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DIFFERENT NUMBER OF BIDDERS IN SEQUENTIAL AUCTIONS

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Abstract

We analyze the correct selling order in a second-price sequential auction for two heterogeneous synergistic goods with local and global bidders. We prove that as the number of local bidders in the second auction approaches infinity, the outcome is always efficient. However, as the number of local bidders in the first auction approaches infinity, the outcome is inefficient with a positive probability. By using simulations, we show that selling the good with more finite bidders in the second auction results in a more efficient outcome. If the selling order is incorrect, the probability of an inefficient outcome is around 19%.

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I. *Introduction*

There are some sequential auctions in which local bidders bid for one specific good and

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global bidders bid for all synergistic goods. Some examples are highway procurement auctions (De Silva 2005) and the procurement auctions of various firms (Elmaghraby 2005). In this paper, we allow for a different number of local bidders on each leg of the sequential auction, and show that the selling order has implications for the efficiency of the outcome. Our paper intends to help policy makers on how to determine the order of selling goods when they are demanded by a different number of bidders.

The global bidder overbids his stand-alone valuation in the first auction due to the hope of winning the second auction and enjoying the synergy. The equilibrium bid is not affected by the number of local bidders in the first auction as the equilibrium calculations are conditional on winning or losing the first auction. However, it is affected by the number of local bidders in the second auction as it changes the continuation payoff by impacting the probability of winning the second good, and hence, enjoying the synergy. As the number of local bidders in the second auction increases, the global bidder's first-auction bid decreases towards their true valuation as it is more likely to lose the second auction. In the limit, the global bidder just bids their valuation truthfully. Hence, the auction results in an efficient outcome. However, as the number of local bidders in the first auction approaches infinity, the global bidder keeps overbidding as there is always a chance of enjoying the synergy by winning the second good as there are finite bidders in the second auction. The outcome is inefficient with a positive probability in this case.

In the second part of the paper, we use simulation methods to calculate the probability of inefficient outcomes and compare the efficiency difference between the sequential auctions and the VCG (Vickrey-Clarke-Groves) auctions to show that our results are robust to a finite number of bidders. We show that the efficiency difference between the VCG auction and the sequential auction is around 1.4 % and the probability of an inefficient outcome is around 19 % when the order of selling is incorrect with a finite number of bidders. However, the efficiency difference between the two auctions is only 0.1 percent in most cases when the order of selling is correct.

In the auction literature with synergy, it has been assumed that the goods are either equivalent (Branco, 1997), or the second good becomes more valuable to the winner of the first good (Jeitschko and Wolfstetter, 2002). In these papers, it is naturally assumed that the number of local bidders are equal in each auction. Krishna and Rosenthal (1996) studies the effect of varying the (finite) number of local bidders for simultaneous auctions but they do not find conclusive results as they write that "*..it appears difficult to say anything in general*". We show, with simulations, that as the number of finite local bidders gets large for one good, then selling that good last is efficient. Elmaghraby (2003) studies the relation between the order of auctioning and efficiency in a second-price sequential procurement auction when capacity constrained bidders have different cost functions. In her model, she links the cost function of the bidders to the ordering of auctions. In our model, we link the number of bidders to the ordering of auctions. Gentry et. al. (2019) show that as the number of bidders approaches infinity, the outcome becomes efficient in a *simultaneous* auction setting. We show that the outcome might be inefficient with a positive probability in sequential auctions depending on the order of selling. Meng and Gunay (2017) calculate the probability of inefficient outcomes in simultaneous auctions for finite bidders. We calculate the same probability for sequential auctions, which is relevant for the policy makers.

There are some papers in the literature that study the (expected) price trend in second-price sequential auctions. Monteiro and Menezes (2003) show that expected prices decrease when there are (positive) synergies in a sequential auction when goods are identical. Sorensen (2006) shows that when goods are stochastically equivalent but not identical, expected prices might not decrease.[1] In our paper, since goods are not stochastically equivalent, the price depends on when the stochastically more valuable good is sold, especially if the difference between the expected value of the two goods is large. Hence, we do not analyze price trends in this paper as it is easy to see that depending on the distributions we find price might decrease or increase.

The novelty of our paper is showing the correct order of selling in sequential auctions when there are different number of bidders in each leg of the auctions. We theoretically show that selling the good with more bidders last is efficient asymptotically. Selling it first, however, is inefficient with a positive probability. By using simulation methods, we also find support for these results when bidders are finite. Therefore, our paper is relevant for public policy when the government is holding sequential auctions and mainly care about efficiency.

II. *The Model*

Consider two goods, A and B, that has zero value to the seller, who sells them with a second-price sequential auction. There is one risk-neutral global bidder, G.[2] If she wins both goods, she enjoys a synergy of $\theta>0$.[3] There are also $N_i>0$ risk neutral local bidders bidding for good $i=A, B$. N_i+1 independent draws from the distribution function F_i determines the private valuation, v_{ki}, for each bidder, $k=G, 1, 2.., N_i$, and $i=A, B$. The distribution function F_i, has a twice differentiable density function $f_i>0$ on the interval $(0, 1]$ with $f_i(0)\geq 0$. F_A and F_B are independent.

We use symmetric subgame perfect Bayesian equilibrium in weakly undominated strate-gies.[4] It is well-known that local bidders bid their valuations truthfully in both auctions. The global bidder's second-auction equilibrium strategy is bidding the marginal valuation for the second good truthfully as this is the last stage of the game. That is, bidding $v_{Gj}+\theta$ if won good i in the first auction, and v_{Gj} otherwise, where $i, j=A, B$ and $i\neq j$.

To derive the global bidder's equilibrium strategy in the first auction for good i, we maximize her payoff given the sequential rationality. Let $p_i=max\{v_{ki}\}$, $k=1, 2.., N_i$, denote the maximum valuation of local bidders for good $i=A, B$. This p_i is the price that the global bidder pays if he wins good i. The distribution function for p_i is $G_i(.)=[F_i(.)]^{N_i}$.

The expected payoff for the global bidder when bidding p is

$$Max_p \int_0^p (v_{Gi}-p)dG_i(p_i)+G_i(p)\int_0^{min\{v_{Gj}+\theta,1\}}(v_{Gj}+\theta-p_j)dG_j(p_j)$$

[1] We thank a referee for pointing out the aforementioned papers in this paragraph.

[2] Multiple global bidders' strategies have not been defined in the literature yet when types are multi-dimensional for moderate synergy levels (see Meng and Gunay, 2017 and the references therein). Hence, assuming one global bidder is not uncommon in the literature for such cases.

[3] We assume θ to be a public information but it will be clear from the proofs that assuming that it is private has no effects on any of the results.

[4] In describing the model below, we closely follow Gunay and Meng, 2017.

$$+(1-G_i(p))\int_0^{v_{Gj}}(v_{Gj}-p_j)dG_j(p_j) \qquad (1)$$

The first integral is the expected profit from winning i in the first auction, second is the expected profit from winning j after winning i, and the third is the expected profit from winning j after losing i. Note that, $Pr(p>p_i)=G_i(p)$, is the probability of the global bidder winning auction i.

Equation 2 is the first order condition, and gives the equilibrium bidding price, p_{ij}, when good i is auctioned first, and j second

$$\frac{dG_i}{dp_{ij}}\underbrace{\left[(v_{Gi}-p_{ij})+\int_0^{min\{v_{Gj}+\theta,1\}}(v_{Gj}+\theta-p_j)dG_j(p_j)\right]}_{\text{Expected profit from winning the first auction at } p_{ij}}=\frac{dG_i}{dp_{ij}}\underbrace{\left[\int_0^{v_{Gj}}(v_{Gj}-p_j)dG_j(p_j)\right]}_{\text{and losing the first auction}} \qquad (2)$$

In the equation, the only term dependent on the number of bidders in the first auction, $\frac{dG_i}{dp_{ij}}$, cancels out from both sides. By using integration by parts and equation 2, we derive the global bidder's equilibrium bid.

Proposition 1. *The global bidder's first-auction equilibrium bid*, p_{ij}, *for good i is*

$$a)\ \textit{If } v_{Gj}+\theta<1, \textit{ then } p_{ij}(v_{Gi}, v_{Gj}, N_j)=v_{Gi}+\int_{v_{Gj}}^{v_{Gj}+\theta}G_j(p_j, N_j)dp_j$$

$$b)\ \textit{If } v_{Gj}+\theta\geq 1, \textit{ then } p_{ij}(v_{Gi}, v_{Gj}, N_j)=v_{Gi}+(v_{Gj}+\theta-1)+\int_{v_{Gj}}^{1}G_j(p_j, N_j)dp_j$$

Proof is in the Appendix. The global bidder's bid p_{ij} is the highest price he is willing to pay to win good i, and at that price she is indifferent between winning and losing. In other words, the global bidder's equilibrium incentives are conditional on winning the first auction; hence, only the number of local bidders in the second auction matters as this affects the probability of winning the second good and the synergy. This is explained in the first part of the corollary below.

Corollary 2. *i*) *The number of local bidders in the first auction has no effect on the bidding price.*

ii) *As the number of local bidders in the second auction approaches infinity*, *the global bidder's bid is*

$p_{ij} \rightarrow v_{Gi}$, *if* $v_{Gj}+\theta<1$.

$p_{ij} \rightarrow v_{Gi}+v_{Gj}+\theta-1$, *if* $v_{Gj}+\theta>1$.

As the number of local bidders in the second auction, N_j, approaches infinity, $G_j=F_j[.]^{N_j}$ approaches to zero, and hence the integrals in proposition 1 approaches zero. When $v_{Gj}+\theta<1$, the global bidder bids his valuation, v_{Gi} for the first good. As he knows that he cannot win the second good given that the maximum of local bidders valuation approaches 1. When $v_{Gj}+\theta>1$, the global bidder wins the second auction for sure if he wins the first good. Hence, he bids truthfully for the first good, $v_{Gi}+v_{Gj}+\theta-1$, which is his total valuation minus the price of second good which is 1. In short, the global bidder bids truthfully when the number of local bidders in the second auction approaches infinity.

The corollary implies that the ordering of auctions has implications for efficiency. When the number of local bidders in the first auction approaches infinity the outcome might be inefficient as the global bidder does not bid truthfully. However, as the number of bidders in the second auction approaches infinity, the outcome is always efficient, as all bidders bid truthfully.

Proposition 3. *i) Assume that* $0<\theta<2$. *As the number of local bidders in the first auction approaches infinity, the outcome of the auction might be inefficient with a positive probability.*

ii) As the number of local bidders in the second auction approaches infinity, the outcome of the sequential auction is efficient.

When the number of local bidders in the first auction approaches infinity but finite in the second auction, the global bidder wins the first auction only by bidding over 1. However, if the global bidder loses the second auction, there is an ex-post loss. We prove that such inefficient outcomes occur with a positive probability. We also show that when the global bidder wins both goods, the efficient outcome might require the local bidders to win them. We also prove that such inefficient outcomes occur with a positive probability. These two types of inefficient outcomes are sufficient to prove the first part of proposition 3. If $\theta=0$, then the global bidder becomes a local bidder as there is no synergy, and all bidders bid truthfully. There cannot be any inefficient outcome. If $\theta\geq 2$, then the global bidder wins both goods. This is the efficient outcome as $v_{Gi}+v_{Gj}+\theta\geq 2\geq p_i+p_j$. The proof is in the appendix.

III. *Simulations*

To calculate the probability of inefficient outcomes, and to show that our results are robust to a finite number of bidders, we use simulation methods by using ex-post valuations. We choose to use this method because calculating the ex-ante probability of inefficient outcomes is not possible or extremely difficult as it is evident by the lack of papers in the literature.

We draw bidders' valuations for good A and B from specified distribution functions. By using our theoretical findings, we calculate the equilibrium bids for all players, and hence, the outcome of the auction. We classify the outcomes as efficient and inefficient for each of 20,000 draws/auctions. Then, we calculate the probability of inefficient outcomes by dividing the number of inefficient outcomes to 20,000 . Moreover, to gain insight, we decompose the inefficient outcomes into two categories by using Meng and Gunay (2017). In the first one, global bidders win one or two goods with an ex-post loss, and in the second one, local bidders win one or both goods when the global bidder is supposed to win both goods.[5]

In Table 1, our simulation results show that Proposition 3 holds even for a finite number of bidders (for the specified distribution). We use uniform distributions to draw valuations for both goods and a synergy level of 0.5.[6] When we have 1 local bidder in the first auction but

[5] There are actually four categories of inefficient outcomes. The first two are the global bidder wins one good with loss and the global bidder wins both goods with loss. We combine them as one category as the global bidder wins the good or goods inefficiently with an ex-post loss. The last two are the local bidder wins one good and the global bidder wins the other with a profit, and the local bidders win both goods. We combine them as one category as the local bidder wins one or two goods inefficiently.

TABLE 1. PROBABILITY OF AN INEFFICIENT OUTCOME

Probability	$NA=2000$ *and* $NB=1$
$p_{AB}^{Ineff.(Total)}\%$	11.4250
$p_{BA}^{Ineff.(Total)}\%$	0.0150

$p_{AB}^{Ineff.(Total)}\%$: Probability of total inefficient outcome in AB auction.
$p_{BA}^{Ineff.(Total)}\%$: Probability of total inefficient outcome in BA auction.
F_A and F_B are uniform distributions.
Synergy with $\theta=0.5$.

TABLE 2. PROBABILITY OF INEFFICIENT OUTCOMES AND COMPARISON OF SOCIAL WELFARE BETWEEN VCG AND SEQUENTIAL AUCTIONS

	$N_A=1$ $N_B=1$	$N_A=1$ $N_B=10$	$N_A=1$ $N_B=50$	$N_A=1$ $N_B=100$
		Synergy Level $\theta=0.2$		
$p_{AB}^{Ineff.(total)}\%$	8.3380	1.9400	0.3880	0.2080
$\Delta SW_{AV}\%$	-0.3449	-0.0485	-0.0035	-0.0011
$p_{BA}^{Ineff.(total)}\%$	5.4000	5.0740	6.0140	6.1920
$\Delta SW_{BV}\%$	-0.2195	-0.1962	-0.1979	-0.2052
		Synergy Level $\theta=0.5$		
$p_{AB}^{Ineff.(total)}\%$	9.1490	3.4440	0.7080	0.3760
$\Delta SW_{AV}\%$	-0.7065	-0.1059	-0.0062	-0.0019
$p_{BA}^{Ineff.(total)}\%$	6.4820	6.1600	9.7700	9.5320
$\Delta SW_{BV}\%$	-0.4614	-0.6187	-0.6066	-0.5818
		Synergy Level $\theta=0.8$		
$p_{AB}^{Ineff.(total)}\%$	4.5400	3.2520	0.7500	0.3440
$\Delta SW_{AV}\%$	-0.3789	-0.0983	-0.0057	-0.0012
$p_{BA}^{Ineff.(total)}\%$	3.9000	4.8000	10.2740	10.2440
$\Delta SW_{BV}\%$	-0.2745	-0.6318	-0.6315	-0.6330

F_A is Beta distribution with $\alpha=1$, $\beta=3$ and F_B is uniform distribution.
$p_{AB}^{Ineff.(Total)}\%$: Probability of an inefficient outcome in AB auction.
$p_{BA}^{Ineff.(Total)}\%$: Probability of an inefficient outcome in BA auction.
$\Delta SW_{AV}\%$: Percentage change in social welfare between VCG and sequential AB auction.
$\Delta SW_{BV}\%$: Percentage change in social welfare between VCG and sequential BA auction.

2000 local bidders in the second one, the probability of an inefficient outcome is almost 0 per cent (0.01 per cent to be exact). If we reverse the selling order, the probability of an inefficient outcome jumps to 11.42 percent as expected by part i of Proposition 3.

In Table 2 and 3 we draw valuations from Beta Distribution for good A and Uniform Distribution for good B. In Table 2, we calculate the probability of an inefficient outcome and compare the efficiency difference between VCG (Vickrey-Clarke-Groves) auction and our sequential auction by keeping the number of first-auction bidders constant, and varying the

[6] We have also used different distributions and get similar results. We do not present them here.

Table 3. Probability of Inefficient Outcomes and Comparison of Social Welfare between VCG and Sequential Auctions

	$N_B=1$ $N_A=1$	$N_B=1$ $N_A=10$	$N_B=1$ $N_A=50$	$N_B=1$ $N_A=100$
Synergy Level $\theta=0.2$				
$p_{AB}^{Ineff.(total)}\%$	8.3380	4.9880	2.2900	1.6640
$\Delta SW_{AV}\%$	-0.3449	-0.1714	-0.0717	-0.0497
$p_{BA}^{Ineff.(total)}\%$	5.4000	1.7360	0.5800	1.3320
$\Delta SW_{BV}\%$	-0.2195	-0.0592	-0.0117	-0.0060
Synergy Level $\theta=0.5$				
$p_{AB}^{Ineff.(total)}\%$	9.1490	15.1840	10.4540	8.6460
$\Delta SW_{AV}\%$	-0.7065	-1.0632	-0.6164	-0.5027
$p_{BA}^{Ineff.(total)}\%$	6.4820	4.6960	2.6620	1.6600
$\Delta SW_{BV}\%$	-0.4614	-0.2980	-0.0755	-0.0422
Synergy Level $\theta=0.8$				
$p_{AB}^{Ineff.(total)}\%$	4.5400	14.5180	18.9020	19.5980
$\Delta SW_{AV}\%$	-0.3789	-1.3227	-1.4900	-1.3859
$p_{BA}^{Ineff.(total)}\%$	3.9000	2.8940	4.6920	5.0460
$\Delta SW_{BV}\%$	-0.2745	-0.3144	-0.1460	-0.0971

F_A is Beta distribution with $\alpha=1$, $\beta=3$ and F_B is uniform distribution.
$p_{AB}^{Ineff.(Total)}\%$: Probability of an inefficient outcome in AB auction.
$p_{BA}^{Ineff.(Total)}\%$: Probability of an inefficient outcome in BA auction.
$\Delta SW_{AV}\%$: Percentage change in social welfare between VCG and sequential AB auction.
$\Delta SW_{BV}\%$: Percentage change in social welfare between VCG and sequential BA auction.

number of second-auction bidders as 1, 10, 50, and 100. In Table 3, we reverse the selling order by keeping the number of second-auction bidders constant but varying the number of first-auction bidders.

To compare the (total) efficiency level of the sequential auction with that of the VCG auction, we, first, calculate the total efficiency for each of the sequential auction and the VCG auction for the same valuations. We do this for 20,000 repetitions/draws. Then, we average the efficiency for the sequential and then the VCG auction separately. Table 2 shows that the efficiency level of the sequential auction is only 0.001% lower than the VCG auction's for most cases, and at most 0.11 % lower when there are more bidders in the second auction. However, Table 3 shows that it is 1.49 % lower than the VCG auction's in some cases when there are more bidders in the first auction. A comparison of the same cases in Table 2 and Table 3 shows that the efficiency difference between VCG and sequential auctions is smaller with the correct order of selling even with finite number of bidders.

Figures 1,2, and 3 summarize the probability of inefficient outcomes calculated in Table 2 and 3. They show that the probability of an inefficient outcome is lower if there are more bidders in the second auction for the synergy levels $\theta=0.2, 0.5, 0.8$, respectively. One interesting result is that in Figure 3, when synergy level is at 0.8, the probability of inefficiency increases as the number of first-auction bidders increases until leveling off around 20 %. Since the synergy level is 0.8, the global bidder bids more than 1 in the first auction even when it has relatively low stand-alone valuations for the goods. As a result, the global bidder wins the first auction with a loss. Hence, they are more likely exposed to a (potential) loss in this case, which

FIGURE 1. PROBABILITY OF INEFFICIENCY COMPARISON WHEN θ=0.2

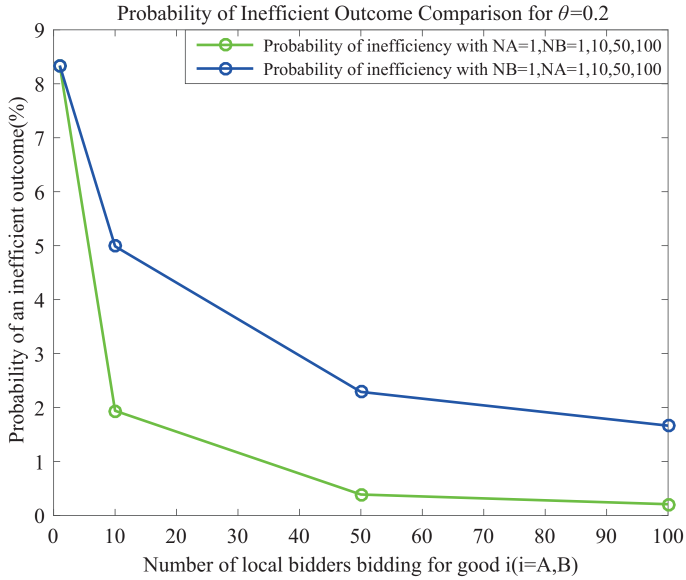


FIGURE 2. PROBABILITY OF INEFFICIENCY COMPARISON WHEN θ=0.5

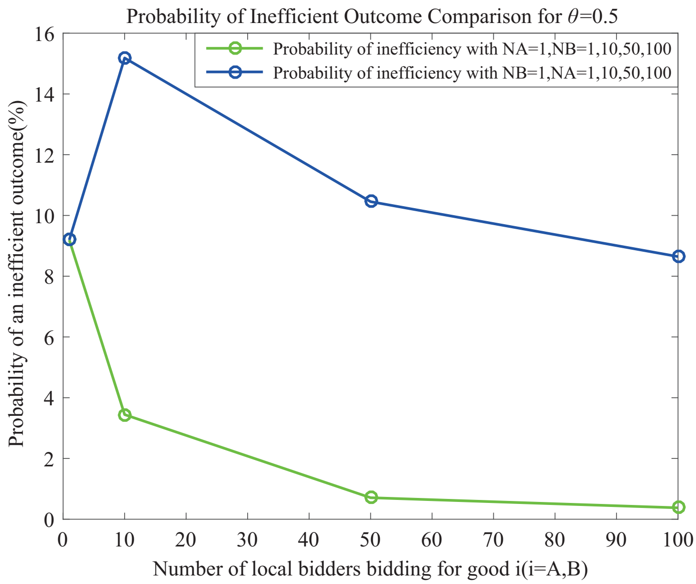
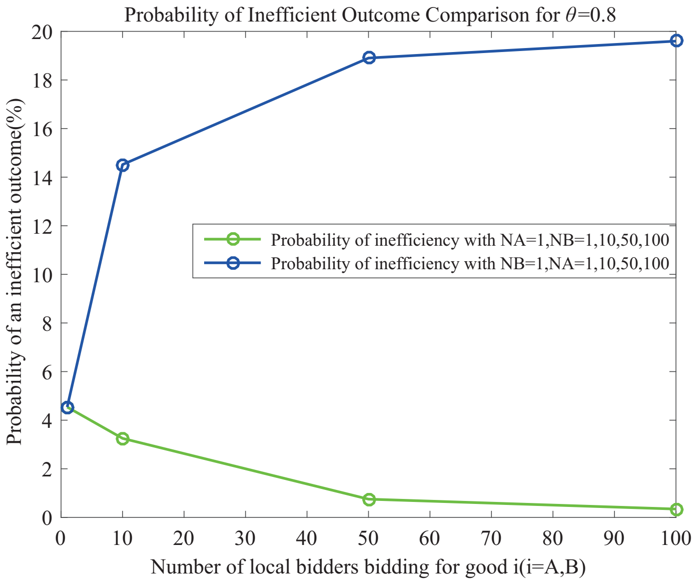


FIGURE 3. PROBABILITY OF INEFFICIENCY COMPARISON WHEN θ=0.8



is an inefficient outcome. Also, there are cases in which the global bidder bids less than 1 when its stand-alone valuations are very small. In such cases, as the number of first-auction bidders increases, the global bidder loses the first auction with a higher probability. However, the efficiency might require that the global bidder must have won both goods. These types of inefficiencies explain why the probability is increasing in Figure 3.

IV. *Conclusion*

In this paper, we show that the correct order of selling two goods when the number of bidders are different in each auction. Selling the good with more bidders last guarantees an efficient outcome asymptotically.

One might think that the result is due to the fact that uncertainty is resolved in the second auction as the winning price for the global bidder becomes deterministic. To show that this is not sufficient for our result, assume that the value of the first good is fixed (for all bidders) say $v_f \in [0, 1]$. Clearly, selling this good second would be efficient. However, selling the good in the first auction would still be efficient although the global bidder does not bid truthfully. To see this, note that the global bidder will overbid for the first object and win. However, everyone values the item same. For the second object, everyone bids truthfully and hence, the global bidder wins the object only if his valuation of $v_{GB}+\theta$ is greater than all local bidder's valuations. If not, he will lose. Coupled with the fact that the first object is valued same for each bidder and global bidder wins or loses the second object only if it is efficient ends up in an efficient outcome. Hence, to get our result that the outcome is inefficient with a positive probability even if there are infinitely many bidders in the first auction, one needs a gap between the global and local bidder's valuations. This is something novel in the literature as there are hardly any papers studying heterogeneous goods or studying different number of bidders.[7]

By using simulations methods, we find support that our results hold for finite number of bidders. Specifically, selling the good with more bidders last is more efficient in the simulations we have used. While we acknowledge that the simulations cannot be enough to prove the result, the message is that when the bidders are different for each good, the correct selling order can be determined with the simulations as selling the good with more bidders first or second makes a difference. Governments who care about efficiency should take this into account.

V. *Appendix*

Proof of Proposition 1.

From equation 2, we have, $p_{ij}=v_{Gi}+\int_0^{min\{v_{Gj}+\theta,1\}}(v_{Gj}+\theta-p_j)dG_j(p_j, N_j)-\int_0^{v_{Gj}}(v_{Gj}-p_j)$ $dG_j(p_j, N_j)$ By using integration by parts, we find the global bidder's first-auction equi-librium bid, p_{ij}, for good i when $v_{Gj}+\theta<1$ is,

$$
\begin{aligned}
p_{ij}&=v_{Gi}+\int_0^{v_{Gj}+\theta}(v_{Gj}+\theta-p_j)dG_j(p_j, N_j)-\int_0^{v_{Gj}}(v_{Gj}-p_j)dG_j(p_j, N_j)\\
&=v_{Gi}+(v_{Gj}+\theta-p_j)G_j(p_j, N_j)\,|_0^{v_{Gj}+\theta}-\int_0^{v_{Gj}+\theta}G_j(p_j, N_j)d(v_{Gj}+\theta-p_j)\\
&\quad-(v_{Gj}-p_j)G_j(p_j, N_j)\,|_0^{v_{Gj}}+\int_0^{v_{Gj}}G_j(p_j, N_j)d(v_{Gj}-p_j)\\
&=v_{Gi}+\int_0^{v_{Gj}+\theta}G_j(p_j, N_j)dp_j-\int_0^{v_{Gj}}G_j(p_j, N_j)dp_j=v_{Gi}+\int_{v_{Gj}}^{v_{Gj}+\theta}G_j(p_j, N_j)dp_j
\end{aligned}
$$

And the global bidder's first-auction equilibrium bid, p_{ij}, for good i when $v_{Gj}+\theta>1$ is,

$$
\begin{aligned}
p_{ij}&=v_{Gi}+\int_0^{1}(v_{Gj}+\theta-p_j)dG_j(p_j, N_j)-\int_0^{v_{Gj}}(v_{Gj}-p_j)dG_j(p_j, N_j)\\
&=v_{Gi}+(v_{Gj}+\theta-p_j)G_j(p_j, N_j)\,|_0^{1}-\int_0^{1}G_j(p_j, N_j)d(v_{Gj}+\theta-p_j)\\
&\quad-(v_{Gj}-p_j)G_j(p_j, N_j)\,|_0^{v_{Gj}}+\int_0^{v_{Gj}}G_j(p_j, N_j)d(v_{Gj}-p_j)\\
&=v_{Gi}+(v_{Gj}+\theta-1)+\int_0^{1}G_j(p_j, N_j)dp_j-\int_0^{v_{Gj}}G_j(p_j, N_j)dp_j=v_{Gi}+(v_{Gj}+\theta-1)+\\
&\quad\int_{v_{Gj}}^{1}G_j(p_j, N_j)dp_j
\end{aligned}
$$

Given the FOC equation 2, and $G_i(.)=[F_i(.)]^{N_i}$, we show that the SOC is satisfied. Since the FOC is met, then

$$
\frac{dG_i}{dp_{ij}}\underbrace{\left[(v_{Gi}-p_{ij})+\int_0^{min\{v_{Gj}+\theta,1\}}(v_{Gj}+\theta-p_j)dG_j(p_j)\right]-\left[\int_0^{v_{Gj}}(v_{Gj}-p_j)dG_j(p_j)\right]}_{\text{Equal to zero}}=0
$$

[7] Muramoto and Sano (2016) studies a sequential auction for heterogeneous objects but their focus is in price anomaly.

Taking the derivative of the FOC with respect to dp_{ij} and the SOC, calculated at the optimum p_{ij} is,

$$\frac{d^2G_i}{dp_{ij}^2}\underbrace{\left[(v_{Gi}-p_{ij})+\int_0^{min\{v_{Gj}+\theta,1\}}(v_{Gj}+\theta-p_j)dG_j(p_j)\right]-\left[\int_0^{v_{Gj}}(v_{Gj}-p_j)dG_j(p_j)\right]}_{\text{Equal to zero at } p_{ij}}-\frac{dG_i}{dp_{ij}}=-\frac{dG_i(p_{ij},N_i)}{dp_{ij}}$$

$=-\frac{dF_i(p_{ij},N_i)^{N_i}}{dp_{ij}}=N_iF_i(p_{ij},N_i)^{N_i-1}f_i(p_{ij},N_i)\leq 0$ since N_i is finite, f is positive by assumption except at zero and F is positive except at zero. Since there is a unique p_{ij} satisfying the FOC and SOC, we get our maximizer. Note that the corner solution of bidding zero cannot be the solution as the global bidder will always lose. This ends the proof.

Proof of Proposition 3.

To prove part i, we need two lemmas. We will state and prove them first.

Lemma 4 *Let X and Y be two independent random variables with the continuous probability density functions $f_x(x)$ and $f_y(y)$ on the support* $[0, 1]$. *Then $Z=X+Y$ has a continuous density function $f_z(z)$ on the support* $[0, 2]$.

Proof of Lemma 4. It is well-known that Z is the convolution of two densities. Specifically, we get the density function

$$f_z(z)=\int f_x(z-y)f_y(y)dy$$

where this is equal to 0 outside the support $[0, 2]$. As f_x and f_y are continuous, this is a continuous density function on the support. It is easy to see that the support is on $[0, 2]$.

Lemma 5 *Let X, Y, and T be three independent random variables with continuous density functions on the support* $[0, 1]$. *Then $X+Y-T$ and $Z-T$ have continuous density functions on the support* $[-1, 2]$.

Proof of Lemma 5. By lemma 4, we know that $Z=X+Y$ has a continuous density function. So we have to show that $H=Z-T$ has a continuous density function. We will start finding the cumulative distribution function, F_h of H:

$$Pr\ (Z-T\leq h)=F_h(h)=\int_{-\infty}^{\infty}\int_{-\infty}^{h+t}f_h(z,t)dzdt=\int_{-\infty}^{\infty}\int_{-\infty}^{h+t}f_t(t)f_z(z)dzdt$$

$$=\int_{-\infty}^{\infty}f_t(t)dt\int_{-\infty}^{h+t}f_z(z)dz=\int_{-\infty}^{\infty}f_t(t)F_z(h+t)dt$$

The third equality follows by the independence of the random variables, and we use Fubini's theorem for the change of order of integrals as the integrals are bounded representing the cumulative distribution function. We also note that f_i's are 0 outside their support. If we take the derivative of $F_h(h)$, then we get the density function:

$$f_h(h)=\frac{d}{dh}F_h(h)=\frac{d}{dh}\int_{-\infty}^{\infty}f_t(t)F_z(h+t)dt=\int_{-\infty}^{\infty}f_t(t)f_z(h+t)dt$$

Since the density functions f_i are continuous, $f_h(h)$ is a continuous density function on its support. It is easy to see that the support is on $[-1, 2]$ as the support of X, Y, T are on $[0, 1]$. This ends the proof of lemma.

Now, we can start proving the proposition. We will show this for two cases when $1 \leq \theta < 2$ and then when $0 < \theta < 1$. We show that in both cases, there might be an inefficient outcome with a positive probability. We also remind the readers that all draws are independent in our model.

A) $1 \leq \theta < 2$

To prove this, we show that there is an inefficient outcome with a positive probability that the global bidder wins both goods but the efficient outcome is the one where the local bidders should have won them. The global bidder should bid $p_{ij} > 1$ in the first auction, and automatically wins the second auction as the second auction bid is $v_{Gj} + \theta \geq 1 \geq p_j$ given the range of θ.[8] The outcome will be inefficient if the inequality $v_{Gi} + v_{Gj} + \theta < 1 + p_j$ holds. In short, we want to show that $Pr\ ((p_{ij} \geq 1) \cap (v_{Gi} + v_{Gj} + \theta < 1 + p_j)) > 0$.

By Proposition 1, since $v_{Gj} + \theta \geq 1$, the global bidder's bid $p_{ij} > 1$, is:

$$1 < p_{ij} = v_{Gi} + v_{Gj} + \theta - 1 + \int_{v_{Gj}}^{1} G_j(p_j, N_j) dp_j$$

If we combine this with the inefficient outcome inequality, we get:

$$2 - \int_{v_{Gj}}^{1} G_j(p_j, N_j) dp_j < v_{Gi} + v_{Gj} + \theta < 1 + p_j$$

We will show that these two inequalities simultaneously hold when p_j approaches to 1, and when $v_{Gi} + v_{Gj}$ approaches to zero. Specifically, we assume that $p_j \in (1 - \epsilon, 1]$ and $v_{Gi} + v_{Gj} \in [0, 2 - \theta - \epsilon)$, where ϵ is a very small positive number and $\epsilon < 2 - \theta$. Both of these events have positive probabilities as $G_j(p_j, N_j)$ has a continuous density function on $[0, 1]$ since N_j is finite, and the density function representing $v_{Gi} + v_{Gj}$ is continuous on $[0, 2]$ by lemma 4. Moreover, the ranges guarantee that $p_j > 1 - \epsilon > v_{Gj}$ since $\theta \geq 1$. We also assume that $v_{Gi} \neq 0$, which is a zero probability event.[9]

As p_j approaches to 1, $G_j(p_j, N_j)$ also approaches to 1 as it is a cumulative distribution function and the support of p_j is on $[0, 1]$. In the limit, $2 - \int_{v_{Gj}}^{1} G_j(p_j, N_j) dp_j = 2 - (1 - v_{Gj}) = 1 + v_{Gj}$. Let us assume that when $p_j = 1 - \epsilon_1$, then $G_j(p_j, N_j) = 1 - \delta_1$, where ϵ_1 and δ_1 are very small positive numbers.[10] Then, we can re-write our inequalities for this case:

$$2 - \int_{v_{Gj}}^{1} (1 - \delta_1) dp_j = 1 + v_{Gj} + \delta_1 (1 - v_{Gj}) < v_{Gi} + v_{Gj} + \theta < 1 + p_j$$

By making p_j arbitrarily close to 1, we make δ_1 arbitrarily close to 0 since G_j is continuous, which makes the term $\delta_1(1 - v_{Gj})$ arbitrarily close to 0; hence, $1 + v_{Gj} + \delta_1(1 - v_{Gj}) < v_{Gi} + v_{Gj} + \theta$ holds for some δ_1 as $1 \leq \theta$ and $v_{Gi} \neq 0$. We also get

[8] There is a zero probability outcome where $v_{GB} = 0$, then in the second auction bids might be equal. We assume that the global bidder wins the auction when bids are tie.

[9] If this assumption does not hold, then there might be a case where $p_{ij} = 1$. Even then, an inefficient outcome occurs if we assume that the global bidder wins the auction equally likely when bids are tie.

[10] While the term δ_1 is a function of N_j, we skip writing in the function notation in order not to complicate the notation.

$1+v_{Gj}+\delta_1(1-v_{Gj})<1+p_j$ as $v_{Gj}<1-\epsilon<p_j$ for some δ_1. The minimum of the two aforementioned δ_1 satisfying each inequality will satisfy both inequalities.

Finally, we have to show that $v_{Gi}+v_{Gj}+\theta<1+p_j$. Since $v_{Gi}+v_{Gj}\in[0, 2-\theta-\epsilon)$, we have $v_{Gi}+v_{Gj}+\theta<2-\epsilon<1+p_j$ since $1-\epsilon<p_j$. Therefore, the equations hold simultaneously and all the events have positive probabilities by continuity of the density functions of the aforementioned random variables. We conclude that $Pr\ ((p_{ij}\geq 1)\cap(v_{Gi}+v_{Gj}+\theta<1+p_j))>0$.

It is easy to see that if $2\leq\theta$, then the global bidder wins both goods given proposition 1 and this will be the efficient outcome.

B) $0<\theta<1$

Since N_i approaches infinity, and hence, p_i, the maximum valuation of the local bidders, tends to 1 in the limit, the global bidder wins the first good only if it bids over 1.

$$\underbrace{Pr\left(1<v_{Gi}+\int_{v_{Gj}}^{v_{Gj}+\theta}G_j(p_j, N_j)dp_j\right)}_{\text{Probability of winning the first auction over the satnd}-\text{alone vlue conditional on } v_{Gj}+\theta<1} \tag{3}$$

Let $\epsilon=\int_{v_{Gj}}^{v_{Gj}+\theta}G_j(p_j, N_j)dp_j$. The number ϵ is positive since N_j is finite which implies $G_j>0$, and $\theta>0$. But then the probability $Pr\ (1-\epsilon<v_{Gi})>0$ holds as the probability density function f_i is continuous. This proves equation 3 is positive.

The global bidder loses the second good after winning the first good if $v_{Gj}+\theta<p_j$ so we have to show that the probability $Pr\ (v_{Gj}+\theta<p_j)=Pr\ (v_{Gj}-p_j<-\theta)>0$. By lemma 5, the probability density function of $v_{Gj}-p_j$ is continuous. But then $Pr\ (v_{Gj}-p_j<-\theta)>0$ holds as long as $\theta<1$. The probability of winning only one license with an ex-post loss is the multiplication of equation 3 and $Pr\ (v_{Gj}-p_j<-\theta)$, which is positive. Note that when the condition $v_{Gj}+\theta<p_j$ is satisfied, we also get $v_{Gj}+\theta\leq 1$, the condition in equation 3, is automatically satisfied as p_j can at most be 1. This proves that when $\theta<1$, there is a positive probability that the outcome might be inefficient.

If θ is zero, all bidders bid truthfully, then the outcomes are efficient. Equivalently, equation 3 will be zero, which shows the outcomes cannot be inefficient if $\theta=0$.

ii) All bidders bid truthfully in this case by our corollary and the explanations in the text. When all bidders bid truthfully, the outcome is efficient.

The global bidder bids $p_{AB}\rightarrow v_{GA}$, when $v_{GB}+\theta<1$. This is truthful bidding since the global bidder should win only A in this case if $v_{LA}<v_{GA}$ and lose B as the maximum bid is 1 by a local bidder. This ends up in an efficient outcome. The global bidder loses both licenses if $v_{LA}>v_{GA}$, which is the efficient outcome.

The global bidder bids $p_{AB}\rightarrow v_{GA}+v_{GB}+\theta-1$ when $v_{GB}+\theta>1$. The global bidder wins both license A and B if $v_{GA}+v_{GB}+\theta-1>v_{LA}$. But the inequality can be written as $v_{GA}+v_{GB}+\theta>1+v_{LA}$ which shows that the outcome is efficient. Conversely, the global bidder loses license A (and B) if $v_{GA}+v_{GB}+\theta-1<v_{LA}$. Re-writing the last inequality, $v_{GA}+v_{GB}+\theta<1+v_{LA}$, shows that the outcome is efficient.

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