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Abstract

This paper discusses issues of axiomatic bargaining problems over opportunity assignments. The fair arbitrator uses the principle of "equal opportunity" for all players to make the recommendation on resource allocations. A framework in such a context is developed and the egalitarian solution to standard bargaining problems is reformulated and axiomatically characterized.

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1 Introduction

In standard axiomatic bargaining models originated from Nash (1950), a typical interpretation of the solution to bargaining problems is the recommendation made by a "*fair arbitrator*" such as the Judge in civil trials, or the function of the Dispute Settlement Body in the WTO mechanism, etc. In such models, this recommendation is based solely on players' utilities. In many contexts, however, the "fair arbitrator" may have other principles in mind when making a recommendation.

For instance, consider the distribution issue of a father's inheritance among his children. The father, as a "fair arbitrator," may have the principle of "*equal opportunities*" for his children and would like to distribute his wealth among his children giving them equal opportunities to do well in their respective lives. Likewise, when educational resources are to be allocated among local public schools, the local government's board of education, as the "fair arbitrator," may propose an allocation that "equalizes" school children's *opportunity sets* for future jobs, skills, college admissions, lives, etc. In both of the above examples, each recommendation of a resource allocation by the "fair arbitrator" effectively identifies a profile of "*opportunities*" or opportunity sets for the individuals involved. The crucial difference from standard axiomatic bargaining models in these examples is that the recommendation made by the arbitrator is not based on utilities of the individuals involved, but on opportunity sets that the recommended resource allocation may give rise to the involved individuals.

This departure from considerations of utilities of individuals to concerns of opportunity sets of individuals is well in line with the recent literature on opportunities and equality of opportunities. One branch of the literature is in political philosophy such as Sen (1980, 1985), Arneson (1989), and Cohen (1993), while the other is in economics, see, for example, Sen (2002), Pattanaik and Xu (1990), Kranich (1996), and Herrero (1997). In the latter branch of the literature, each individual is characterized by his opportunity sets, from which his well-being or welfare is evaluated.

An opportunity set of an individual is interpreted as a set of feasible options or alternatives available to the individual for living. Depending on the context, those alternatives can be commodity bundles, or bundles of characteristics *à la* Gorman (1956, 1980) and Lancaster (1966), or bundles of functionings *à la* Sen (1980, 1985), and Nussbaum (1988, 1993, 2000).[1] A resource allocation in an economy then identifies a collection of opportunity sets, one for each individual in the economy. Note that, for a given resource allocation, opportunity sets of individuals are necessarily interdependent. Note also that different resource allocations can give rise to various collections of opportunity sets for the individuals in the economy.

The question we address is, among various collections of opportunity sets for the individuals involved, how the "fair arbitrator" should make the recommendation on a resource allocation that yields a profile of opportunity sets for individuals in the economy, which is deemed as "fair." For this purpose, we extend standard bargaining models to the setting in which each individual is endowed with his opportunity sets, which are generated by his consumption bundles given his individual characteristics, and, in which the fair arbitrator makes recommendations based on profiles of opportunity sets for the individuals in the economy.[2] We present two related formulations of extended bargaining models. In the first place, we formulate axioms in terms of profiles of opportunity sets. This formulation corresponds to standard bargaining models. The advantage of this formulation is to have a general and abstract framework to discuss bargaining problems in our setting. Since an important component of our primitive information about individuals is their opportunity sets, this formulation appeals directly to our intuition regarding this important component. To have a better understanding of the underlying allocations proposed by a solution to our extended bargaining models and with the above general formulation in hand, we next formulate our axioms in economic environments directly. For both formulations, we introduce the egalitarian solution for extended bargaining models and study it axiomatically.

The remainder of the paper is organized as follows. Section 2 introduces our economic environments and our problem. Section 3 defines and axiomat-

[1]The functioning and capability approach to human well-being developed by Sen (1985, 1987) and Nussbaum (1988, 1993, 2000) starts by identifying aspects of life that are of intrinsic value to people and then considers them to be attributes of an individual's well-being. The identified attributes such as being well-nourished, being healthy, interactions with family and friends, are called 'functionings'. An individual's well-being is then assessed on the basis of the achieved functioning bundle and the opportunity set of various functioning bundles available to the individual.

[2]Gotoh and Yoshihara (2003) discuss allocation mechanisms which assign individuals capability sets through distributing outputs produced by them. Their approach is quite different from the approach based on bargaining that this paper addresses.

ically characterizes the egalitarian solution in our context. Section 4 defines and axiomatically characterizes the egalitarian allocation rule in economic environments. We conclude the paper in Section 5 by briefly commenting on our approach and the results.

2 Economic environments and bargaining problems on opportunity assignments

2.1 Economic environments

There are infinitely many types of goods (commodities). The universe of "potential goods" is denoted by Ξ, and the class of non-empty and finite subsets of Ξ is designated by $\mathcal{M}$, with generic elements, K, L, M,..., each is to be called a finite list of commodities. The cardinality of $M \in \mathcal{M}$ is denoted by $\#M = m$. For each $M \in \mathcal{M}$, let us denote a generic commodity bundle in $\mathbb{R}^m_+$ by x.

The population in the economy is given by the set $N = \{1, \cdots, n\}$, where $2 \leq n < +\infty$. Given a finite list of commodities $M \in \mathcal{M}$, every individual has a common consumption space $\mathbb{R}^m_+$. There are k *basic living conditions* in the economy, which are relevant for all individuals for the purpose of describing their objective well-beings attainable by means of their consumption vectors. These basic living conditions can be interpreted broadly. For example, they can be skills that individuals can develop through education, or they can be occupations which individuals can engage in after the graduation at school. Or they can be characteristics of commodities in the sense of Gorman (1956, 1980) and Lancaster (1966), or they can be various functionings according to Sen (1980, 1985) and Nussbaum (1988, 1993, 2000). For our formal analysis, we do not need to stick to a particular interpretation though a certain interpretation may be more appropriate than other interpretations for a given context.

Thus, an achievement of living condition f, where $f = 1, 2, \cdots, k$, by individual i is denoted by $b_{if} \in \mathbb{R}_+$. Individual i's achievement of basic living conditions is given by listing b_{if}: $\mathbf{b}_i = (b_{i1}, \cdots, b_{ik}) \in \mathbb{R}^k_+$. There are two crucial factors that determine the achievement of individual's basic living conditions: one is the amount of resources or commodities she can access for attaining these living conditions, and the other is the individual's *ability* to realize these living conditions by utilizing commodities. Note that, given

$M \in \mathcal{M}$ and each individual $i \in N$, the latter is formulated as i's *opportunity correspondence* $c_i^m : \mathbb{R}_+^m \twoheadrightarrow \mathbb{R}_+^k$ which associates to every commodity vector $x_i \in \mathbb{R}_+^m$ a non-empty subset $c_i^m(x_i)$ of $\mathbb{R}_+^k$. The intended interpretation is that i is able to have access to each living-condition vector $\mathbf{b}_i \in c_i^m(x_i)$ by means of his commodity vector x_i.

As a matter of notation, for any sets $C, C' \subseteq \mathbb{R}_+^k$, we write $C > C'$ if for all $\mathbf{b}' \in C'$, there exists $\mathbf{b} \in C$ such that $\mathbf{b} \gg \mathbf{b}'$. Each opportunity correspondence satisfies the following requirements:

(a) For all $x_i, x_i' \in \mathbb{R}_+^m$, if $x_i \leq x_i'$, then $c_i^m(x_i) \subseteq c_i^m(x_i')$.[3]
(b) $c_i^m(\mathbf{0}) = \{\mathbf{0}\}$ and there exists $x_i \in \mathbb{R}_+^m \backslash \{\mathbf{0}\}$ such that $c_i^m(x_i) \cap \mathbb{R}_{++}^k \neq \varnothing$;
(c) For all $x_i \in \mathbb{R}_+^m$, $c_i^m(x_i)$ is *compact* and *comprehensive* in $\mathbb{R}_+^k$; and
(d) c_i^m is *continuous* on $\mathbb{R}_+^m$.

Requirement (**a**) is a monotonicity property: more commodities generate "no smaller" opportunity sets. Requirement (**b**) essentially says that commodities are "desirable": they can help individuals in achieving positive levels of basic living conditions. Requirement (**c**) stipulates that any given commodity bundle generates a bounded opportunity set. And finally, requirement (**d**) says that "small" changes in commodity bundles lead to "small" changes in opportunity sets.

Let $\mathfrak{C}^M$ be the set of all possible opportunity correspondences defined on $\mathbb{R}_+^m$, which satisfy the above four requirements. Given $M \in \mathcal{M}$, an *economy with* $\overline{x}$ *endowments of* M*-goods* is described by a list $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) = (M, (c_i^m)_{i \in N}, \overline{x})$, where $\mathbf{c}^m \in \mathfrak{C}^{Mn}$, $\overline{x} \in \mathbb{R}_+^m$, and $\mathfrak{C}^{Mn}$ stands for the n-fold Cartesian product of $\mathfrak{C}^M$. Let $\mathcal{E}^M$ be the class of all such economies with endowments of M-goods. Let $\mathcal{E} \equiv \cup_{M \in \mathcal{M}} \mathcal{E}^M$. Given $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}^M$, a vector $\mathbf{x} = (x_i)_{i \in N} \in \mathbb{R}_+^{mn}$ is *feasible for* $\mathbf{e} \in \mathcal{E}^M$ if for all $i \in N$, $x_i \in \mathbb{R}_+^m$, and $\sum x_i \leq \overline{x}$. We denote by $A(\mathbf{e})$ the set of feasible allocations for $\mathbf{e} \in \mathcal{E}^M$. Let $A(\mathcal{E}) \equiv \cup_{\mathbf{e} \in \mathcal{E}} A(\mathbf{e})$.

For each individual $i \in N$, given $M \in \mathcal{M}$ and given i's consumption vector x_i, $c_i^m(x_i)$ generates an opportunity set $C_i = c_i^m(x_i)$ for i. An opportunity assignment is a list of n opportunity sets one for each individual in the society. Given $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}$, the *set of possible opportunity assignments for* $\mathbf{e} \in \mathcal{E}$ is:

$$\mathcal{C}(\mathbf{e}) \equiv \{\mathbf{C} = (C_i)_{i \in N} \subseteq \mathbb{R}_+^{kn} \mid \exists \mathbf{x} = (x_i)_{i \in N} \in A(\mathbf{e}) : C_i = c_i^m(x_i) (\forall i \in N)\}.$$

[3] For all vectors $\mathbf{a} = (a_1, \ldots, a_p)$ and $\mathbf{b} = (b_1, \ldots, b_p) \in \mathbb{R}^p$, $\mathbf{a} \geq \mathbf{b}$ if and only if $a_i \geq b_i$ $(i = 1, \ldots, p)$; $\mathbf{a} > \mathbf{b}$ if and only if $\mathbf{a} \geq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$; $\mathbf{a} \gg \mathbf{b}$ if and only if $a_i > b_i$ $(i = 1, \ldots, p)$.

Note that for any $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}$, any $\mathbf{C} = (C_i)_{i \in N} \in \mathcal{C}(\mathbf{e})$, and any $i \in N$, the opportunity C_i is a compact, comprehensive set in $\mathbb{R}^k_+$ containing the origin. For each $i \in N$ and each living condition $f = 1, \ldots, k$, let $\max_f (C_i)$ be the maximum amount of living condition f by i that he can achieve under his opportunity set C_i ; that is, $\max_f (C_i) \equiv \max \{b_f \mid (b_1, \cdots, b_f, \cdots, b_k) \in C_i\}$. Let $\Sigma \equiv \{\mathcal{C} \mid \exists \mathbf{e} \in \mathcal{E} : \mathcal{C} = \mathcal{C}(\mathbf{e})\}$ be the class of all such possible sets of opportunity assignments. Note that each set $\mathcal{C}$ in Σ is compact in terms of Hausdorff metric by the assumption (d) of the opportunity correspondence and the fact that $A(\mathbf{e})$ is compact for every $\mathbf{e} \in \mathcal{E}$. Also, for any $\mathcal{C} \in \Sigma$, if $\mathbf{C} = (C_i)_{i \in N} \in \mathcal{C}$, then for each $j \in N$, every living condition $f = 1, \ldots, k$, and any $b_f \leq \max_f (C_j)$, there exists $\mathbf{C}' = \left(C'_j, \mathbf{C}_{-j}\right) \in \mathcal{C}$ such that $b_f = \max_f \left(C'_j\right)$ and $C'_j \subseteq C_j$ by the assumption of (a), (b), and (d) of opportunity correspondences.

In the above formulation, while the number of basic living conditions is fixed, the number of goods can vary. This approach is theoretically appropriate, because the basic living conditions are the attributes that the society is concerned about, and the improvement in these attributes is a goal of the society. In contrast, the goods are the main means to realize such improvement, and the types of such means would increase or change, according to technological change and innovation. Note that the increase of the types of goods does not necessarily lead to the improvement in the basic living conditions, as discussed by Sen (1980, 1985).

2.2 Opportunity sets and their ranking

Let $\mathcal{K}$ be the universal class of compact, comprehensive subsets in $\mathbb{R}^k_+$ containing the origin. Thus, $C \in \mathcal{K}$ implies that for any $M \in \mathcal{M}$, there exists $c^m \in \mathfrak{C}^M$ such that for some $x \in \mathbb{R}^m_+$, $c^m(x) = C$. Note that for each $\mathcal{C} \in \Sigma$ and every $i \in N$, there exists $C_i^* \in \mathcal{K}$ such that for every $\mathbf{C} \in \mathcal{C}$, $C_i^* \supseteq C_i$ holds, and $\left(C_i^*, \mathbf{C}^0_{-i}\right) \in \mathcal{C}$ with $C_j^0 \equiv \{\mathbf{0}\}$ for any $j \neq i$. This is followed from the requirements of opportunity correspondences introduced in Section 2.1 and the definition of Σ. Given $\mathcal{C} \in \Sigma$, let us denote such C_i^* by $m_i(\mathcal{C})$ for each $i \in N$. Note that the profile $(m_i(\mathcal{C}))_{i \in N}$ is analogous to what is called in the standard bargaining theory the *ideal point*, which is necessary for defining the Kalai-Smorodinsky solution [Kalai and Smorodinsky (1975)].

How are various opportunity sets measured by individuals in the economy? We assume that there is an objective way of ranking various opportunity sets by individuals, where this objective measure of alternative opportunity sets is formalized as a binary relation $R \subseteq \mathcal{K} \times \mathcal{K}$. The relation R satisfies reflexivity: [for all $C \in \mathcal{K}$, $(C, C) \in R$], completeness: [for all $C, C' \in \mathcal{K}$, $(C, C') \in R$ or $(C', C) \in R$], and transitivity: [for all $C, C', C'' \in \mathcal{K}$, if $(C, C') \in R$ & $(C', C'') \in R$, then $(C, C'') \in R$]. Thus, R is an ordering over $\mathcal{K}$. Note P and I are respectively the asymmetric and symmetric parts of R.

Such an objective measure of various opportunity sets can be given different interpretations here. For example, it may be interpreted as a *standard evaluation* by the society (see Sen, 1987). Through open discussions and debates about the relevance of the basic living conditions and the use of democratic voting procedures, the society decides on the "standard" to be used for ranking various opportunity sets and then uses this standard to come up with a common way of measuring opportunity sets for different individuals. Alternatively, such an objective measure of various opportunity sets can be regarded as the result of aggregating the personal measures of opportunity sets of different individuals in the society. That is, instead of assuming that an objective measure of opportunity sets primitively exists and is acknowledged by all of the members in the society, we may start from assuming that individuals have their personal measures of opportunity sets. Even in such an alternative framework, however, it is plausible to assume that interpersonal comparability of opportunity sets is embedded therein. For instance, if two opportunity sets, C and C' with $C' < C$, are assigned to two individuals, then the members of the society can reasonably believe that whoever having the opportunity set C is better off than whoever having the opportunity set C' (the dominance criterion termed by Sen (1987)). If such a property of interpersonal comparability holds, then, as shown in Pattanaik and Xu (2007), all of the personal rankings over opportunity sets by the individuals are identical and can be regarded as a common ranking over opportunity sets. Thus, in this case, the common ranking is derived from individuals' personal rankings.

Note that, given the comprehensiveness of opportunity sets in $\mathcal{K}$, when $C > C'$, necessarily, we have C' as a proper subset of C. In this paper, we assume that an ordering R on $\mathcal{K}$ satisfies the following three properties:

Monotonicity: For all $C, C' \in \mathcal{K}$, if $C \supseteq C'$ then $(C, C') \in R$, and if $C > C'$, then $(C, C') \in P$.

Dominance: For all $C, C', C'' \in \mathcal{K}$, if $(C, C') \in P, (C, C'') \in P$ then

$(C, C' \cup C'') \in P$.

Continuity: For all $C \in \mathcal{K}$, the sets $\{C' \in \mathcal{K} \mid (C', C) \in P\}$ and $\{C' \in \mathcal{K} \mid (C, C') \in P\}$ are respectively open in $\mathcal{K}$ with respect to the Hausdorf topology.

It may be noted that, in our context, Monotonicity is a fairly non-controversial property and it essentially requires that a "bigger" opportunity set be ranked higher than a "smaller" opportunity set. Similar conditions have been used in the literature on ranking opportunity sets, see for example, Gaertner and Xu (2006), Pattanaik and Xu (2007), and Xu (2002, 2003, 2004). The property of Dominance requires that, when an opportunity set C is ranked higher than each of the opportunity sets C' and C'', then C is ranked higher than the union of C' and C''. It was proposed in Xu (2003). Continuity is a technical requirement for ranking opportunity sets.

When an ordering R satisfies the above three properties, we have the following result due to Xu (2003):

Proposition 1. If R on $\mathcal{K}$ satisfies Monotonicity, Dominance and Continuity, then there exists a continuous and increasing function $g : \mathbb{R}^m_+ \to \mathbb{R}_+$ such that for all $C, C' \in \mathcal{K}$,

$$(C, C') \in R \Leftrightarrow \max_{\mathbf{b} \in C} g(\mathbf{b}) \geq \max_{\mathbf{b}' \in C'} g(\mathbf{b}').$$

Therefore, under Monotonicity, Dominance and Continuity, the ranking of opportunity sets can be viewed as based on "*indirect utilities*" of opportunity sets: the 'indirect utility' of an opportunity being the maximum 'well-being' obtainable in that opportunity set.

2.3 Bargaining problems on opportunity assignments

The formal problem that we are interested in is the bargaining problem over opportunity assignments among individuals. Analogous to the standard bargaining model, we can interpret each $\mathcal{C} \in \Sigma$ as a *bargaining problem* and Σ as the *domain* of bargaining problems, and a solution to the problem is to pick up a subset of opportunity assignments $\{\mathbf{C} = (C_i)_{i \in N}\}$ from $\mathcal{C}$. Then, a *bargaining solution* in this context is a correspondence F which associates to every $\mathcal{C} \in \Sigma$, a non-empty subset $F(\mathcal{C}) \subseteq \mathcal{C}$.

How is our model related to the motivation discussed in the Introduction? The following examples may help us in understanding our approach.

Example 1: Let k be the number of skills that an individual can develop through education, and let $x \in \mathbf{R}_+^m$ be an educational resource. Then, the k dimensions of the opportunity set $\mathbf{c}_i^m(x) \subseteq \mathbb{R}_+^k$ represent the types of skills, and each element $\mathbf{b}_i = (b_{if})_{f \in \{1,\ldots,k\}} \in \mathbf{c}_i^m(x)$ implies that individual i can develop the level of each skill f up to b_{if}, whenever he is educated with the educational support x and some amount of his own effort. The difference of native talents among individuals is reflected in the difference of opportunity correspondences among them. In this setting, the bargaining problem would be to assign opportunities for future skills by allocating educational resources.

Example 2: The WTO consists of many member countries and one of its functions is to settle disputes among its member countries. Disputes between or among member countries are really about net trades of goods, services or capital. The Dispute Settlement Body of the WTO thus makes recommendations as how to structure net trades among the affected member countries.[4] Each member country is concerned about, for example, the aggregate employment rate, the growth rates of several sectors like manufacturing, agriculture, and service, and the health condition of its population. These concerns correspond to our notion of achievements. Each member country's interests can be captured by the country's *opportunity sets* representing opportunities to achieve a degree of employment rate, to have reasonable growth rates for its concerned sectors, and to offer its population a good health. The bargaining problem can then be interpreted as follows. The Dispute Settlement Body in the WTO mechanism acts as the fair arbitrator and it recommends the settlements that affect net trade based on equal opportunities for the disputed member countries along the factors that we discussed above.

Example 3: Our last example concerns the allocation of the budget by a central government to its several local jurisdictions. In many cases, the

[4] Quite often, disputes seemingly are about things like *access to member countries' markets and information*, *legal protection concerning trades from member countries*, or *pricing rules*. These are rules governing trade between and among nations and they have direct effect on net trade between member countries. As a consequence, we can interpret that disputes are really about net trade.

allocation of this budget intends for different localities to have equal opportunities for growth and for access to clean water, for example. Growth and access to clean water are two of the many factors that different local jurisdictions are concerned about, and local governments are concerned about their *opportunities* along these factors. The bargaining problem in this example can thus be viewed as how the fair arbitrator, the central government, makes budgetary allocations on the basis of equal opportunities for different local jurisdictions along those factors such as growth and environmental quality of each region.

3 The egalitarian solution: a first characterization

Given a social evaluation of opportunity sets R satisfying Monotonicity, Continuity and Dominance, the egalitarian solution we consider in the paper is defined as follows:

Egalitarian Solution: *A bargaining solution* F^E *is the* egalitarian solution *if and only if: for every* $\mathcal{C} \in \Sigma$, $F^E(\mathcal{C}) = \{\mathbf{C} = (C_1, \cdots, C_n) \in \mathcal{C} \mid (C_i, C_j) \in I$ *holds for any* i,$j \in N$ *and there is no other* $\mathbf{C}' = (C'_1, \cdots, C'_n) \in \mathcal{C}$ *such that* $(C'_i, C_i) \in P$ *for all* $i \in N\}$.

Thus, the solution F^E selects all the undominated assignments, such that in each of these assignments, everyone's opportunity is indifferent with any other's in terms of R.

3.1 Axioms on bargaining solutions

In this subsection, we shall present and discuss axioms on bargaining solutions over opportunity assignments. All of the axioms introduced below are formulated in terms of opportunity assignments. Such axioms are considerably weak and can appeal to our intuitions directly.

The first axiom is the corresponding weak efficiency axiom in standard bargaining models.

Weak Efficiency (WE): *For each* $\mathcal{C} \in \Sigma$ *and each* $\mathbf{C} = (C_i)_{i\in N} \in F(\mathcal{C})$, *there is no* $\mathbf{C}' = (C'_i)_{i\in N} \in \mathcal{C}$ *such that for every* $i \in N$, $C'_i > C_i$.

Therefore, the axiom (WE) requires that the solution should not select an opportunity assignment that is strictly dominated by another feasible opportunity assignment. It may be reminded that, when an opportunity assignment $\mathbf{C}'$ strictly dominates an opportunity assignment $\mathbf{C}$, we have $[C'_i > C_i$ for every $i \in N]$, which requires that, for each $i \in N$, C'_i is obtained from C_i by expanding it "outwardly"; as a consequence, necessarily, each C_i is a proper subset of C'_i.

To introduce our next axiom, we first define a symmetric problem. We say that $\mathcal{C} \in \Sigma$ is *symmetric* if for every permutation $\pi : N \to N$, and for every $\mathbf{C} = (C_i)_{i \in N} \in \mathcal{C}$, $\pi(\mathbf{C}) \equiv \left(C_{\pi(i)}\right)_{i \in N} \in \mathcal{C}$ holds.

Symmetry (S): *For each $\mathcal{C} \in \Sigma$, if* (i) $\mathcal{C}$ *is* symmetric, *and* (ii) *there exists a $\mathbf{C} \in \mathcal{C}$ such that it is* weakly efficient in $\mathcal{C}$ *and $C_i = C_j$ for all $i, j \in N$, then there exists some $\mathbf{C}^* \in F(\mathcal{C})$ such that $C^*_i = C^*_j$ for all $i, j \in N$, and there is no $\mathbf{C}' \in F(\mathcal{C})$ such that $\left(C'_i, C'_j\right) \in P$ for some $i, j \in N$.*

The axiom (S) stipulates that, for each symmetric problem with at least one weakly efficient and identical opportunity assignment, the solution selects at least one identical opportunity assignment, and further, no opportunity assignment selected by the solution is such that one individual's opportunity set strictly dominates another individual's opportunity set.

The following axiom is analogous to the axiom of contraction independence in standard bargaining models:

Contraction Independence (CI): *For each $\mathcal{C}, \mathcal{C}' \in \Sigma$ with $\mathcal{C} \supseteq \mathcal{C}'$, if $F(\mathcal{C}) \cap \mathcal{C}' \neq \varnothing$, then $F(\mathcal{C}') = F(\mathcal{C}) \cap \mathcal{C}'$.*

The axiom (CI) corresponds to Nash's Independence of Irrelevant Alternatives in standard bargaining models. It requires that if an opportunity assignment is chosen from a "larger" problem and is still available when the larger problem shrinks to a smaller problem, then it should be chosen from the smaller problem as well.

Our final axiom is an informational requirement on a solution to a problem and is stated below:

Informational Invariance (II) : *For each $\mathcal{C} \in \Sigma$ and each $\mathbf{C} = (C_i)_{i \in N}, \mathbf{C}' = (C'_i)_{i \in N} \in \mathcal{C}$, if $\mathbf{C} \in F(\mathcal{C})$ and $(C'_i, C_i) \in I$ for all $i \in N$, then $\mathbf{C}' \in F(\mathcal{C})$.*

According to the axiom (II), if two opportunity assignments are "equivalent" in the sense that the two opportunity sets for each and every individual specified by the corresponding opportunity assignments are ranked equally, then whenever one opportunity assignment is chosen by the solution, the other opportunity assignment should be chosen by the solution as well. The axiom (II) thus implies that the informational requirement in our context is contained exclusively in the social evaluation ordering R. A similar axiom, called *No Discrimination*,[5] is discussed by Thomson (1983) in the context of fair allocation problems.

3.2 A characterization of the egalitarian solution

Before we present our characterization result, the following observations are useful throughout this subsection. Given the social ordering R satisfying Monotonicity, Continuity, and Dominance, let $G : \mathcal{K} \to \mathbb{R}_+$ be a real-valued, ordinal representation of the social ordering R such that for any $C \in \mathcal{K}$,

$$G(C) = \max_{\mathbf{b} \in C} g(\mathbf{b}).$$

Then, for each bargaining problem $\mathcal{C} \in \Sigma$, we define

$$G(\mathcal{C}) \equiv \left\{ G(\mathbf{C}) = (G(C_i))_{i \in N} \in \mathbb{R}^n_+ \mid \mathbf{C} \in \mathcal{C} \right\}.$$

Let $\partial G(\mathcal{C})$ be the upper boundary of $G(\mathcal{C})$. Since $\mathcal{C}$ is derived from an underlying economic environment $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}$, where $\mathbf{c}^m$ is a profile of opportunity correspondences satisfying the requirements (a), (b), (c), and (d), and G is continuous on $\mathcal{K}$, $\partial G(\mathcal{C})$ constitutes a connected set in $\mathbb{R}^n_+$. Moreover, since $\mathcal{C}$ is *comprehensive*[6] by the requirements (a), (b), and (d) of opportunity correspondences, $G(\mathcal{C})$ must be comprehensive. Finally, by choosing $G(\{\mathbf{0}\}) = 0$ for the zero vector $\mathbf{0} \in \mathbb{R}^k_+$, $G(\mathcal{C})$ has $\mathbf{0} \in \mathbb{R}^n_+$ as its element, since $\underbrace{(\{\mathbf{0}\}, \ldots, \{\mathbf{0}\})}_{n-\text{times}} \in \mathcal{C}$. Therefore, $G(\mathcal{C})$ corresponds to a standard normalized, non-convex, and comprehensive bargaining problem.

Given these observations, we can easily see that the solution F^E is *well-defined* in the sense that for each $\mathcal{C} \in \Sigma$, $F^E(\mathcal{C})$ is non-empty. This is

[5]No Discrimination requires that if, for any allocation recommended by a solution, there exists another allocation whose corresponding utility allocation is identical to that of the first allocation, then the second allocation should be recommended by the solution as well.

[6]$\mathcal{C}$ is comprehensive if, for each $\mathbf{C} \in \mathcal{C}$ and each $i \in N$ with $C_i \neq \{\mathbf{0}\}$, there exists $\mathbf{C}' \in \mathcal{C}$ such that $C'_i < C_i$ and $C'_j \subseteq C_j$ for all $j \in N \setminus \{i\}$.

because, for each $\mathcal{C} \in \Sigma$, its corresponding $\partial G(\mathcal{C})$ always contains the vector of equal real numbers, and the inverse image of this vector constitutes the set $F^E(\mathcal{C})$. It may be remarked that, in general, F^E is multi-valued.

We now give a characterization result of the solution F^E.

Theorem 1: *The* egalitarian solution F^E *is the unique solution satisfying* (WE), (S), (CI) and (II).

Proof. First, it may be checked that F^E satisfies the four axioms of the theorem.

Next, we show that if a solution F satisfies (WE), (S), (CI) and (II), then $F = F^E$. Consider a bargaining problem $\mathcal{C} \in \Sigma$, which is derived from an underlying economic environment $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}$. Suppose $F \neq F^E$. By (II), $G(F(\mathcal{C})) \neq G\left(F^E(\mathcal{C})\right)$. Then, there exists $\mathbf{r}^* \in G(F(\mathcal{C})) \backslash G\left(F^E(\mathcal{C})\right)$. For each $i \in N$, let

$$\mathcal{C}(i) \equiv \left\{C_i' \in \mathcal{K} \mid \exists \mathbf{C}_{-i} \in \mathcal{K}^{n-1} : (C_i', \mathbf{C}_{-i}) \in \mathcal{C}\right\},$$

and for each $r \in \mathbb{R}_+$, define

$$\mathcal{C}(i;r) \equiv \left\{C_i' \in \mathcal{C}(i) \mid G(C_i') = r\right\}.$$

Consider $comp\{\mathbf{r}^*\} \equiv \{\mathbf{r} \in \mathbb{R}^n_+ \mid \mathbf{r} \leq \mathbf{r}^*\}$, and $\mathcal{C}^* \equiv G^{-1}(comp\{\mathbf{r}^*\}) \cap \mathcal{C}$.

Insert Figure 1 around here

Since G is continuous, we can choose a subset $\mathcal{C}^*_s \subseteq \mathcal{C}^*$ so that (i) for each $\mathbf{r} \in comp\{\mathbf{r}^*\}$, there exists a $(C_i^{r_i})_{i \in N} \in \mathcal{C}^*_s$ such that $[C_i^{r_i} \in \mathcal{C}(i; r_i)$ for all $i \in N]$; and (ii) for any $\mathbf{r}, \mathbf{r}' \in comp\{\mathbf{r}^*\}$, and each $i \in N$, $C_i^{r_i} = C_i^{r_i'}$ holds only if $r_i = r_i'$, and $C_i^{r_i} \supset C_i^{r_i'}$ holds if $r_i > r_i'$. Indeed, for $\mathbf{r}^*$, there exists a feasible allocation $\mathbf{x}^* = (x_i^*)_{i \in N} \in A(\mathbf{e})$ such that $C_i^{r_i^*} = c_i^m(x_i^*)$ with $G\left(C_i^{r_i^*}\right) = r_i^*$ for each $i \in N$. Then, for each $i \in N$, for any $r_i < r_i^*$ with $r_i \geq 0$, there exists a suitable $\lambda^{r_i} \in [0, 1)$ such that $C_i^{r_i} \equiv c_i^m(\lambda^{r_i} x_i^*)$ with $G(C_i^{r_i}) = r_i$. This is because, first, it follows from the monotonicity property (a) that either $c_i^m(\lambda x_i^*) \subset c_i^m(\lambda' x_i^*)$ or $c_i^m(\lambda x_i^*) = c_i^m(\lambda' x_i^*)$ holds for any $\lambda, \lambda' \in [0, 1]$ with $\lambda < \lambda'$. In particular, for $r_i = 0$, $\lambda^{r_i} = 0$ ensures that $C_i^{r_i} = c_i^m(\lambda^{r_i} x_i^*) = \{\mathbf{0}\}$ with $G(C_i^{r_i}) = 0$ by the property (b) and $G(\{\mathbf{0}\}) = 0$. Then, by the monotonicity of G, $G(c_i^m(\lambda x_i^*)) < G(c_i^m(\lambda' x_i^*))$ or

$G\left(c_i^m(\lambda x_i^*)\right) = G\left(c_i^m(\lambda' x_i^*)\right)$ holds for any $\lambda, \lambda' \in [0,1]$ with $\lambda < \lambda'$. Second, since $G \circ c_i^m : \mathbb{R}_+^m \to \mathbb{R}_+$ is continuous (by the continuity of c_i^m and the continuity of G)[7], for any $r_i < r_i^*$ with $r_i > 0$, there exists a suitable $\lambda^{r_i} \in (0,1)$ such that $C_i^{r_i} \equiv c_i^m(\lambda^{r_i} x_i^*)$ with $G\left(C_i^{r_i}\right) = r_i$.

In this way, a subset $\left\{C_i^{r_i} \mid 0 \le r_i \le r_i^*\right\}$ can be constructed for each $i \in N$. By the construction of each $C_i^{r_i}$, if $C_i^{r_i} = c_i^m(\lambda x_i^*)$ and $C_i^{r'_i} = c_i^m(\lambda' x_i^*)$, then $c_i^m(\lambda x_i^*) \subset c_i^m(\lambda' x_i^*)$ holds if $r_i < r'_i$, and $c_i^m(\lambda x_i^*) = c_i^m(\lambda' x_i^*)$ holds only if $r_i = r'_i$. Thus, it follows that for any $\mathbf{r}, \mathbf{r}' \in comp\left\{\mathbf{r}^*\right\}$, and each $i \in N$, $C_i^{r_i} = C_i^{r'_i}$ holds only if $r_i = r'_i$, and $C_i^{r_i} \supset C_i^{r'_i}$ holds if $r_i > r'_i$. Moreover, by the construction, for any $\mathbf{r} \in comp\left\{\mathbf{r}^*\right\}$, $\left(C_i^{r_i}\right)_{i \in N} \in \mathcal{C}$ holds. In this way, $\mathcal{C}_s^* \subseteq \mathcal{C}^*$ is constructed. By condition (i) of the definition of $\mathcal{C}_s^*$, $G\left(\mathcal{C}_s^*\right) = comp\left\{\mathbf{r}^*\right\}$.

Now, by using the information of $\mathcal{C}_s^*$, let us construct a new economy $\mathbf{e}^* = (M^*, \widehat{\mathbf{c}}^1, \overline{x}^*) \in \mathcal{E}$. Firstly, let $M^* \cap M = \varnothing$, $\#M^* = 1$, and $\overline{x}^* = 1$. Secondly, for each $i \in N$, let the opportunity correspondence $\widehat{c}_i^1$ be given as follows:
(i) for all $x \in \left[0, \frac{1}{n}\right]$, $\widehat{c}_i^1(x) = c_i^m(\lambda^{nx} x_i^*)$ with $\lambda^{nx} = nx \in [0,1]$; and
(ii) for all $x \in \left(\frac{1}{n}, 1\right]$, $\widehat{c}_i^1(x) = C_i^{r_i^*}$.
Then, consider $\mathcal{C}^{**} \equiv \mathcal{C}\left(\mathbf{e}^*\right) \in \Sigma$. Since $\mathcal{C}^{**} \subseteq \mathcal{C}^* \subseteq \mathcal{C}$ and $G\left(\mathcal{C}^{**}\right) = G\left(\mathcal{C}^*\right)$, we obtain $\mathbf{r}^* \in G\left(F\left(\mathcal{C}^{**}\right)\right)$ from (CI) and (I). Next, consider $\mathcal{C}^{\triangle} \equiv \cup_{\pi \in \Pi} \pi\left(\mathcal{C}^{**}\right)$.

Insert Figure 2 around here.

In the following discussion, we will construct a new economy $\mathbf{e}^{\triangle} \in \mathcal{E}$ such that $\mathcal{C}\left(\mathbf{e}^{\triangle}\right) \supseteq \mathcal{C}^{\triangle}$ and $G\left(\mathcal{C}\left(\mathbf{e}^{\triangle}\right)\right) = G\left(\mathcal{C}^{\triangle}\right)$. Firstly, given $\mathbf{r}^* \in G\left(\mathcal{C}^{\triangle}\right)$, let $r_{\max}^* \equiv \max_{i \in N}\left\{r_i^*\right\}$ and $r_{\min}^* \equiv \min_{i \in N}\left\{r_i^*\right\}$. Define, for each $i \in N$, an opportunity correspondence $\widehat{c}_i^{*1} : [0,1] \to \mathbb{R}_+^k$ by:

(I) for all $x \in \left[0, \frac{1}{n}\right]$, $\widehat{c}_i^{*1}(x) = c_i^m(nx\lambda^{r_{\min}^*} x_i^*)$ where $\lambda^{r_{\min}^*} \equiv \max\left\{\lambda \in [0,1] \mid c_i^m(\lambda x_i^*) = C_i^{r_{\min}^*}\right\}$; and

(II) for all $x \in \left(\frac{1}{n}, 1\right]$, $\widehat{c}_i^{*1}(x) = c_i^m(\lambda^x x_i^*)$ where $\lambda^x \equiv \frac{n\left(1-\lambda^{r_{\min}^*}\right)}{n-1}x + \frac{n\lambda^{r_{\min}^*}-1}{n-1}$,
where $C_i^{r_{\min}^*}$, for each $i \in N$, comes from $\mathcal{C}^{**}$.

[7]By the definition of $G(C) = \max_{\mathbf{b} \in C} g(\mathbf{b})$, the continuity of $G \circ c_i^m$ follows from Berge's Maximum Theorem.

Secondly, let us define $c^{\triangle n} : [0,1]^n \to \mathbb{R}^k_+$ by $c^{\triangle n}(x) \equiv \cup_{i\in N} \widehat{c}_i^{*1}(x_i)$ for each $x = (x_i)_{i\in N} \in [0,1]^n$. Thirdly, define $\mathbf{e}^{\triangle} \equiv (M^{*(n)}, \mathbf{c}^{\triangle n}, (\overline{x}^*)^n) \in \mathcal{E}$, where $M^{*(n)} \cap M = \varnothing$ and $\#M^{*(n)} = n$, $\mathbf{c}^{\triangle n} = \underbrace{(c^{\triangle n}, \ldots, c^{\triangle n})}_{n\text{-times}}$, and $(\overline{x}^*)^n \equiv \underbrace{(1, \ldots, 1)}_{n\text{-times}}$.

Note that for each $x = (x_i)_{i\in N} \in [0,1]^n$, if $x = (x_i, \mathbf{0}_{-i})$, $c^{\triangle n}(x) = \widehat{c}_i^{*1}(x_i)$, and in particular if $x_i = \frac{1}{n}$, then $c^{\triangle n}(x) = C_i^{r^*_{\min}}$, whereas if $x_i = 1$, then $c^{\triangle n}(x) = C_i^{r_i^*}$; if $x = \underbrace{\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)}_{n\text{-times}}$, then $c^{\triangle n}(x) = C^{r^*_{\min}} \equiv \cup_{i\in N} C_i^{r^*_{\min}}$ with $G\left(C^{r^*_{\min}}\right) = r^*_{\min}$; if x is such that $x_i = 1$ for some $i \in N$, then $G\left(c^{\triangle n}(x)\right) \geq r_i^*$; and if $x = (\overline{x}^*)^n$, then $c^{\triangle n}(x) = \cup_{i\in N} C_i^{r_i^*}$ and $G\left(c^{\triangle n}(x)\right) = r^*_{\max}$. Note that the last equation follows from the definition of G (recall that the common ordering R satisfies Dominance). By these properties followed from the definition of $c^{\triangle n}$, we can see that $\mathcal{C}\left(\mathbf{e}^{\triangle}\right) \supseteq \mathcal{C}^{\triangle}$ and $G\left(\mathcal{C}\left(\mathbf{e}^{\triangle}\right)\right) = G\left(\mathcal{C}^{\triangle}\right)$. Moreover, $\mathcal{C}\left(\mathbf{e}^{\triangle}\right)$ is a symmetric problem containing a weakly efficient identical opportunity assignment $\mathbf{C}^{\mathbf{r}^*_{\min}} \equiv \underbrace{\left(C^{r^*_{\min}}, \ldots, C^{r^*_{\min}}\right)}_{n\text{-times}}$.

From the construction of $\mathcal{C}\left(\mathbf{e}^{\triangle}\right)$, by (WE), (S), and (II), we must have $G\left(F\left(\mathcal{C}\left(\mathbf{e}^{\triangle}\right)\right)\right) = \left\{G\left(\mathbf{C}^{\mathbf{r}^*_{\min}}\right)\right\}$. Let $\mathbf{r}^{\triangle E} \equiv \underbrace{(r^*_{\min}, \ldots, r^*_{\min})}_{n\text{-times}} = G\left(\mathbf{C}^{\mathbf{r}^*_{\min}}\right)$, which is the egalitarian outcome for the problem $\mathcal{C}\left(\mathbf{e}^{\triangle}\right)$. Consider a feasible allocation $\mathbf{x} = (x^i)_{i\in N} \in A\left(\mathbf{e}^{\triangle}\right)$ such that $x^i = (x_i, \mathbf{0}_{-i})$ with $x_i = \frac{1}{n}$ for all $i \in N$. Then, $c^{\triangle n}(x^i) = C_i^{r^*_{\min}}$ for all $i \in N$. Therefore, $\left(C_i^{r^*_{\min}}\right)_{i\in N} = \left(c^{\triangle n}(x^i)\right)_{i\in N} \in \mathcal{C}\left(\mathbf{e}^{\triangle}\right)$. Then, as $G\left(\left(C_i^{r^*_{\min}}\right)_{i\in N}\right) = \mathbf{r}^{\triangle E}$, $\left(C_i^{r^*_{\min}}\right)_{i\in N} \in F\left(\mathcal{C}\left(\mathbf{e}^{\triangle}\right)\right)$ holds by (II).

Insert Figure 3 around here

Note that by the definition of $\widehat{c}_i^1$, there exists a feasible allocation $(x_i^{**})_{i\in N} \in A(\mathbf{e}^*)$ such that $x_i^{**} \leq \frac{1}{n}$ and $\widehat{c}_i^1(x_i^{**}) = C_i^{r^*_{\min}}$ for all $i \in N$. Therefore, $\left(C_i^{r^*_{\min}}\right)_{i\in N} \in \mathcal{C}^{**}$ holds. Then, by $G\left(F\left(\mathcal{C}\left(\mathbf{e}^{\triangle}\right)\right)\right) = \left\{\mathbf{r}^{\triangle E}\right\}$, $\mathcal{C}\left(\mathbf{e}^{\triangle}\right) \supseteq \mathcal{C}^{\triangle} \supseteq \mathcal{C}^{**}$, and $\left(C_i^{r^*_{\min}}\right)_{i\in N} \in F\left(\mathcal{C}\left(\mathbf{e}^{\triangle}\right)\right) \cap \mathcal{C}^{**}$, it follows from (CI) that $\left(C_i^{r^*_{\min}}\right)_{i\in N} \in F\left(\mathcal{C}^{**}\right)$ and $\left\{\mathbf{r}^{\triangle E}\right\} = G\left(F\left(\mathcal{C}^{**}\right)\right)$ hold. However, as $\mathbf{r}^* \in G\left(F\left(\mathcal{C}^{**}\right)\right)$, this is a contradiction.

Insert Figure 4 around here

Thus, $G(F(\mathcal{C})) \backslash G\left(F^E(\mathcal{C})\right) = \varnothing$. By (II), clearly, $F = F^E$. ■

It is easy to check the independence of axioms in Theorem 1, so we only note the results and omit the detailed proofs here: Regarding (II), any *proper subsolution of* F^E satisfies all the axioms except (II); regarding (WE), a solution which solely selects $\underbrace{(\{\mathbf{0}\}, \ldots, \{\mathbf{0}\})}_{n\text{-times}}$ from any bargaining problem satisfies all the axioms but (WE); regarding (S), the *Nash solution* which is introduced by Xu and Yoshihara (2006a) for (opportunity)-bargaining problems satisfies all the axioms except (S); and regarding (CI), the *Kalai-Smorodinsky solution* which is introduced by Xu and Yoshihara (2006a) for (opportunity)-bargaining problems satisfies all the axioms but (CI).

Note that the (utility)-egalitarian solution to standard *convex* and comprehensive (utility)-bargaining problems was axiomatically studied by Kalai (1977), where the solution is characterized by weak efficiency, symmetry, and strong monotonicity. Our result of Theorem 1 suggests that the (utility)-egalitarian solution to standard *non-convex* and comprehensive utility-bargaining problems can be characterized by the corresponding axioms of weak efficiency, symmetry and contraction independence. This indeed is the case, see Xu and Yoshihara (2006). It may be noted that, in the standard convex (*resp.* non-convex) and comprehensive (utility)-bargaining problems, the corresponding axiom to our axiom (II) becomes redundant.

It may be remarked that, in our framework, the available class of bargaining problems over opportunity assignments is derived from the specified economies explicitly. As such, no external assumption on the richness of the domain of bargaining problems is made. The axiom (II) is to ensure the full correspondence property of a bargaining solution. As we remarked earlier, the G function that is used for representing the ranking of opportunity set is not a utility function. In view of these, our approach differs from utility-based bargaining problems, where a solution would equalize utilities across individuals. For example, in our approach, F^E may select an opportunity assignment $\mathbf{C} = (C_1, \cdots, C_n)$ such that $C_1 = \cdots = C_n$, and yet, for any individuals i and j, i's achieved bundle from her opportunity set C_i may be very different from j's achieved bundle from his opportunity set C_j, and consequently, their 'utilities' from their respective achieved bundles may not be equalized.

4 The egalitarian allocation rule in economic environments

Though Theorem 1 gives us a good general understanding of the egalitarian solution to bargaining problems in our setting, it does not say anything directly about properties of resource allocation mechanisms which realize the egalitarian opportunity assignments in each economy. Quite often, it would be useful to know properties of such mechanisms when the egalitarian solution is applied to the concrete bargaining problems on economic resource allocations. Moreover, in the context of *fair allocation problems*, characterizing such mechanisms would be useful as well if the egalitarian opportunity assignments are deemed to be desirable outcomes. For these purposes, in this section, we reformulate our bargaining problems directly in economic environments.[8] We start the analysis by introducing some additional definitions.

An *allocation rule* is a correspondence φ which associates to every $\mathbf{e} \in \mathcal{E}$, a non-empty subset $\varphi(\mathbf{e}) \subseteq A(\mathbf{e})$. An allocation rule φ *attains* a bargaining solution F if and only if for every $\mathbf{e} \in \mathcal{E}$, $\mathbf{c}\left(\varphi(\mathbf{e})\right) = F(\mathcal{C}(\mathbf{e}))$, where $\mathbf{c}\left(\varphi(\mathbf{e})\right) \equiv \{\mathbf{C} = (C_i)_{i\in N} \in \mathcal{C}(\mathbf{e}) \mid \exists\ \mathbf{x} = (x_i)_{i\in N} \in \varphi(\mathbf{e}) : C_i = c_i^m(x_i)\ (\forall i \in N)\}$. Then, the egalitarian allocation rule is introduced as follows.

Egalitarian Allocation Rule: *An allocation rule φ^E is the* egalitarian rule *if it attains the egalitarian solution: for all $\mathbf{e} \in \mathcal{E}$, $\mathbf{c}\left(\varphi^E(\mathbf{e})\right) = F^E(\mathcal{C}(\mathbf{e}))$.*

We now present and discuss relevant axioms on allocation rules that attain bargaining solutions over opportunity assignments. We first introduce an axiom that is similar to weak efficiency in standard bargaining models. Its intuition is straightforward.

Weak Economic Efficiency (WEE): *For each $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}$ and each $\mathbf{x} \in \varphi(\mathbf{e})$, there is no $\mathbf{x}' \in A(\mathbf{e})$ such that for every $i \in N$, $c_i^m(x_i') > c_i^m(x_i)$.*

We shall denote the set of weakly economic efficient allocations for $\mathbf{e}$ by $WE(\mathbf{e})$.

The next two axioms correspond to the axioms of symmetry and contraction independence introduced in Section 3.

[8] A parallel analysis was also developed in the standard (utility)-bargaining problems by Roemer (1988) and Yoshihara (2003, 2006).

Economic Symmetry (ES): *For each* $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}$ *with* $c_i^m = c_j^m$ for all $i, j \in N$, *there exists* $\mathbf{x} \in \varphi(\mathbf{e})$ *such that for any* $i, j \in N$, $c_i^m(x_i) = c_j^m(x_j)$, *and there is no* $\mathbf{x}' \in \varphi(\mathbf{e})$ *such that* $\left(c_i^m(x_i'), c_j^m\left(x_j'\right)\right) \in P$ *for some* $i, j \in N$.

Economic Contraction Independence (ECI): *For each* $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}), \mathbf{e}' = (M, \mathbf{c}^m, \overline{x}') \in \mathcal{E}$ *with* $\overline{x} \geq \overline{x}'$, *if* $\mathbf{c}(\varphi(\mathbf{e})) \cap \mathcal{C}(\mathbf{e}') \neq \varnothing$, *then* $\mathbf{c}(\varphi(\mathbf{e}')) = \mathbf{c}(\varphi(\mathbf{e})) \cap \mathcal{C}(\mathbf{e}')$.

Informational requirements on allocation rules in the current setting are stated in the following axioms.

Strong Economic Informational Invariance (SEII) : *For each* $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}), \mathbf{e}' = (L, \mathbf{c}^l, \overline{x}') \in \mathcal{E}$ *with* $\mathcal{C}(\mathbf{e}) = \mathcal{C}(\mathbf{e}')$, *for any* $\mathbf{x} \in \varphi(\mathbf{e})$ *and any* $\mathbf{x}' \in A(\mathbf{e}')$, *if* $\left(c_i^l(x_i'), c_i^m(x_i)\right) \in I$ *holds for all* $i \in N$, *then* $\mathbf{x}' \in \varphi(\mathbf{e}')$.

Economic Informational Invariance (EII) : *For each* $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}$, *for any* $\mathbf{x} \in \varphi(\mathbf{e})$ *and any* $\mathbf{x}' \in A(\mathbf{e})$, *if* $(c_i^m(x_i'), c_i^m(x_i)) \in I$ *holds for all* $i \in N$, *then* $\mathbf{x}' \in \varphi(\mathbf{e})$.

Full Correspondence (F) : *For each* $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}$, *for any* $\mathbf{x} \in \varphi(\mathbf{e})$ *and any* $\mathbf{x}' \in A(\mathbf{e})$, *if* $c_i^m(x_i') = c_i^m(x_i)$ *holds for all* $i \in N$, *then* $\mathbf{x}' \in \varphi(\mathbf{e})$.

It may be noted that (SEII) implies (EII) and (EII) implies (F). (SEII) requires that, for any two economies having the same set of opportunity assignments, if an allocation $\mathbf{x}$ is chosen by the allocation rule for the first economy and if an allocation $\mathbf{x}'$ is feasible in the second economy, then the allocation $\mathbf{x}'$ should be chosen for the second economy as long as every individual views the opportunity set generated under $\mathbf{x}$ being indifferent to the opportunity set generated under $\mathbf{x}'$. (EII) is weaker than (SEII) in that (EII) is confined to the same economy, and (F) is weaker than (EII) in that the allocation $\mathbf{x}'$ should be chosen for the second economy as long as (i) $\mathbf{x}$ is chosen for the first economy, (ii) $\mathbf{x}'$ is feasible, and (iii) $\mathbf{x}'$ generates the same opportunity set for every individual as $\mathbf{x}$.

(SEII) can be further decomposed. It turns out that (SEII) embodies an element relating to dimensional changes in endowments of commodities. To capture this idea formally, we introduce a definition first. Letting $x \in \mathbb{R}_+^m$ and $c_i^m \in \mathfrak{C}^M$ and letting K be a proper subset of M, we say that *each good in* K *is useless for individual* $i \in N$ *at* x if, for all $x_K' \equiv (x_f')_{f \in K} \in$

$\mathbb{R}^k_+$, $c_i^m(x'_K, x_{M\setminus K}) = c_i^m(x_K, x_{M\setminus K})$, where $x_K \equiv (x_f)_{f\in K}$. Therefore, a commodity is useless for an individual at a commodity bundle x if it does not "contribute" anything to this individual's opportunity set under x.

Independence of Useless New Commodities (INC): *Let* $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}^M$*, and let* $\mathbf{e}' = (M \cup L, \mathbf{c}^{m+l}, (\overline{x}, \overline{y})) \in \mathcal{E}^{M\cup L}$*, where* $M \cap L = \varnothing$*, be such that (1) for any* $\mathbf{x} = (x_i)_{i\in N} \in WE(\mathbf{e})$*, there exists* $(y_i^{\mathbf{x}})_{i\in N} \in \mathbb{R}^{nl}_+$ *such that*

$$c_i^{m+l}(x_i, y_i^{\mathbf{x}}) = c_i^m(x_i)(\forall i \in N) \text{ and } (x_i, y_i^{\mathbf{x}})_{i\in N} \in WE(\mathbf{e}'),$$

and (2) there exists $\widehat{\mathbf{x}} = (\widehat{x}_i)_{i\in N} \in WE(\mathbf{e})$ *such that each good of* L *is useless for every agent* $i \in N$ *at* $(\widehat{x}_i, \mathbf{0}) \in \mathbb{R}^{m+l}_+$*. Then,* $\widehat{\mathbf{x}} \in \varphi(\mathbf{e})$ *if and only if* $(\widehat{x}_i, \mathbf{0})_{i\in N} \in \varphi(\mathbf{e}')$*.*

(INC) essentially requires that, by adding useless new commodities to an economy, those allocations that "preserve" the original allocations chosen by an allocation rule for the original economy and use none of the useless new commodities should continue to be chosen from the "enlarged" economy by the allocation rule.

The characterization of the egalitarian allocation rule is summarized in the following theorem, Theorem 2.

Theorem 2: *The* egalitarian allocation rule φ^E *is the unique rule satisfying* (WEE), (ES), (ECI), (EII), *and* (INC).

To prove Theorem 2, we first prove the following series of lemmas. The proof of Theorem 2 will then follow. In Lemma 1, we shall establish the following result: Given any two economies that generate the same set of possible opportunity assignments, if an allocation rule satisfies (WEE), (F) and (INC), then the sets of all opportunity assignments attained by the allocations chosen by the allocation rule under the two economies must be the same. Formally:

Lemma 1: *Let* $\mathbf{e}_1$*,* $\mathbf{e}_2 \in \mathcal{E}$ *be such that* $\mathbf{e}_1 = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}^M$*,* $\mathbf{e}_2 = (L, \mathbf{c}^l, \overline{y}) \in \mathcal{E}^L$*, and* $\mathcal{C}(\mathbf{e}_1) = \mathcal{C}(\mathbf{e}_2)$*. Then, the allocation rule* φ *which satisfies* (WEE), (F), and (INC) has the following property: $\mathbf{c}(\varphi(\mathbf{e}_1)) = \mathbf{c}(\varphi(\mathbf{e}_2))$.

Proof. Let $\mathbf{e}_1$, $\mathbf{e}_2 \in \mathcal{E}$ be such that $\mathbf{e}_1 = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}^M$, $\mathbf{e}_2 = (L, \mathbf{c}^l, \overline{y}) \in \mathcal{E}^L$, and $\mathcal{C}(\mathbf{e}_1) = \mathcal{C}(\mathbf{e}_2) = \mathcal{C}$. In the following, we will construct another economy $\mathbf{e}_3 = (S, \mathbf{c}^s, \bar{z}) \in \mathcal{E}^S$ such that $s = m + l$ and $\mathcal{C}(\mathbf{e}_3) \supseteq \mathcal{C}(\mathbf{e}_1) = \mathcal{C}(\mathbf{e}_2)$, and will show that (WEE), (F), and (INC) together imply

$$\mathbf{c}\left(\varphi(\mathbf{e}_1)\right) = \mathbf{c}\left(\varphi(\mathbf{e}_3)\right) \cap \mathcal{C},$$

and

$$\mathbf{c}\left(\varphi(\mathbf{e}_2)\right) = \mathbf{c}\left(\varphi(\mathbf{e}_3)\right) \cap \mathcal{C}.$$

With the above implications, $\mathbf{c}\left(\varphi(\mathbf{e}_1)\right) = \mathbf{c}\left(\varphi(\mathbf{e}_2)\right)$ will then follow.

Let $S = M \cup L$ so that $\#S = s \equiv m + l$. For each $\mathbf{C} \in \mathcal{C}$, there exist $\mathbf{x} \in A(\mathbf{e}_1)$ and $\mathbf{y} \in A(\mathbf{e}_2)$ such that $[\forall i \in N : c_i^m(x_i) = C_i = c_i^l(y_i)]$. Let $[\mathbf{0}, \bar{x}] \equiv \prod_{h \in M} [0, \bar{x}_h]$ and $[\mathbf{0}, \bar{y}] \equiv \prod_{h \in L} [0, \bar{y}_h]$ with $(\bar{x}_h)_{h \in M} = \bar{x}$ and $(\bar{y}_h)_{h \in L} = \bar{y}$. For each $i \in N$, each $x \in \mathbb{R}_+^m$, and each $C_i = c_i^m(x)$, $\left(c_i^l\right)^{-1}(C_i) \equiv \left\{y \in \mathbb{R}_+^l \mid c_i^l(y) = C_i\right\}$. Let, for each $x \in \mathbb{R}_+^m$,

$$\tau_i(x) \equiv \left(c_i^l\right)^{-1} \circ c_i^m(x) \equiv \left\{y \in \mathbb{R}_+^l \mid c_i^l(y) = c_i^m(x)\right\}$$

and, for each $y \in \mathbb{R}_+^l$,

$$\tau_i^{-1}(y) \equiv \left(c_i^m\right)^{-1} \circ c_i^l(y) \equiv \left\{x \in \mathbb{R}_+^m \mid c_i^m(x) = c_i^l(y)\right\}.$$

Note that $\left(c_i^l\right)^{-1}(C_i)$ is non-empty for $C_i = c_i^m(x)$ if $x \in [0, \bar{x}]$, and $\left(c_i^m\right)^{-1}(C_i)$ is non-empty for $C_i = c_i^l(y)$ if $y \in [\mathbf{0}, \bar{y}]$. Moreover, $\tau_i(x) \cap [\mathbf{0}, \bar{y}] \neq \varnothing$ for $x \in [0, \bar{x}]$, and $\tau_i^{-1}(y) \cap [0, \bar{x}] \neq \varnothing$ for $y \in [\mathbf{0}, \bar{y}]$.

For each $i \in N$ and each $(x, y) \in \mathbb{R}_+^s$, we define

$$c_i^*(x, y) \equiv c_i^m(x) \cap c_i^l(y).$$

Note that, for each $x \in [\mathbf{0}, \bar{x}]$, $\tau_i(x)$ is non-empty, and for each $y(x) \in \tau_i(x)$, $c_i^*(x, y(x)) = c_i^m(x)$. From the construction, it follows that $c_i^* \in \mathfrak{C}^S$ for each $i \in N$. Define $\mathbf{e}_3 = (S, \mathbf{c}^*, (\bar{x}, \bar{y}))$ with $\mathbf{c}^* = (c_i^*)_{i \in N}$. Then, $\mathcal{C}(\mathbf{e}_3) \supseteq \mathcal{C}$.[9] Next, for each $i \in N$, let $\widehat{c}_i : \mathbb{R}_+^s \twoheadrightarrow \mathbb{R}_+^k$ be defined as $\widehat{c}_i(x_i, y) = c_i^m(x_i)$ for all $x_i \in \mathbb{R}_+^m$ and all $y \in \mathbb{R}_+^l$. Let $\widehat{\mathbf{e}}_1 = (S, \widehat{\mathbf{c}}, (\bar{x}, \bar{y}))$ with $\widehat{\mathbf{c}} = (\widehat{c}_i)_{i \in N}$. Then, $\mathcal{C}(\widehat{\mathbf{e}}_1) = \mathcal{C}$.

Compare $\mathbf{e}_3$ and $\widehat{\mathbf{e}}_1$. By construction, for any $(x, y) \in \mathbb{R}_+^s$, $\widehat{c}_i(x, y) \supseteq c_i^*(x, y)$ holds for each $i \in N$. Moreover, for any $(x, y) \in \mathbb{R}_+^s$ with $(x, y) \leq (\bar{x}, \bar{y})$, there exists $(\delta(x), \delta(y)) \in \mathbb{R}_+^s$ with $(x, y) \leq (\delta(x), \delta(y)) \leq (\bar{x}, \bar{y})$

[9] Note that $c_i^*(x, y) \in \mathcal{C}(\mathbf{e}_3) \backslash \mathcal{C}$ if and only if $c_i^*(x, y) \neq c_i^m(x)$, $c_i^*(x, y) \neq c_i^l(y)$, and $c_i^m(x) \cap c_i^l(y) \notin \mathcal{C}$.

such that $\widehat{c}_i(x, y) = c_i^*(\delta(x), \delta(y))$. Indeed, if $\delta(x) = x$ and $\delta(y) = (y \vee y')$ for $y' \in \tau_i(x) \cap [\mathbf{0}, \overline{y}]$ where $y \vee y' \equiv (\max\{y_h, y'_h\})_{h \in L}$, then $\widehat{c}_i(x, y) = c_i^*(\delta(x), \delta(y))$ holds. Thus, $\mathcal{C}(\mathbf{e}_3) \supseteq \mathcal{C}(\widehat{\mathbf{e}}_1)$.

Next, for each $i \in N$, we define a new commodity space $\mathbb{R}^s_+ \times \mathbb{R}^{t(i)}_+$, with $t(i) = l$ and $\overline{w}^i \in \mathbb{R}^{t(i)}_+ \setminus \{0\}$, and a new correspondence $\widehat{c}_i^* : \mathbb{R}^s_+ \times \mathbb{R}^{t(i)}_+ \twoheadrightarrow \mathbb{R}^k_+$ such that, for any $(x, y) \in \mathbb{R}^s_+$,
(i) $\widehat{c}_i^*\left(x, y, \overline{w}^i\right) = \widehat{c}_i(x \wedge \overline{x}, y \wedge \overline{y})$;
(ii) $\widehat{c}_i^*(x, y, 0) = c_i^*(x \wedge \overline{x}, y \wedge \overline{y})$;
(iii) for any $w \leq \overline{w}^i$, there exists $(x', y') \in \mathbb{R}^s_+$ with $x' = x \wedge \overline{x} = \delta(x \wedge \overline{x})$ and $y \wedge \overline{y} \leq y' \leq \delta(y \wedge \overline{y})$ such that $\widehat{c}_i^*(x, y, w) = c_i^*(x', y')$;
(iv) for any $w, w' \leq \overline{w}^i$ with $w \geq w'$, $\widehat{c}_i^*(x, y, w) \supseteq \widehat{c}_i^*(x, y, w')$; and
(v) for any w with $w \nleq \overline{w}^i$, $\widehat{c}_i^*(x, y, w) = \widehat{c}_i^*\left(x, y, w \wedge \overline{w}^i\right)$.
Note that, for each $i \in N$, $\widehat{c}_i^*$ defined above can be constructed to meet $\widehat{c}_i^* \in \mathfrak{C}^{S \cup T(i)}$, where $T(i)$ denotes the set of commodities in $\mathbb{R}^{t(i)}_+$.

To see this, we note that there exist a permutation $\rho : L \to T(i)$ and a bijection $\beta_i^y : \left[\mathbf{0}, \overline{w}^i\right] \to [y, \delta(y)]$ such that $\beta_{ki}^y(w) \equiv \frac{w_{\rho(k)}}{\overline{w}^i_{\rho(k)}}(\delta_k(y) - y_k) + y_k$ holds for any $w \in \left[\mathbf{0}, \overline{w}^i\right]$ and every $k \in L$. Then, we can define the correspondence $\widehat{c}_i^* : \mathbb{R}^s_+ \times \mathbb{R}^{t(i)}_+ \twoheadrightarrow \mathbb{R}^k_+$ as follows: for any $(x, y) \in \mathbb{R}^s_+$, if $(x, y) \leq (\overline{x}, \overline{y})$, then $\widehat{c}_i^*(x, y, w) = c_i^*\left(x, \beta_i^y(w)\right)$ for any $w \leq \overline{w}^i$, while if $(x, y) \nleq (\overline{x}, \overline{y})$, then $\widehat{c}_i^*(x, y, w) = c_i^*\left(x \wedge \overline{x}, \beta_i^{y \wedge \overline{y}}(w)\right)$ for any $w \leq \overline{w}^i$; and $\widehat{c}_i^*(x, y, w) = \widehat{c}_i^*\left(x, y, w \wedge \overline{w}^i\right)$ for any w with $w \nleq \overline{w}^i$. Note that by the construction of β_i^y, the conditions (i), (ii), and (iv) of $\widehat{c}_i^*$ are satisfied in this definition. As $c_i^* \in \mathfrak{C}^S$, we can see that this definition of the correspondence ensures that $\widehat{c}_i^* \in \mathfrak{C}^{S \cup T(i)}$.

Given $(\widehat{c}_i^*)_{i \in N}$, let us define a new commodity space $\mathbb{R}^s_+ \times \left(\prod_{i \in N} \mathbb{R}^{t(i)}_+\right)$ with $\left(\overline{w}^i\right)_{i \in N} \in \prod_{i \in N} \mathbb{R}^{t(i)}_+$, and a profile of new correspondences $(\widehat{c}_i^{**})_{i \in N}$ as follows: for each $i \in N$, $\widehat{c}_i^{**} : \mathbb{R}^s_+ \times \left(\prod_{i \in N} \mathbb{R}^{t(i)}_+\right) \twoheadrightarrow \mathbb{R}^k_+$ is such that, for each $(x, y, w_1, \ldots, w_n) \in \mathbb{R}^s_+ \times \left(\prod_{i \in N} \mathbb{R}^{t(i)}_+\right)$, $\widehat{c}_i^{**}(x, y, w_1, \ldots, w_n) = \widehat{c}_i^*(x, y, w_i)$. Let T denote the set of commodities in $\prod_{i \in N} \mathbb{R}^{t(i)}_+$ and define a new economy $\widehat{\mathbf{e}}^{**} \equiv (S \cup T, \widehat{\mathbf{c}}^{**}, (\overline{x}, \overline{y}, \left(\overline{w}^i\right)_{i \in N})) \in \mathcal{E}$ with $\widehat{\mathbf{c}}^{**} \equiv (\widehat{c}_i^{**})_{i \in N}$. Then, $\mathcal{C}(\widehat{\mathbf{e}}^{**}) \supseteq \mathcal{C}$ holds, since $\mathcal{C}(\widehat{\mathbf{e}}^{**}) = \mathcal{C}(\mathbf{e}_3) \supseteq \mathcal{C}(\widehat{\mathbf{e}}_1)$.[10]

[10]Note that $\mathcal{C}(\widehat{\mathbf{e}}^{**}) = \mathcal{C}(\mathbf{e}_3)$ holds by the following reasonings. First, by the property (ii) of $\widehat{c}_i^*$, $\mathcal{C}(\widehat{\mathbf{e}}^{**}) \supseteq \mathcal{C}(\mathbf{e}_3)$ holds. Second, by the properties (iii) and (v) of $\widehat{c}_i^*$, $\mathcal{C}(\widehat{\mathbf{e}}^{**}) \subseteq \mathcal{C}(\mathbf{e}_3)$ holds.

Let $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \widehat{\mathbf{w}}) \in \varphi(\widehat{\mathbf{e}}^{**})$. By (WEE), we can assume that each $i \in N$ receives $\left(\widehat{x}_i, \widehat{y}_i, \left(\overline{w}^i, \mathbf{0}_{-i}\right)\right)$ in the allocation $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \widehat{\mathbf{w}})$, since, by the construction of $\widehat{\mathbf{c}}^{**}$, every commodity in $T(i)$ is useless for any other individual $i' \in N \setminus \{i\}$ at $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \widehat{\mathbf{w}})$ and is therefore only useful for i. Moreover, by (WEE) and (F), without loss of generality, we can assume that $\widehat{c}_i^{**}\left(\widehat{x}_i, \widehat{y}_i, \left(\overline{w}^i, \mathbf{0}_{-i}\right)\right) = \widehat{c}_i^{**}(\widehat{x}_i, \widehat{y}_i, \mathbf{0}) = c_i^*(\widehat{x}_i, \widehat{y}_i)$ for each $i \in N$. Indeed, for each $i \in N$, note that

$$\widehat{c}_i^{**}\left(\widehat{x}_i, \widehat{y}_i, \left(\overline{w}^i, \mathbf{0}_{-i}\right)\right) = \widehat{c}_i^{*}\left(\widehat{x}_i, \widehat{y}_i, \overline{w}^i\right) = \widehat{c}_i(\widehat{x}_i, \widehat{y}_i) = c_i^m(\widehat{x}_i).$$

Then, since $\mathcal{C}(\mathbf{e}_1) = \mathcal{C}(\mathbf{e}_2)$, there exists $(\widetilde{y}_i)_{i \in N} \in \times_{i \in N}(\tau_i(\widehat{x}_i)) \cap A(\mathbf{e}_2)$ such that $c_i^l(\widetilde{y}_i) = c_i^m(\widehat{x}_i)$ for each $i \in N$. Therefore, for each $i \in N$,

$$c_i^m(\widehat{x}_i) = c_i^*(\widehat{x}_i, \widetilde{y}_i) = \widehat{c}_i^*(\widehat{x}_i, \widetilde{y}_i, 0) = \widehat{c}_i^{**}(\widehat{x}_i, \widetilde{y}_i, \mathbf{0}).$$

Note that by the definition, $\widehat{c}_i^{**}\left(\widehat{x}_i, \widehat{y}_i, \left(\overline{w}^i, \mathbf{0}_{-i}\right)\right) = \widehat{c}_i^{**}\left(\widehat{x}_i, \widetilde{y}_i, \left(\overline{w}^i, \mathbf{0}_{-i}\right)\right)$ holds for each $i \in N$. Therefore, by (F), $\left(\widehat{x}_i, \widetilde{y}_i, \left(\overline{w}^i, \mathbf{0}_{-i}\right)\right)_{i \in N} \in \varphi(\widehat{\mathbf{e}}^{**})$ follows from $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \widehat{\mathbf{w}}) \in \varphi(\widehat{\mathbf{e}}^{**})$. Thus, without loss of generality, let $\widehat{y}_i \equiv \widetilde{y}_i$ for each $i \in N$. Then, we have $\widehat{c}_i^{**}\left(\widehat{x}_i, \widehat{y}_i, \left(\overline{w}^i, \mathbf{0}_{-i}\right)\right) = \widehat{c}_i^{**}(\widehat{x}_i, \widehat{y}_i, \mathbf{0}) = c_i^*(\widehat{x}_i, \widehat{y}_i)$ for each $i \in N$. Consequently, $\left(\widehat{x}_i, \widehat{y}_i, \left(\overline{w}^i, \mathbf{0}_{-i}\right)\right)_{i \in N} \in \varphi(\widehat{\mathbf{e}}^{**})$ implies $(\widehat{x}_i, \widehat{y}_i, \mathbf{0})_{i \in N} \in \varphi(\widehat{\mathbf{e}}^{**})$ by (WEE) and (F), and each of the commodities T is useless for each $i \in N$ at $(\widehat{x}_i, \widehat{y}_i, \mathbf{0})$.

Compare $\widehat{\mathbf{e}}^{**}$ and $\widehat{\mathbf{e}}_1$. For any $(\mathbf{x}, \mathbf{y}) \in WE(\widehat{\mathbf{e}}_1)$, $\left(\mathbf{x}, \mathbf{y}, \left(\overline{w}^i, \mathbf{0}_{-i}\right)_{i \in N}\right) \in WE(\widehat{\mathbf{e}}^{**})$ holds, and $\widehat{c}_i^{**}\left(x_i, y_i, \left(\overline{w}^i, \mathbf{0}_{-i}\right)\right) = \widehat{c}_i(x_i, y_i)$ holds for every $i \in N$. Then, as $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \mathbf{0}) \in \varphi(\widehat{\mathbf{e}}^{**})$ and each of the commodities T is useless for each $i \in N$ at $(\widehat{x}_i, \widehat{y}_i, \mathbf{0})$ by the last argument in the previous paragraph, $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \in \varphi(\widehat{\mathbf{e}}_1)$ holds by (INC). Thus, by (F), we have

$$\mathbf{c}(\varphi(\widehat{\mathbf{e}}^{**})) \cap \mathcal{C} = \mathbf{c}(\varphi(\widehat{\mathbf{e}}_1)).$$

Compare $\widehat{\mathbf{e}}^{**}$ and $\mathbf{e}_3$. For any $(\mathbf{x}, \mathbf{y}) \in WE(\mathbf{e}_3)$, $(\mathbf{x}, \mathbf{y}, \mathbf{0}) \in WE(\widehat{\mathbf{e}}^{**})$ holds, and $\widehat{c}_i^{**}(x_i, y_i, \mathbf{0}) = c_i^*(x_i, y_i)$ holds for every $i \in N$. Then, by (INC), $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \in \varphi(\mathbf{e}_3)$. Thus, by (F), we have

$$\mathbf{c}(\varphi(\widehat{\mathbf{e}}^{**})) = \mathbf{c}(\varphi(\mathbf{e}_3)).$$

Compare $\widehat{\mathbf{e}}_1$ and $\mathbf{e}_1$. For any $\mathbf{x} \in WE(\mathbf{e}_1)$, $(\mathbf{x}, \mathbf{y}) \in WE(\widehat{\mathbf{e}}_1)$ and $\widehat{c}_i(x_i, y_i) = c_i^m(x_i)$ hold for any $\mathbf{y} \in \mathbb{R}_+^{nl}$ and every $i \in N$. Moreover, any commodity in L is useless for each $i \in N$ at $(x_i, \mathbf{0})$ under $\widehat{\mathbf{e}}_1$. Thus, by (INC), $\widehat{\mathbf{x}} \in \varphi(\mathbf{e}_1)$. By (F), we have

$$\mathbf{c}(\varphi(\widehat{\mathbf{e}}_1)) = \mathbf{c}(\varphi(\mathbf{e}_1)).$$

Therefore, from the above, $\mathbf{c}(\varphi(\mathbf{e}_1)) = \mathbf{c}(\varphi(\widehat{\mathbf{e}}_1)) = \mathbf{c}(\varphi(\widehat{\mathbf{e}}^{**})) \cap \mathcal{C} = \mathbf{c}(\varphi(\mathbf{e}_3)) \cap \mathcal{C}$ follows, and consequently, $\mathbf{c}(\varphi(\mathbf{e}_1)) = \mathbf{c}(\varphi(\mathbf{e}_3)) \cap \mathcal{C}$. ■

Lemma 2: (WEE), (EII), and (INC) *imply* (SEII).

Proof. Let $\mathbf{e}_1$, $\mathbf{e}_2 \in \mathcal{E}$ be such that $\mathbf{e}_1 = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}^M$, $\mathbf{e}_2 = (L, \mathbf{c}^l, \overline{y}) \in \mathcal{E}^L$, and $\mathcal{C}(\mathbf{e}_1) = \mathcal{C}(\mathbf{e}_2)$. Noting that (EII) implies (F), by **Lemma 1**, $\mathbf{c}(\varphi(\mathbf{e}_1)) = \mathbf{c}(\varphi(\mathbf{e}_2))$. Then, it follows that (EII) implies (SEII). ■

Lemma 3: (WEE), (F), (ECI), and (INC) *imply* (CI).

Proof. Let $\mathcal{C}, \mathcal{C}' \in \Sigma$ be such that $\mathcal{C} \supseteq \mathcal{C}'$ and $F(\mathcal{C}) \cap \mathcal{C}' \neq \varnothing$. Then, there exist $\mathbf{e}_1 = (M, \mathbf{c}^m, \overline{x}), \mathbf{e}_2 = (L, \mathbf{c}^l, \overline{y}) \in \mathcal{E}$ such that $\mathcal{C}(\mathbf{e}_1) = \mathcal{C}$ and $\mathcal{C}(\mathbf{e}_2) = \mathcal{C}'$. Moreover, $F(\mathcal{C}) \cap \mathcal{C}' \neq \varnothing$ implies that $\mathbf{c}(\varphi(\mathbf{e}_1)) \cap \mathcal{C}(\mathbf{e}_2) \neq \varnothing$. Let $\widehat{\mathbf{x}} \in \varphi(\mathbf{e}_1)$ be such that $(c_i^m(\widehat{x}_i))_{i \in N} \in \mathcal{C}(\mathbf{e}_2)$.

We will construct two new economies $\widehat{\mathbf{e}}^{**} = (K, \mathbf{c}^k, \overline{z}), \mathbf{e}^{**} = (K, \mathbf{c}^k, \overline{z}') \in \mathcal{E}$ such that $\overline{z} \geq \overline{z}'$, $\mathcal{C}(\widehat{\mathbf{e}}^{**}) \supseteq \mathcal{C}$, and $\mathcal{C}(\mathbf{e}^{**}) \supseteq \mathcal{C}'$. By this construction, it will follow that $\mathcal{C}(\widehat{\mathbf{e}}^{**}) \supseteq \mathcal{C}(\mathbf{e}^{**})$. Then, (WEE), (F), (ECI), and (INC) will imply that $F(\mathcal{C}(\mathbf{e}^{**})) = F(\mathcal{C}(\widehat{\mathbf{e}}^{**})) \cap \mathcal{C}(\mathbf{e}^{**})$. In what follows, we will show $F(\mathcal{C}(\mathbf{e}^{**})) \cap \mathcal{C}' = F(\mathcal{C}')$ and $F(\mathcal{C}(\widehat{\mathbf{e}}^{**})) \cap \mathcal{C}' = F(\mathcal{C}) \cap \mathcal{C}'$, and will conclude that $F(\mathcal{C}') = F(\mathcal{C}) \cap \mathcal{C}'$.

Let S be a set of commodities such that $S = M \cup L$ so that $\#S = s \equiv m + l$. Because $\mathcal{C}(\mathbf{e}_1) \supseteq \mathcal{C}(\mathbf{e}_2)$, for each $\mathbf{C} \in \mathcal{C}(\mathbf{e}_2)$, there exist $\mathbf{x} \in A(\mathbf{e}_1)$ and $\mathbf{y} \in A(\mathbf{e}_2)$ such that $c_i^m(x_i) = C_i = c_i^l(y_i)$ for each $i \in N$. Then, let $\mathbf{c}^* = (c_i^*)_{i \in N}$ be defined as in the proof of **Lemma 1**. Define $\mathbf{e}_3 = (S, \mathbf{c}^*, (\overline{x}, \overline{y}))$ with $\mathbf{c}^* = (c_i^*)_{i \in N}$. Then, $\mathcal{C}(\mathbf{e}_3) \supseteq \mathcal{C}(\mathbf{e}_2)$ holds. Defining $\widehat{\mathbf{e}}_1 = (S, \widehat{\mathbf{c}}, (\overline{x}, \overline{y}))$ with $\widehat{\mathbf{c}} = (\widehat{c}_i)_{i \in N}$ as in the proof of **Lemma 1**, we then have $\mathcal{C}(\widehat{\mathbf{e}}_1) = \mathcal{C}(\mathbf{e}_1)$.

Next, define $\widehat{\mathbf{e}}^{**} \equiv (S \cup T, \widehat{\mathbf{c}}^{**}, (\overline{x}, \overline{y}, (\overline{w}^i)_{i \in N})) \in \mathcal{E}$ with $\widehat{\mathbf{c}}^{**} \equiv (\widehat{c}_i^{**})_{i \in N}$ in a similar way to construct the economy $\widehat{\mathbf{e}}^{**}$ in the proof of **Lemma 1**. First, given $\mathcal{C}(\mathbf{e}_1) \supseteq \mathcal{C}(\mathbf{e}_2)$, it follows that for any $y \in [\mathbf{0}, \overline{y}]$, there exists $\gamma^i(y) \in [\mathbf{0}, \overline{x}] \cap \tau_i^{-1}(y)$ such that $c_i^m(\gamma^i(y)) = c_i^l(y)$ for each $i \in N$. Then, define a new commodity space $\mathbb{R}_+^s \times \mathbb{R}_+^{t(i)}$, with $t(i) = m$ and $\overline{w}^i \in \mathbb{R}_+^{t(i)} \setminus \{0\}$, and a new correspondence $\widehat{c}_i^* : \mathbb{R}_+^s \times \mathbb{R}_+^{t(i)} \twoheadrightarrow \mathbb{R}_+^k$ such that, for any $(x, y) \in \mathbb{R}_+^s$,

$$\widehat{c}_i^*(x, y, w) \equiv \begin{cases} c_i^m(x \wedge \overline{x}) \cap c_i^m\left(\beta_i^{\gamma^i(y \wedge \overline{y})}(w)\right) & \text{for any } w \leq \overline{w}^i; \\ \widehat{c}_i^*\left(x, y, w \wedge \overline{w}^i\right) & \text{for any } w \nleq \overline{w}^i, \end{cases}$$

where, for any $(x, y) \in [\mathbf{0}, \overline{x}] \times [\mathbf{0}, \overline{y}]$, a bijection $\beta_i^{\gamma^i(y)} : \left[\mathbf{0}, \overline{w}^i\right] \rightarrow [\gamma^i(y), (x \vee \gamma^i(y))]$ is defined as: for any $w \in \left[\mathbf{0}, \overline{w}^i\right]$ and every $k \in M$,

$$\beta_{ki}^{\gamma^i(y)}(w) \equiv \frac{w_{\rho(k)}}{\overline{w}^i_{\rho(k)}} \left(\left(x \vee \gamma^i(y)\right)_k - \gamma_k^i(y)\right) + \gamma_k^i(y).$$

By this definition, this correspondence $\widehat{c}_i^*$ also satisfies the properties (i), (ii), (iv), and (v) discussed in the proof of **Lemma 1**. Correspondingly, T is defined as in the proof of **Lemma 1**. That is, T is the set of commodities in $\prod_{i \in N} \mathbb{R}_+^{t(i)}$, and $\left(\overline{w}^i\right)_{i \in N} \in \prod_{i \in N} \mathbb{R}_+^{t(i)}$. Then, define $\widehat{c}_i^{**}$ as in the proof of Lemma 1, so that for each $(x, y, w_1, \ldots, w_n) \in \mathbb{R}_+^s \times \left(\prod_{i \in N} \mathbb{R}_+^{t(i)}\right)$, $\widehat{c}_i^{**}(x, y, w_1, \ldots, w_n) = \widehat{c}_i^*(x, y, w_i)$. In this way, the economy $\widehat{\mathbf{e}}^{**} = (S \cup T, \widehat{\mathbf{c}}^{**}, (\overline{x}, \overline{y}, \left(\overline{w}^i\right)_{i \in N})) \in \mathcal{E}$ is constructed, so that $\mathcal{C}(\widehat{\mathbf{e}}^{**}) \supseteq \mathcal{C}(\mathbf{e}_1)$ follows then. Moreover, define another new economy $\mathbf{e}^{**} \equiv (S \cup T, \widehat{\mathbf{c}}^{**}, (\overline{x}, \overline{y}, \mathbf{0})) \in \mathcal{E}$ with $\widehat{\mathbf{c}}^{**} \equiv (\widehat{c}_i^{**})_{i \in N}$. We then have $\mathcal{C}(\mathbf{e}^{**}) = \mathcal{C}(\mathbf{e}_3)$.

Note that, for any $\widehat{\mathbf{x}} \in \varphi(\mathbf{e}_1)$ with $(c_i^m(\widehat{x}_i))_{i \in N} \in \mathcal{C}(\mathbf{e}_2)$, there exists $\widehat{\mathbf{y}} \in A(\mathbf{e}_2)$ such that $(c_i^m(\widehat{x}_i))_{i \in N} = \left(c_i^l(\widehat{y}_i)\right)_{i \in N}$. Then, as in the proof of **Lemma 1**, $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}) \in \varphi(\widehat{\mathbf{e}}_1)$ and $(\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \mathbf{0}) \in \varphi(\widehat{\mathbf{e}}^{**})$ follow from (WEE), (F), and (INC). As $(\widehat{c}_i(\widehat{x}_i, \widehat{y}_i))_{i \in N} = (\widehat{c}_i^{**}(\widehat{x}_i, \widehat{y}_i, \mathbf{0}))_{i \in N}$ by the definition of any $\widehat{\mathbf{x}} \in \varphi(\mathbf{e}_1)$ with $(c_i^m(\widehat{x}_i))_{i \in N} \in \mathcal{C}(\mathbf{e}_2)$, we have $\mathbf{c}(\varphi(\widehat{\mathbf{e}}_1)) \cap \mathcal{C}(\mathbf{e}_2) = \mathbf{c}(\varphi(\widehat{\mathbf{e}}^{**})) \cap \mathcal{C}(\mathbf{e}_2)$. Moreover, as $(\widehat{c}_i^{**}(\widehat{x}_i, \widehat{y}_i, \mathbf{0}))_{i \in N} \in \mathcal{C}(\mathbf{e}^{**})$, we have $\mathbf{c}(\varphi(\widehat{\mathbf{e}}^{**})) \cap \mathcal{C}(\mathbf{e}^{**}) \neq \varnothing$. Thus, by (ECI), we have $\mathbf{c}(\varphi(\mathbf{e}^{**})) = \mathbf{c}(\varphi(\widehat{\mathbf{e}}^{**})) \cap \mathcal{C}(\mathbf{e}^{**})$.

As $\mathcal{C}(\mathbf{e}^{**}) = \mathcal{C}(\mathbf{e}_3)$, we have $\mathbf{c}(\varphi(\mathbf{e}^{**})) = \mathbf{c}(\varphi(\mathbf{e}_3))$ from **Lemma 2**. Moreover, $\mathbf{c}(\varphi(\mathbf{e}_3)) \cap \mathcal{C}(\mathbf{e}_2) = \mathbf{c}(\varphi(\mathbf{e}_2))$ holds by the proof of **Lemma 1**. Thus, $\mathbf{c}(\varphi(\mathbf{e}^{**})) \cap \mathcal{C}(\mathbf{e}_2) = \mathbf{c}(\varphi(\mathbf{e}_3)) \cap \mathcal{C}(\mathbf{e}_2) = \mathbf{c}(\varphi(\mathbf{e}_2))$.

Then, by using the results in the previous two paragraphs, we conclude

$$\begin{aligned}
\mathbf{c}(\varphi(\mathbf{e}_2)) &= \mathbf{c}(\varphi(\mathbf{e}^{**})) \cap \mathcal{C}(\mathbf{e}_2) = \mathbf{c}(\varphi(\widehat{\mathbf{e}}^{**})) \cap \mathcal{C}(\mathbf{e}^{**}) \cap \mathcal{C}(\mathbf{e}_2) \\
&= \mathbf{c}(\varphi(\widehat{\mathbf{e}}^{**})) \cap \mathcal{C}(\mathbf{e}_2)(\text{by } \mathcal{C}(\mathbf{e}^{**}) = \mathcal{C}(\mathbf{e}_3) \supseteq \mathcal{C}(\mathbf{e}_2)) \\
&= \mathbf{c}(\varphi(\widehat{\mathbf{e}}_1)) \cap \mathcal{C}(\mathbf{e}_2).
\end{aligned}$$

Noting that $F(\mathcal{C}') = \mathbf{c}(\varphi(\mathbf{e}_2))$, $F(\mathcal{C}) = \mathbf{c}(\varphi(\mathbf{e}_1))$, and $\mathcal{C}(\mathbf{e}_2) = \mathcal{C}'$, we have therefore obtained the desired result: $F(\mathcal{C}') = \mathbf{c}(\varphi(\mathbf{e}_2)) = \mathbf{c}(\varphi(\mathbf{e}_1)) \cap \mathcal{C}(\mathbf{e}_2) = F(\mathcal{C}) \cap \mathcal{C}'$. ■

Lemma 4: (WEE), (F), (ECI), (ES), and (INC) *imply* (S).

Proof. Let $\mathcal{C}$ be a symmetric problem such that it contains identical assignments among which there is one, say $\mathbf{C}^*$, that is weakly efficient. Then, there exists $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}^M$ such that $\mathcal{C}(\mathbf{e}) = \mathcal{C}$. In the following discussion, we will construct a new economy $\mathbf{e}' = (L, \mathbf{c}^l, \overline{y}) \in \mathcal{E}^L$ such that $c_i^l = c_j^l$ for all $i, j \in N$ and $\mathcal{C}(\mathbf{e}') \supseteq \mathcal{C}$, and $G(\mathcal{C}(\mathbf{e}')) = G(\mathcal{C})$. Once such an economy is given, we will show that (WEE), (F), (ES), and (INC) imply $\mathbf{C}^* \in F(\mathcal{C}(\mathbf{e}'))$ and $\{G(\mathbf{C}^*)\} = G(F(\mathcal{C}(\mathbf{e}')))$. Furthermore, since $\mathbf{C}^* \in \mathcal{C}$, by (WEE), (F), (ECI), and (INC), $\mathbf{C}^* \in F(\mathcal{C})$ and $\{G(\mathbf{C}^*)\} = G(F(\mathcal{C}))$ will follow from **Lemma 3**. This will imply that (S) holds.

Given $\mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}$ with $\mathcal{C}(\mathbf{e}) = \mathcal{C}$, let us consider $\mathbf{e}_{\{1,2\}} = \left(M, \left(c_2^m, c_1^m, \mathbf{c}_{-\{1,2\}}^m\right), \overline{x}\right) \in \mathcal{E}$. Then, since $\mathcal{C}(\mathbf{e})$ is symmetric, $\mathcal{C}(\mathbf{e}_{\{1,2\}})$ is also symmetric, and $\mathcal{C}(\mathbf{e}_{\{1,2\}}) = \mathcal{C}$. Then, define $\mathbf{e}^{1,2} = \left(M^{1,2}, (c_i^{2m})_{i\in N}, (\overline{x}, \overline{x})\right)$ with $\# M^{1,2} = 2m$ as follows: for each $i \in \{1,2\}$, let, for each $(x_1, x_2) \in \mathbb{R}_+^{2m}$, $c_i^{2m}(x_1, x_2) \equiv c_1^m(x_1) \cap c_2^m(x_2)$. In addition, for any other $i \in N \setminus \{1,2\}$, let, for each $(x_1, x_2) \in \mathbb{R}_+^{2m}$, $c_i^{2m}(x_1, x_2) = c_i^m(x_1)$. Then, $\mathcal{C}(\mathbf{e}^{1,2}) \supseteq \mathcal{C}$ and $G(\mathcal{C}(\mathbf{e}^{1,2})) = G(\mathcal{C})$. If $N = \{1,2\}$, then $\mathbf{e}^{1,2}$ is an economy with $c_i^{2m} = c_j^{2m}$ for all $i, j \in N$.

Given, $\mathbf{e}^{1,2} \in \mathcal{E}$, consider $\mathbf{e}_{\{1,3\}}^{1,2} = \left(M^{1,2}, \left(c_3^{2m}, c_1^{2m}, c_1^{2m}, \mathbf{c}_{-\{1,2,3\}}^{2m}\right), (\overline{x}, \overline{x})\right)$ and $\mathbf{e}_{\{2,3\}}^{1,2} = \left(M^{1,2}, \left(c_1^{2m}, c_3^{2m}, c_1^{2m}, \mathbf{c}_{-\{1,2,3\}}^{2m}\right), (\overline{x}, \overline{x})\right)$. Then, $\mathcal{C}(\mathbf{e}_{\{1,3\}}^{1,2}) = \mathcal{C}(\mathbf{e}_{\{2,3\}}^{1,2}) = \mathcal{C}(\mathbf{e}^{1,2})$. Define $\mathbf{e}^{1,2,3} = \left(M^{1,2,3}, (c_i^{4m})_{i\in N}, (\overline{x}, \overline{x}, \overline{x}, \overline{x})\right)$ with $\# M^{1,2,3} = 4m$ as follows: for each $i \in \{1,2,3\}$, let, for each $(x_1, x_2, x_3, x_4) \in \mathbb{R}_+^{4m}$, $c_i^{4m}(x_1, x_2, x_3, x_4) = c_1^{2m}(x_1, x_2) \cap c_3^{2m}(x_3, x_4)$. In addition, for any other $i \in N \setminus \{1,2,3\}$, let, for each $(x_1, x_2, x_3, x_4) \in \mathbb{R}_+^{4m}$, $c_i^{4m}(x_1, x_2, x_3, x_4) = c_i^{2m}(x_1, x_2)$. Then, $\mathcal{C}(\mathbf{e}^{1,2,3}) \supseteq \mathcal{C}(\mathbf{e}^{1,2})$ and $G(\mathcal{C}(\mathbf{e}^{1,2,3})) = G(\mathcal{C}(\mathbf{e}^{1,2}))$. If $N = \{1,2,3\}$, then $\mathbf{e}^{1,2,3}$ is an economy with $c_i^{4m} = c_j^{4m}$ for all $i, j \in N$.

By repeating this procedure up to n, we obtain

$$\mathbf{e}^{(1,\ldots,n)} = \left(M^{1,\ldots,n}, \left(c_i^{2^{(n-1)}m}\right)_{i\in N}, (\overline{x}, \ldots, \overline{x})\right)$$

with $(\overline{x}, \ldots, \overline{x}) \in \mathbb{R}_+^{2^{(n-1)}m}$ and an identical profile $\left(c_i^{2^{(n-1)}m}\right)_{i\in N}$ such that $\mathcal{C}(\mathbf{e}^{(1,\ldots,n)}) \supseteq \mathcal{C}$ and $G\left(\mathcal{C}(\mathbf{e}^{(1,\ldots,n)})\right) = G(\mathcal{C})$ hold. Then, since $\mathbf{C}^*$ is weakly efficient in $\mathcal{C}$, the identical assignment $\mathbf{C}^*$ is weakly efficient in $\mathcal{C}(\mathbf{e}^{(1,\ldots,n)})$. ■

Proof of Theorem 2. It is easy to check that φ^E satisfies (WEE), (ES), (ECI), (EII), and (INC). Suppose that φ satisfies (WEE), (ES), (ECI), (EII), and (INC). Then, by **Lemmas 2**, **3**, and **4**, φ attains a bargaining solution which satisfies (WE), (S), (CI), and (II). By **Theorem 1**, φ attains F^E. ■

In Theorem 2, the independence of the axioms can be checked as follows. For (WEE), the allocation rule φ such that $\varphi(\mathbf{e}) = \{\mathbf{0}\}$ for each $\mathbf{e}$ satisfies all of the axioms but (WEE); for (ES), the Nash allocation rule φ^N which attains the Nash solution [Xu and Yoshihara (2006a)] to (opportunity)-bargaining problems satisfies all of the axioms but (ES); for (ECI), the Kalai-Smorodinsky allocation rule φ^K which attains the Kalai-Smorodinsky solution [Xu and Yoshihara (2006a)] to (opportunity)-bargaining problems satisfies all of the axioms but (ECI); for (EII), an allocation rule which attains a proper subsolution of the egalitarian solution F^E satisfies all of the axioms but (EII); and finally, for (INC), let us consider the following rule φ^*:

$$\varphi^*(\mathbf{e}) = \begin{cases} \varphi^E(\mathbf{e}) & \text{if } c_i^m = c_j^m \text{ for all } i, j \in N \text{ in } \mathbf{e} = (M, \mathbf{c}^m, \overline{x}) \in \mathcal{E}; \\ \varphi^N(\mathbf{e}) & \text{otherwise.} \end{cases}$$

This φ^* satisfies (WEE), (ES), (ECI), and (EII), but (INC). This is because $\mathbf{c}(\varphi^*(\mathbf{e}_1)) \neq \mathbf{c}(\varphi^*(\mathbf{e}_2))$ generally holds for $\mathcal{C}(\mathbf{e}_1) = \mathcal{C}(\mathbf{e}_2)$ if $c_i^m = c_j^m$ for all $i, j \in N$ in $\mathbf{e}_1$ and $c_i^l \neq c_j^l$ for some $i, j \in N$ in $\mathbf{e}_2$. Then, by **Lemma 1**, φ^* violates (INC).

5 Discussion and Conclusion

In this paper, we have extended the standard bargaining model to situations in which players are characterized by their opportunity sets rather than by their utilities[11] and in which the fair arbitrator makes the recommendation with the guiding principle of equal opportunity for all players. In such a setting, we have formulated our problems in terms of bargaining problems among players on opportunity assignments, defined the egalitarian solution in our context and studied it axiomatically. Most of the axioms used in our axiomatic characterization of the proposed solutions correspond to their counterparts in standard bargaining models, but formulated either in terms of opportunity assignments or directly in economic environments. We have discussed and commented on the axioms that are unique in our context.

Note that the results of this paper are obtained under the presumption of a common ranking R over opportunity sets. What would happen if individuals are allowed to have different rankings of opportunity sets? Note that, with

[11]It may be remarked that, when $k = 1$ (there is only one basic living condition), our model reduces to the standard non-convex bargaining problem.

individuals having possibly different rankings of opportunity sets, we still need a way of making interpersonal comparisons of these rankings in order to define the egalitarian solution:[12] $F^E(\mathcal{C}) = \{(C_i)_{i\in N} \in \mathcal{C} \mid G_i(C_i) = G_j(C_j)$ for any $i,j \in N$ and $\nexists(C_i')_{i\in N} \in \mathcal{C}$ such that $G_i(C_i') > G_i(C_i)$ for all $i \in N\}$. With a properly formulated way of comparing different individuals' rankings that cannot allow the individual rankings to be identical, Theorem 1 may no longer hold as the egalitarian solution may not satisfy the axiom (S). This is because, with possibly different rankings for different individuals, it is possible to construct a symmetric bargaining problem $\mathcal{C}$, in which there is an efficient and symmetric opportunity assignment $(C_i)_{i\in N} = (C, \ldots, C) \in \mathcal{C}$ such that $G_i(C) \neq G_j(C)$ for some $i, j \in N$.

In what follows, we shall comments on a connection between our egalitarian solution and the notion of equality of opportunity proposed by Roemer (1998).

Structurally, the egalitarian solution of bargaining over opportunity assignments and the standard egalitarian solution to the problems of utility allocations are similar. However, their conceptual implications are quite different.

First, our φ^E attempts to capture the idea of equality of opportunity, whereas the egalitarian solution for the standard bargaining problems is based on the idea of equality of outcome or equality of welfare. The egalitarian principle based on outcomes or welfares has been under critical scrutiny in recent years and rival theories of equality based on opportunities have been advocated by several people. For instance, Sen (1980) proposed equality of capabilities, which requires equalizing opportunities among individuals to realize their functionings, and contrasted and emphasized the difference between equality of outcome in any form and equality of capabilities in their respective performances. Our φ^E can be viewed as a mechanism to implement the idea of equality of capabilities.

Secondly, as we discussed earlier in the Introduction and in Section 2, there are situations in which bargaining problems are about opportunity assignments rather than final outcomes or payoffs of players. In such cases, one plausible guiding principle is "leveling players' playing fields as much as

[12] One could formulate an alternative notion of an egalitarian solution based on envy-freeness or egalitarian-equivalence studied in the literature on resource allocations without explicitly introducing interpersonal comparisons of individual rankings. This seems an interesting project that deserves further exploration and we leave it for future scrutiny.

possible" (Roemer (1998)). This guiding principle does not necessarily guarantee the equalization of players' final outcomes, and our egalitarian solution is a way to ensure that players have the same playing field. The egalitarian solution to the problems of bargaining over opportunity assignments seems to fit those situations well.

Thirdly, our φ^E does not actually guarantee equal final outcomes for players. To see this, let us consider the situation discussed in **Example 1**. Given an allocation $\mathbf{x}$ recommended by φ^E in an economy $\mathbf{e} = (M, \mathbf{c}^m, \overline{x})$, each individual $i \in N$ is guaranteed to acquire any level of skills within $c_i^m(x_i)$ by devoting his effort appropriately. However, the exact levels of skills he actually acquires depend on his effort level. Suppose that every individual has the common utility function $u(\mathbf{b}, a)$, which is a function of realized skill vector $\mathbf{b}$ and the effort level a. Each individual i chooses his effort level a_i and $\mathbf{b}_i$ in order to maximize $u(\mathbf{b}, a)$ subject to $\mathbf{b} \in c_i^m(x_i)$. Let $v(c_i^m(x_i)) \equiv \max_{\mathbf{b} \in c_i^m(x_i),\ a} u(\mathbf{b}, a)$. The standard egalitarian solution recommends an allocation $\mathbf{x}'$ in order to guarantee $v(c_i^m(x_i')) = v\left(c_j^m\left(x_j'\right)\right)$ for any $i, j \in N$. On the other hand, φ^E guarantees that, for a given ordering R over opportunity sets, $\left(c_i^m(x_i), c_j^m(x_j)\right) \in I$ for any $i, j \in N$, but $[v(c_i^m(x_i)) = v\left(c_j^m(x_j)\right)]$ may not hold for some individuals $i, j \in N$. Therefore, the recommendation by φ^E is much different from the recommendation of the standard egalitarian solution. In the context of **Example 1**, we believe that φ^E is more plausible than the standard egalitarian solution. This is because the choices of a_i and $\mathbf{b}_i$ are a matter of personal responsibility, and φ^E delegates this personal responsibility to individual players while the standard egalitarian solution does not.

How is our approach in the paper related to the *theory of equality of opportunity* discussed by Roemer (1998)? There are some differences between our approach in this paper and Roemer's model (1998). For example, for Roemer (1998), the task is to propose a social welfare function that determines the optimal equal opportunity policy, while in this paper, we define and characterize "fair" solutions for bargaining problems based on players' opportunity sets. There are also similarities between our model and Roemer's model. In Roemer (1998), the resource allocation determined by the optimal policy is to guarantee any two individuals the equal opportunity of access to the same level of "advantages" regardless of their "types", if their effort rankings within their own "types" are identical. In our model, an individual's type in the sense of Roemer is reflected in the individual's opportunity correspondence, and a bargaining solution such as F^E determines a resource allocation to guarantee an equitable assignment of opportunities among individuals, under which every individual may access to the same level of living-condition vectors.

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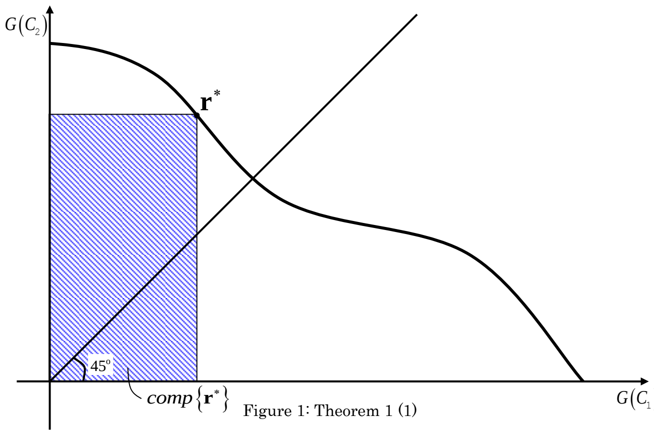

Figure 1: Theorem 1 (1)

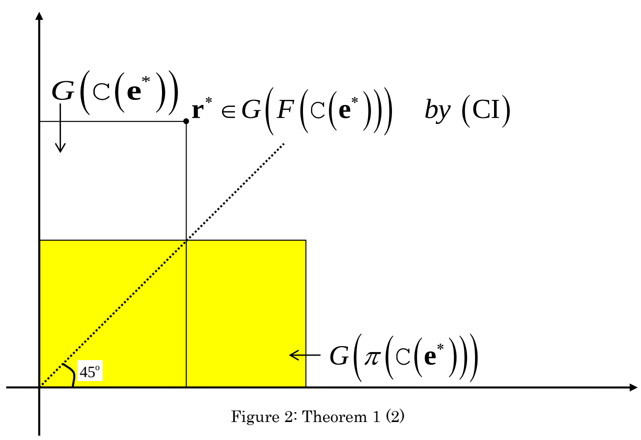

Figure 2: Theorem 1 (2)

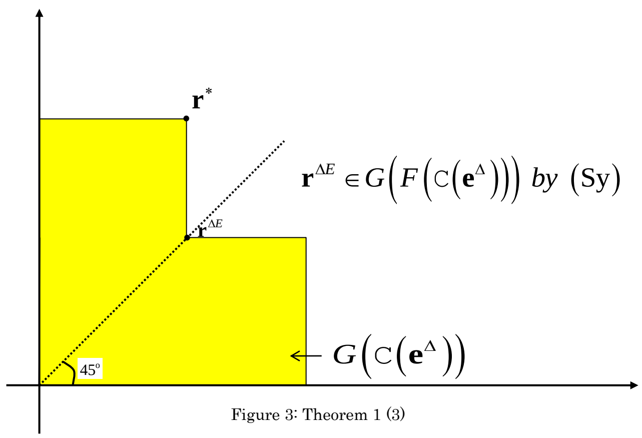

Figure 3: Theorem 1 (3)

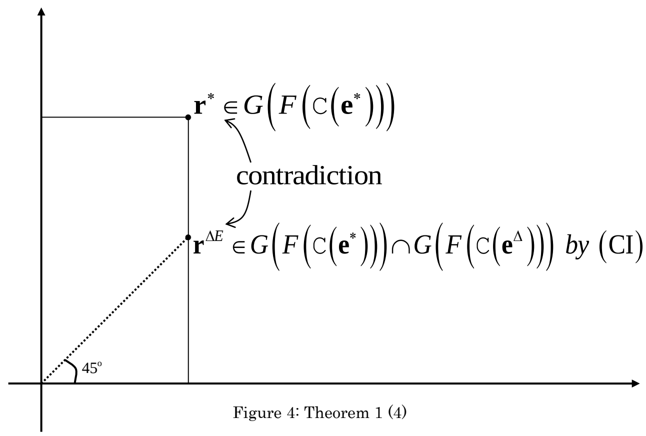

Figure 4: Theorem 1 (4)