_i' Z_{1i}^{\Delta} Z_{1i}^{\Delta\prime} u_i \\
& + \frac{1}{T(k_1^0 - k^*)} \sum_{i=1}^{N} u_i' X_i (X_i' X_i)^{-1} X_i' Z_{1i}^{\Delta} Z_{1i}^{\Delta\prime} X_i (X_i' X_i)^{-1} X_i' u_i \qquad \text{(B.20)}
\end{aligned}
$$

Consider the first term in (B.20),

$$
\frac{1}{T(k_1^0 - k^*)} \sum_{i=1}^{N} u_i' Z_{1i}^{\Delta} Z_{1i}^{\Delta\prime} u_i = O_p \left( \frac{N}{T} \right).
$$

Similarly, it can be shown that the second term,

$$
\begin{aligned}
& \frac{1}{T(k_1^0 - k^*)} \sum_{i=1}^{N} u_i' X_i (X_i' X_i)^{-1} X_i' Z_{1i}^{\Delta} Z_{1i}^{\Delta\prime} u_i \\
= & \sqrt{\frac{k_1^0 - k^*}{T}} \frac{1}{T} \sum_{i=1}^{N} \frac{u_i' X_i}{\sqrt{T}} \left( \frac{X_i' X_i}{T} \right)^{-1} \frac{X_i' Z_{1i}^{\Delta}}{k_1^0 - k^*} \frac{Z_{1i}^{\Delta\prime} u_i}{\sqrt{k_1^0 - k^*}} = O_p \left( \frac{N}{T} \right),
\end{aligned}
$$

and the third term,

$$
\begin{aligned}
& \frac{1}{T(k_1^0 - k^*)} \sum_{i=1}^{N} u_i' X_i (X_i' X_i)^{-1} X_i' Z_{1i}^{\Delta} Z_{1i}^{\Delta\prime} X_i (X_i' X_i)^{-1} X_i' u_i \\
= & \frac{k_1^0 - k^*}{T} \frac{1}{T} \sum_{i=1}^{N} \frac{u_i' X_i}{\sqrt{T}} \left( \frac{X_i' X_i}{T} \right)^{-1} \frac{X_i' Z_{1i}^{\Delta}}{k_1^0 - k^*} \frac{Z_{1i}^{\Delta\prime} X_i}{k_1^0 - k^*} \left( \frac{X_i' X_i}{T} \right)^{-1} \frac{X_i' u_i}{\sqrt{T}} = O_p \left( \frac{N}{T} \right).
\end{aligned}
$$

Thus, we have

$$\sum_{i=1}^{N} \frac{1}{k_1^0 - k^*} u_i' M_i Z_{1i}^{\Delta} \left(Z_i' M_i Z_i\right)^{-1} Z_{1i}^{\Delta\prime} M_i u_i = O_p\left(\frac{N}{T}\right).$$

(v) By expanding $M_i$, we show that

$$\begin{aligned}
& \sum_{i=1}^{N} \frac{1}{k_1^0 - k^*} u_i' M_i Z_{1i}^{\Delta} \left(Z_i' M_i Z_i\right)^{-1} Z_{1i}^{0\prime} M_i u_i \\
= & \sum_{i=1}^{N} \frac{1}{k_1^0 - k^*} u_i' Z_{1i}^{\Delta} \left(Z_i' M_i Z_i\right)^{-1} Z_{1i}^{0\prime} M_i u_i - \sum_{i=1}^{N} \frac{1}{k_1^0 - k^*} u_i' X_i (X_i' X_i)^{-1} X_i' Z_{1i}^{\Delta} \left(Z_i' M_i Z_i\right)^{-1} Z_{1i}^{0\prime} M_i u_i \\
= & O_p\left(\frac{N}{\sqrt{T}}\right) + O_p\left(\frac{N}{T}\right).
\end{aligned}$$

(vi) We show that

$$\begin{aligned}
& \sum_{i=1}^{N} \frac{1}{k_1^0 - k^*} u_i{}' M_i Z_{1i}^{0} \left(Z_i{}' M_i Z_i\right)^{-1} \left(Z_{1i}^{0\prime} M_i Z_{1i}^{0} - Z_i{}' M_i Z_i\right) \left(Z_{1i}^{0\prime} M_i Z_{1i}^{0}\right)^{-1} Z_{1i}^{0\prime} M_i u_i \\
= & \sum_{i=1}^{N} \frac{1}{k_1^0 - k^*} u_i{}' M_i Z_{1i}^{0} \left(Z_i{}' M_i Z_i\right)^{-1} \left(-Z_{1i}^{\Delta\prime} M_i Z_{1i}^{\Delta} - Z_{1i}^{\Delta\prime} M_i Z_{1i}^{0} - Z_{1i}^{0\prime} M_i Z_{1i}^{\Delta}\right) \left(Z_{1i}^{0\prime} M_i Z_{1i}^{0}\right)^{-1} Z_{1i}^{0\prime} M_i u_i \\
= & O_p\left(\frac{N}{T}\right).
\end{aligned}$$

The proof of Lemma B.2 is complete.■

**Proof of Proposition 1**. We first show that for any given $\epsilon > 0$,

$$P\left(\sup_{K(C_1)} \left| \frac{\sum_{i \in G_1} H_{1i}(k^*) + \sum_{i \in G_2} H_{2i}(k^*)}{k_1^0 - k^*} \right| \geq \lambda \phi_N \right) < \epsilon. \tag{B.21}$$

Using (B.5) and (B.9), we see that the sum of $H_{1i}(k^*) + H_{2i}(k^*)$ can be decomposed into three parts:

$$
\begin{aligned}
& \frac{1}{k_1^0-k^*}\left(\sum_{i\in G_1} H_{1i}(k^*)+\sum_{i\in G_2} H_{2i}(k^*)\right) \\
= & \frac{1}{k_1^0-k^*}\left[\sum_{i\in G_1} 2\delta_i^{0\prime} Z_{1i}^{0\,\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i - \sum_{i\in G_1} 2\delta_i^{0\prime} Z_{1i}^{0\,\prime} M_i u_i\right] \\
& + \frac{1}{k_1^0-k^*}\left[\sum_{i\in G_2} 2\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i - \sum_{i\in G_2} 2\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^0 (Z_{1i}^{0\,\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^0 M_i u_i\right] \\
& + \frac{1}{k_1^0-k^*}\left[\sum_{i=1}^N u_i' M_i Z_i' (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i - \sum_{i=1}^N u_i' M_i Z_{1i}^0 (Z_{1i}^{0\,\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^{0\,\prime} M_i u_i\right] \\
= & H_1 + H_2 + H_3.
\end{aligned}
$$

Consider the first term by replacing $Z_{1i}^0$ with $Z_i - Z_{1i}^\Delta$,

$$
\begin{aligned}
|H_1| &= 2\left|\frac{1}{k_1^0-k^*}\sum_{i\in G_1}\left[\delta_i^{0\prime} Z_{1i}^{0\,\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i - \delta_i^{0\prime} Z_{1i}^{0\,\prime} M_i u_i\right]\right| \\
&= 2\left|\frac{1}{k_1^0-k^*}\sum_{i\in G_1}\left[\delta_i^{0\prime} Z_i' M_i u_i - \delta_i^{0\prime} Z_{1i}^{\Delta\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i - \delta_i^{0\prime} Z_i{}' M_i u_i + \delta_i^{0\prime} Z_{1i}^{\Delta\prime} M_i u_i\right]\right| \\
&= 2\left|\frac{1}{k_1^0-k^*}\sum_{i\in G_1}\left[\delta_i^{0\prime} Z_{1i}^{\Delta\prime} M_i u_i - \delta_i^{0\prime} Z_{1i}^{\Delta\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i\right]\right| \\
&\le 2\left|\frac{1}{k_1^0-k^*}\sum_{i\in G_1}\delta_i^{0\prime} Z_{1i}^{\Delta\prime} u_i\right| + 2\left|\frac{1}{k_1^0-k^*}\sum_{i\in G_1}\delta_i^{0\prime} Z_{1i}^{\Delta\prime} X_i (X_i' X_i)^{-1} X_i' u_i\right| \\
&\quad +2\left|\frac{1}{k_1^0-k^*}\sum_{i\in G_1}\delta_i^{0\prime} Z_{1i}^{\Delta\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i\right| \\
&= O_p(\sqrt{\phi_{N_1}}) + O_p\left(\sqrt{\frac{\phi_{N_1}}{T}}\right) + O_p\left(\sqrt{\frac{\phi_{N_1}}{T}}\right), \qquad \text{(B.22)}
\end{aligned}
$$

where the inequality is obtained by expanding $M_i$, and the final equality uses the orders in (i)–(iii) of Lemma B.2.

For the second term $H_2$, replacing $Z_i$ with $Z_{1i}^{\Delta} + Z_{1i}^0$, and using (iii) of Lemma B.2,

$$
\begin{aligned}
|H_2| &= 2\left|\frac{1}{k_1^0 - k^*}\sum_{i\in G_2}\left[2\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_i (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i - 2\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^0 (Z_{1i}^{0\,\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^{0\,\prime} M_i u_i\right]\right| \\
&= 2\left|\frac{1}{k_1^0 - k^*}\sum_{i\in G_2}\left[\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^{\Delta} (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i + \delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^0 (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i\right.\right. \\
&\quad \left.\left. -\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^0 (Z_{1i}^{0\,\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^0 M_i u_i\right]\right| \\
&= 2\left|\frac{1}{k_1^0 - k^*}\sum_{i\in G_2}\left[\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^{\Delta} (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i + \delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^0 (Z_i' M_i Z_i)^{-1} Z_{1i}^{\Delta\prime} M_i u_i\right.\right. \\
&\quad \left.\left. +\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^0 \left[\left(Z_i{}' M_i Z_i\right)^{-1} - \left(Z_{1i}^{0\,\prime} M_i Z_{1i}^0\right)^{-1}\right] Z_{1i}^{0\,\prime} M_i u_i\right]\right| \\
&\leq 2\left|\frac{1}{k_1^0 - k^*}\sum_{i\in G_2}\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^{\Delta} (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i\right| \\
&\quad +2\left|\frac{1}{k_1^0 - k}\sum_{i\in G_2}\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^0 (Z_i' M_i Z_i)^{-1} Z_{1i}^{\Delta\prime} M_i u_i\right| \qquad \text{(B.23)} \\
&\quad +2\left|\frac{1}{k_1^0 - k^*}\sum_{i\in G_2}\delta_i^{0\prime} Z_{2i}^{0\,\prime} M_i Z_{1i}^0 \left[\left(Z_i{}' M_i Z_i\right)^{-1} - \left(Z_{1i}^{0\,\prime} M_i Z_{1i}^0\right)^{-1}\right] Z_{1i}^{0\,\prime} M_i u_i\right| \\
&= O_p\left(\sqrt{\frac{\phi_{N_2}}{T}}\right) + O_p(\sqrt{\phi_{N_2}}) + O_p\left(\sqrt{\frac{\phi_{N_2}}{T}}\right) + O_p\left(\sqrt{\frac{\phi_{N_2}}{T}}\right). \qquad \text{(B.24)}
\end{aligned}
$$

By (iv)–(vi) of Lemma B.2, the order of the third term is

$$
\begin{aligned}
|H_3| &= \left|\frac{1}{k_1^0 - k^*}\sum_{i=1}^{N}\left[u_i' M_i Z_i' (Z_i' M_i Z_i)^{-1} Z_i' M_i u_i - u_i' M_i Z_{1i}^0 (Z_{1i}^{0\,\prime} M_i Z_{1i}^0)^{-1} Z_{1i}^{0\,\prime} M_i u_i\right]\right| \\
&\leq \left|\frac{1}{k_1^0 - k^*}\sum_{i=1}^{N} u_i' M_i Z_{1i}^{\Delta\prime} (Z_i' M_i Z_i)^{-1} Z_{1i}^{\Delta\prime} M_i u_i\right| + \left|\frac{2}{k_1^0 - k^*}\sum_{i=1}^{N} u_i' M_i Z_{1i}^{\Delta\prime} (Z_i' M_i Z_i)^{-1} Z_{1i}^{0\,\prime} M_i u_i\right| \\
&\quad + \left|\sum_{i=1}^{N}\frac{1}{k_1^0 - k^*} u_i' M_i Z_{1i}^0 \left[\left(Z_i' M_i Z_i\right)^{-1} - \left(Z_{1i}^{0\,\prime} M_i Z_{1i}^0\right)^{-1}\right] Z_{1i}^{0\,\prime} M_i u_i\right| \\
&= O_p\left(\frac{N}{T}\right) + O_p\left(\frac{N}{T}\right) + O_p\left(\frac{N}{\sqrt{T}}\right) + O_p\left(\frac{N}{T}\right). \qquad \text{(B.25)}
\end{aligned}
$$

Combining (B.22), (B.24), and (B.25) under Assumption 2, the term,

$$\begin{aligned}
& \frac{1}{\phi_N}\left|\frac{\sum\limits_{i\in G_1} H_{1i}(k^*)+\sum\limits_{i\in G_2} H_{2i}(k^*)}{k_1^0-k^*}\right| \\
= \quad & \frac{1}{\phi_N}\left|\left[O_p(\sqrt{\phi_{N_1}})+O_p(\sqrt{\phi_{N_2}})+O_p\left(\sqrt{\frac{\phi_{N_1}}{T}}\right)+O_p\left(\sqrt{\frac{\phi_{N_2}}{T}}\right)+O_p\left(\frac{N}{\sqrt{T}}\right)+O_p\left(\frac{N}{T}\right)\right]\right| \\
\to \quad & 0,
\end{aligned}$$

will vanish for any $k^* \in K(C_1)$. On the other hand, the part $\phi_N^{-1}(k_1^0-k^*)^{-1}$ $\left|\sum\limits_{i\in G_1} J_{1i}(k^*)+\sum\limits_{i\in G_2} J_{2i}(k^*)\right|$ has a lower bound from Lemma B.1. Hence, for any $\epsilon>0$,

$$P\left(\sup_{K(C_1)}\left|\frac{\sum\limits_{i\in G_1} H_{1i}(k^*)+\sum\limits_{i\in G_2} H_{2i}(k^*)}{k_1^0-k^*}\right| \geq \sup_{K(C_1)}\left|\frac{\sum\limits_{i\in G_1} J_{1i}(k^*)+\sum\limits_{i\in G_2} J_{2i}(k^*)}{k_1^0-k^*}\right|\right)<\epsilon,$$

which implies that

$$\begin{aligned}
P\left(\sup_{K(C_1)}\sum_{i\in G_1} -J_{1i}(k^*)+H_{1i}(k^*)+\sum_{i\in G_2} -J_{2i}(k^*)+H_{2i}(k^*)\geq 0\right) \quad &< \quad \epsilon, \\
P\left(\sup_{K(C_1)}\sum_{i=1}^{N}\left[SV_i(k^*)-SV_i(k_1^0)\right]\geq 0\right) \quad &< \quad \epsilon.
\end{aligned}$$

Finally, we obtain that for any given $\epsilon>0$, and both large $N$ and $T$,

$$P(\hat{k}\in K(C_1))<\epsilon.$$

In other words, the total sum of squared residuals cannot be maximized in the case of $k^* \in K(C_1)$. By symmetry, the estimation of the common break point (3) can be transformed into

$$\hat{k}=\arg\max_{1\leq k^*\leq T-1}\left[\sum_{i\in G_1}(SV_i(k^*)-SV_i(k_2^0))+\sum_{i\in G_2}(SV_i(k^*)-SV_i(k_2^0))\right].$$

Similarly, we can show that, for any given $\epsilon>0$,

$$P(\hat{k}\in K(C_2))<\epsilon.$$

The common break point estimator is obtained in set $K(C_2)$ with probability tending to zero. Thus, we complete the proof of Proposition 1. ■

Proposition 1 indicates that the estimated common break is stochastically bounded by either true break points or located between $k_1^0$ and $k_2^0$. Then, we can say that

$$\frac{k_1^0 - \hat{k}}{T} = O_p\left(\frac{1}{T}\right), \quad \text{if} \quad \hat{k} \leq k_1^0, \tag{B.26}$$

$$\frac{\hat{k} - k_2^0}{T} = O_p\left(\frac{1}{T}\right), \quad \text{if} \quad \hat{k} \geq k_2^0. \tag{B.27}$$

Using this property of the common break estimator under the alternative, we next show that the numerator of the statistic will diverge under $H_{1A}$.

**Proof of Proposition 2**. Under the alternative, from (A.4), the CUSUM of the residuals for individuals in group $j$ ($j = 1, 2$) are calculated as

$$\begin{aligned}
&\frac{1}{\sqrt{NT}} \sum_{i \in G_j} \sum_{t=1}^{k} \hat{u}_{it} \\
&= \frac{1}{\sqrt{NT}} \sum_{i \in G_j} \sum_{t=1}^{k} u_{it} - \frac{1}{\sqrt{NT}} \sum_{i \in G_j} \sum_{t=1}^{k} x_{it}'(\hat{\beta}_i - \beta_i^0) - \frac{1}{\sqrt{NT}} \sum_{i \in G_j} \sum_{t=\hat{k}+1}^{k} x_{it}'(\hat{\delta}_i - \delta_i^0) 1_{\{k > \hat{k}\}} \\
&+ \frac{1}{\sqrt{NT}} \sum_{i \in G_j} \sum_{t=k_j^0+1}^{k} x_{it}' \delta_i^0 1_{\{k_j^0 < k \leq \hat{k}\}} + \frac{1}{\sqrt{NT}} \sum_{i \in G_j} \sum_{t=k_j^0+1}^{\hat{k}} x_{it}' \delta_i^0 1_{\{k_j^0 < \hat{k} < k\}} \\
&- \frac{1}{\sqrt{NT}} \sum_{i \in G_j} \sum_{t=\hat{k}+1}^{k} x_{it}' \delta_i^0 1_{\{\hat{k} < k \leq k_j^0\}} - \frac{1}{\sqrt{NT}} \sum_{i \in G_j} \sum_{t=\hat{k}+1}^{k_j^0} x_{it}' \delta_i^0 1_{\{\hat{k} < k_j^0 < k\}}.
\end{aligned}$$

Then, the total sum of the squared residuals $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}\hat{u}_{it}$ is expressed as

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}u_{it}-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{k}x_{it}'(\hat{\beta}_i(\hat{k})-\beta_i^0)-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{k}x_{it}'(\hat{\delta}_i(\hat{k})-\delta_i^0)1_{\{k>\hat{k}\}}\\
&+\frac{1}{\sqrt{NT}}\sum_{i\in G_1}\sum_{t=k_1^0+1}^{k}x_{it}'\delta_i^0 1_{\{k_1^0<k\leq\hat{k}\}}+\frac{1}{\sqrt{NT}}\sum_{i\in G_1}\sum_{t=k_1^0+1}^{\hat{k}}x_{it}'\delta_i^0 1_{\{k_1^0<\hat{k}<k\}}\\
&-\frac{1}{\sqrt{NT}}\sum_{i\in G_1}\sum_{t=\hat{k}+1}^{k}x_{it}'\delta_i^0 1_{\{\hat{k}<k\leq k_1^0\}}-\frac{1}{\sqrt{NT}}\sum_{i\in G_1}\sum_{t=\hat{k}+1}^{k_1^0}x_{it}'\delta_i^0 1_{\{\hat{k}<k_1^0<k\}}\\
&+\frac{1}{\sqrt{NT}}\sum_{i\in G_2}\sum_{t=k_2^0+1}^{k}x_{it}'\delta_i^0 1_{\{k_2^0<k\leq\hat{k}\}}+\frac{1}{\sqrt{NT}}\sum_{i\in G_2}\sum_{t=k_2^0+1}^{\hat{k}}x_{it}'\delta_i^0 1_{\{k_2^0<\hat{k}<k\}}\\
&-\frac{1}{\sqrt{NT}}\sum_{i\in G_2}\sum_{t=\hat{k}+1}^{k}x_{it}'\delta_i^0 1_{\{\hat{k}<k\leq k_2^0\}}-\frac{1}{\sqrt{NT}}\sum_{i\in G_2}\sum_{t=\hat{k}+1}^{k_2^0}x_{it}'\delta_i^0 1_{\{\hat{k}<k_2^0<k\}}\\
&=U_1^{H_1}-U_2^{H_1}-U_3^{H_1}+U_4^{H_1}+U_5^{H_1}-U_6^{H_1}-U_7^{H_1}+U_8^{H_1}+U_9^{H_1}-U_{10}^{H_1}-U_{11}^{H_1}. \qquad \text{(B.28)}
\end{aligned}
$$

Since $\hat{k}<k_1^0$ in $U_6^{H_1},U_7^{H_1}$, and $\hat{k}>k_2^0$ in $U_8^{H_1},U_9^{H_1}$, using the orders (B.26) and (B.27), we have

$$
U_6^{H_1}=O_p\left(\sqrt{\frac{N}{T}}\right),U_7^{H_1}=O_p\left(\sqrt{\frac{N}{T}}\right),U_8^{H_1}=O_p\left(\sqrt{\frac{N}{T}}\right),U_9^{H_1}=O_p\left(\sqrt{\frac{N}{T}}\right). \qquad \text{(B.29)}
$$

From (A.10), we know that for $i\in G_j, j=1,2$,

$$
\begin{aligned}
&\sqrt{T}(\hat{\beta}_i-\beta_i^0)=\sqrt{T}\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{\hat{k}}x_{it}u_{it}+\sqrt{T}\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=k_j^0+1}^{\hat{k}}x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}>k_j^0\}}\\
&=\begin{cases}\sqrt{T}\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{\hat{k}}x_{it}u_{it}+\sqrt{T}\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=k_1^0+1}^{\hat{k}}x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}>k_1^0\}} & \text{if } j=1,\\ \sqrt{T}\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{\hat{k}}x_{it}u_{it}+O_p\left(\frac{1}{\sqrt{T}}\right) & \text{if } j=2,\end{cases}
\end{aligned}
$$

using order (B.27). Then, the second term $U_2^{H_1}$ becomes

$$\begin{aligned}
&\frac{1}{\sqrt{N}}\sum_{i\in G_1}\frac{1}{T}\sum_{t=1}^{k}x_{it}'\sqrt{T}\left[\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{\hat{k}}x_{it}u_{it}+\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=k_1^0+1}^{\hat{k}}x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}>k_1^0\}}\right]\\
&\quad+\frac{1}{\sqrt{N}}\sum_{i\in G_2}\frac{1}{T}\sum_{t=1}^{k}x_{it}'\left[\sqrt{T}\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{\hat{k}}x_{it}u_{it}+O_p\left(\frac{1}{\sqrt{T}}\right)\right]\\
&=O_p(1)+\frac{1}{\sqrt{N}}\sum_{i\in G_1}\frac{1}{T}\sum_{t=1}^{k}x_{it}'\left(\frac{1}{T}\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\frac{1}{\sqrt{T}}\sum_{t=k_1^0+1}^{\hat{k}}x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}>k_1^0\}}\\
&\quad+O_p(1)+O_p\left(\sqrt{\frac{N}{T}}\right)\\
&=O_p(1)+U_{21}^{H_1}1_{\{\hat{k}>k_1^0\}}+O_p(1)+O_p\left(\sqrt{\frac{N}{T}}\right).
\end{aligned}\tag{B.30}$$

Considering the third term $U_3^{H_1}$, for individuals $i\in G_j$, the coefficient estimator is

$$\begin{aligned}
\hat{\delta}_i &= \left(\sum_{t=\hat{k}+1}^{T}x_{it}x_{it}'\right)^{-1}\sum_{t=\hat{k}+1}^{T}x_{it}y_{it}-\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{\hat{k}}x_{it}y_{it}\\
&= \left(\sum_{t=\hat{k}+1}^{T}x_{it}x_{it}'\right)^{-1}\left\{\sum_{t=\hat{k}+1}^{T}x_{it}(x_{it}'\beta_i^0+x_{it}'\delta_i^0+u_{it})1_{\{\hat{k}\geq k_j^0\}}\right.\\
&\quad\left.+\left[\sum_{t=\hat{k}+1}^{k_j^0}x_{it}(x_{it}'\beta_i^0+u_{it})+\sum_{t=k_j^0+1}^{T}x_{it}(x_{it}'\beta_i^0+x_{it}'\delta_i^0+u_{it})\right]1_{\{\hat{k}<k_j^0\}}\right\}\\
&\quad-\left[\beta_i^0+\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{\hat{k}}x_{it}u_{it}+\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=k_j^0+1}^{\hat{k}}x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}>k_j^0\}}\right]\\
&= \delta_i^0+\left(\sum_{t=\hat{k}+1}^{T}x_{it}x_{it}'\right)^{-1}\sum_{t=\hat{k}+1}^{T}x_{it}u_{it}-\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=1}^{\hat{k}}x_{it}u_{it}\\
&\quad-\left(\sum_{t=\hat{k}+1}^{T}x_{it}x_{it}'\right)^{-1}\sum_{t=\hat{k}+1}^{k_j^0}x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}<k_j^0\}}-\left(\sum_{t=1}^{\hat{k}}x_{it}x_{it}'\right)^{-1}\sum_{t=k_j^0+1}^{\hat{k}}x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}>k_j^0\}},
\end{aligned}$$

where the fourth term in the final equality is $O_p(1/T)$ for individuals in group 1 using order (B.26), while the fifth term is $O_p(1/T)$ for individuals in group 2 using order (B.27). Then, the third term $U_3^{H_1}$ can be rewritten as

$$\left\{ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{1}{T} \sum_{t=\hat{k}+1}^{k} x_{it}' \sqrt{T} \left[ \left( \sum_{t=\hat{k}+1}^{T} x_{it}x_{it}' \right)^{-1} \sum_{t=\hat{k}+1}^{T} x_{it}u_{it} - \left( \sum_{t=1}^{\hat{k}} x_{it}x_{it}' \right)^{-1} \sum_{t=1}^{\hat{k}} x_{it}u_{it} \right] \right.$$
$$-O_p\left(\sqrt{\frac{N}{T}}\right) - \frac{1}{\sqrt{N}} \sum_{i\in G_1} \frac{1}{T} \sum_{t=\hat{k}+1}^{k} x_{it}' \left( \sum_{t=1}^{\hat{k}} x_{it}x_{it}' \right)^{-1} \sum_{t=k_1^0+1}^{\hat{k}} x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}>k_1^0\}}$$
$$\left. -\frac{1}{\sqrt{N}} \sum_{i\in G_2} \frac{1}{T} \sum_{t=\hat{k}+1}^{k} x_{it}' \left( \frac{1}{T} \sum_{t=\hat{k}+1}^{T} x_{it}x_{it}' \right)^{-1} \frac{1}{\sqrt{T}} \sum_{t=\hat{k}+1}^{k_2^0} x_{it}x_{it}'\delta_i^0 1_{\{\hat{k}<k_2^0\}} - O_p\left(\sqrt{\frac{N}{T}}\right) \right\} 1_{\{k>\hat{k}\}}$$
$$= \left[ O_p(1) - O_p\left(\sqrt{\frac{N}{T}}\right) - U_{31}^{H_1} 1_{\{\hat{k}>k_1^0\}} - U_{32}^{H_1} 1_{\{\hat{k}<k_2^0\}} - O_p\left(\sqrt{\frac{N}{T}}\right) \right] 1_{\{k>\hat{k}\}}. \quad \text{(B.31)}$$

Thus, from (B.29), (B.30), and (B.31), (B.28) can be rewritten as

$$\begin{aligned}
& \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{k} \hat{u}_{it} \\
= \; & O_p(1) - \left[ O_p(1) + U_{21}^{H_1} 1_{\{\hat{k}>k_1^0\}} + O_p(1) + O_p\left(\sqrt{\frac{N}{T}}\right) \right] \\
& - \left[ O_p(1) - O_p\left(\sqrt{\frac{N}{T}}\right) - U_{31}^{H_1} 1_{\{\hat{k}>k_1^0\}} - U_{32}^{H_1} 1_{\{\hat{k}<k_2^0\}} - O_p\left(\sqrt{\frac{N}{T}}\right) \right] 1_{\{k>\hat{k}\}} \\
& + U_4^{H_1} + U_5^{H_1} + O_p\left(\sqrt{\frac{N}{T}}\right) - U_{10}^{H_1} - U_{11}^{H_1} \\
= \; & -U_{21}^{H_1} 1_{\{\hat{k}>k_1^0\}} + \left[ U_{31}^{H_1} 1_{\{\hat{k}>k_1^0\}} + U_{32}^{H_1} 1_{\{\hat{k}<k_2^0\}} \right] 1_{\{k>\hat{k}\}} + U_4^{H_1} + U_5^{H_1} - U_{10}^{H_1} - U_{11}^{H_1} \\
& + O_p(1) + O_p\left(\sqrt{\frac{N}{T}}\right). \qquad \text{(B.32)}
\end{aligned}$$

Next, we show that (B.32) diverges at the rate of $\sqrt{NT}$ under the alternative in the following three cases.

**Case (i).** Suppose that $\hat{k} < k_1^0 < k_2^0$, we have

$$U_{21}^{H_1} 1_{\{\hat{k}>k_1^0\}} = 0, \; U_{31}^{H_1} 1_{\{\hat{k}>k_1^0\}} = 0, \; U_4^{H_1} = 0, \; U_5^{H_1} = 0.$$

Choosing $k \in (k_1^0 + C_1, k_2^0]$, we can see that $U_{11}^{H_1} = 0$, and

$$\begin{aligned}
& -U_{10}^{H_1} + U_{32}^{H_1} 1_{\{\hat{k} < k_2^0\}} 1_{\{k > \hat{k}\}} \\
= & -\frac{1}{\sqrt{NT}} \sum_{i \in G_2} \sum_{t=\hat{k}+1}^{k} x_{it}' \delta_i^0 + \frac{1}{\sqrt{NT}} \sum_{i \in G_2} \sum_{t=\hat{k}+1}^{k} x_{it}' \left( \sum_{t=\hat{k}+1}^{T} x_{it} x_{it}' \right)^{-1} \sum_{t=\hat{k}+1}^{k_2^0} x_{it} x_{it}' \delta_i^0 \\
= & -\sqrt{\frac{T}{N}} \sum_{i \in G_2} \frac{1}{T} \sum_{t=\hat{k}+1}^{k} x_{it}' \left( \frac{1}{T} \sum_{t=\hat{k}+1}^{T} x_{it} x_{it}' \right)^{-1} \frac{1}{T} \sum_{t=k_2^0+1}^{T} x_{it} x_{it}' \delta_i^0 \\
= & \ O_p(\sqrt{NT}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sup_{k \in [1, T-1]} US_{NT}(k, \hat{k}) & \geq \sup_{k \in (k_1^0 + C_1, k_2^0]} US_{NT}(k, \hat{k}) \\
& = \sup_{k \in (k_1^0 + C_1, k_2^0]} \left( U_{32}^{H_1} - U_{10}^{H_1} + O_p(1) + O_p\left( \sqrt{\frac{N}{T}} \right) \right)^2 = O_p(NT).
\end{aligned}$$

**Case (ii).** Suppose that $k_1^0 \leq \hat{k} \leq k_2^0$. If $\hat{k} \in (k_1^0 + C_1, k_2^0]$, choosing $k \in [k_1^0, k_1^0 + C_1]$, we have

$$U_{31}^{H_1} 1_{\{k > \hat{k}\}} = 0, \ U_{32}^{H_1} 1_{\{k > \hat{k}\}} = 0, \ U_5^{H_1} = 0, \ U_{10}^{H_1} = 0, \ U_{11}^{H_1} = 0,$$

$$U_4^{H_1} = \frac{1}{\sqrt{NT}} \sum_{i \in G_1} \sum_{t=k_1^0+1}^{k} x_{it}' \delta_i^0 = O_p\left( \sqrt{\frac{N}{T}} \right),$$

since $k < \hat{k}$, and

$$U_{21}^{H_1} = \sqrt{\frac{T}{N}} \sum_{i \in G_1} \frac{1}{T} \sum_{t=1}^{k} x_{it}' \left( \frac{1}{T} \sum_{t=1}^{\hat{k}} x_{it} x_{it}' \right)^{-1} \frac{1}{T} \sum_{t=k_1^0+1}^{\hat{k}} x_{it} x_{it}' \delta_i^0 = O_p(\sqrt{NT}).$$

Thus, we have

$$\begin{aligned}
\sup_{k \in [1, T-1]} US_{NT}(k, \hat{k}) & \geq \sup_{k \in [k_1^0, k_1^0 + C_1]} US_{NT}(k, \hat{k}) \\
& = \sup_{k \in [k_1^0, k_1^0 + C_1]} \left( U_{21}^{H_1} + O_p(1) + O_p\left( \sqrt{\frac{N}{T}} \right) \right)^2 = O_p(NT).
\end{aligned}$$

If $\hat{k} \in [k_1^0, k_1^0 + C_1]$, since $(\hat{k} - k_1^0)/T = O_p(1/T)$,

$$U_{21}^{H_1} 1_{\{\hat{k} > k_1^0\}} = O_p\left( \sqrt{\frac{N}{T}} \right), \ U_{31}^{H_1} 1_{\{\hat{k} > k_1^0\}} = O_p\left( \sqrt{\frac{N}{T}} \right), \ U_4^{H_1} = O_p\left( \sqrt{\frac{N}{T}} \right), \ U_5^{H_1} = O_p\left( \sqrt{\frac{N}{T}} \right).$$

Choosing $k = k_2^0$, we have $U_{11}^{H_1} = 0$, and

$$\begin{aligned}
& U_{32}^{H_1} 1_{\{\hat{k}<k_2^0\}} - U_{10}^{H_1} \\
= & \frac{1}{\sqrt{NT}} \sum_{i\in G_2} \sum_{t=\hat{k}+1}^{k_2^0} x_{it}' \left( \sum_{t=\hat{k}+1}^{T} x_{it}x_{it}' \right)^{-1} \sum_{t=\hat{k}+1}^{k_2^0} x_{it}x_{it}'\delta_i^0 - \frac{1}{\sqrt{NT}} \sum_{i\in G_2} \sum_{t=\hat{k}+1}^{k_2^0} x_{it}'\delta_i^0 \\
= & -\sqrt{\frac{T}{N}} \sum_{i\in G_2} \frac{1}{T} \sum_{t=\hat{k}+1}^{k_2^0} x_{it}' \left( \frac{1}{T} \sum_{t=\hat{k}+1}^{T} x_{it}x_{it}' \right)^{-1} \frac{1}{T} \sum_{t=k_2^0+1}^{T} x_{it}x_{it}'\delta_i^0 \\
= & O_p\left(\sqrt{NT}\right).
\end{aligned}$$

Thus, we have

$$\sup_{k\in[1,T-1]} US_{NT}(k,\hat{k}) \geq US_{NT}(k_2^0,\hat{k}) = \left( U_{32}^{H_1} - U_{10}^{H_1} + O_p(1) + O_p\left(\sqrt{\frac{N}{T}}\right)\right)^2 = O_p(NT).$$

**Case (iii).** Suppose that $k_1^0 < k_2^0 < \hat{k}$, then we have

$$U_{32}^{H_1} 1_{\{\hat{k}<k_2^0\}} = 0,\ U_{10}^{H_1} = 0,\ U_{11}^{H_1} = 0.$$

Choosing $k \in (k_1^0, k_1^0 + C_1]$, we can see that $U_{31}^{H_1} 1_{\{k>\hat{k}\}} = 0$, $U_5^{H_1} = 0$, $U_4^{H_1} = O_p\left(\sqrt{N/T}\right)$, and

$$\begin{aligned}
U_{21}^{H_1} 1_{\{\hat{k}>k_1^0\}} &= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{t=1}^{k} x_{it}' \left( \sum_{t=1}^{\hat{k}} x_{it}x_{it}' \right)^{-1} \sum_{t=k_1^0+1}^{\hat{k}} x_{it}x_{it}'\delta_i^0 \\
&= \sqrt{\frac{T}{N}} \sum_{i=1}^{N} \frac{1}{T} \sum_{t=1}^{k} x_{it}' \left( \frac{1}{T} \sum_{t=1}^{\hat{k}} x_{it}x_{it}' \right)^{-1} \frac{1}{T} \sum_{t=k_1^0+1}^{\hat{k}} x_{it}x_{it}'\delta_i^0 \\
&= O_p(\sqrt{NT}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sup_{k\in[1,T-1]} US_{NT}(k,\hat{k}) &\geq \sup_{k\in(k_1^0,k_1^0+C_1]} US_{NT}(k,\hat{k}) \\
&= \sup_{k\in(k_1^0,k_1^0+C_1]} \left( U_{21}^{H_1} + O_p(1) + O_p\left(\sqrt{\frac{N}{T}}\right)\right)^2 = O_p(NT).
\end{aligned}$$

The proof of Proposition 2 is complete. ■

**Proof of Proposition 3**. From Proposition 1, under the alternative $H_{1A}$, the estimated common break $\hat{k}$ takes a value in $[k_1^0 - C_1, k_2^0 + C_2]$ with probability approaching one, for arbitrary positive constants $C_1, C_2$. Thus, we investigate the limiting properties of the normalization factor in three cases that $k_1^0 - C_1 \le \hat{k} < k_1^0$, $k_1^0 \le \hat{k} \le k_2^0$, and $k_2^0 < \hat{k} \le k_2^0 + C_2$.

**Case (i).** Suppose that $k_1^0 - C_1 \le \hat{k} < k_1^0$, we have,

$$\inf_{(k_1,k_2)\in\Omega(\epsilon)} V_{NT}(k_1, \hat{k}, k_2) \le V_{NT}(k_1, \hat{k}, k_2^0), \text{ for } k_1 \in \Omega(\epsilon).$$

To show that the minimum value of $V_{NT}(k_1, \hat{k}, k_2)$ is stochastically bounded, it is sufficient to show that for any $k_1 \in \Omega(\epsilon)$,

$$V_{NT}(k_1, \hat{k}, k_2^0) = O_p(1).$$

In this case, the model is estimated by regressing $Y_i$ on $[X_i, X_{1i}(k_1, \hat{k}), X_{2i}(\hat{k}, k_2^0), X_{3i}(k_2^0)]$, which is expressed as

$$\begin{aligned} Y_i &= [X_i, X_{1i}(k_1, \hat{k}), X_{2i}(\hat{k}, k_2^0), X_{3i}(k_2^0)] \begin{bmatrix} \beta_i \\ \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{bmatrix} + u_i \\ &= \tilde{X}_i(k_1, \hat{k}, k_2^0) b_{1i} + u_i, \end{aligned} \tag{B.33}$$

while the true model with distinct common breaks is defined by

$$\begin{aligned} Y_i &= [X_i, X_{1i}(\hat{k}, k_1^0), X_{2i}(k_1^0, k_2^0), X_{3i}(k_2^0)] b_{1i}^0 + u_i \\ &= \tilde{X}_i(\hat{k}, k_1^0, k_2^0) b_{1i}^0 + u_i \qquad\qquad \text{(B.34)} \\ b_{1i}^0 &= \begin{cases} [\beta_i^{0\prime}, 0, \delta_i^{0\prime}, \delta_i^{0\prime}]' & \text{if } i \in G_1, \\ [\beta_i^{0\prime}, 0, 0, \delta_i^{0\prime}]' & \text{if } i \in G_2. \end{cases} \end{aligned}$$

Replacing $Y_i$ in (B.33) by (B.34), the residuals can be written by, for individuals in group 1,

$$\begin{aligned} \tilde{u}_i &= \tilde{X}_i(\hat{k}, k_1^0, k_2^0) b_{1i}^0 + u_i - \tilde{X}_i(k_1, \hat{k}, k_2^0) \tilde{b}_{1i}(\hat{k}) \\ &= u_i - \tilde{X}_i(k_1, \hat{k}, k_2^0)[\tilde{b}_{1i}(\hat{k}) - b_{1i}^0] + [\tilde{X}_i(\hat{k}, k_1^0, k_2^0) - \tilde{X}_i(k_1, \hat{k}, k_2^0)] b_{1i}^0 \\ &= u_i - \tilde{X}_i(k_1, \hat{k}, k_2^0) \begin{bmatrix} \tilde{\beta}_i - \beta_i^0 \\ \tilde{\delta}_{1i} \\ \tilde{\delta}_{2i} - \delta_i^0 \\ \tilde{\delta}_{3i} - \delta_i^0 \end{bmatrix} + [0, X_{1i}(\hat{k}, k_1^0) - X_{1i}(k_1, \hat{k}), X_{2i}(k_1^0, k_2^0) - X_{2i}(\hat{k}, k_2^0), 0] \begin{bmatrix} \beta_i^0 \\ 0 \\ \delta_i^0 \\ \delta_i^0 \end{bmatrix} \\ &= u_i - X_i(\tilde{\beta}_i - \beta_i^0) - X_{1i}(k_1, \hat{k})\tilde{\delta}_{1i} - X_{2i}(\hat{k}, k_2^0)(\tilde{\delta}_{2i} - \delta_i^0) - X_{3i}(k_2^0)(\tilde{\delta}_{3i} - \delta_i^0) \\ &\quad + [X_{2i}(k_1^0, k_2^0) - X_{2i}(\hat{k}, k_2^0)]\delta_i^0. \end{aligned} \tag{B.35}$$

For individuals in group 2, we have

$$\begin{aligned}
\tilde{u}_i &= u_i - \tilde{X}_i(k_1,\hat{k},k_2^0)\begin{bmatrix}\tilde{\beta}_i-\beta_i^0\\ \tilde{\delta}_{1i}\\ \tilde{\delta}_{2i}\\ \tilde{\delta}_{3i}-\delta_i^0\end{bmatrix} + [0, X_{1i}(\hat{k},k_1^0)-X_{1i}(k_1,\hat{k}), X_{2i}(k_1^0,k_2^0)-X_{2i}(\hat{k},k_2^0),0]\begin{bmatrix}\beta_i^0\\ 0\\ 0\\ \delta_i^0\end{bmatrix}\\
&= u_i - X_i(\tilde{\beta}_i-\beta_i^0) - X_{1i}(k_1,\hat{k})\tilde{\delta}_{1i} - X_{2i}(\hat{k},k_2^0)\tilde{\delta}_{2i} - X_{3i}(k_2^0)(\tilde{\delta}_{3i}-\delta_i^0). \qquad \text{(B.36)}
\end{aligned}$$

By the definition of the denominator, $V_{NT}(k_1,\hat{k},k_2^0)$ can be decomposed into four parts $V_1^{H_1}$, $V_2^{H_1}$, $V_3^{H_1}$, and $V_4^{H_1}$, defined by

$$\begin{aligned}
V_1^{H_1} &= \frac{1}{T}\sum_{s=1}^{k_1}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{s}\tilde{u}_{it}\right)^2, \; V_2^{H_1} = \frac{1}{T}\sum_{s=k_1+1}^{\hat{k}}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{\hat{k}}\tilde{u}_{it}\right)^2,\\
V_3^{H_1} &= \frac{1}{T}\sum_{s=\hat{k}+1}^{k_2^0}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{s}\tilde{u}_{it}\right)^2, \; V_4^{H_1} = \frac{1}{T}\sum_{s=k_2^0+1}^{T}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=s}^{T}\tilde{u}_{it}\right)^2.
\end{aligned}$$

From (B.35) and (B.36), for $t \le \hat{k}$, the residuals $\tilde{u}_{it}$ are calculated on the basis of subsamples $\{x_{i1},\cdots,x_{ik_1}\}$, and $\{x_{i(k_1+1)},\cdots,x_{i\hat{k}}\}$, which are the same as those in (A.22) under the null hypothesis. Using the asymptotic distribution of (A.23)–(A.27) and $k_1^0-\hat{k}=O_p(1)$, we can derive the limiting distributions of the terms $V_1^{H_1}$ and $V_2^{H_1}$ as follows:

$$V_1^{H_1}+V_2^{H_1} \Rightarrow \sigma^2\int_0^{\tau_1}\left(W(r)-\frac{r}{\tau_1}W(\tau_1)\right)^2 dr+\sigma^2\int_{\tau_1}^{\tau_1^0}\left[W(\tau_1^0)-W(r)-\frac{\tau_1^0-r}{\tau_1^0-\tau_1}(W(\tau_1^0)-W(\tau_1))\right]^2.$$

We next consider the third term, which can be rewritten as

$$\begin{aligned}
V_3^{H_1} &= \frac{1}{T}\sum_{s=\hat{k}+1}^{k_2^0}\Bigg[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{s}u_{it} - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{s}x_{it}'(\tilde{\beta}_i-\beta_i^0) - \frac{1}{\sqrt{NT}}\sum_{i\in G_1}\sum_{t=\hat{k}+1}^{s}x_{it}'(\tilde{\delta}_{2i}-\delta_i^0)\\
&\quad -\frac{1}{\sqrt{NT}}\sum_{i\in G_1}\left(\sum_{t=\hat{k}+1}^{s}x_{it}'\delta_i^0 1_{\{s\le k_1^0\}}+\sum_{t=\hat{k}+1}^{k_1^0}x_{it}'\delta_i^0 1_{\{s>k_1^0\}}\right)-\frac{1}{\sqrt{NT}}\sum_{i\in G_2}\sum_{t=\hat{k}+1}^{s}x_{it}'\tilde{\delta}_{2i}\Bigg]^2 \qquad \text{(B.37)}\\
&= \frac{1}{T}\sum_{s=\hat{k}+1}^{k_2^0}\Bigg[\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{s}u_{it} - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=\hat{k}+1}^{s}x_{it}'(\tilde{\beta}_i-\beta_i^0) - \frac{1}{\sqrt{NT}}\sum_{i\in G_1}\sum_{t=\hat{k}+1}^{s}x_{it}'(\tilde{\delta}_{2i}-\delta_i^0)\\
&\quad - O_p\left(\sqrt{\frac{N}{T}}\right)-\frac{1}{\sqrt{NT}}\sum_{i\in G_2}\sum_{t=\hat{k}+1}^{s}x_{it}'\tilde{\delta}_{2i}\Bigg]^2\\
&= \frac{1}{T}\sum_{s=\hat{k}+1}^{k_2^0}\left(V_{31}^{H_1}-V_{32}^{H_1}-V_{33}^{H_1}-O_p\left(\sqrt{\frac{N}{T}}\right)-V_{34}^{H_1}\right)^2.
\end{aligned}$$

Since $k_1^0 - \hat{k} = O_p(1)$, the terms in parentheses in (B.37) are $o_p(1)$ and will vanish as $N, T \to \infty$. Similar to $V_{31}$ and $V_{32}$, we have

$$\begin{aligned} V_{31}^{H_1} &\Rightarrow \sigma(W(r) - W(\tau_1^0)), && \text{(B.38)} \\ V_{32}^{H_1} &\Rightarrow \sigma(r - \tau_1^0)\frac{W(\tau_1)}{\tau_1}. && \text{(B.39)} \end{aligned}$$

The coefficient estimator $\tilde{\delta}_{2i}$ is calculated by, for $i \in G_1$,

$$\begin{aligned} \tilde{\delta}_{2i} &= \left(\sum_{t=\hat{k}+1}^{k_2^0} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{k_2^0} x_{it}y_{it} - \left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{k_1} x_{it}y_{it} \\ &= \left(\sum_{t=\hat{k}+1}^{k_2^0} x_{it}x_{it}'\right)^{-1} \left[\sum_{t=\hat{k}+1}^{k_1^0} x_{it}(x_{it}'\beta_i^0 + u_{it}) + \sum_{t=k_1^0+1}^{k_2^0} x_{it}(x_{it}'\beta_i^0 + x_{it}'\delta_i^0 + u_{it})\right] \\ &\quad - \left[\beta_i^0 + \left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{k_1} x_{it}u_{it}\right] \\ &= \delta_i^0 + \left(\sum_{t=\hat{k}+1}^{k_2^0} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{k_2^0} x_{it}u_{it} - \left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{k_1} x_{it}u_{it} - O_p\left(\frac{1}{T}\right), \end{aligned}$$

for $i \in G_2$,

$$\begin{aligned} \tilde{\delta}_{2i} &= \left(\sum_{t=\hat{k}+1}^{k_2^0} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{k_2^0} x_{it}(x_{it}'\beta_i^0 + u_{it}) - \left[\beta_i^0 + \left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{k_1} x_{it}u_{it}\right] \\ &= \left(\sum_{t=\hat{k}+1}^{k_2^0} x_{it}x_{it}'\right)^{-1} \sum_{t=\hat{k}+1}^{k_2^0} x_{it}u_{it} - \left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{k_1} x_{it}u_{it}. \end{aligned}$$

Then, we have

$$\begin{aligned} V_{33}^{H_1} + V_{34}^{H_1} &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(\frac{1}{T}\sum_{t=k_1^0+1}^{s} x_{it}' + O_p\left(\frac{1}{T}\right)\right) \sqrt{T} \left[\left(\sum_{t=k_1^0+1}^{k_2^0} x_{it}x_{it}'\right)^{-1} \sum_{t=k_1^0+1}^{k_2^0} x_{it}u_{it} \right. \\ &\quad \left. - \left(\sum_{t=1}^{k_1} x_{it}x_{it}'\right)^{-1} \sum_{t=1}^{k_1} x_{it}u_{it} + O_p\left(\frac{1}{T}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{T}\right)\right] \\ &\Rightarrow \sigma(r - \tau_1^0)\left(\frac{W(\tau_2^0) - W(\tau_1^0)}{\tau_2^0 - \tau_1^0} - \frac{W(\tau_1)}{\tau_1}\right). \end{aligned}$$

Thus, we can find that the limiting distribution of $V_3^{H_1}$ is

$$\sigma^2 \int_{\tau_1^0}^{\tau_2^0} \left[ W(r) - W(\tau_1^0) - \frac{r - \tau_1^0}{\tau_2^0 - \tau_1^0} (W(\tau_2^0) - W(\tau_1^0)) \right]^2 dr.$$

Since the coefficient estimator $\hat{\delta}_{3i}$ remains the same in groups 1 and 2, we have

$$\begin{aligned} V_4^{H_1} &= \frac{1}{T} \sum_{s=k_2^0+1}^{T} \left\{ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \left[ \sum_{t=s}^{T} u_{it} - \sum_{t=s}^{T} x_{it}'(\hat{\beta}_{1i} - \beta_i^0) - \sum_{t=s}^{T} x_{it}'(\hat{\delta}_{3i} - \delta_i^0) \right] \right\}^2 \\ &\Rightarrow \sigma^2 \int_{\tau_2^0}^{1} \left[ W(1) - W(r) - \frac{1-r}{1-\tau_2^0} \left( W(1) - W(\tau_2^0) \right) \right]^2 dr. \end{aligned}$$

Thus, we can say that

$$\inf_{(k_1,k_2)\in\Omega(\epsilon)} V_{NT}(k_1, \hat{k}, k_2) \le V_{NT}(k_1, \hat{k}, k_2^0) = V_1^{H_1} + V_2^{H_1} + V_3^{H_1} + V_4^{H_1} = O_p(1).$$

The proof of Proposition 2(i) is complete.

**Case (ii)** Suppose that $k_1^0 \le \hat{k} \le k_2^0$. In this case, we have

$$\inf_{(k_1,k_2)\in\Omega(\epsilon)} V_{NT}(k_1, \hat{k}, k_2) \le V_{NT}(k_1^0, \hat{k}, k_2^0).$$

We can easily find that the term $V_{NT}(k_1^0, \hat{k}, k_2^0)$ estimated using true break points will have a finite limiting distribution.

**Case (iii)** Suppose that $k_2^0 < \hat{k} \le k_2^0 + C_2$. In this case, from (B.27), we have $\hat{k} - k_2^0 = O_p(1)$. Similar to the proof of case (i), we can show that

$$\inf_{(k_1,k_2)\in\Omega(\epsilon)} V_{NT}(k_1, \hat{k}, k_2) \le V_{NT}(k_1^0, \hat{k}, k_2) = O_p(1), \text{ for any } k_2 \in \Omega(\epsilon).$$

Thus, we complete the proof of Proposition 3. ■

**Proof of Theorem 2**. From Proposition 1, we show that $P(\hat{k} \in [k_1^0 - C_1, k_2^0 + C_2]) \to 1$. Furthermore, for any $\hat{k} \in [k_1^0 - C_1, k_2^0 + C_2]$,

$$\begin{aligned} \sup_{k\in\Omega(\epsilon)} US_{NT}(k, \hat{k}) &= O_p(NT), \\ \sup_{(k_1,k_2)\in\Omega(\epsilon)} V_{NT}^{-1}(k_1, \hat{k}, k_2) &= O_p(1) \quad (\text{or} \quad \infty), \end{aligned}$$

from Propositions 2 and 3. Thus, the proof of Theorem 2 is complete. ■

Table 1: Critical values

| $c_{\tau^0}$ | 10% | 5% | 1% |
|---|---|---|---|
| $c_{0.1}$ | 43.425 | 56.822 | 92.840 |
| $c_{0.2}$ | 43.912 | 57.249 | 92.341 |
| $c_{0.3}$ | 45.501 | 57.962 | 93.689 |
| $c_{0.4}$ | 45.427 | 57.997 | 90.335 |
| $c_{0.5}$ | 45.540 | 57.842 | 85.984 |
| $c_{0.6}$ | 45.250 | 57.276 | 90.397 |
| $c_{0.7}$ | 46.489 | 59.175 | 93.728 |
| $c_{0.8}$ | 45.201 | 59.248 | 94.886 |
| $c_{0.9}$ | 43.515 | 57.203 | 92.908 |

Table 2: Size of the test DGP.1

| T | N | 10% | 5% | 1% |
|---|---|---|---|---|
| (a) $\rho = 0$ | | | | |
| 20 | 10 | 0.145 | 0.089 | 0.034 |
| | 50 | 0.136 | 0.074 | 0.026 |
| | 100 | 0.098 | 0.053 | 0.015 |
| 50 | 10 | 0.086 | 0.048 | 0.011 |
| | 50 | 0.076 | 0.036 | 0.009 |
| | 100 | 0.063 | 0.027 | 0.006 |
| 100 | 10 | 0.073 | 0.032 | 0.005 |
| | 50 | 0.072 | 0.034 | 0.006 |
| | 100 | 0.064 | 0.033 | 0.004 |
| 200 | 10 | 0.083 | 0.037 | 0.008 |
| | 50 | 0.075 | 0.032 | 0.006 |
| | 100 | 0.086 | 0.041 | 0.008 |
| (b) $\rho = 0.4$ | | | | |
| 20 | 10 | 0.226 | 0.146 | 0.060 |
| | 50 | 0.234 | 0.153 | 0.069 |
| | 100 | 0.231 | 0.143 | 0.058 |
| 50 | 10 | 0.134 | 0.068 | 0.022 |
| | 50 | 0.151 | 0.084 | 0.028 |
| | 100 | 0.145 | 0.084 | 0.024 |
| 100 | 10 | 0.113 | 0.063 | 0.017 |
| | 50 | 0.105 | 0.058 | 0.016 |
| | 100 | 0.101 | 0.055 | 0.016 |
| 200 | 10 | 0.107 | 0.048 | 0.013 |
| | 50 | 0.091 | 0.043 | 0.009 |
| | 100 | 0.091 | 0.050 | 0.013 |

Table 3: Power of the test DGP.2 (under $H_{1A}$)

| T | N | 10% | 5% | 1% |
|---|---|---|---|---|
| (a) $\rho = 0$ | | | | |
| 20 | 10 | 0.149 | 0.088 | 0.021 |
| | 50 | 0.586 | 0.455 | 0.222 |
| | 100 | 0.846 | 0.741 | 0.474 |
| 50 | 10 | 0.281 | 0.174 | 0.046 |
| | 50 | 0.918 | 0.847 | 0.614 |
| | 100 | 0.993 | 0.982 | 0.911 |
| 100 | 10 | 0.561 | 0.425 | 0.190 |
| | 50 | 0.996 | 0.980 | 0.916 |
| | 100 | 1.000 | 1.000 | 0.992 |
| (b) $\rho = 0.4$ | | | | |
| 20 | 10 | 0.304 | 0.216 | 0.100 |
| | 50 | 0.803 | 0.722 | 0.519 |
| | 100 | 0.965 | 0.929 | 0.789 |
| 50 | 10 | 0.390 | 0.276 | 0.121 |
| | 50 | 0.950 | 0.901 | 0.742 |
| | 100 | 0.997 | 0.991 | 0.946 |
| 100 | 10 | 0.611 | 0.491 | 0.258 |
| | 50 | 0.994 | 0.985 | 0.932 |
| | 100 | 1.000 | 1.000 | 0.994 |

[1] $k_1^0 = [T/4]$, $k_2^0 = [3T/4]$, $N_1 : N_2 = 5 : 5$.

Table 4: Power of the test DGP.2 (under $H_{1A}$)

| $\rho$ | $\delta_{1i}$, $\delta_{2i}$ | 10% | 5% | 1% |
|---|---|---|---|---|
| 0 | U(0,0.1) | 0.130 | 0.064 | 0.012 |
| | U(0.1,0.2) | 0.538 | 0.401 | 0.167 |
| | U(0.2,0.3) | 0.918 | 0.850 | 0.611 |
| | U(0.3,0.4) | 0.995 | 0.984 | 0.919 |
| | U(0.4,0.5) | 1.000 | 0.999 | 0.986 |
| | U(0.5,0.6) | 1.000 | 1.000 | 0.998 |
| | U(0.6,0.7) | 1.000 | 1.000 | 1.000 |
| | U(0.7,0.8) | 1.000 | 1.000 | 1.000 |
| | U(0.8,0.9) | 1.000 | 1.000 | 1.000 |
| | U(0.9,1.0) | 1.000 | 1.000 | 1.000 |
| | U(1.4,1.5) | 1.000 | 1.000 | 1.000 |
| 0.4 | U(0,0.1) | 0.206 | 0.138 | 0.037 |
| | U(0.1,0.2) | 0.652 | 0.540 | 0.292 |
| | U(0.2,0.3) | 0.955 | 0.913 | 0.750 |
| | U(0.3,0.4) | 0.997 | 0.992 | 0.949 |
| | U(0.4,0.5) | 1.000 | 1.000 | 0.993 |
| | U(0.5,0.6) | 1.000 | 1.000 | 1.000 |
| | U(0.6,0.7) | 1.000 | 1.000 | 1.000 |
| | U(0.7,0.8) | 1.000 | 1.000 | 1.000 |
| | U(0.8,0.9) | 1.000 | 1.000 | 1.000 |
| | U(0.9,1.0) | 1.000 | 1.000 | 1.000 |
| | U(1.4,1.5) | 1.000 | 1.000 | 1.000 |

[1] $T = 50, N = 50$.
[2] $k_1^0 = [T/4]$, $k_2^0 = [3T/4]$, $N_1 : N_2 = 5 : 5$.

Table 5: Power of the test DGP.2 (under $H_{1A}$)

| $k_1^0$ | $k_2^0$ | 10% | 5% | 1% |
|---|---|---|---|---|
| [0.2T] | [0.25T] | 0.120 | 0.066 | 0.018 |
| | [0.3T] | 0.325 | 0.225 | 0.070 |
| | [0.4T] | 0.801 | 0.695 | 0.465 |
| | [0.5T] | 0.941 | 0.891 | 0.734 |
| | [0.6T] | 0.955 | 0.918 | 0.764 |
| | [0.7T] | 0.925 | 0.875 | 0.692 |
| | [0.8T] | 0.859 | 0.771 | 0.558 |

[1] $N = T = 50$, $\rho = 0.4$, $N_1 : N_2 = 5 : 5$.

Table 6: Power of the test DGP.2 (under $H_{1A}$)

| $N_1 : N_2$ | 10% | 5% | 1% |
|---|---|---|---|
| 2:N-2 | 0.168 | 0.105 | 0.037 |
| 1:9 | 0.262 | 0.187 | 0.073 |
| 2:8 | 0.546 | 0.447 | 0.241 |
| 3:7 | 0.811 | 0.719 | 0.500 |
| 4:6 | 0.938 | 0.888 | 0.737 |
| 5:5 | 0.978 | 0.950 | 0.840 |

[1] $N = T = 50$, $\rho = 0.4$.
[2] $k_1^0 = [0.3T]$, $k_2^0 = [0.7T]$.

Table 7: Power of the test ($\rho = 0.4$ under $H_{2A}$)

| $k_1^0$ | $k_2^0$ | $k_3^0$ | $N_1 : N_2 : N_3$ | T | N | 10% | 5% | 1% |
|---|---|---|---|---|---|---|---|---|
| [T/6] | [3T/6] | [4T/6] | 3:3:4 | 50 | 10 | 0.296 | 0.204 | 0.080 |
| | | | | | 50 | 0.646 | 0.522 | 0.284 |
| | | | | | 100 | 0.740 | 0.642 | 0.445 |
| [0.4T] | [0.5T] | [0.6T] | 3:3:4 | 50 | 10 | 0.248 | 0.165 | 0.058 |
| | | | | | 50 | 0.485 | 0.381 | 0.211 |
| | | | | | 100 | 0.629 | 0.506 | 0.296 |
| [0.2T] | [0.25T] | [0.5T] | 3:3:4 | 50 | 10 | 0.342 | 0.239 | 0.109 |
| | | | | | 50 | 0.835 | 0.750 | 0.562 |
| | | | | | 100 | 0.964 | 0.923 | 0.809 |
| [0.2T] | [0.3T] | [0.8T] | 3:3:4 | 50 | 10 | 0.334 | 0.235 | 0.087 |
| | | | | | 50 | 0.704 | 0.591 | 0.363 |
| | | | | | 100 | 0.851 | 0.772 | 0.583 |
| [0.2T] | [0.5T] | [0.8T] | 1:4:5 | 50 | 10 | 0.334 | 0.224 | 0.087 |
| | | | | | 50 | 0.851 | 0.758 | 0.566 |
| | | | | | 100 | 0.925 | 0.866 | 0.725 |

Table 8: Power of the test (under $H_{3A}$)

| T | N | 10% | 5% | 1% |
|---|---|---|---|---|
| (a) $\rho = 0$ | | | | |
| 20 | 10 | 0.147 | 0.089 | 0.022 |
| | 50 | 0.548 | 0.412 | 0.191 |
| | 100 | 0.654 | 0.497 | 0.255 |
| 50 | 10 | 0.281 | 0.173 | 0.050 |
| | 50 | 0.688 | 0.515 | 0.247 |
| | 100 | 0.884 | 0.770 | 0.474 |
| 100 | 10 | 0.468 | 0.329 | 0.129 |
| | 50 | 0.908 | 0.815 | 0.541 |
| | 100 | 0.969 | 0.895 | 0.637 |
| (b) $\rho = 0.4$ | | | | |
| 20 | 10 | 0.344 | 0.243 | 0.109 |
| | 50 | 0.725 | 0.584 | 0.352 |
| | 100 | 0.842 | 0.740 | 0.493 |
| 50 | 10 | 0.300 | 0.204 | 0.076 |
| | 50 | 0.842 | 0.730 | 0.477 |
| | 100 | 0.903 | 0.810 | 0.538 |
| 100 | 10 | 0.517 | 0.392 | 0.183 |
| | 50 | 0.942 | 0.874 | 0.642 |
| | 100 | 0.879 | 0.761 | 0.438 |

Table 9: Power of the test DGP.3 (under $H_{4A}$)

| T | N | 10% | 5% | 1% |
|---|---|---|---|---|
| (a) $\rho = 0$ | | | | |
| 20 | 10 | 0.316 | 0.219 | 0.087 |
| | 50 | 0.701 | 0.600 | 0.374 |
| | 100 | 0.876 | 0.831 | 0.685 |
| 50 | 10 | 0.555 | 0.428 | 0.218 |
| | 50 | 0.964 | 0.939 | 0.807 |
| | 100 | 0.996 | 0.994 | 0.973 |
| 100 | 10 | 0.771 | 0.676 | 0.448 |
| | 50 | 0.999 | 0.995 | 0.981 |
| | 100 | 1.000 | 1.000 | 1.000 |
| (b) $\rho = 0.4$ | | | | |
| 20 | 10 | 0.552 | 0.447 | 0.267 |
| | 50 | 0.920 | 0.867 | 0.706 |
| | 100 | 0.991 | 0.975 | 0.914 |
| 50 | 10 | 0.684 | 0.571 | 0.351 |
| | 50 | 0.994 | 0.984 | 0.908 |
| | 100 | 0.999 | 0.999 | 0.987 |
| 100 | 10 | 0.815 | 0.728 | 0.531 |
| | 50 | 0.999 | 0.998 | 0.984 |
| | 100 | 1.000 | 1.000 | 1.000 |

[1] $k_1^0 = [T/4]$, $k_2^0 = [3T/4]$.

Table 10: Detection of common break dates during the period 2005M02 – 2011M12

| Category | N | T | Statistic $S_{NT}(k, k_1, k_2)$ | Estimated common break date |
|---|---|---|---|---|
| (a) 2005M02 – 2011M12 | | | | |
| Foreign Large Blend | 58 | 83 | 270.2507*** | 2008M01 |
| Foreign Small/Mid Blend | 7 | 83 | 98.1177*** | 2008M02 |
| Foreign Large Growth | 39 | 83 | 299.1527*** | 2008M01 |
| Foreign Small/Mid Growth | 11 | 83 | 8.5283 | 2008M11 |
| Large Blend | 205 | 83 | 278.6359*** | 2008M01 |
| Mid-Cap Blend | 54 | 83 | 305.5441*** | 2008M01 |
| Small Blend | 76 | 83 | 240.9046*** | 2008M01 |
| Large Growth | 186 | 83 | 100.4435*** | 2008M02 |
| Mid-Cap Growth | 88 | 83 | 103.6767*** | 2008M02 |
| Small Growth | 92 | 83 | 110.178*** | 2008M02 |
| (b) 2005M02 – 2008M05 | | | | |
| Foreign Large Blend | 58 | 40 | 146.0774*** | 2007M12 |
| Foreign Small/Mid Blend | 7 | 40 | 146.2134*** | 2007M12 |
| Foreign Large Growth | 39 | 40 | 145.7293*** | 2007M12 |
| Foreign Small/Mid Growth | 11 | 40 | 116.7004*** | 2007M12 |
| Large Blend | 205 | 40 | 150.0121*** | 2007M12 |
| Mid-Cap Blend | 54 | 40 | 146.7684*** | 2007M12 |
| Small Blend | 76 | 40 | 137.2884*** | 2007M12 |
| Large Growth | 186 | 40 | 157.4825*** | 2007M12 |
| Mid-Cap Growth | 88 | 40 | 151.1206*** | 2007M12 |
| Small Growth | 92 | 40 | 136.7168*** | 2007M12 |
| (c) 2008M06 – 2011M12 | | | | |
| Foreign Large Blend | 58 | 43 | 52.1541* | 2008M12 |
| Foreign Small/Mid Blend | 7 | 43 | 26.9158 | 2008M12 |
| Foreign Large Growth | 39 | 43 | 31.5376 | 2008M12 |
| Foreign Small/Mid Growth | 11 | 43 | 34.1773 | 2008M12 |
| Large Blend | 205 | 43 | 10.6933 | 2008M12 |
| Mid-Cap Blend | 54 | 43 | 9.4507 | 2008M12 |
| Small Blend | 76 | 43 | 13.8338 | 2008M12 |
| Large Growth | 186 | 43 | 16.8967 | 2008M12 |
| Mid-Cap Growth | 88 | 43 | 20.9836 | 2008M12 |
| Small Growth | 92 | 43 | 9.853 | 2008M12 |

[1] * reject at the 10% significance level.
[2] ** reject at the 5% significance level.
[3] *** reject at the 1% significance level