# Essays on Stationary and Nonstationary Common Factor Models in Econometrics

by

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# Abstract

Stationary and nonstationary common factor models are a driving force of recent empirical studies in various fields of economics. For example, in macroeconomics, the model represents the traditional idea of summarizing a large set of macroeconomic time-series by a small number of factors related to fundamental concepts such as real and nominal factors. In asset pricing, there is a long history of modeling common factors in investigating cross-sections of asset returns. In this dissertation, I make further contributions to theoretical developments in stationary and nonstationary common factor models under the perspective of developing useful applications in the fields of empirical macroeconomics and asset pricing.

Chapter 1 proposes tests to investigate whether the skewed property of time-series data is attributed to the economy-wide common components and/or the idiosyncratic components. To this end, I apply the formal econometric test based on the coefficient of skewness proposed by Bai and Ng (2005) to the large dimensional common factor model. I propose the Wald-type and max-type test statistics for the space spanned by the common components and a test for idiosyncratic components. The results show that these tests have the standard asymptotic distributions. Monte Carlo simulations confirm that all tests have good size and power in finite samples. Furthermore, I apply the tests to a common factor model using 127 U.S. macroeconomic time-series data from 1960 to 2019. Strong evidence of skewness is found in the common components as well as some idiosyncratic components related to housing, labor market, and uncertainty. Finally, results suggest empirical relevance of incorporating the skewed dynamics in business cycle modeling in a general equilibrium context or in specific factor markets.

Chapter 2 assesses the size and power properties of the right-tailed version of the Panel Analysis of Nonstationarity in Idiosyncratic and Common Components (PANIC) of Bai and Ng (2004) tests when the common and/or the idiosyncratic components are moderately explosive. I find that, when the idiosyncratic component is moderately explosive, the tests for the common components may have considerable size-distortions, and those for the idiosyncratic component may suffer from the nonmonotonic power problem. I provide an analytic explanation under the moderately local to unity framework developed by Phillips and Magdalinos (2007). I then propose a new cross-sectional (CS) approach to disentangle the common and idiosyncratic components in a relatively short explosive window. Monte Carlo simulations show that the CS approach is robust to the nonmonotonic power problem.

Chapter 3 applies the date-stamping methodology for the origination of explosive behaviors proposed in the seminal work of Phillips et al. (2011) to the large dimensional factor model. To this end, I compare two methods of identifying the common and idiosyncratic components: PANIC and CS investigated in the previous chapter. Monte Carlo simulations show that, when the explosive behavior lies only in the common component, the origination date is precisely estimated by either method. However, when the explosive behaviors exist in the idiosyncratic components, the PANIC method loses its power of detection and provides inaccurate origination dates. These problems are resolved through the CS method.

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# List of Abbreviations and Symbols

| | |
|---|---|
| ADF | Augmented Dickey–Fuller |
| AR | Autoregressive |
| CPI | Consumer price index |
| CS | Cross-sectional |
| FEDFUNDS | Federal funds rate |
| GDP | Gross domestic product |
| GNP | Gross national product |
| HAC | Heteroskedasticity and autocorrelation consistent |
| $H_0$ | Null hypothesis |
| $H_1$ | Alternative hypothesis |
| IP | Industrial production |
| LTU | Local to unity |
| MLTU | Moderately local to unity |
| MSE | Mean squared error |
| OECD | Organisation for Economic Co-operation and Development |
| $O(\cdot)$, $o(\cdot)$ | The standard asymptotic orders of sequences |
| $O_p(\cdot)$, $o_p(\cdot)$ | The orders of convergence in probability under $P$ as $N, T \to \infty$ or $N, T, h \to \infty$. |
| PANIC | Panel analysis of nonstationarity in idiosyncratic and common components |
| PPI | Producer price index |
| $\xrightarrow{p}$ | The convergence in probability under the probability measure $P$ |
| $\Rightarrow$ | The convergence in distribution |

| | |
|---|---|
| SIC | Standard industrial classification |
| WTI | West Texas intermediate |
| $\|\|x\|\|$ | The Euclidean norm of vector $x$. For the matrices, the vector-induced norm is used |
| $x \approx y$ | $\|x - y\| = o_p(1)$ for two vectors of random variables $x$ and $y$ |
| $x \otimes y$ | The Kronecker product of two vectors $x$ and $y$ |
| $x^{\otimes l}$ | The $l$th Kronecker power of vector $x$ |

# Chapter 1

# Testing Skewed Dynamics in the Common Factor Model

## 1.1 Introduction

Asymmetric properties in the U.S. macroeconomic time-series have been considered as an important element in understanding business cycles since Mitchell (1927) and Burns and Mitchell (1946). Previously, Neftçi (1984) showed statistical evidence of the asymmetric behavior exhibited by the unemployment rate using a Markov-switching model. Hamilton (1989) reached a similar conclusion for the GNP using Markov-switching model. Morley and Piger (2012) used a nonlinear regime-switching model and provided an empirical support for highly asymmetric business cycles with large negative recessions using the cycle component of the GDP. Compared with such ample evidence of asymmetric business cycles, empirical studies that dealt with skewness have been relatively scarce, even though these two terms are typically used interchangeably. De Long and Summers (1984) is one of the few studies that used a simple coefficient of skewness for the GNP, industrial production, and unemployment rate over the OECD countries. They found no evidence of skewness in all variables, except for the U.S. unemployment rate. They concluded that asymmetry is not a phenomenon of first order importance in understanding business cycles. Later, Bai and Ng (2005) developed a formal econometric test based on the coefficient of skewness to assess whether a univariate time-series exhibits skewed dynamics by accounting for serial dependence. Using data up to 1997, they found that the U.S. unemployment rate did not show skewed dynamics and strengthened the view of De Long and Summers (1984). However, evidence of skewness in exchange rates, CPI inflation, and stock returns is newly obtained.

The discussion has changed focus with the recent short but deep recessions such as the Great Recession of 2007-2008. Jensen et al. (2020) highlighted increasing negative skewness in the U.S. business cycles over the last three decades. The phenomenon is linked to financial frictions emanated from borrowing constraints of households and firms during recessions. In particular, they emphasized the importance of increasing financial leverages due to financial liberalization.[1] Plagborg-Møller et al. (2020) attempted to forecast higher moments, including skewness of the real GDP, using financial variables. They found that financial variables contributed slightly to forecasting the skewness of the GDP growth, that is, growth risk. Additionally, another strand of research investigates micro-level or cross-sectional skewness to uncover mechanism of business cycles. Salgado et al. (2019) studied firm-level panel data of almost fifty countries and found a procyclical skewness in the growth rates of firm sales, productivity, and employment. Ilut et al. (2018) focused on the fact that firms with concave hiring rules respond more to negative shocks than to positive shocks. Busch et al. (2018) used panel data of individual labor income in the United States, Germany, and Sweden and showed that the skewness in income distribution is procyclical. Dew-Becker et al. (2021) reported that the production network model matched the sectoral data in the United States, suggesting the existence of missing common factors in the model. Kent et al. (2019) used data of the GDP, consumption, and investment of 154 countries and quantified the importance of skewness shock. All these works are suggestive to the goal of this chapter.

In this study, I aim to provide methods to investigate the long-standing question whether the U.S. macroeconomic time-series exhibits skewed dynamics, particularly focusing on whether the skewed dynamics is detected in the economy-wide common factors and/or the idiosyncratic components. To this end, I extend the formal econometric test proposed by Bai and Ng (2005) to a large dimensional common factor model. A large body of literature applied the common factor model to macroeconomic panel data set (Bernanke et al., 2005; McCracken & Ng, 2016; Stock & Watson, 2016) with the presumption that the concepts of important macroeconomic variables may not be

[1] For details of the mechanism of financial frictions as a source of skewed business cycles, see, for example, Brunnermeier et al. (2013) and reference therein.

directly observed. Instead, they are latent factors that drive the set of observed data. If I find skewed dynamics in the common components, it should support incorporating fundamental mechanisms such as concave decision rules and borrowing constraints in the business cycle modeling. Furthermore, identifying skewed dynamics in the idiosyncratic components may offer important information, for example, that related to production network model.

Thus, I propose two types of tests: the Wald-type and max-type to assess the skewed property in the common components. The former is expected to have a higher power when all the common factors are uniformly skewed and the latter when only one factor is skewed. I also propose a test to investigate the skewness in the idiosyncratic component. I derive their asymptotic distributions under $N, T \to \infty$ with $\sqrt{T}/N \to 0$, where $N$ and $T$ are the cross-section and time dimensions, respectively. Technically, the latter condition warrants the effect of factor estimation errors to diminish in the limit. Monte Carlo simulation shows that the two tests for the common component and the test for the idiosyncratic components have very good size properties, especially when $N$ and $T$ are large. All tests have a power close to their observed counterparts. The effects of factor estimation errors are minor even if $N$ is relatively smaller than $T$ in finite samples. As an empirical analysis, I apply the tests to 127 U.S. macroeconomic time-series data obtained from 1960 to 2019.

When I investigate the raw observed data, the results match those obtained by De Long and Summers (1984) and Bai and Ng (2005). In addition, I find strong evidence of skewness in data related to housing because the sample includes the period of the Great Recession and is consistent with the literature that emphasize financial frictions. More importantly, I obtained evidence that the space spanned by the common factors are skewed by using either the Wald-type or max-type test. Furthermore, skewed dynamics is found in several variables related to employment, which is consistent with previous studies. Overall, results suggest empirical relevance of incorporating the skewed dynamics in business cycle modeling.

The remainder of this chapter is organized as follows. Section 1.2 explains the model, hypotheses, and test statistics. Section 1.3 provides the asymptotic distributions of the proposed tests and examines the regularity conditions to derive them. Section 1.4 conducts Monte Carlo simulations to assess the finite sample size and power of the proposed tests. Section 1.5 provides an empirical application using the U.S. macroeconomic time-series data, and Section 1.6 concludes. The following notations are used throughout the chapter: the Euclidean norm of vector $x$ is denoted by $||x||$. For the matrices, the vector-induced norm is used. The Kronecker product of two vectors $x$ and $y$ is denoted by the symbol $x \otimes y$. Especially, the $l$th Kronecker power of vector $x$ is denoted by the symbol $x^{\otimes l}$.[2] The symbol $\xrightarrow{p}$ represents convergence in probability under the probability measure $P$ and the symbol $\Rightarrow$ denotes convergence in distribution. Finally, $O_p(\cdot)$ and $o_p(\cdot)$ are the orders of convergence in probability under $P$ as $N, T \to \infty$.

## 1.2 Model and test statistics

I consider the following common factor model.

$$x_{i,t} = \lambda_i' f_t + e_{i,t}, \quad \text{for } i = 1, \dots, N \text{ and } t = 1, \dots, T, \tag{1.2.1}$$

where $x_{i,t}$ is a scalar of the observed random variable; $f_t = (f_{1,t}, \dots, f_{r,t})'$ represents an $r \times 1$ vector of the common factors; $\lambda_i = (\lambda_{1,i}, \dots, \lambda_{r,1})'$ indicates an $r \times 1$ vector of the factor loadings; and $e_{i,t}$ is a scalar idiosyncratic component. Subscripts $i$ and $t$ denote the cross-section and time dimensions, respectively. I am interested in a type of data in which both $N$ and $T$ are large while the number of common factors $r$ is small, so that a few important factors drive a large set of panel data. The model may include cross-sectional specific intercepts; however, it would be trivial to consider them for estimation and inference, so I suppress them to focus on the essence of the problem. I may also use a matrix representation of (1.2.1) by writing $X = F\Lambda' + e$, where $X$ is a $T \times N$ matrix whose $(t, i)$th element is $x_{i,t}$, $F = (f_1, \dots, f_T)'$ is a $T \times r$ matrix of the common factors, $\Lambda = (\lambda_1, \dots, \lambda_N)'$ is an $N \times r$ matrix of the factor loadings, and $e$ is a $T \times N$ matrix whose $(t, i)$th element is $e_{i,t}$. In the absence of ambiguity, notations $e_i = (e_{i,1}, \dots, e_{i,T})'$, a $T \times 1$ vector, and $e_t = (e_{1,t}, \dots, e_{N,t})'$, an $N \times 1$ vector, may also be used.

I estimate $f_t$ and $\lambda_i$ using the principal component method (Bai, 2003; Bai & Ng, 2002). In other word, I obtain $\hat{F} = (\hat{f}_1, \dots, \hat{f}_T)'$, an estimate for $F$, as the $\sqrt{T}$ times eigenvectors corresponding to the $r$ largest eigenvalues of $(NT)^{-1}XX'$ with normalization of $\hat{F}'\hat{F}/T = I_r$, where $I_r$ is an $r \times r$ identity matrix. The factor loadings are estimated via the least squares principle $\hat{\lambda}_i = T^{-1} \sum_{t=1}^{T} \hat{f}_t x_{i,t}$. The idiosyncratic components are also estimated by $\hat{e}_{i,t} = x_{i,t} - \hat{\lambda}_i' \hat{f}_t$. In this study, I assume that $r$ is known for simplicity, although the standard methods of Bai and Ng (2002)

[2] For example, $(x_1, x_2)'^{\otimes 2} = (x_1, x_2)' \otimes (x_1, x_2)' = (x_1^2, x_1x_2, x_2x_1, x_2^2)'$.

and Ahn and Horenstein (2013) are applied in practice.

Previous studies have applied the common factor model to a large macroeconomic panel data set (Bernanke et al., 2005; McCracken & Ng, 2016; Stock & Watson, 2016) with the presumption that concepts of important macroeconomic variables might not directly be observed. Instead, they might be latent factors that drive the set of observed data. In this study, I focus on the issue that skewed dynamics observed in macroeconomic time series data are attributed to the economy-wide common components such as policies or the idiosyncratic components of individual variables. This could contribute to macroeconomic modeling that appropriate frictions should be included as the source of business cycle.

To this end, I employ the standard type of coefficient of skewness for a univariate time series (Bai & Ng, 2005; De Long & Summers, 1984). If I take the $k$th common factor $\{f_{k,t}\}_{t=1}^{T}$, it is defined by the following equation:

$$\tau_k^f = \frac{\mu_{k,3}^f}{\left(\mu_{k,2}^f\right)^{\frac{3}{2}}},$$

where $\mu_{k,l}^f = E[(f_{k,t} - \mu_{k,1}^f)^l]$ for $l = 2, 3$, and $\mu_{k,1}^f = E(f_{k,t})$. I use the letter $\mu$ to denote population centered moments supplemented with the superscript $f$ indicating the common factor, with the first subscript $k$ indicating the $k$th factor ($k = 1, \ldots, r$), and the second subscript $l$ indicating the $l$th moment ($l = 1, 2, \cdots$). Additionally, an $r \times 1$ vector of the population coefficients of skewness is expressed by $\tau^f = (\tau_1^f, \ldots, \tau_r^f)'$. I employ the sample counterparts of the population coefficients of skewness $\hat{\tau}^f = (\hat{\tau}_1^f, \ldots, \hat{\tau}_r^f)'$, where

$$\hat{\tau}_k^f = \frac{\hat{\mu}_{k,3}^f}{\left(\hat{\mu}_{k,2}^f\right)^{\frac{3}{2}}},$$

with $\hat{\mu}_{k,l}^f = T^{-1} \sum_{t=1}^{T} (\hat{f}_{k,t} - \hat{\mu}_{k,1}^f)^l$ for $l = 2, 3$, and $\hat{\mu}_{k,1}^f = T^{-1} \sum_{t=1}^{T} \hat{f}_{k,t}$.

Moreover, I use the null hypothesis ($H_0$) and alternative hypothesis ($H_1$)

$$H_0 : \tau^f = 0_{r\times 1}, \tag{2-N}$$

$$H_1 : \tau^f \neq 0_{r\times 1}. \tag{2-A}$$

I aim to test whether the entire space spanned by the $r$ common factors is skewed; hence, I collectively deal with the coefficients of skewness for the $r$ common factors. I do not aim to examine a specific individual factor because it cannot be identified without appropriate restrictions. Furthermore, I focus on whether the $\tau^f$ is zero, that is, whether the dynamics in the common component is not skewed positively or negatively. This is because I cannot identify the specific $\tau^f$ without the true data generating process; similarly, I cannot verify specific individual factors. To that effect, I propose two types of test statistics based on $\hat{\tau}^f$.

**Wald-type test:**

$$W_{NT} = T\left[\hat{\tau}^{f\prime}\hat{V}^{-1}\hat{\tau}^f\right],$$

where

$$\hat{V} = \hat{M}_2^{-3}\hat{\Gamma}^f,$$

with $\hat{M}_2 = diag(\hat{\mu}^f_{1,2}, \ldots, \hat{\mu}^f_{r,2})$ and $\hat{\Gamma}^f$ is the consistent estimate for the covariance matrix of $(\hat{a}_1'\hat{z}_{1,t}, \ldots, \hat{a}_r'\hat{z}_{r,t})'$ and

$$\begin{aligned} \hat{a}_k &= \left(1, -3\hat{\mu}^f_{k,2}\right)', \\ \hat{z}_{k,t} &= \left(\left[\left(\hat{f}_{k,t} - \hat{\mu}^f_{k,1}\right)^3 - \hat{\mu}^f_{k,3}\right], \left(\hat{f}_t - \hat{\mu}^f_{k,1}\right)\right)'. \end{aligned}$$

**Max-type test:**

$$M_{NT} = \max_{1\le k\le r} abs\left[\sqrt{T}\left(\hat{\mu}^f_{k,2}\right)^{\frac{3}{2}}\left(\hat{\Gamma}^f\right)^{-\frac{1}{2}}\hat{\tau}^f\right].$$

First, the use of the max-type test is intuitively justified because the estimated factors by the principal components are orthogonal and thus produce sample coefficients of skewness that are asymptotically independent. Second, it is expected that the Wald-type test is more powerful when all $r$ factors are uniformly skewed; thus, the max-type test is used when there is heterogeneity in the coefficient of skewness among the common factors. Third, I can construct $\hat{\Gamma}^f$ by the standard heteroskedasticity and autocorrelation consistent (HAC) estimator, in line with Newey and West (1987) or Andrews (1991) if some serial dependence is potentially present in $z_t$, where

$$z_t = \left(\left[\left(f_t - \mu^f_1\right)^{\otimes 3} - \mu^f_3\right]', \left(f_t - \mu^f_1\right)'\right)',$$

with $\mu^f_3 = E[(f_t - \mu^f_1)^{\otimes 3}]$ and $\mu^f_1 = E(f_t)$, which stacks the elements of the first and third centered cross moments.[3]

Investigating the $i$th idiosyncratic components, the null and the alternative hypotheses are as follows:

$$H_0 : \tau_i^e = 0, \tag{3-N}$$

$$H_1 : \tau_i^e \neq 0, \tag{3-A}$$

where $\tau_i^e$ is the population coefficient of skewness of $\{e_{i,t}\}_{t=1}^T$ defined by

$$\tau_i^e = \frac{\mu_{i,3}^e}{\left(\mu_{i,2}^e\right)^{\frac{3}{2}}},$$

where $\mu_{i,l}^e = E[(e_{i,t} - \mu_{i,1}^e)^l]$ for $l = 2, 3$ and $\mu_{i,1}^e = E(e_{i,t})$. These population centered moments of $e_{i,t}$ also employ $\mu$ with the same notations as those of the common factors but by replacing $f$ by $e$. Additionally, $i$ is used as the first lower subscript to denote the cross-sectional unit.

The test statistic is constructed based on the sample coefficient of skewness computed by the estimated $i$th idiosyncratic component, as given below:

$$\hat{\tau}_i^e = \frac{\hat{\mu}_{i,3}^e}{\left(\hat{\mu}_{i,2}^e\right)^{\frac{3}{2}}},$$

where $\hat{\mu}_{i,l}^e = T^{-1}\sum_{t=1}^T(\hat{e}_{i,t} - \hat{\mu}_{i,1}^e)^l$ for $l = 2, 3$ and $\hat{\mu}_{i,1}^e = T^{-1}\sum_{t=1}^T \hat{e}_{i,t}$. Since the idiosyncratic component is one-dimensional, I propose the test statistic as follows:

$$S_{i,NT} = \sqrt{T}\left(\hat{\mu}_{i,2}^e\right)^{\frac{3}{2}}\left(\hat{\Gamma}_i^e\right)^{-\frac{1}{2}}\hat{\tau}_i^e,$$

where $\hat{\Gamma}_i^e$ is the consistent estimate for the variance of $\hat{b}_i'\hat{v}_{i,t}$ and

$$\hat{b}_i = \left(1, -3\hat{\mu}_{i,2}^e\right)',$$
$$\hat{v}_{i,t} = \left(\left[\left(\hat{e}_{i,t} - \hat{\mu}_{i,1}^e\right)^3 - \hat{\mu}_{i,3}^e\right], \left(\hat{e}_{i,t} - \hat{\mu}_{i,1}^e\right)\right)'.$$

[3] Lemmas A.2.3 and A.4 in Appendix A.1 show that the estimation error of the sample mean in the sample third moment remains asymptotically. This error corresponds to the second element of $z_t$.

If dependence is considered in $v_{i,t}$, where

$$v_{i,t} = \left(\left[\left(e_{i,t} - \mu^e_{i,1}\right)^3 - \mu^e_{i,3}\right], \left(e_{i,t} - \mu^e_{i,1}\right)\right)',$$

I replace $\hat{\Gamma}^e_i$ with the HAC estimator. In the next section, I derive the asymptotic distributions of the suggested tests.

## 1.3 Theoretical results

### 1.3.1 Assumptions

In this section, I derive the asymptotic distributions under the null hypothesis. I follow the regularity assumptions of Bai (2003) for the common factor models and those of Bai and Ng (2005) to guarantee the property of the skewness tests. Let $m$ be a generic constant.

**Assumption 1.1.**

1. *$E(||f_t||^{6+\delta}) < \infty$ with some $\delta > 0$ and $T^{-1}\sum_{t=1}^{T} f_t f_t' \to \Sigma_F$ for some $r \times r$ positive definite matrix.*
2. *$\{f_t\}_{t=1}^{T}$ is stationary up to the 6th order.*
3. *$T^{-1/2}\sum_{t=1}^{T} z_t \xrightarrow{d} N(0, \Omega)$, where $\Omega = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T} E(z_t z_s')$.*

**Assumption 1.2.** *$||\lambda_i|| \leq m$ and $N^{-1}\Lambda'\Lambda \to \Sigma_\Lambda$ as $N \to \infty$ for some $r \times r$ positive definite matrix.*

Assumptions 1.1.1 and 1.2 are standard in the literature of common factor model. Assumption 1.1.1 excludes factors with a time trend. Assumption 1.2 ensures that every common factor has a certain contribution to the infinitely observed variables. Assumption 1.1.2 restricts the moments to be time-invariant for the common factors up to the 6th order and, Assumption 1.1.3 warrants the central limit theorem for up to the third moment. These follow the skewness test examined by Bai and Ng (2005). For the idiosyncratic components, I require the following regularity conditions.

**Assumption 1.3.**

1. *$E(e_{i,t}) = 0$ for all $i$ and $t$ and $E(|e_{i,t}|^{8+\delta}) < \infty$ with some $\delta > 0$.*

2. $\{e_{i,t}\}_{t=1}^{T}$ *for* $t = 1, \ldots, T$ *is stationary up to 6th order for all* $i$.

3. $T^{-1/2}\sum_{t=1}^{T} v_{i,t} \xrightarrow{d} N(0, \Omega_i)$, *where* $\Omega_i = \lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T} E(v_{i,t}v_{i,s}')$.

4. *Let* $\gamma_{st} = E(N^{-1}\sum_{i=1}^{N} e_{i,s}e_{i,t})$.

   *Then,* $|\gamma_{ss}| \leq m$ *and* $T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T} |\gamma_{st}| \leq m$ *for all* $s$ *and* $t$.

5. *Let* $\phi_{ij,t} = E(e_{i,t}e_{j,t})$. *Then,* $|\phi_{ij,t}| \leq |\phi_{ij}|$ *for all* $t$ *with some* $\phi_{ij}$ *such that* $N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} |\phi_{ij}| \leq m$.

6. *Let* $\phi_{ij,ts} = E(e_{i,t}e_{j,s})$. *Then,* $(NT)^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} |\phi_{ij,ts}| \leq m$.

7. *For every* $(t, s)$, $E|N^{-1/2}\sum_{i=1}^{N}[e_{i,t}e_{i,s} - E(e_{i,t}e_{i,s})]|^4 \leq m$.

**Assumption 1.4.** $f_t$ *and* $e_{i,s}$ *are mutually independent for all leads and lags.*

Assumption 1.3.2 requires time-invariant moments of the idiosyncratic components similar to Assumption 1.1.2. Assumptions 1.3.4–1.3.7 allow weak cross-sectional and time dependence in the idiosyncratic components, following the standard setup of the common factor model such as Bai (2003). Assumption 1.4 ensures independence between the two components.

### 1.3.2 Asymptotic distributions

I derive the asymptotic distributions of the proposed test statistics under the null hypotheses. I obtain the following results for the common components.

**Theorem 1.1.** *Suppose Assumptions 1.1–1.4 hold and* $\{z_t\}_{t=1}^{T}$ *has no serial dependence. Under the null hypothesis (2-N), the following hold as* $N, T \to \infty$ *with* $\sqrt{T}/N \to 0$,

1. *Wald-type test:* $W_{NT} \Rightarrow \chi_r^2$.

2. *Max-type test:* $M_{NT} \Rightarrow \Theta$ *where* $\Theta \sim \max(abs(N(0_{r\times 1}, I_r)))$.

These results imply a direct use of Bai and Ng's (2005) idea in the common factor model with an added condition $\sqrt{T}/N \to 0$. This condition is similar to the asymptotic normality of the factor loading estimate in Bai (2003) and ensures that the factor estimation errors diminish asymptotically and the estimated common components are regarded as factual. Although I do not assume serial dependence in $z_t$ to obtain the theoretical results, it is allowed in practice using the standard HAC estimator proposed by Newey and West (1987) or Andrews (1991). I study the finite sample properties of the test in the next section via Monte Carlo simulation. Furthermore, I obtain the following theorem for the idiosyncratic components.

**Theorem 1.2.** *Suppose Assumptions 1.1–1.4 hold and $\{v_{i,t}\}_{t=1}^{T}$ has no serial dependence. Under the null hypothesis (3-N), $S_{i,NT} \Rightarrow N(0,1)$ for any $i$ as $N, T \to \infty$ with $\sqrt{T}/N \to 0$.*

Here, the same remarks apply as Theorem 1. Importantly, I investigate the power of these tests under various specifications in the next section.

## 1.4 Monte Carlo simulations

In this section, I investigate finite sample properties of the proposed tests via Monte Carlo simulation. In particular, I am interested in two issues. First, I examine whether the properties are affected by the behaviors of the other components. For example, I am interested in whether the tests for the common components present a good size when the idiosyncratic components are skewed, and whether the test for the idiosyncratic components yields a good size when the common components are skewed. Such an interaction between the common and idiosyncratic components is concerned if the effects of factor estimation errors are relevant in finite samples. Notably, these effects disappear under $\sqrt{T}/N \to 0$ as $N, T \to \infty$ in the limit. Second, I investigate if the tests have a good power against various types of skewed distributions. To this end, I follow the setup of Bai and Ng (2005) and consider four different distributions: the log normal distribution, chi-squared distribution, exponential distribution, and generalized lambda distribution.[4] I also consider four variations of the generalized lambda distributions with respect to the parameter values. These are summarized in Table 1.1 as D1–D7. When I consider a symmetric distribution, the standard normal distribution is used and is labeled as D0.

[4]The generalized lambda distribution can take on various shapes depending on the parameters. The inverse of the cumulative distribution function is known as $F^{-1}(u) = \lambda_1 + [u^{\lambda_3} - (1-u)^{\lambda_4}]/\lambda_2, 0 < u < 1$.

Table 1.1: Probability distributions and coefficient of skewness.

| | Probability Distributions | $\tau$ |
|---|---|---|
| D0 | standard normal distribution | 0 |
| D1 | log normal distribution | 6.18 |
| D2 | chi-squared distribution with 2 degree of freedom | 2 |
| D3 | exponential distribution with parameter 1 | 2 |
| D4 | generalized lambda distribution with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, -1, -0.0075, 0.03)$ | 1.5 |
| D5 | generalized lambda distribution with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, -1, -0.1, -0.18)$ | 2 |
| D6 | generalized lambda distribution with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, -1, -0.001, 0.13)$ | 3.16 |
| D7 | generalized lambda distribution with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, -1, -0.0001, 0.17)$ | 3.8 |

I generate data from the following model:

$$x_{i,t} = \lambda_i' f_t + e_{i,t},$$

$$f_t = 0.5 f_{t-1} + u_t,$$

$$e_{i,t} = 0.5 e_{i,t-1} + \varepsilon_{i,t},$$

for $i = 1, \ldots, N$ and $t = 1, \ldots, T$. I draw $\lambda_i$ from the standard normal distribution and $u_t$ and $\varepsilon_{i,t}$ for $t = 1, \ldots, T$ as well as $f_0$ and $e_{i,0}$ independently from the distributions presented in Table 1.1. I use a simple autoregressive (AR) model of order one with the AR coefficients 0.5 for $f_t$ and $e_{i,t}$ for all $i$, although using other values for the AR coefficients does not change results qualitatively. To assess the effects of sample size, I consider that $N$ and $T$ are either 100 or 300, and the number of factors $r$ is 2. Throughout this section, I use the nominal level 0.05, and the results are assessed based on 2,000 Monte Carlo replications. All tests are constructed using the HAC variance proposed by Newey and West (1987).

I first investigate the size of the Wald-type and max-type tests for the common components. Table 1.2 reports the size of the Wald-type and max-type tests. In both experiments, I independently draw the two components of $u_t$ from the standard normal distributions. The column D0 shows the size when the idiosyncratic components $\varepsilon_{i,t}$ are also drawn from the standard normal distribution. The columns D1–D7 present the size of the tests when the idiosyncratic components $\varepsilon_{i,t}$ are drawn from the skewed distributions D1–D7. In the column labeled as "observed," the size is reported for the true common factors ($f_t$) instead of the estimated factors ($\hat{f}_t$); hence, this corresponds to the

Table 1.2: Size of the Wald-type and max-type tests.

(%)

| $T$ | $N$ | | D0 | D1 | D2 | D3 | D4 | D5 | D6 | D7 | observed |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 100 | Wald | 3.05 | 2.45 | 2.50 | 2.95 | 2.80 | 2.10 | 1.90 | 2.80 | 3.00 |
| | | Max | 2.90 | 2.55 | 2.60 | 2.25 | 2.85 | 2.40 | 2.00 | 3.15 | 3.05 |
| 100 | 300 | Wald | 2.80 | 3.30 | 3.00 | 3.10 | 2.20 | 3.20 | 3.10 | 3.20 | 3.00 |
| | | Max | 2.75 | 3.15 | 2.60 | 2.55 | 2.50 | 2.30 | 2.70 | 3.05 | 3.05 |
| 300 | 100 | Wald | 4.25 | 4.40 | 4.15 | 4.25 | 4.00 | 4.05 | 4.30 | 4.05 | 5.10 |
| | | Max | 4.05 | 4.15 | 4.10 | 3.70 | 3.85 | 3.70 | 3.75 | 3.50 | 5.15 |
| 300 | 300 | Wald | 4.15 | 4.10 | 5.35 | 4.60 | 4.70 | 4.00 | 4.05 | 4.40 | 5.10 |
| | | Max | 3.95 | 4.25 | 5.15 | 4.15 | 4.35 | 4.25 | 4.25 | 4.25 | 5.15 |

test that is not affected by the factor estimation errors.[5] The size of the Wald-type test is relatively close to 5% when $T = 300$. When $T = 100$, the size is somewhat conservative, but this occurs even in the "observed" column; hence, these size-distortions are not caused by the factor estimation errors. The larger sample size, the less distinct the difference from "observed" column. I suspect that the factor estimation errors would have an effect, but with a minor magnitude. The max-type test is similar to that of the Wald-type test. This table confirms the relevance of Theorem 1.1.

I then investigate the size of the test for idiosyncratic components. Table 1.3 reports the size of the $S_{i,NT}$ test for $i = 1$ when all the idiosyncratic components $\varepsilon_{i,t}$ are drawn from the standard normal distribution. The two common components of $u_t$ are drawn from the distributions D0–D7, so that the common components may or may not be skewed. The results suggest that the size is close to 5% in all cases, validating the results of Theorem 1.2 in finite samples. Importantly, the effects of factor estimation errors also remain immaterialized in this case. In addition, I conducted an experiment in which the idiosyncratic components for $i = 2, \dots, N$ are drawn from the same skewed distributions as $u_t$. The results are similar to those of Table 1.3; thus, they are suppressed.

I now focus on the power of the proposed tests. To consider the tests for the common components, I draw the two components of $u_t$ from the skewed distributions D1–D7. The idiosyncratic components $\varepsilon_{i,t}$ are drawn from the standard normal distribution; however, using the skewed distributions has not changed the qualitative results. Table 1.4 reports the power of the Wald-type test, and Table 1.5 presents the power of the max-type test. The power using the true series is

[5]The results for the true series directly reflect the characteristics of Bai and Ng's (2005) test, since there are no issues regarding identification and estimation of the common factors, Thus, they provide a benchmark result.

Table 1.3: Size of the idiosyncratic test.

(%)

| $T$ | $N$ | D0 | D1 | D2 | D3 | D4 | D5 | D6 | D7 | observed |
|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 100 | 3.55 | 3.55 | 3.50 | 3.10 | 3.15 | 3.55 | 4.25 | 3.95 | 4.40 |
| 100 | 300 | 5.35 | 4.85 | 4.70 | 4.15 | 4.00 | 4.75 | 5.35 | 5.10 | 4.40 |
| 300 | 100 | 3.65 | 4.00 | 3.70 | 3.55 | 4.75 | 4.45 | 4.00 | 3.50 | 4.90 |
| 300 | 300 | 4.20 | 4.50 | 5.50 | 5.40 | 4.50 | 5.00 | 5.35 | 4.10 | 4.90 |

Table 1.4: Power of the Wald-type test.

(%)

| $T$ | $N$ | D1 | D2 | D3 | D4 | D5 | D6 | D7 |
|---|---|---|---|---|---|---|---|---|
| 100 | 100 | 58.25 | 71.40 | 55.30 | 75.70 | 61.15 | 67.75 | 64.15 |
| | | (62.20) | (89.60) | (80.10) | (84.90) | (66.15) | (78.35) | (73.30) |
| 100 | 300 | 57.90 | 69.95 | 53.25 | 75.10 | 65.90 | 63.55 | 65.10 |
| | | (62.75) | (88.25) | (78.60) | (85.35) | (69.50) | (76.75) | (73.70) |
| 300 | 100 | 81.05 | 97.40 | 95.30 | 97.45 | 85.60 | 92.80 | 90.90 |
| | | (83.00) | (99.70) | (99.30) | (98.80) | (88.45) | (94.85) | (93.05) |
| 300 | 300 | 80.75 | 97.50 | 95.35 | 97.20 | 85.65 | 93.25 | 89.85 |
| | | (84.10) | (99.60) | (99.50) | (98.90) | (88.05) | (95.75) | (93.55) |

Note: The values in parentheses show the powers when the true common factors are used.

reported in parentheses. I observe that; although the power becomes slightly lower than that using the true series, both tests have a good power in all cases. Furthermore, the Wald-type test has a higher power than the max-type test because the two factors are uniformly skewed in this experiment. Additionally, I set an experiment in which the first component in $u_t$ is generated from the skewed distributions D1–D7 but the second component in $u_t$ is drawn from the standard normal distribution. The other specifications are similar to the previous experiment. Table 1.6 shows the power of the Wald-type test in the upper row and that of the max-type test in the lower row. I observe that the max-type test has a higher power than the Wald-type test, especially when $T$ is large. This is expected because only one factor is skewed in this experiment.

Finally, I investigate the power of the idiosyncratic test. The idiosyncratic components $\varepsilon_{i,t}$ are drawn from the skewed distributions D1–D7. The common components $u_t$ are drawn from the standard normal distributions, although the results do not change even if $u_t$ are drawn from the skewed distributions. Table 1.7 reports the power of the idiosyncratic test for $i = 1$ and suggests that the idiosyncratic tests have a decent power in all cases. Indeed, the power is close to that of

Table 1.5: Power of the max-type test.

(%)

| $T$ | $N$ | D1 | D2 | D3 | D4 | D5 | D6 | D7 |
|---|---|---|---|---|---|---|---|---|
| 100 | 100 | 40.25 | 58.30 | 41.60 | 61.70 | 45.80 | 50.85 | 50.25 |
| | | (48.10) | (79.80) | (62.70) | (73.70) | (54.40) | (64.30) | (62.00) |
| 100 | 300 | 43.20 | 56.65 | 41.75 | 61.50 | 49.05 | 51.55 | 50.85 |
| | | (49.00) | (78.70) | (63.50) | (75.35) | (55.60) | (64.65) | (61.40) |
| 300 | 100 | 64.70 | 94.55 | 91.60 | 93.00 | 71.80 | 82.65 | 79.45 |
| | | (71.20) | (98.50) | (97.70) | (95.80) | (77.55) | (90.10) | (85.20) |
| 300 | 300 | 63.95 | 94.65 | 92.60 | 92.35 | 71.85 | 83.10 | 77.60 |
| | | (72.30) | (98.40) | (98.45) | (96.15) | (77.65) | (90.85) | (86.00) |

Note: The values in parentheses show the powers when the true common factors are used.

Table 1.6: Power of the Wald-type and max-type tests : When only one factor is skewed.

(%)

| $T$ | $N$ | | D1 | D2 | D3 | D4 | D5 | D6 | D7 |
|---|---|---|---|---|---|---|---|---|---|
| 100 | 100 | Wald | 25.40 | 34.30 | 24.00 | 35.75 | 26.25 | 29.05 | 27.75 |
| | | Max | 25.10 | 34.20 | 23.35 | 34.90 | 26.25 | 29.45 | 27.45 |
| 100 | 300 | Wald | 23.00 | 35.00 | 23.75 | 37.80 | 26.70 | 31.05 | 29.35 |
| | | Max | 21.80 | 35.65 | 23.50 | 37.40 | 25.25 | 31.30 | 29.40 |
| 300 | 100 | Wald | 46.20 | 81.95 | 72.55 | 76.90 | 54.60 | 64.35 | 58.70 |
| | | Max | 48.25 | 81.50 | 72.50 | 78.30 | 56.15 | 66.05 | 60.30 |
| 300 | 300 | Wald | 46.55 | 82.10 | 71.35 | 77.25 | 52.80 | 64.15 | 56.60 |
| | | Max | 47.00 | 81.45 | 72.70 | 78.50 | 55.00 | 66.75 | 60.35 |

Table 1.7: Power of the idiosyncratic test.

(%)

| $T$ | $N$ | D1 | D2 | D3 | D4 | D5 | D6 | D7 |
|---|---|---|---|---|---|---|---|---|
| 100 | 100 | 42.55 | 68.00 | 54.25 | 64.85 | 46.75 | 56.50 | 52.10 |
| | | (43.60) | (70.80) | (59.35) | (66.65) | (47.15) | (58.00) | (53.30) |
| 100 | 300 | 59.75 | 93.25 | 92.25 | 88.50 | 66.75 | 80.25 | 73.85 |
| | | (60.10) | (93.55) | (92.60) | (88.40) | (67.00) | (80.55) | (74.25) |
| 300 | 100 | 45.40 | 68.20 | 56.95 | 66.05 | 47.25 | 56.00 | 51.60 |
| | | (45.60) | (70.15) | (60.20) | (66.85) | (48.30) | (56.95) | (52.85) |
| 300 | 300 | 60.15 | 92.80 | 92.80 | 89.25 | 68.30 | 78.90 | 72.95 |
| | | (60.00) | (92.95) | (93.05) | (89.20) | (68.30) | (78.75) | (72.95) |

Note: The value in parentheses shows the power when the true common factor is used.

the test using the true series reported in the lower row.

## 1.5 Empirical application

In this section, I apply the proposed tests for the common and idiosyncratic components to U.S. macroeconomic time-series data of FRED-MD provided by McCracken and Ng (2016). I also present results of the original test proposed by Bai and Ng (2005) when applied to the updated observed data. The data set contains 127 series from April, 1960, and I use the sample period up to September, 2019. I follow Stock and Watson (2002b) to clean outliers. The 127 series are categorized into eight groups: "output and income," "labor market," "housing," "consumption, orders and inventories," "money and credit," "interest and exchange rates," "prices," and "stock market." All variables are transformed to achieve stationarity, as suggested by McCracken and Ng (2016). I determine the number of factors using the $IC_{p2}$ of Bai and Ng (2002) and obtain $r = 7$. Through a casual investigation, I find that the first factor has relatively large factor loadings for the output and labor market related series. The second to the fourth factors are more related to price, housing and financial variables. The fifth to the seventh factors are related to unemployment, hours worked, and asset pricing variables such as stock prices and term spreads of interest rates. To implement the proposed tests, I use the HAC variance estimate with the bandwidth selected by Newey and West (1987) to account for potential serial correlations in the first three moments.

Table 1.8 reports the results of the tests. Table 1.8(a) contains the two tests for the space spanned by the common factors. Table 1.8(b) provides the results of the idiosyncratic components as well as those of the original Bai and Ng's (2005) tests applied to the observed data. I start with the tests using the observed data. Consistent with De Long and Summers (1984) and Bai and Ng (2005), relatively scarce evidence on the skewness is obtained. However, I clearly observed the skewed property in some series of "labor market." These accord with De Long and Summers's (1984) finding using the U.S. unemployment rate. I also observe that many series of "stock market" and "interest and exchange rates" exhibit the skewed property, which is consistent with the finding of Bai and Ng (2005). In addition, I find strong evidence of skewness in many series of "housing," which remained unexplored in previous studies.

This study analyzed whether the skewed dynamics are attributed to the economy-wide common factors. Table 1.8(a), I observe that both the Wald-type and the max-type tests give significance at the 5% level. My casual observation into the test statistics for each common factor shows that every test statistic for common factor, except for the second to the fourth factors, takes a relatively large value.[6] These justify the mechanism of skewed dynamics of the nation-wide business cycles in terms of a general equilibrium.

Some interesting findings in the tests for idiosyncratic components are as follows. First, many series of "housing" such as "Housing Starts (Total New Privately Owned)" and "New Private Housing Permit" and some series of "employment" such as "All Employees: Financial Activities" and "Avg Weekly Hours: Goods-Producing" have skewed properties even after the common components are controlled. Second, "Japan/U.S. Foreign Exchange Rate" and "Crude Oil Price, spliced WTI and Cushing" reveal evidence of idiosyncratic skewness. These are likely produced by some international factors, which may not be captured by the common components estimated by this data set. Third, skewness in the idiosyncratic component is also found in the "S&P's Volatility Index," which is related to uncertainty or risk perception in financial markets.

Overall, results suggest empirical relevance of incorporating skewed dynamics in business cycle modeling either in a general equilibrium context or in specific factor markets. These include recent studies that emphasize the role of financial frictions (Jensen et al., 2020), concave decision rules in labor markets (Ilut et al., 2018), and uncertainty shock (Plagborg-Møller et al., 2020; Salgado et al., 2019), among others.

[6]The test statistics for individual factor estimates are -2.256 (the first), -0.768 (the second), -1.282 (the third), -1.517 (the fourth), -2.075 (the fifth), 2.770 (the sixth), and -1.825 (the seventh).

Table 1.8: Tests for skewed dynamics for common and idiosyncratic components of the U.S. macroeconomic time-series.

(a) Tests for common component

| Wald-type test | | Max-type test | |
|---|---|---|---|
| 15.006 | ** | 2.770 | ** |

Note: "***", "**" and "*" denote the significance levels at 1%, 5% and 10%, respectively.

(b) Tests for idiosyncratic components and tests for observed variables

| tcode | Description | Observed | | Idiosyncratic | |
|---|---|---|---|---|---|
| | Group 1. Output and Income | | | | |
| 5 | Real Personal Income | 1.400 | - | 1.387 | - |
| 5 | Real Personal Income Ex Transfer Receipts | 0.251 | - | 0.092 | - |
| 5 | IP Index | -1.703 | * | -1.519 | - |
| 5 | IP: Final Products and Nonindustrial Supplies | -1.501 | - | 0.510 | - |
| 5 | IP: Final Products (Market Group) | -1.239 | - | 1.118 | - |
| 5 | IP: Consumer Goods | -0.058 | - | 0.465 | - |
| 5 | IP: Durable Consumer Goods | -0.049 | - | 0.839 | - |
| 5 | IP: Nondurable Consumer Goods | -0.973 | - | -0.428 | - |
| 5 | IP: Business Equipment | -1.496 | - | -0.053 | - |
| 5 | IP: Materials | -1.813 | * | -1.629 | - |
| 5 | IP: Durable Materials | -2.336 | ** | -1.666 | * |
| 5 | IP: Nondurable Materials | -1.626 | - | -1.222 | - |
| 5 | IP: Manufacturing (SIC) | -1.715 | * | -1.684 | * |
| 5 | IP: Residential Utilities | -1.476 | - | -0.903 | - |
| 5 | IP: Fuels | 1.319 | - | 1.366 | - |
| 2 | Capacity Utilization: Manufacturing | -1.779 | * | -2.340 | ** |
| | Group 2. Labor Market | | | | |
| 2 | Help-Wanted Index for United States | 0.493 | - | 1.034 | - |
| 2 | Ratio of Help Wanted/No. Unemployed | -1.558 | - | -0.402 | - |
| 5 | Civilian Labor Force | 0.956 | - | 0.770 | - |
| 5 | Civilian Employment | -0.096 | - | 1.056 | - |
| 2 | Civilian Unemployment Rate | 1.885 | * | -0.589 | - |
| 2 | Average Duration of Unemployment (Weeks) | -0.256 | - | -0.760 | - |
| 5 | Civilians Unemployed - Less Than 5 Weeks | 0.499 | - | 0.705 | - |
| 5 | Civilians Unemployed for 5-14 Weeks | 2.028 | ** | 1.710 | * |
| 5 | Civilians Unemployed - 15 Weeks & Over | 2.061 | ** | -0.083 | - |
| 5 | Civilians Unemployed for 15-26 Weeks | 0.265 | - | 0.128 | - |
| 5 | Civilians Unemployed for 27 Weeks and Over | 2.158 | ** | 0.681 | - |
| 5 | Initial Claims | 1.408 | - | 0.790 | - |
| 5 | All Employees: Total Nonfarm | -1.519 | - | 0.114 | - |
| 5 | All Employees: Goods-Producing Industries | -2.368 | ** | 0.111 | - |
| 5 | All Employees: Mining and Logging: Mining | -0.734 | - | -0.300 | - |
| 5 | All Employees: Construction | -0.249 | - | -0.087 | - |
| 5 | All Employees: Manufacturing | -2.412 | ** | -0.803 | - |
| 5 | All Employees: Durable Goods | -2.127 | ** | -1.068 | - |
| 5 | All Employees: Nondurable Goods | -1.573 | - | 0.943 | - |
| 5 | All Employees: Service-Providing Industries | -0.211 | - | 0.492 | - |
| 5 | All Employees: Trade, Transportation & Utilities | -1.243 | - | 0.457 | - |
| 5 | All Employees: Wholesale Trade | -1.645 | * | 0.637 | - |
| 5 | All Employees: Retail Trade | -0.805 | - | -0.417 | - |
| 5 | All Employees: Financial Activities | -1.344 | - | -2.427 | ** |
| 5 | All Employees: Government | 1.616 | - | 1.645 | * |
| 1 | Avg Weekly Hours: Goods-Producing | -1.291 | - | -2.554 | ** |
| 2 | Avg Weekly Overtime Hours: Manufacturing | -0.524 | - | -0.111 | - |
| 1 | Avg Weekly Hours: Manufacturing | -1.763 | * | -2.394 | ** |
| 6 | Avg Hourly Earnings: Goods-Producing | -0.279 | - | -0.268 | - |
| 6 | Avg Hourly Earnings: Construction | -0.723 | - | -0.571 | - |
| 6 | Avg Hourly Earnings: Manufacturing | 0.999 | - | 0.209 | - |
| | Group 3. Housing | | | | |
| 4 | Housing Starts: Total New Privately Owned | -2.994 | *** | -2.756 | *** |
| 4 | Housing Starts, Northeast | -2.294 | ** | 2.427 | ** |
| 4 | Housing Starts, Midwest | -4.246 | *** | -2.802 | *** |
| 4 | Housing Starts, South | -2.207 | ** | -0.325 | - |
| 4 | Housing Starts, West | -3.356 | *** | -3.115 | *** |
| 4 | New Private Housing Permits | -2.740 | *** | -3.014 | *** |
| 4 | New Private Housing Permits, Northeast | -1.800 | * | 3.369 | *** |
| 4 | New Private Housing Permits, Midwest | -3.685 | *** | -3.047 | *** |
| 4 | New Private Housing Permits, South | -1.754 | * | -3.834 | *** |
| 4 | New Private Housing Permits, West | -3.258 | *** | -2.187 | ** |
| | Group 4. Consumption, Orders and Inventories | | | | |
| 5 | Real Personal Consumption Expenditures | -1.014 | - | -1.188 | - |
| 5 | Real Manu. and Trade Industries Sales | -0.740 | - | -1.411 | - |
| 5 | Retail and Food Services Sales | -0.936 | - | -0.880 | - |
| 5 | New Orders for Consumer Goods | -1.072 | - | -1.290 | - |
| 5 | New Orders for Durable Goods | -1.021 | - | 0.108 | - |
| 5 | New Orders for Nondefense Capital Goods | -0.315 | - | 0.062 | - |
| 5 | Unfilled Orders for Durable Goods | 2.284 | ** | 1.932 | * |
| 5 | Total Business Inventories | -0.754 | - | 0.978 | - |
| 2 | Total Business: Inventories to Sales Ratio | 1.171 | - | 1.070 | - |
| 2 | Consumer Sentiment Index | -0.044 | - | 0.619 | - |
| | Group 5. Money and Credit | | | | |
| 6 | M1 Money Stock | -0.682 | - | -0.781 | - |
| 6 | M2 Money Stock | -1.657 | * | -1.703 | * |
| 5 | Real M2 Money Stock | 1.273 | - | 1.545 | - |
| 6 | Monetary Base | 0.905 | - | 0.769 | - |
| 6 | Total Reserves of Depository Institutions | 0.254 | - | -0.205 | - |
| 7 | Reserves of Depository Institutions | -0.018 | - | -0.721 | - |
| 6 | Commercial and Industrial Loans | -1.412 | - | -1.370 | - |
| 6 | Real Estate Loans at All Commercial Banks | 0.834 | - | 0.505 | - |
| 6 | Total Nonrevolving Credit | 0.426 | - | 0.328 | - |
| 2 | Nonrevolving Consumer Credit to Personal Income | 0.574 | - | 0.552 | - |
| 6 | Consumer Motor Vehicle Loans Outstanding | 0.829 | - | 0.813 | - |
| 6 | Total Consumer Loans and Leases Outstanding | -0.214 | - | -0.258 | - |
| 6 | Securities in Bank Credit at All Commercial Banks | -0.940 | - | -0.991 | - |
| | Group 6. Interest and Exchange Rates | | | | |
| 2 | Effective Federal Funds Rate | -0.730 | - | -0.540 | - |
| 2 | 3-Month AA Financial Commercial Paper Rate | -1.834 | * | -1.304 | - |
| 2 | 3-Month Treasury Bill | -1.049 | - | 1.685 | * |
| 2 | 6-Month Treasury Bill | -0.765 | - | 1.700 | * |
| 2 | 1-Year Treasury Rate | -0.419 | - | 1.604 | - |
| 2 | 5-Year Treasury Rate | -0.943 | - | 0.767 | - |
| 2 | 10-Year Treasury Rate | -1.008 | - | -0.266 | - |
| 2 | Moody's Seasoned Aaa Corporate Bond Yield | -0.751 | - | 0.910 | - |
| 2 | Moody's Seasoned Baa Corporate Bond Yield | 0.978 | - | 1.444 | - |
| 1 | 3-Month Commercial Paper Minus FEDFUNDS | -1.144 | - | 2.166 | ** |
| 1 | 3-Month Treasury Bill Minus FEDFUNDS | -2.616 | *** | -2.063 | ** |
| 1 | 6-Month Treasury Bill Minus FEDFUNDS | -2.494 | ** | -0.374 | - |
| 1 | 1-Year Treasury C Minus FEDFUNDS | -2.434 | ** | 1.699 | * |
| 1 | 5-Year Treasury C Minus FEDFUNDS | -2.718 | *** | 0.621 | - |
| 1 | 10-Year Treasury C Minus FEDFUNDS | -2.752 | *** | 0.506 | - |
| 1 | Moody's Aaa Corporate Bond Minus FEDFUNDS | -2.586 | *** | -0.157 | - |
| 1 | Moody's Baa Corporate Bond Minus FEDFUNDS | -2.061 | ** | -1.909 | * |
| 5 | Trade Weighted U.S. Dollar Index: Major Currencies | -0.207 | - | -1.925 | * |
| 5 | Switzerland/U.S. Foreign Exchange Rate | -0.748 | - | -0.855 | - |
| 5 | Japan/U.S. Foreign Exchange Rate | -2.496 | ** | -2.452 | ** |
| 5 | U.S./U.K. Foreign Exchange Rate | -1.413 | - | -1.728 | * |
| 5 | Canada/U.S. Foreign Exchange Rate | 0.614 | - | -0.860 | - |
| | Group 7. Prices | | | | |
| 6 | PPI: Finished Goods | -0.628 | - | 0.698 | - |
| 6 | PPI: Finished Consumer Goods | -0.849 | - | 0.729 | - |
| 6 | PPI: Intermediate Processed Goods | -1.817 | * | -1.038 | - |
| 6 | PPI: Intermediate Unprocessed Goods | -0.713 | - | -0.546 | - |
| 6 | PPI: Metals and Metal Products: | -1.080 | - | -0.903 | - |
| 6 | Crude Oil, spliced WTI and Cushing | 2.493 | ** | 2.319 | ** |
| 6 | CPI: All Items | 0.107 | - | -0.672 | - |
| 6 | CPI: Apparel | 1.594 | - | 1.786 | * |
| 6 | CPI: Transportation | -0.722 | - | -1.286 | - |
| 6 | CPI: Medical Care | -0.817 | - | -1.058 | - |
| 6 | CPI: Commodities | -0.977 | - | -0.885 | - |
| 6 | CPI: Durables | 0.596 | - | 0.274 | - |
| 6 | CPI: Services | 0.964 | - | 0.663 | - |
| 6 | CPI: All Items Less Food | -1.166 | - | 0.481 | - |
| 6 | CPI: All Items Less Shelter | -0.834 | - | -0.798 | - |
| 6 | CPI: All Items Less Medical care | -0.779 | - | -0.978 | - |
| 6 | Personal Cons. Expend.: Chain Index | -1.297 | - | 0.883 | - |
| 6 | Personal Cons. Exp: Durable Goods | -0.729 | - | -0.713 | - |
| 6 | Personal Cons. Exp: Nondurable Goods | -1.259 | - | -1.481 | - |
| 6 | Personal Cons. Exp: Services | 1.069 | - | 1.047 | - |
| | Group 8. Stock Market | | | | |
| 5 | S&P's Common Stock Price Index: Composite | -2.340 | ** | -0.881 | - |
| 5 | S&P's Common Stock Price Index: Industrials | -2.407 | ** | -0.755 | - |
| 2 | S&P's Composite Common Stock: Dividend Yield | 1.492 | - | -0.660 | - |
| 5 | S&P's Composite Common Stock: Price-Earnings Ratio | 0.037 | - | -0.777 | - |
| 1 | S&P's Volatility Index | 1.915 | * | 3.027 | *** |

Note: "***", "**" and "*" denote the significance levels at 1%, 5% and 10%, respectively. The column tcode denotes the following data transformation for a series to achieve stationarity: (1) no transformation; (2) $\Delta x_t$; (3) $\Delta^2 x_t$; (4) $\log(x_t)$; (5) $\Delta \log(x_t)$; (6) $\Delta^2 \log(x_t)$; (7) $\Delta(x_t/x_{t-1} - 1.0)$.

## 1.6 Conclusions

A rapidly growing body of literature have been investigating the skewed property in business cycles. Some studies have emphasized the role of financial frictions while others have focused more on concave decision rules in labor market. Importance of the role of uncertainty in business cycles have been attracting much attention. However, statistical evidence of skewed dynamics in the U.S. macroeconomic time-series has been scarce. De Long and Summers (1984) found no evidence of skewness, except for the U.S. unemployment rate. Bai and Ng (2005) intensified this view, although evidence of skewness in exchange rates, CPI inflation, and stock returns is newly obtained.

In this study, I proposed methods to investigate whether the skewed property observed in previous studies is attributed to the economy-wide common components and/or idiosyncratic components. I considered two types of test statistics for the space spanned by the common components and a test for idiosyncratic components. Additionally, I derived the asymptotic distributions under the null hypothesis and investigated their finite sample size and power using the Monte Carlo simulations. Furthermore, I applied these tests to a common factor model using 127 U.S. macroeconomic time-series data of McCracken and Ng (2016). I found strong evidence of skewness in the common components. Additionally, some idiosyncratic components related to housing, labor market, and uncertainty exhibited skewed properties. These results suggest empirical relevance of incorporating the skewed dynamics in business cycle modeling.

The results of this study suggest directions for future research. First, it would be beneficial, if the methods could incorporate time-varying characteristics of the skewed property, as indicated by many authors including Jensen et al. (2020) and Plagborg-Møller et al. (2020). Second, my methods may potentially be extended to produce cross-section data set that is free from the common components as pointed out by Dew-Becker et al. (2021).

# Chapter 2

# A Cross-Sectional Method for Right-Tailed PANIC Tests under a Moderately Local to Unity Framework[1]

## 2.1 Introduction

Large dimensional common factor models are a driving force in recent empirical analysis in various fields of economics. Bai (2003) and Bai and Ng (2006) provide sufficient conditions under which the principal component estimator is consistent for the common and idiosyncratic components when the series have no time trends. When the series have stochastic trends of integration of the order one, the standard practice is to induce stationarity by transforming the original data by first differencing before identifying and estimating the common and idiosyncratic components.[2] If one is interested in identifying whether the stochastic trends lie in the common or idiosyncratic components, Bai and Ng (2004) suggest applying augmented Dickey–Fuller (ADF) tests (Dickey & Fuller, 1979) for these components estimated by first-differenced data. This method is called the panel analysis of nonstationarity in idiosyncratic and common components (PANIC). One main advantage of this

[1] This chapter is a joint work with Yohei Yamamoto. The published version is Yamamoto, Y., & Horie, T. (2022). A cross-sectional method for right-tailed PANIC tests under a moderately local to unity framework. *Econometric Theory*, *forthcoming*. (Available at `http://doi.org/10.1017/S0266466622000044`).

[2] Bai (2004) proposes estimating common stochastic trends by using the principal components of level data when none of the idiosyncratic errors have stochastic trends, but the common factors do. See also the seminal work of Stock and Watson (2002a, 2005) for empirical examples.

method is that the common and idiosyncratic components are separately identified under the null hypothesis of a random walk. In addition, the ADF test has nontrivial power when testing against the alternative hypothesis of stationarity (hereafter, the left-tailed test) because, under such a hypothesis, the first-differenced series may be over-differenced, but has no time trends; hence, the common and idiosyncratic components are correctly identified. Bai and Ng (2004) show that the test for the common components has good size and power despite stationary or random walk idiosyncratic components. The same can be said of the test for the idiosyncratic components. Therefore, the PANIC approach successfully disentangles the common and idiosyncratic components.

In this study, I investigate whether this convenient property of the PANIC approach is available even when the right-tailed version of the ADF test (hereafter, the right-tailed test) is used. The right-tailed unit root tests are used in various applications. For example, testing for speculative bubbles in asset prices is a long-standing problem for which numerous econometric techniques have been developed. The most recent studies include the seminal work of Phillips et al. (2011), in which they pay attention to the link between speculative bubbles and the explosive behaviors of asset price data. Their strategy is to fit a univariate AR model and test whether the root is greater than unity. The present study considers situations where speculative bubbles may be present in large dimensional panel data of financial assets. It is important to investigate whether these bubbles are an economy-wide phenomenon or market-specific events. This study takes a step towards answering such a question.

Consistent with Becheri and van den Akker (2015) and Westerlund (2015), I first show that both left- and right-tailed PANIC tests for common and idiosyncratic explosive behaviors exhibit the standard local asymptotic power when the AR coefficient shrinks to one at a fast rate of $T^{-1}$, where $T$ is the time dimension of the panel data set (see Appendix B.1). A potential problem of such a local to unity (LTU) asymptotic framework is that it only considers small deviations from the unit root. The recent literature establishes that the asymptotic results under such local asymptotic frameworks may not adequately approximate the finite sample behaviors of the test statistics (see, e.g., Deng & Perron, 2008). With this caveat in mind, I take an approach that considers the AR root that shrinks to one at a slower rate than $T^{-1}$. In particular, I use the moderately local to unity (MLTU) framework developed by Phillips and Magdalinos (2007). Under this framework, I find that the explosive idiosyncratic components may be identified as the common component. As a result, the tests for the common and idiosyncratic components have size distortions and power loss.

Monte Carlo simulations illustrate the analytic findings. I first confirm Bai and Ng's (2004) results — that is, as far as the left-tailed tests are concerned, the PANIC approach provides good size and power. However, the right-tailed tests behave very differently from the left-tailed tests. First, the test for the common components shows significant size distortions when some idiosyncratic components are explosive because the explosive idiosyncratic components are misidentified as the common factor. Second, the test for the idiosyncratic components suffers from size distortions when the common components are explosive for the same reason. Finally, and most importantly, the test for the idiosyncratic component shows an upward power function when the AR coefficient is slightly larger than one. However, the power function starts to decline toward zero as the AR coefficient further increases. This phenomenon is the well-known nonmonotonic power problem widely documented in the context of structural change tests (Perron, 1991; Vogelsang, 1999). What is new in this study is that the source of nonmonotonic power is the identification failure between the common factors and explosive idiosyncratic components under the alternative hypothesis.

This study provides a new method of testing for explosive behavior in the common and idiosyncratic components. In many empirical situations, explosive behaviors appear only in a certain subperiod and the series are not explosive in the remaining sample period — I take advantage of this fact. Therefore, I can set a training sample during which no, or only weak, explosive behavior exists. I then use cross-sectional (CS) regressions to estimate the common components in the explosive window as the coefficients attached to the factor loadings, while the factor loadings are estimated in the training sample. I call this the CS method. It is shown that the tests for the common components and the tests for the idiosyncratic components achieve the correct asymptotic size and are consistent under the MLTU framework. A Monte Carlo simulation shows that the CS test for common components considerably reduces size distortions. More importantly, the CS test for idiosyncratic components is robust to the nonmonotonic power problem.

The structure of the remaining chapter is as follows. Section 2.2 introduces the model, assumptions, and existing PANIC tests. Section 2.3 presents the finite sample size and power of the right-tailed PANIC tests and investigates their theoretical properties under the MLTU framework. Section 2.4 proposes a new CS method and investigates its theoretical and finite sample properties.

Section 2.5 concludes the chapter. The details of technical derivations and additional results are provided in Appendix B, including further details on the Results under the LTU Framework (Appendix B.1), Proof of Theorem SA-1 and Theorem 2.1 (Appendix B.2), Proof of Factor Estimation Errors in Theorem 2.1 (i) (Appendix B.3), and Proof of Theorem SA-2 and Theorem 2.2 (Appendix B.4). Throughout the chapter, the following notations are used. The Euclidean norm of vector $x$ is denoted by $\|x\|$. For the matrices, the vector-induced norm is used. The symbols $O(\cdot)$ and $o(\cdot)$ denote the standard asymptotic orders of sequences. The symbol $\xrightarrow{p}$ represents convergence in probability under the probability measure $P$ and the symbol $\Rightarrow$ denotes convergence in distribution. $O_p(\cdot)$ and $o_p(\cdot)$ are the orders of convergence in probability under $P$ as $N, T \to \infty$ (or $N, T, h \to \infty$). I use the symbol $x \approx y$ when $\|x - y\| = o_p(1)$, for two vectors of random variables $x$ and $y$.

## 2.2 Model and test statistics

I consider the common factor model:

$$X_{i,t} = \mu_i + \lambda_i' F_t + U_{i,t}, \quad \text{for } i = 1, ..., N \text{ and } t = 1, ..., T, \tag{2.2.1}$$

where $X_{i,t}$ is a scalar of the observed random variable, $\mu_i$ is an intercept, $F_t$ and $\lambda_i$ are the $r \times 1$ vectors of the common factors and factor loadings, respectively, and $U_{i,t}$ is a scalar idiosyncratic component. I focus on the essence of the problem by assuming the number of factors is one with no loss of substance and is known by the econometrician so the estimation of $r$ is not needed.[3,4] The common factor follows $(1 - \alpha L)F_t = C(L)e_t$ where $C(L) = \sum_{j=0}^{\infty} C_j L^j$ with $C_0 = 1$ and $e_t$ is a white noise disturbance. The idiosyncratic components follow $(1 - \rho_i L)U_{i,t} = D_i(L)z_{i,t}$, where $\rho_i$ is the AR coefficient of the $i$th cross-section, $D_i(L) = \sum_{j=0}^{\infty} D_{ij} L^j$, $D_{i0} = 1$, and $z_{i,t}$ is a white noise disturbance.

I consider the following assumptions in this model. Let $M < \infty$ be a generic constant.

[3] When $r > 1$, one can implement the right-tailed test series-by-series with the individual factors to investigate whether the common factor space is explosive. This is adequate because at least one rejection implies that the whole space is explosive. This is in contrast to the left-tailed tests. As Bai and Ng (2004) contemplate, rejections of individual factors do not necessarily imply a rejection for the common factor space if they have a cointegration relationship. Note that since the estimated factors are uncorrelated with each other, the size of the testing for series-by-series is controlled.

[4] The cross-section-specific intercepts, $\mu_i$, are eliminated in the first-differenced data such that they do not affect inference on $\alpha$ and $\rho_i$. When (2.2.1) includes linear time trends, one can work with the demeaned first differenced data and the equivalent principal components are obtained.

**Assumption 2.1.** *For every* $t = 0, 1, ..., T$, $e_t \sim i.i.d.(0, \sigma^2)$, $\mathbb{E}\,|e_t|^4 \leq M$, *and* $\sum_{j=0}^{\infty} j\,|C_j| < M$. *Furthermore,* $\mathbb{E}|F_0| \leq M$.

**Assumption 2.2.**

1. $\lambda_i$ *is a nonrandom quantity satisfying* $|\lambda_i| \leq M$ *or a random quantity satisfying* $\mathbb{E}\,|\lambda_i|^2 \leq M$.

2. $N^{-1} \sum_{i=1}^{N} \lambda_i^2 \xrightarrow{p} \sigma_\lambda^2$, *where* $\sigma_\lambda$ *is a positive constant.*

**Assumption 2.3.** *For every* $t, s = 0, 1, ..., T$ *and* $i = 1, ..., N$, *the following hold.*

1. $z_{i,t} \sim i.i.d.(0, \sigma_i^2)$, $\mathbb{E}\,|z_{i,t}|^8 \leq M$, *and* $\sum_{j=0}^{\infty} j\,|D_{ij}| < M$.

2. *Let* $\phi_{i,j} = \mathbb{E}(z_{i,t} z_{j,t})$. *Then,* $\sum_{i=1}^{N} |\phi_{i,j}| \leq M$ *for all* $j$ *and* $N^{-1} \sum_{i=1}^{N} \sum_{j=1}^{N} |\phi_{i,j}| \leq M$.

3. *Let* $\zeta_{s,t} = \mathbb{E}\left|N^{-1/2} \sum_{i=1}^{N} [z_{i,s} z_{i,t} - \mathbb{E}(z_{i,s} z_{i,t})]\right|^4$. *Then,* $\zeta_{s,t} \leq M$.

4. $\mathbb{E}\,|U_{i,0}| \leq M$ *for every* $i = 1, \cdots, N$.

**Assumption 2.4.** $z_{i,s}$, $e_t$, *and* $\lambda_j$ *are mutually independent for every* $(i, j, s, t)$.

The model and assumptions follow those of Bai and Ng (2004). In particular, Assumption 2.3.1 permits weak serial correlations in the idiosyncratic errors $(1 - \rho_i L) U_{i,t}$, while Assumptions 2.3.2 and 2.3.3 allow weak cross-sectional correlations. Bai and Ng (2004) consider the unit root test against the alternative hypothesis of stationarity for the common and idiosyncratic components. In this study, I am interested in the test against the alternative hypothesis of an explosive process. For the common component, $H_0 : \alpha = 1$ versus $H_1 : \alpha > 1$, and for the $i$th idiosyncratic component, $H_0 : \rho_i = 1$ versus $H_1 : \rho_i > 1$. Under the restriction of $\alpha = 1$, the model is the same as Bai and Ng's (2004) PANIC. They propose a method of separately identifying the common factors and idiosyncratic errors under the null hypothesis that the common factors follow random walks. This is based on first-differenced data; therefore, $x_{i,t} = \lambda_i f_t + u_{i,t}$ where $x_{i,t} = X_{i,t} - X_{i,t-1}$, $f_t = F_t - F_{t-1}$, and $u_{i,t} = U_{i,t} - U_{i,t-1}$. In the following, I assume that there are $T+1$ observations $t = 0, 1, ..., T$ for $X_{i,t}$ (so that $F_t$ and $U_{i,t}$) for notational simplicity. The common factors and factor loadings can be estimated by using $x_{i,t}$ following the principal component method such that

$$(\hat{f}_t, \hat{\lambda}_i) = \arg \min_{\{\lambda_i\}_{i=1}^{N}, \{f_t\}_{t=1}^{T}} \sum_{i=1}^{N} \sum_{t=1}^{T} (x_{i,t} - \lambda_i f_t)^2, \qquad (2.2.2)$$

with normalization $T^{-1}\sum_{t=1}^{T}\hat{f}_t^2 = 1$. This minimization problem provides a common factor estimate $\hat{f} = [\hat{f}_1, ..., \hat{f}_T]'$ as the $\sqrt{T}$-times eigenvectors of $xx'$ corresponding to the largest eigenvalue, where $x$ is a $T \times N$ matrix with the $(t, i)$th element being $x_{i,t}$. The factor loadings are estimated by $\hat{\lambda}_i = \frac{1}{T}\sum_{t=1}^{T}\hat{f}_t x_{i,t}$, the level common factor is estimated by $\hat{F}_t = \sum_{s=1}^{t}\hat{f}_s$, and the level idiosyncratic errors are estimated by $\hat{U}_{i,t} = \sum_{s=1}^{t}\hat{u}_{i,s}$, where $\hat{u}_{i,s} = x_{i,s} - \hat{\lambda}_i\hat{f}_s$.

The unit root test for the common component (hereafter, the common test) can be implemented by using a $t$-test for $H_0 : \delta = 0$ in the regression $\hat{f}_t = \delta\hat{F}_{t-1} + error$ such that $t_{\hat{F}} = \hat{\delta}/se(\hat{\delta})$, where $\hat{\delta}$ is an ordinary least squares estimator for $\delta$ and $se(\hat{\delta})$ represents its standard errors. The regression may include an intercept and a time trend with appropriate adjustment to the critical values to take account of the time trend. When an intercept is included, I denote the $t$-test by $\bar{t}_{\hat{F}}$. When the errors are suspected of being serially correlated, I can include the lags of $\hat{f}_t$ in the regression. However, a model with no lags is relevant for asset price data in which no serial correlations are present in their first differences.[5] If necessary, I can extend the framework to the model with $p$ lags. The lag order selection follows the conventional method, such as the information criteria based on estimated components. As shown in Said and Dickey (1984), the asymptotic distributions of the $t$-tests are not affected by including $p$ lags if $p^3/T \to \infty$. Here, this condition must consider that factor estimation errors vanish if $N, T \to \infty$; hence, I require $p^3/\min\{N, T\} \to 0$. When $r > 1$, I propose testing the estimated common factors series-by-series to determine whether any of the common factors are explosive. This is a sufficient treatment for the present study because I am only interested in the space spanned by the common factors.[6] The unit root test for the $i$th idiosyncratic component (hereafter, the idiosyncratic test) is implemented by using a $t$-test for $H_0 : \delta_i = 0$ in the regression $\hat{u}_{i,t} = \delta_i\hat{U}_{i,t-1} + error$ so that $t_{\hat{U}}(i) = \hat{\delta}_i/se(\hat{\delta}_i)$ where the same note as $t_{\hat{F}}$ applies. When an intercept is included, I denote the $t$-test by $\bar{t}_{\hat{U}}(i)$.

As Bai and Ng (2004) note, this approach is convenient because the common and idiosyncratic components are separately identified by using the first-differenced data. This way, both common and idiosyncratic tests have the standard Dickey and Fuller (1979) distribution under the null hypothesis. If the alternative hypothesis of stationarity is true, the series become over-differenced,

[5] Phillips and Yu (2011) also consider only the model with $p = 0$.

[6] Bai and Ng (2004) consider the method proposed by Stock and Watson (1988) to determine the number of common trends in the factor space in the setting of $I(0)$ and $I(1)$. However, the number of explosive common trends is not direct interest. Hence, their method is not used in this study.

but they remain stationary; hence, the tests have nontrivial power. Further, their simulation study shows that the common test demonstrates good size and power despite stationary or random walk idiosyncratic components. The same can be said of the idiosyncratic test. Therefore, the PANIC approach successfully disentangles the common and idiosyncratic components.

*Remark* 2.1 (Bai & Ng, 2004). Let Assumptions 2.1–2.4 hold. (i) Under $\alpha = 1$ and $|\rho_i| \leq 1$ for all $i$, $t_{\hat{F}} \Rightarrow [\int_0^1 W(r)dW(r)]/[\int_0^1 W(r)^2 dr]^{1/2}$ and $\bar{t}_{\hat{F}} \Rightarrow [\int_0^1 \bar{W}(r)dW(r)]/[\int_0^1 \bar{W}(r)^2 dr]^{1/2}$ as $N, T \to \infty$, where $W(r)$ is the standard Wiener process defined on $r \in [0, 1]$ and $\bar{W}(r) = W(r) - \int_0^1 W(r)dr$. (ii) Under $\rho_i = 1$, $\alpha = 1$, and $|\rho_j| \leq 1$ for all $j \neq i$, $t_{\hat{U}}(i) \Rightarrow (\int_0^1 W_i(r)dW_i(r))/[\int_0^1 W_i(r)^2 dr]^{1/2}$ and $\bar{t}_{\hat{U}}(i) \Rightarrow [\int_0^1 \bar{W}_i(r)dW_i(r)]/[\int_0^1 \bar{W}_i(r)^2 dr]^{1/2}$ as $N, T \to \infty$, where $W_i(r)$ are standard Wiener processes defined on $r \in [0, 1]$ and $\bar{W}_i(r) = W_i(r) - \int_0^1 W_i(r)dr$.

These null distributions are applicable to both left- and right-tailed tests, as long as the common and idiosyncratic components are consistently estimated. This is warranted in Bai and Ng's (2004) framework, where all components are $I(1)$ or $I(0)$ such that their first differences are stationary. However, this is not necessarily the case if explosive processes are present. When some idiosyncratic components are moderately explosive, the common components may not be consistently estimated, and a consistent estimate for the idiosyncratic components is not warranted either. Hence, size distortions in the common and idiosyncratic tests are concerned. I discuss the properties of PANIC tests in explosive environments in the next section.

## 2.3 Properties of the PANIC tests

### 2.3.1 Finite sample properties

I begin analysis by investigating the finite sample properties of the PANIC tests via Monte Carlo simulations. Although I focus on the empirical size and power of the right-tailed tests, those of the left-tailed tests are also presented for reference. While the latter experiment overlaps with Bai and Ng's (2004) results, it is instructive to illustrate how differently the left-tailed and right-tailed tests behave. The data are generated from (2.2.1) with $F_t = \alpha F_{t-1} + e_t$ and $U_{i,t} = \rho_i U_{i,t-1} + z_{i,t}$ with $r = 1$, where $\lambda_i$, $e_t$, $z_{it}$, $F_0$, and $U_{0,i}$ are independently drawn from the standard normal

Table 2.1: Size of the PANIC tests.

| Common tests | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Left-tailed tests | | | | | Right-tailed tests | | | |
| $N$ | 100 | 150 | 100 | 150 | $N$ | 100 | 150 | 100 | 150 |
| $T$ | 100 | 100 | 150 | 150 | $T$ | 100 | 100 | 150 | 150 |
| $\rho_i = 1.0$ | 0.049 | 0.050 | 0.048 | 0.054 | $\rho_i = 1.0$ | 0.049 | 0.050 | 0.048 | 0.045 |
| 0.8 | 0.043 | 0.046 | 0.049 | 0.046 | 1.02 | 0.069 | 0.063 | 0.243 | 0.206 |
| 0.6 | 0.051 | 0.046 | 0.043 | 0.051 | 1.04 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.4 | 0.047 | 0.050 | 0.050 | 0.049 | 1.06 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.2 | 0.053 | 0.048 | 0.049 | 0.052 | 1.08 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.0 | 0.051 | 0.044 | 0.049 | 0.045 | 1.10 | 1.000 | 1.000 | 1.000 | 1.000 |
| Idiosyncratic tests | | | | | | | | | |
| | Left-tailed tests | | | | | Right-tailed tests | | | |
| $N$ | 100 | 150 | 100 | 150 | $N$ | 100 | 150 | 100 | 150 |
| $T$ | 100 | 100 | 150 | 150 | $T$ | 100 | 100 | 150 | 150 |
| $\alpha = 1.0$ | 0.047 | 0.049 | 0.052 | 0.050 | $\alpha = 1.00$ | 0.048 | 0.049 | 0.045 | 0.051 |
| 0.8 | 0.050 | 0.054 | 0.052 | 0.051 | 1.02 | 0.035 | 0.039 | 0.025 | 0.025 |
| 0.6 | 0.050 | 0.050 | 0.052 | 0.050 | 1.04 | 0.011 | 0.010 | 0.002 | 0.002 |
| 0.4 | 0.050 | 0.045 | 0.053 | 0.052 | 1.06 | 0.001 | 0.001 | 0.002 | 0.001 |
| 0.2 | 0.049 | 0.049 | 0.053 | 0.044 | 1.08 | 0.001 | 0.001 | 0.005 | 0.006 |
| 0.0 | 0.051 | 0.056 | 0.051 | 0.050 | 1.10 | 0.002 | 0.002 | 0.010 | 0.008 |

quasi-random variables in each replication.[7] To evaluate size and power, I vary the values of $\alpha$ and $\rho_i$ from 1.0 to 1.1 for the right-tailed test and from 1.0 to 0.0 for the left-tailed test. Results using the regression models that include (A) no deterministic components, (B) an intercept but no time trend, and (C) an intercept and a linear time trend are produced. Since they are almost identical, I only report case (B). I use the first idiosyncratic component to evaluate the idiosyncratic tests; however, this choice is trivial because the Monte Carlo design is symmetric for any $i$. I use $(N, T) = (100, 100), (100, 150), (150, 100), (150, 150)$ to investigate size and power. The number of replications is 5,000 and the nominal level 5% is used.

I first consider size. I set $\alpha = 1.0$ to evaluate the size of the common test and $\rho_i = 1.0$ to evaluate that of the idiosyncratic test. The upper panel of Table 2.1 reports the size of the common test as a function of $\rho_i$ and the lower panel shows the size of the idiosyncratic test as a function of

[7] I also computed the size and power of the right-tailed PANIC test using models with $p = 4[\frac{\min\{N,T\}}{100}]^{1/4}$. The results are qualitatively the same and are, thus, not reported.

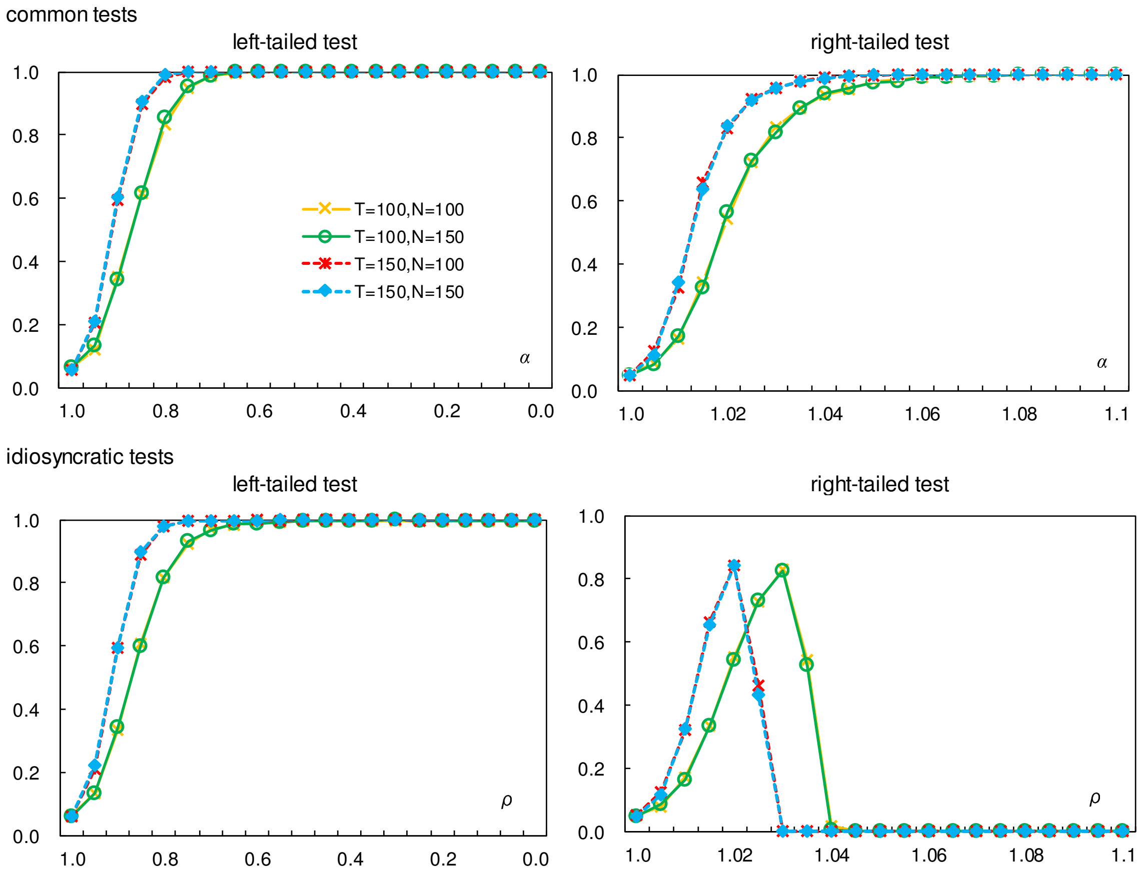


Figure 2.3.1: Power of the PANIC tests.

$\alpha$. The left-tailed tests exhibit good size properties — along the same lines as in Bai and Ng (2004) — confirming that the PANIC approach successfully disentangles the common and idiosyncratic components. However, the results of the right-tailed tests are markedly different. The size of the common test is close to the nominal level when $\rho_i$ is approximately smaller than 1.02; however, it quickly reaches one as $\rho_i$ increases. Further, the size of the idiosyncratic test is also distorted toward zero as $\alpha$ increases. These size distortions suggest that the convenient property of Bai and Ng (2004), that is, the common and idiosyncratic components are separately identified no longer applies to the right-tailed tests. Regarding the effect of sample size, the size of the left-tailed test is good regardless of $N$ and $T$, while the size of the right-tailed test deteriorates as $T$ increases.

Next, I consider power. The upper panels in Figure 2.3.1 report the power functions of the common test under $\rho_i = 1$ for all $i$ and show that the common test has a standard power function.[8]

[8] Setting at $\rho_i > 1$ does not show any unique power features of the common tests, except for the size distortions already reported in Table 2.1. That is, the power functions of the right-tailed test in the case of $\rho_i > 1$ start at a point above the nominal level, but draw an upward curve.

Table 2.2: Size of the PANIC right-tailed tests when one idiosyncratic component is explosive.

| | Common tests | | | | | Idyosyncratic tests | | | |
|---|---|---|---|---|---|---|---|---|---|
| $N$ | 100 | 150 | 100 | 150 | $N$ | 100 | 150 | 100 | 150 |
| $T$ | 100 | 100 | 150 | 150 | $T$ | 100 | 100 | 150 | 150 |
| $\rho_N = 1.00$ | 0.047 | 0.050 | 0.048 | 0.048 | $\alpha = 1.00$ | 0.050 | 0.047 | 0.047 | 0.046 |
| 1.02 | 0.050 | 0.054 | 0.045 | 0.047 | 1.02 | 0.043 | 0.039 | 0.025 | 0.026 |
| 1.04 | 0.059 | 0.056 | 0.471 | 0.389 | 1.04 | 0.011 | 0.012 | 0.003 | 0.003 |
| 1.06 | 0.529 | 0.445 | 0.966 | 0.962 | 1.06 | 0.002 | 0.002 | 0.001 | 0.003 |
| 1.08 | 0.920 | 0.900 | 0.997 | 0.998 | 1.08 | 0.001 | 0.001 | 0.006 | 0.004 |
| 1.10 | 0.986 | 0.983 | 1.000 | 1.000 | 1.10 | 0.004 | 0.003 | 0.010 | 0.011 |

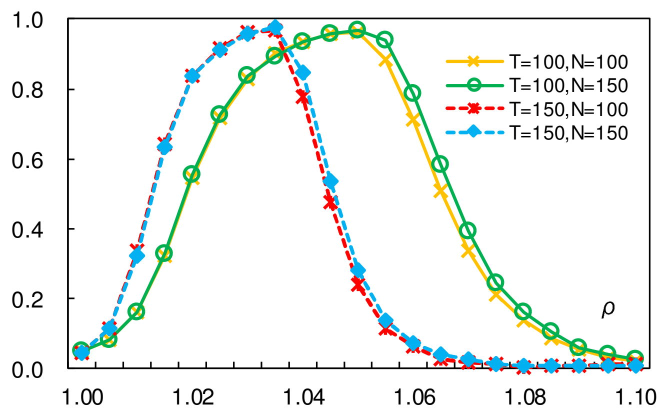


Figure 2.3.2: Power of the PANIC idiosyncratic test when one idiosyncratic component is explosive.

My interest is the power functions of the idiosyncratic test presented in the lower panels. The left-tailed test again has the standard power function; however, the right-tailed test shows a clear nonmonotonic pattern. When the explosive coefficient $\rho_i$ is slightly larger than one, the power increases as $\rho_i$ increases; however, the power function starts to decline toward zero as $\rho_i$ further increases. This means the PANIC approach fails to detect explosive behaviors in an individual idiosyncratic component unless they are quite small.[9] Regarding the effect of sample size, the power increases as $T$ increases when the function is monotonic, but remains the same as $N$ increases. When it is nonmonotonic, the peak of the power shifts leftward as $T$ increases, but again, remains the same as $N$ increases. I also present simulation results in which only one of the idiosyncratic

[9] Bai and Ng (2004) also propose a pooled test for the idiosyncratic components and investigate the properties of the left-tailed tests under the assumption that idiosyncratic components are cross-sectionally independent. This is not direct interest. However, unreported Monte Carlo results show that the pooled version of the right-tailed PANIC tests have qualitatively similar finite sample properties to those of the individual idiosyncratic tests reported in Figures 2.3.1 and 2.4.1.

components is explosive, that is, $\rho_N \geq 1.0$ and $\rho_i = 1.0$ for all $i \neq N$. Table 2.2 shows the size of the common and the idiosyncratic right-tailed tests, while Figure 2.3.2 presents the power of the idiosyncratic test for $i = N$. The results are consistent with the previous case in which all the idiosyncratic components are explosive, that is, the common test has considerable size distortions and the idiosyncratic test shows nonmonotonic power. These findings motivate us to theoretically investigate the PANIC methods under explosive environments in the following subsections.

### 2.3.2 Analytic investigation

Becheri and van den Akker (2015) and Westerlund (2015) derive the standard local asymptotic power of the pooled panel unit root tests in which common factors are extracted by the PANIC method. In doing so, the first-order AR coefficients are assumed to shrink to one at a fast rate of $T^{-1}$.[10] I also investigate the power properties of the common and individual idiosyncratic right-tailed tests by using two complementary asymptotic frameworks. The first approach follows the same lines and assumes that the AR coefficients shrink to one at a fast order $\alpha_T = 1 + \frac{c}{T}$ and $\rho_{i,T} = 1 + \frac{c_i}{T}$, where $c$ and $c_i$ are constants. This LTU asymptotic framework is expected to capture the finite sample properties of the tests when explosiveness is weak. Appendix B.1 shows that the common and the idiosyncratic tests have the standard local asymptotic power for both the left- and right-tailed versions.

It is well known that the results under the LTU framework may not adequately approximate the finite sample behaviors of the test statistics. In the context of structural change tests, a certain type of test statistic may have good power when the magnitude of change is assumed to shrink to zero at a fast rate of $T^{-1/2}$, but it loses power when the magnitude is fixed. This class of tests typically draws a concave-shaped power function, called the nonmonotonic power problem.[11] One reason for this phenomenon is that, under the alternative hypothesis, a change in the conditional mean and a change in the persistence parameter are not separately identified. Yamamoto and Tanaka (2015) further investigate this problem in the factor model, pointing out that the factor loading structural change and appearance of extra factors may not be separately identified under the alternative hypothesis when the structural changes occur at common dates. In such a case, the

[10] The rate also depends on $N$ because they consider the pooled tests.

[11] As far as the authors know, Perron (1991) was the first study to point out this problem in structural change tests. See Vogelsang (1999), Perron and Yamamoto (2016), and the references therein.

standard tests of Breitung and Eickmeier (2011) suffer from the nonmonotonic power problem.

I provide an analytic explanation for why the PANIC tests may have size distortions and nonmonotonic power. I claim that an identification problem between the common and explosive idiosyncratic components occurs under the alternative hypothesis. To this end, I take an approach that assumes the explosive root shrinking to one at a slower rate. In particular, I use the MLTU framework developed by Phillips and Magdalinos (2007).

**Assumption M.** *1. The AR coefficients satisfy* $\alpha_T = 1 + \frac{c}{k_T}$ *and* $\rho_{i,T} = 1 + \frac{c_i}{k_T}$*, where* $c \geq 0$*,* $c_i \geq 0$ *and* $k_T$ *is a deterministic sequence such that* $k_T \to \infty$ *and* $k_T = o(T)$*. 2.* $C(L) = 1$ *and* $D_i(L) = 1$ *for all* $i$*.*

The quantities $c$ and $c_i$ $(i = 1, ..., N)$ are localizing coefficients and take nonnegative values to focus on the explosive case. The scaling factor $k_T$ is an arbitrary deterministic function of $T$ that satisfies $k_T \to \infty$ strictly slower than $T$ to consider stronger explosiveness than that in the local assumption. A typical formulation is $k_T = T^\kappa$, where $0 < \kappa < 1$.

In this setting, the principal component estimate cannot only consistently estimate the common factors, but also misidentify the common components. I illustrate this fact in the following theorem by considering two cases. The first case assumes that $c > 0$ but $c_i = 0$ for all $i$, such that only the common factor is explosive. The second case is $c_i > 0$ for some or all $i$ but $c = 0$. Hence, only the idiosyncratic components are explosive.

**Theorem 2.1.** *Let Assumptions 2.1–2.4 and M hold. If* $k_T$ *grows sufficiently slowly such that* $\alpha_T^T T^{-1/2}$ *and* $\rho_{i,T}^T T^{-1/2}$ *go to infinity as* $T \to \infty$*, then the following equation holds for the factor estimate:*

$$\hat{f}_t = A f_t + N^{-1} \sum_{i=1}^{N} a_i u_{i,t}, \tag{2.3.1}$$

*where* $A \equiv V^{-1}N^{-1}T^{-1}\hat{f}' f \Lambda' \Lambda + V^{-1}N^{-1}T^{-1}\hat{f}' u \Lambda$ *and* $a_i \equiv V^{-1}T^{-1}\hat{f}' f \lambda_i' + V^{-1}T^{-1}\hat{f}' u_i$ *with* $V$ *being the largest eigenvalue of* $N^{-1}T^{-1}xx'$*. Then, the following hold:*

*(i) If* $c > 0$ *and* $c_i = 0$ *for all* $i$*, then* $V = O_p(\alpha_T^T T^{-1/2})$*. Furthermore, if the stochastic order of* $V$ *is* $\alpha_T^T T^{-1/2}$*,* $A = O_p(1)$ *and* $a_i = O_p(1)$*.*

*(ii) If* $c = 0$ *and* $c_i > 0$ *for all* $i$*, then* $V = O_p(\rho_{i,T}^T T^{-1/2})$*. Furthermore, if the stochastic order of* $V$ *is* $\rho_{i,T}^T T^{-1/2}$*,* $A = O_p(1)$ *and* $a_i = O_p(1)$*.*

A proof is provided in Appendix B.2. From part (i), I can deduce that the common test behaves well under the alternative hypothesis. This is because, when the true common component is explosive, the estimation errors of the factor space consist of the second term of (2.3.1). Since the explosive common component dominates the factor estimation errors,[12] the common component estimate continues to be explosive and the power remains.[13]

Part (ii) yields a more interesting case. When the idiosyncratic components are explosive, the second term dominates the first term, because $u_{i,t}$ are explosive, while $f_t$ is not. Hence, $\hat{f}_t$ is dominated by the explosive idiosyncratic components $u_{i,t}$. Therefore, even when the true common component is not explosive, its estimate may be so when some idiosyncratic components are explosive. More intuitively, because the principal component estimator is based on the eigenvectors associated with the largest eigenvalues of the covariance matrix of the panel data, when the idiosyncratic components are explosive, one of the eigenvalues diverges. This causes the idiosyncratic time series to be misidentified as a common component.

To provide intuition of the condition that $\alpha_T^T T^{-1/2}$ (and $\rho_{i,T}^T T^{-1/2}$) tends to infinity, let us consider the case of $k_T = T^\kappa$. In this case, $\alpha_T^T$ is approximated by $\exp(cT^{1-\kappa})$ and the condition requires it grows faster than $T^{1/2}$. Apparently, the LTU ($\kappa = 1$) does not satisfy this condition, because $\exp(cT^{1-\kappa}) = \exp(c)$ is a flat function of $T$. On the contrary, if $\kappa = 0$, then $\alpha_T^T = \exp(cT)$ and this always diverges faster than $T^{1/2}$; hence, $\alpha_T^T T^{-1/2}$ diverges to infinity.[14] Therefore, this condition is more relevant, as $\kappa$ is smaller (or $\alpha_T$ is larger). In my unreported numerical exercise, $\alpha_T^T T^{-1/2}$ increases as $T$ when $\kappa$ is 0.80 or smaller when $c = 1.0$. The value of $\alpha_T$ that corresponds to $\kappa = 0.80$ when $T = 100$ is $\alpha_T = 1.03$. This gives us a rough guide for when the condition $\alpha_T^T T^{-1/2} \to \infty$ holds.[15]

I can derive clear implications of part (ii): when $u_{i,t}$ are explosive, the size of the common test is distorted because $\hat{f}_t$ is dominated by $u_{i,t}$ that are explosive. More interestingly, because $\hat{f}_t$ is dominated by the explosive $u_{i,t}$, the idiosyncratic component estimate $\hat{u}_{i,t}$ becomes nonexplosive for

[12] In Appendix B.3, I show that, in case (i), the factor estimation errors in the differenced factor are $o_p(1)$, but those in the level factor are $O_p(T^{1/2}N^{-1/2})$.

[13] Things are different for the idiosyncratic tests, because the true idiosyncratic components are not explosive and do not dominate the factor estimation errors. Therefore, the idiosyncratic test could suffer from size distortions.

[14] Here, how long it takes for the divergence to be evident depends on $c$. I appreciate this advice from Professor Peter C. B. Phillips.

[15] As shown in the Monte Carlo simulation, the finite sample power of the idiosyncratic test starts to decrease when $T = 100$ and $\rho_i = 1.03$.

the reason in the following remark. This explains the nonmonotonic power of the idiosyncratic test.

*Remark* 2.2. I illustrate the power loss of the idiosyncratic test by taking the special case of (ii), where only the first cross-sectional unit has an explosive idiosyncratic component. I have $\hat{f}_t \approx a_1 u_{1,t}$. By plugging this into the factor loading estimate $\hat{\lambda}_1 = \left(\sum_{t=1}^T \hat{f}_t^2\right)^{-1} \left(\sum_{t=1}^T \hat{f}_t x_{1,t}\right)$ I obtain

$$\begin{aligned} \hat{\lambda}_1 &= \left(\sum_{t=1}^T \hat{f}_t^2\right)^{-1} \left(\sum_{t=1}^T \hat{f}_t x_{1,t}\right) \approx \left(a_1^2 \sum_{t=1}^T u_{1,t}^2\right)^{-1} \left(a_1 \sum_{t=1}^T u_{1,t} x_{1,t}\right), \\ &= \left(a_1^2 T^{-1} \sum_{t=1}^T u_{1,t}^2\right)^{-1} \left(a_1 \lambda_1 T^{-1} \sum_{t=1}^T u_{1,t} f_t + a_1 T^{-1} \sum_{t=1}^T u_{1,t}^2\right) \approx a_1^{-1}, \end{aligned} \quad (2.3.2)$$

because the numerator of the second line is dominated by the second term. By plugging $\hat{f}_t \approx a_1 u_{1,t}$ and (2.3.2) into the idiosyncratic component estimate $\hat{u}_{1,t} = x_{1,t} - \hat{\lambda}_1 \hat{f}_t$ I obtain

$$\begin{aligned} \hat{u}_{1,t} &= x_{1,t} - \hat{\lambda}_1 \hat{f}_t, \\ &\approx u_{1,t} + \lambda_1 f_t - (a_1^{-1})(a_1 u_{1,t}), \\ &= u_{1,t} + \lambda_1 f_t - u_{1,t} = \lambda_1 f_t. \end{aligned} \quad (2.3.3)$$

Therefore, equations $\hat{f}_t \approx a_1 u_{1,t}$ and (2.3.2) imply $\hat{\lambda}_1 \hat{f}_t \approx u_{1,t}$ and equation (2.3.3) implies $\hat{u}_{1,t} \approx \lambda_1 f_t$. These mean that the common and idiosyncratic components are reversely identified by their estimates. Hence, the idiosyncratic test loses power.

I validate this identification problem by investigating the correlation coefficient between $\hat{f}_t$ and $f_t$ and the correlation coefficient between $\hat{f}_t$ and $u_{1,t}$. If the misidentification occurs, the former decreases, but the latter increases as $u_{1,t}$ becomes more explosive. To this end, I generate the same data as in section 2.3.1 and compute the average of the absolute correlation coefficients between the estimated and true common components $\left|Corr(\hat{f}_t, f_t)\right|$ over 5,000 replications. I also compute the average of the absolute correlation coefficients between the estimated common component and true idiosyncratic component $\left|Corr(\hat{f}_t, u_{1,t})\right|$. The left panel of Figure 2.3.3 shows that, as $u_{1,t}$ becomes more explosive, $\hat{f}_t$ becomes less correlated with $f_t$, but more correlated with $u_{1,t}$. This finding is consistent with Theorem 2.1 (ii). Next, as equation (2.3.3) suggests, I compute the average of the absolute correlation coefficients between the estimated and true idiosyncratic components $|Corr(\hat{u}_{1,t}, u_{1,t})|$ and the average of the absolute correlation coefficients between the estimated idiosyncratic component and true common component $|Corr(\hat{u}_{1,t}, f_t)|$. The right panel of Figure

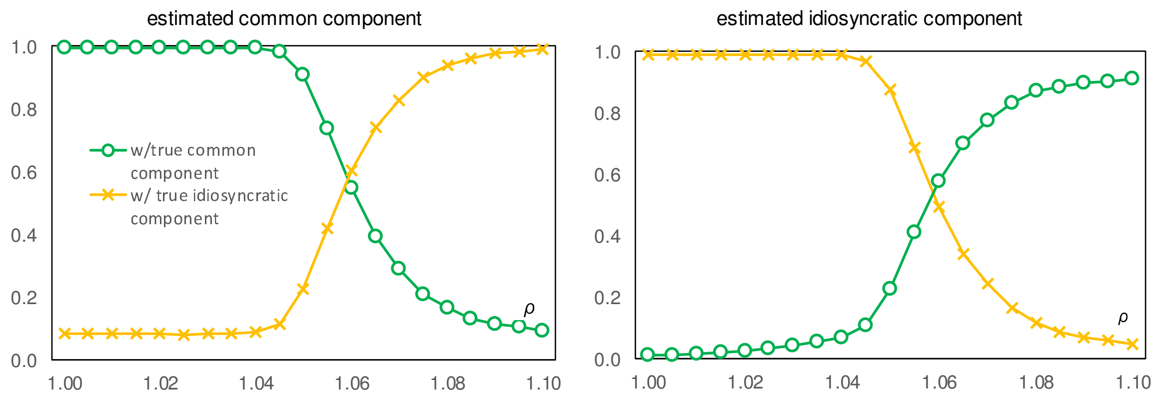


Figure 2.3.3: Absolute values of the correlation coefficients of the estimated components (with the true common and idiosyncratic components).

2.3.3 shows that, as $u_{1,t}$ becomes more explosive, $\hat{u}_{1,t}$ becomes less correlated with $u_{1,t}$, but more correlated with $f_t$, because $\hat{u}_{1,t}$ inherits the time-series properties of $f_t$.

*Remark* 2.3. To make Theorem 2.1 more comprehensive, I may consider the case of $c > 0$ and $c_i > 0$ for some $i$. Then, I have $V = O_p(\max\{\alpha_T^T T^{-1/2}, \rho_{i,T}^T T^{-1/2}\})$, $A = O_p(1)$, and $a_i = O_p(1)$. The explosive idiosyncratic components become dominating components in the estimated factors if they are more strongly explosive than the common components. This case yields the same implication as part (ii), because what matters is the explosive behavior in the idiosyncratic components. In addition, I may also consider the case of $c < 0$ and/or $c_i < 0$ for some $i$. Then, $F_t$ and/or $U_{i,t}$ are $I(0)$ for some $i$ and they would not contaminate the factor estimation as shown in Bai and Ng (2004). Therefore, this case merely gives the same implication as when $c = 0$ and/or $c_i = 0$.

## 2.4 Cross-sectional approach

This section provides a new method of testing explosive behavior in the common and idiosyncratic components. It is based on the following two key ingredients. First, it takes advantage of the fact that explosive behaviors appear only in a certain subperiod and the series are not explosive in the rest of the sample period. If this is the case, I can time-wise localize the explosive behaviors by considering model (2.2.1) with $F_t = \alpha F_{t-1} + e_t$ and $U_{i,t} = \rho_i U_{i,t-1} + z_{i,t}$ where $\alpha = \rho_i = 1$ for $t = 1, ..., T$ and $\alpha$, $\rho_i \geq 1$ for $t = T+1, ..., T+h$, for any $i$, with $h$ being the length of the window, such that the data are assumed to have a certain period $t \in [1, T]$ in which no explosive behaviors exist in either the common or the idiosyncratic components. I call this the training sample.[16] On the contrary, the period of interest $t \in [T+1, T+h]$ is called the explosive window.

### 2.4.1 Algorithm

The second key element is using cross-sectional regressions to estimate the common factors in the explosive window instead of using the principal component estimation of the first-differenced series. This is because the first-differenced series of the explosive process remains explosive and, thus, violates Assumption 2.3. Hence, the common factors are not consistently estimated. To address this problem, I estimate the factor loadings in the training sample in a nonexplosive environment. I then use these loadings as the regressors of the cross-sectional regressions in the explosive window to estimate the common factors as the coefficients attached to the factor loadings. In this way, I can avoid the identification problem between the common and idiosyncratic components investigated in section 2.3.2. An important model assumption is that the factor loadings are constant for the training sample and explosive window. I also keep the assumption that the number of factors remains the same. I call this approach the CS method and it involves the following steps:

**Algorithm:**

**Step 1.** Use the first-differenced data $x_{i,t}$ for $t = 1, ..., T$ to estimate the factor loadings $\lambda_i$ by using the principal component method (2.2.2). Denote the factor loadings estimated in the training sample by $\hat{\lambda}_i^*$.

**Step 2.** At $t = T+1$, estimate the level of the common factors by the CS regression of $\{X_{i,t}\}_{i=1}^N$ on $\left\{\hat{\lambda}_i^*\right\}_{i=1}^N$ so that $\tilde{F}_t = \left(\sum_{i=1}^N \hat{\lambda}_i^* \hat{\lambda}_i^{*\prime}\right)^{-1} \left(\sum_{i=1}^N \hat{\lambda}_i^* X_{i,t}\right)$ and the idiosyncratic components by $\tilde{U}_{i,t} = X_{i,t} - \hat{\lambda}_i^{*\prime} \tilde{F}_t$. Then, repeat this for $t = T+2, ..., T+h$.

**Step 3.** Construct the common test $t_{\tilde{F}}^*$ by using $\tilde{F}_t$ and $\tilde{f}_t = \tilde{F}_t - \tilde{F}_{t-1}$ in the regression $\tilde{f}_t = \delta \tilde{F}_{t-1} + error$ and the idiosyncratic test $t_{\tilde{U}}^*(i)$ by using $\tilde{U}_{i,t}$ and $\tilde{u}_{i,t} = \tilde{U}_{i,t} - \tilde{U}_{i,t-1}$ in the regression $\tilde{u}_{i,t} = \delta_i \tilde{U}_{i,t-1} + error$ for $t = T+1, ..., T+h$. In both regressions, lags of the dependent variable can be included if serial correlations in the errors are concerned. I denote the tests using the regression with an intercept by $\bar{t}_{\tilde{F}}^*$ and $\bar{t}_{\tilde{U}}^*(i)$.

[16] I can easily show that weak (local) explosive processes with AR coefficients $1 + \frac{c}{T}$ and $1 + \frac{c_i}{T}$ can exist in the training sample.

*Remark* 2.4. Although I set $t \in [1, T]$ and $t \in [T+1, T+h]$ as the training sample and the explosive window, respectively, this does not mean that the origination dates of explosive behaviors have to be known in practice for the following reasons. First, the explosive behaviors can start later than $T+1$. If so, I am merely implementing the right-tailed unit root tests for the sample that includes a nonexplosive subsample. Second, explosive behaviors can start before $T$ as long as they are as weak as the LTU. This is because, even in the presence of the explosive behavior, the common and idiosyncratic components are identified as I see in Theorem SA-1 of Appendix B.1 and so are the factor loadings. Third, the origination dates of explosive behaviors in the common components and in any idiosyncratic components are allowed to be heterogeneous because I implement the tests series-by-series. One method of selecting the training sample is to use an existing date-stamping method, such as in Phillips et al. (2011), to the cross-sectionally averaged series $\bar{X}_t = N^{-1} \sum_{i=1}^{N} X_{i,t}$.

### 2.4.2 Theoretical results

I next provide an asymptotic justification of the CS method. Note that the time dimension of the testing period is now $h$ instead of $T$; hence, I consider the following Assumption 2.5 in place of Assumption M.

**Assumption 2.5.** *1. The AR coefficients satisfy $\alpha_h = 1 + \frac{c}{k_h}$ and $\rho_{i,h} = 1 + \frac{c_i}{k_h}$, where $c \geq 0$, $c_i \geq 0$ and $k_h$ is a deterministic sequence such that $k_h \to \infty$ and $k_h = o(h)$. 2. $C(L) = 1$ and $D_i(L) = 1$ for all $i$.*

I obtain the following results. For brevity, again, I provide a proof under i.i.d. assumptions, setting $C(L) = 1$ and $D_i(L) = 1$ in Appendix B.4. The case with an intercept is shown, while the case with no intercept can be similarly given.

**Theorem 2.2.** *Let Assumptions 2.1–2.5 hold. With $c \geq 0$ and $c_j \geq 0$ for any $j = 1, ..., N$, the following hold as $N, T, h \to \infty$.*

*(a: common tests) If $c = 0$,*

$$\bar{t}^*_{\tilde{F}} \Rightarrow \left( \int_0^1 \bar{W}(r) dW(r) \right) / \left[ \int_0^1 \bar{W}(r)^2 dr \right]^{1/2},$$

*if* $\frac{\rho_{j,h}^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}} \to 0$ *for any* $j$. *If* $c > 0$, *with* $\pi \in (0,\infty)$ *and* $\Theta \equiv N(0,\sigma^2/2c)$,

$$\alpha_h^{-h}\bar{t}^*_{\tilde{F}} \approx \begin{cases} \sqrt{\frac{c}{2\sigma^2}}\left|\Theta\right|, & \text{if } T/k_h \to 0 \\ \sqrt{\frac{c}{2\sigma^2}}\left|\frac{F_T}{\sqrt{T}}\sqrt{\pi}+\Theta\right|, & \text{if } T/k_h \to \pi \end{cases},$$

$$\alpha_h^{-h}k_h^{1/2}T^{-1/2}\bar{t}^*_{\tilde{F}} \approx \sqrt{\frac{c}{2\sigma^2}}\left|\frac{F_T}{\sqrt{T}}\right|, \quad \text{if } T/k_h \to \infty,$$

*if* $\frac{\alpha_h^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}} \to 0$ *and* $\frac{\rho_{j,h}^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}} \to 0$ *for any* $j$.

*(b: idiosyncratic tests) If* $c_i = 0$,

$$\bar{t}^*_{\tilde{U}}(i) \Rightarrow \left(\int_0^1 \bar{W}_i(r)dW_i(r)\right) / \left[\int_0^1 \bar{W}_i(r)^2 dr\right]^{1/2}.$$

*if* $\frac{\alpha_h^h h k_h^{-1}}{\min\{N,T^{1/2}\}} \to 0$ *and* $\frac{\rho_{j,h}^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}} \to 0$ *for any* $j$. *If* $c_i > 0$, *with* $\pi \in (0,\infty)$ *and* $\Theta_i \equiv N(0,\sigma_i^2/2c_i)$,

$$\rho_{i,h}^{-h}\bar{t}^*_{\tilde{U}}(i) \approx \begin{cases} \sqrt{\frac{c_i}{2\sigma_i^2}}\left|\Theta_i\right|, & \text{if } T/k_h \to 0 \\ \sqrt{\frac{c_i}{2\sigma_i^2}}\left|\frac{U_{i,T}}{\sqrt{T}}\sqrt{\pi}+\Theta_i\right|, & \text{if } T/k_h \to \pi \end{cases},$$

$$\rho_{i,h}^{-h}k_h^{1/2}T^{-1/2}\bar{t}^*_{\tilde{U}}(i) \approx \sqrt{\frac{c_i}{2\sigma_i^2}}\left|\frac{U_{i,T}}{\sqrt{T}}\right|, \quad \text{if } T/k_h \to \infty,$$

*if* $\frac{\alpha_h^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}} \to 0$ *and* $\frac{\rho_{j,h}^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}} \to 0$ *for any* $j$.

Theorem 2.2 shows that the CS method provides the test statistics that have the correct asymptotic size under the MLTU framework if the stated conditions on the relative rate among $N$, $T$, and $h$ hold. Under the alternative hypothesis, the tests are consistent and behave as follows. If $k_h$ is faster than $T$ so that $T/k_h \to 0$, then I obtain the limit involving the explosive sample thus $|\Theta|$ and the test diverges to positive infinity at a rate of $\alpha_h^h$ with probability one. If $k_h$ is slower than $T$ so that $T/k_h \to \infty$, then the test statistic scaled by $\alpha_h^{-h}k_h^{1/2}T^{-1/2}$ is asymptotically dominated by the absolute value of the initial value term. The test diverges to positive infinity at a rate of $\alpha_h^h k_h^{-1/2}T^{1/2}$ with probability one. If $T$ and $k_h$ grow at the same rate ($T/k_h \to \pi$), then both effects are dominant and the test diverges to positive infinity at a rate of $\alpha_h^h$ with probability one. More importantly, the divergence becomes faster as the localizing coefficient $c$ increases, which ensures the monotonic power property of the CS test.

The added condition $\frac{\alpha_h^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}} \to 0$ (and $\frac{\rho_{j,h}^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}} \to 0$) requires $h$ not to grow very fast to eliminate the effects of factor estimation error. To obtain an intuition behind the condition, I give a parametric example of $k_h = h^\kappa$ where $\kappa$ lies on $(0,1)$ and I let $N = T$ for simplicity. Then, the condition reduces to $\frac{\alpha_h^h h^{1-\kappa}}{T^{1/2}} \to 0$. Taking its logarithm and using $\log(\alpha_h) = ch^{-\kappa} + o(1)$ yield,

$$ch^{1-\kappa} + (1-\kappa)\log(h) - \frac{1}{2}\log(T) \to -\infty.$$

Suppose $c = 0.5$. Then, I can show that this is satisfied when $h$ grows at a rate of $\log(T)$. As another example, I set $k_h = h/\log(h)$. Then, the condition becomes

$$c\log(h) + \log(\log(h)) - \frac{1}{2}\log(T) \to -\infty. \tag{2.4.1}$$

With $c = 0.5$, $h$ is required to grow slower than $T$ only slightly. Because the condition $\frac{\alpha_h^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}} \to 0$ (and $\frac{\rho_{j,h}^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}} \to 0$) requires $h$ not to grow very fast, only the case of $T/k_h \to \infty$ applies in most examples. However, $T/k_h \to \pi$ and $T/k_h \to 0$ are also relevant when $h$ is relatively fast. To see this in the second example, I let $T = h^{1-\epsilon}/\log(h)$ with $\epsilon \geq 0$. Then, $T/k_h \to 1$ when $\epsilon = 0$ and $T/k_h \to 0$ when $\epsilon > 0$. In either case, (2.4.1) holds when $c$ is sufficiently smaller than $\frac{1}{2}$. In the next subsection, I consider the finite sample performance of the test via Monte Carlo simulation using realistic values for the parameters and the sample size.

*Remark* 2.5. The key element of the CS method is estimation errors in the levels and the first differences of the common factors as shown in Lemma B.6 (or Lemma B.11 for the demeaned version) in Appendix B.4:

$$\begin{aligned}
\tilde{F}_t - HF_t &= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right), \\
\tilde{f}_t - Hf_t &= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),
\end{aligned}$$

where $\rho_h = \max_i \rho_{i,h}$. Hence, I extensively use them to prove Theorem 2.2.

### 2.4.3 Finite sample properties

This subsection investigates the finite sample property of the CS method via Monte Carlo simulations. The data are generated by the same model as in section 2.3.1. To investigate the validity of theoretical results more directly, I set the AR coefficients to be $\alpha = 1 + \frac{c}{h^\kappa}$ and $\rho_i = 1 + \frac{c_i}{h^\kappa}$, where $c = c_i = 0$ for $t = 1, ..., T$ and $c$, $c_i \geq 0$ for $t = T+1, ..., T+h$, for any $i$. I use the sample size $N = 100$ and $T = 50$ and two lengths of the explosive window $h = 50$ and 100. All $\lambda_i$, $u_{i,t}$, $z_{i,t}$, $F_0$, and $U_{0,i}$ are independently drawn from the standard normal quasi-random variables in each replication. The size and power of the CS and PANIC tests in the explosive window are computed at the 5% nominal level using 5,000 replications.

Table 2.3 presents the size of the common and idiosyncratic tests in the upper and lower panels, respectively. The left and the right panels correspond to the $h = 50$ and 100 cases, respectively. Consistent with the findings in section 2.3.1, the PANIC common test shows serious size distortions when the idiosyncratic components are explosive and the PANIC idiosyncratic test becomes undersized when the common component is explosive when $\kappa = 0.8$ and $c$ is large. Although the CS common test also shows size distortions, these are considerably smaller than those in the PANIC common tests. As for the CS idiosyncratic test, I now see over-rejections, especially when $\kappa = 0.8$ and $c$ is large. This is consistent with the conditions provided in Theorem 2.2. In my unreported results, I observe that the size of the CS test slightly improves as both $N$ and $T$ increase, although the effect is not discernible. Figure 2.4.1 reports the power of both tests. Most importantly, the bottom panels of Figure 2.4.1 suggest that the CS idiosyncratic test is free of the nonmonotonic power problem. The power functions of the CS and PANIC common tests are similar because the tests are asymptotically equivalent. In summary, the CS tests display size distortions when the explosiveness is strong, but it performs well in general with a moderately explosive process with $\kappa$ being 0.85 or lower. The CS tests outweigh the PANIC tests with respect to the power of idiosyncratic tests.

Finally, the CS method relies on the fundamental model assumptions that the factor loadings and the number of factors are constant even when the explosive regime starts. I investigate the consequences of instabilities pertaining to them. First, the factor loadings have structural changes such that $X_{i,t} = \lambda_i F_t + U_{i,t}$ for $t = 1, ..., T$ and $X_{i,t} = (\lambda_i + \Delta_i)F_t + U_{i,t}$ for $t = T+1, ..., T+h$,

Table 2.3: Size of the CS and PANIC tests.

| Common tests | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $h$ | 50 | 50 | 50 | 50 | 100 | 100 | 100 | 100 |
| $\kappa$ | 0.80 | 0.85 | 0.90 | 0.95 | 0.80 | 0.85 | 0.90 | 0.95 |
| CS | | | | | | | | |
| $c_i = 0.0$ | 0.053 | 0.048 | 0.047 | 0.050 | 0.055 | 0.056 | 0.044 | 0.057 |
| 0.2 | 0.053 | 0.052 | 0.053 | 0.057 | 0.052 | 0.066 | 0.046 | 0.044 |
| 0.4 | 0.059 | 0.057 | 0.053 | 0.058 | 0.054 | 0.053 | 0.046 | 0.063 |
| 0.6 | 0.075 | 0.066 | 0.060 | 0.054 | 0.067 | 0.043 | 0.048 | 0.049 |
| 0.8 | 0.087 | 0.070 | 0.072 | 0.057 | 0.094 | 0.079 | 0.055 | 0.056 |
| 1.0 | 0.142 | 0.095 | 0.075 | 0.058 | 0.190 | 0.089 | 0.063 | 0.059 |
| PANIC | | | | | | | | |
| $ci = 0.0$ | 0.048 | 0.047 | 0.050 | 0.056 | 0.054 | 0.052 | 0.043 | 0.055 |
| 0.2 | 0.056 | 0.054 | 0.053 | 0.056 | 0.054 | 0.073 | 0.044 | 0.043 |
| 0.4 | 0.063 | 0.059 | 0.050 | 0.059 | 0.061 | 0.060 | 0.045 | 0.063 |
| 0.6 | 0.097 | 0.069 | 0.067 | 0.053 | 0.095 | 0.050 | 0.050 | 0.053 |
| 0.8 | 0.238 | 0.114 | 0.092 | 0.064 | 0.209 | 0.108 | 0.067 | 0.054 |
| 1.0 | 0.872 | 0.280 | 0.125 | 0.085 | 0.814 | 0.192 | 0.086 | 0.068 |

| Idiosyncratic tests | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $h$ | 50 | 50 | 50 | 50 | 100 | 100 | 100 | 100 |
| $\kappa$ | 0.80 | 0.85 | 0.90 | 0.95 | 0.80 | 0.85 | 0.90 | 0.95 |
| CS | | | | | | | | |
| $c = 0.0$ | 0.067 | 0.063 | 0.056 | 0.043 | 0.050 | 0.055 | 0.054 | 0.048 |
| 0.2 | 0.068 | 0.046 | 0.057 | 0.043 | 0.051 | 0.062 | 0.041 | 0.054 |
| 0.4 | 0.049 | 0.048 | 0.049 | 0.052 | 0.057 | 0.053 | 0.055 | 0.045 |
| 0.6 | 0.079 | 0.069 | 0.074 | 0.063 | 0.055 | 0.060 | 0.052 | 0.062 |
| 0.8 | 0.108 | 0.066 | 0.076 | 0.064 | 0.116 | 0.090 | 0.065 | 0.058 |
| 1.0 | 0.206 | 0.122 | 0.069 | 0.064 | 0.219 | 0.103 | 0.078 | 0.065 |
| PANIC | | | | | | | | |
| $c = 0.0$ | 0.068 | 0.062 | 0.058 | 0.037 | 0.049 | 0.057 | 0.054 | 0.042 |
| 0.2 | 0.064 | 0.047 | 0.050 | 0.042 | 0.048 | 0.064 | 0.044 | 0.043 |
| 0.4 | 0.044 | 0.044 | 0.055 | 0.057 | 0.048 | 0.043 | 0.055 | 0.047 |
| 0.6 | 0.059 | 0.059 | 0.052 | 0.054 | 0.046 | 0.053 | 0.041 | 0.058 |
| 0.8 | 0.038 | 0.040 | 0.052 | 0.046 | 0.027 | 0.046 | 0.051 | 0.051 |
| 1.0 | 0.031 | 0.039 | 0.042 | 0.044 | 0.025 | 0.037 | 0.052 | 0.048 |

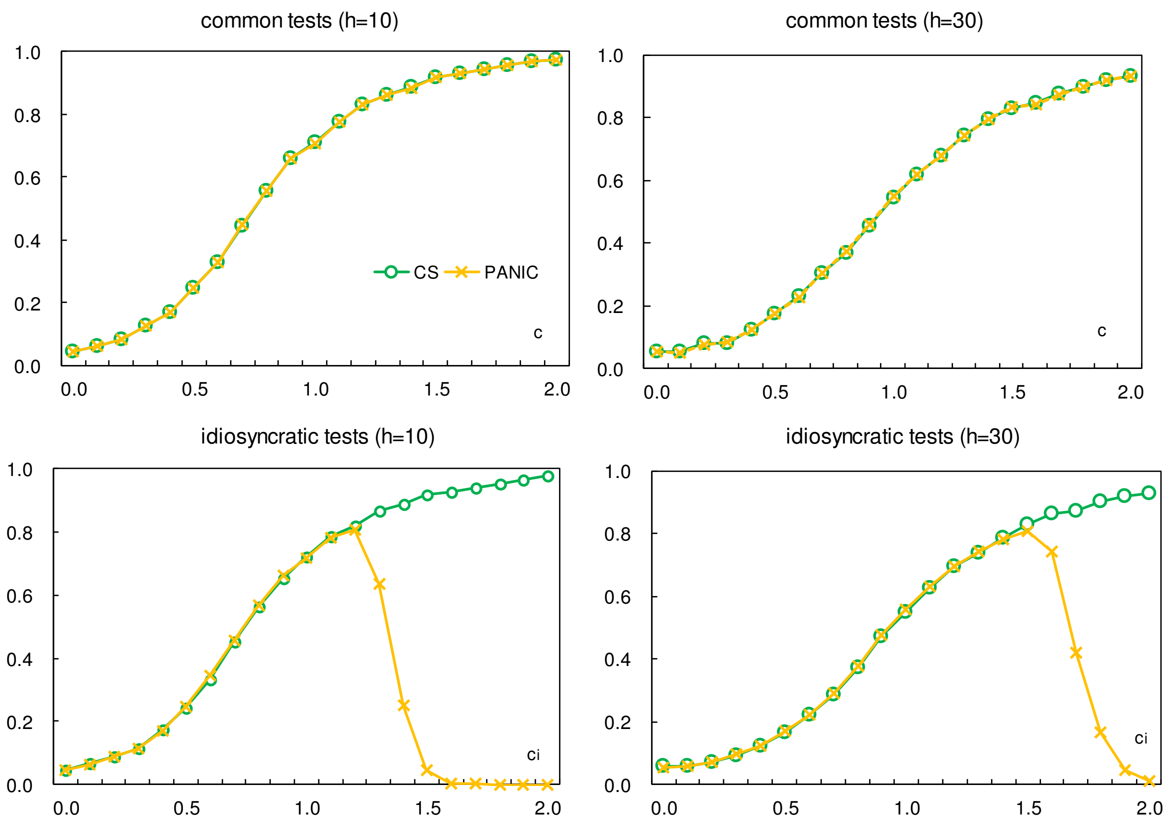


Figure 2.4.1: Power of the CS and PANIC tests.

where the change $\Delta_i \sim i.i.d.U[0,1]$. Second, I generate $F_t$ and $U_{i,t}$ via the same processes, but with an additional common factor $G_t = 0$ for $t \in [1, T]$ and $G_t = \alpha G_{t-1} + v_t$ for $t \in [T+1, T+h]$, where $v_t$ follows $i.i.d.N(0,1)$. Then, $X_{i,t} = \lambda_i F_t + U_{i,t}$ for $t = 1, ..., T$ and $X_{i,t} = \lambda_i F_t + \gamma_i G_t + U_{i,t}$ for $t = T+1, ..., T+h$, where the new factor loadings are generated by $\gamma_i \sim i.i.d.N(0,1)$. In both cases, I implement the PANIC and CS tests in the same manner as the previous case without accounting for such instabilities. Since the structural changes in the factor loadings and the presence of the additional factor are most likely to occur simultaneously when $F_t$ switches to the explosive regime, I focus on the power of the tests. Figure 2.4.2 reports the power of the common and idiosyncratic tests in the case of structural changes and Figure 2.4.3 presents them in the case of the new factor for $N = 100$, $T = 50$, $h = 100$ and $\kappa = 0.85$. They show that the nonmonotonic power of the idiosyncratic tests is still found in the PANIC tests, but it is resolved in the CS tests. The power of the common tests is rarely affected by the structural changes, although I see some power loss in the PANIC tests in Figure 2.4.2 and in the CS tests in Figure 2.4.3.

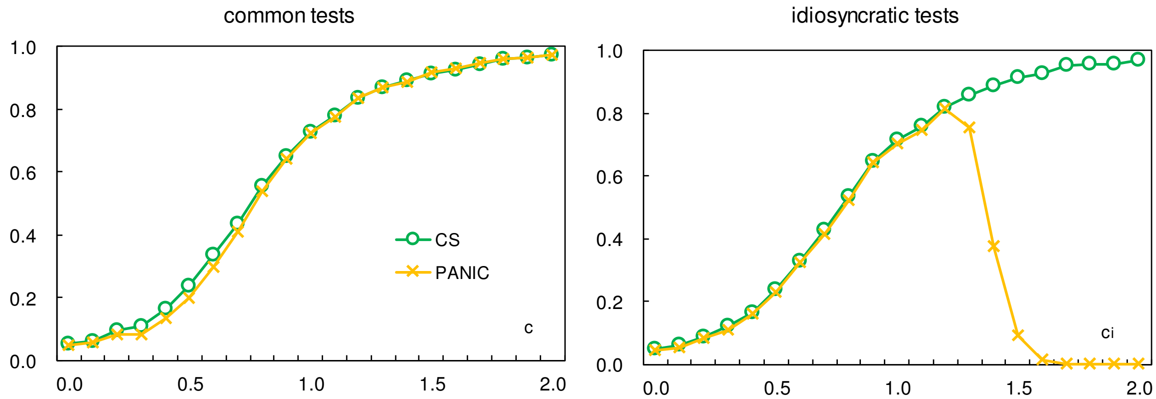


Figure 2.4.2: Power of the CS and PANIC tests when the factor loadings have structural changes.

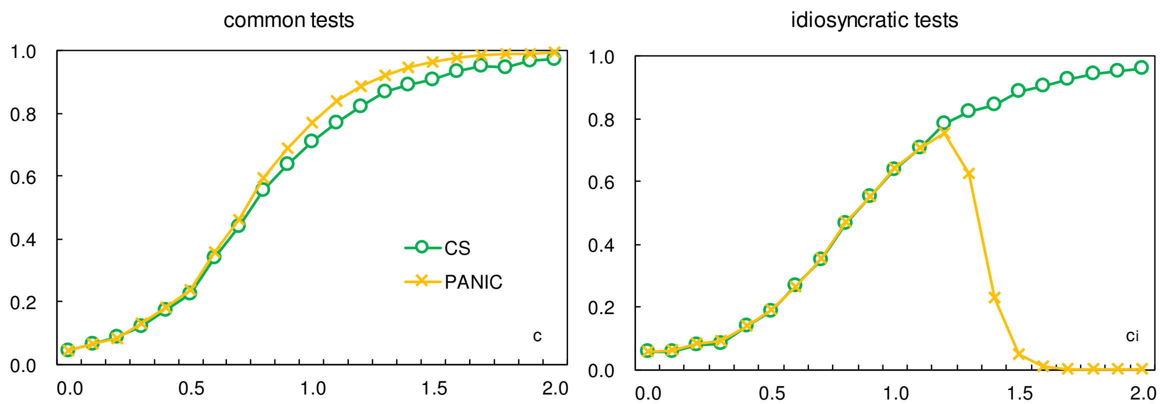


Figure 2.4.3: Power of the CS and PANIC tests when a new factor appears.

## 2.5 Conclusions

In this study, I showed that, when the PANIC tests are applied to the explosive alternative hypothesis, both the common and the idiosyncratic tests may exhibit serious size distortions. More importantly, the idiosyncratic tests suffer from the nonmonotonic power problem. I then provide a new CS method to disentangle the common and idiosyncratic components to obtain standard monotonic power function. The proposed tests achieve the correct asymptotic size and are consistent under the MLTU framework. A Monte Carlo simulation shows that the CS test for common components considerably reduces size distortions and the CS test for idiosyncratic components is robust to the nonmonotonic power problem.

This study has several implications. First, the nonmonotonic power problem can occur not only in certain structural change tests, as shown in Perron and Yamamoto (2016), but also in more general circumstances in which important model parameters are not correctly identified under the alternative hypothesis. Earlier studies such as Müller and Elliott (2003) argued that Elliott et al.'s (1996) efficient unit root tests may have power that drops to zero when the initial value is moderately large. This study uncovers another possibility of the nonmonotonic power problem in unit root testing when unobserved common and idiosyncratic components are misidentified. Second, asymptotic frameworks that allow general deviations from the null hypothesis, such as the MLTU of Phillips and Magdalinos (2007), are extremely useful in approximating such phenomena. Third, the proposed method can potentially extend the right-tailed PANIC tests to various empirical analyses, including testing financial bubbles (see Phillips et al., 2011) in large panel data and factor-augmented regressions (see Stock & Watson, 2016). A caveat is that the proposed method is not free from size distortions when the other nuisance components are strongly explosive. In addition, the relevance of the constant factor loading assumption must be assessed in particular empirical settings. These issues should be carefully incorporated in future studies.

# Chapter 3

# Date-Stamping the Origination of Explosive Behaviors in the Large Dimensional Factor Model[1]

## 3.1 Introduction

Testing for speculative bubbles in asset prices is a long-standing problem for which numerous econometric techniques have been developed. The most recent studies include the seminal work of Phillips et al. (2011) which linked speculative bubbles to explosive behaviors of asset prices.[2] Their strategy is to fit a univariate AR model and test whether the root is greater than unity. Although this chapter is motivated by these studies, it explicitly accounts for empirical facts, such as the speculative bubbles that prevailed in global as well as individual markets during the period of exuberance. It is important to investigate whether these bubbles are an economy-wide phenomenon or market-specific events. To address this question, I formally analyze how panel data of asset prices comove in an explosive environment. A common practice in the empirical finance literature is to assume a linear factor structure in panel data of asset prices. A short list of major works includes Fama and French (1993), Litterman and Scheinkman (1991), and Ang and Piazzesi (2003), who examined stock and

[1] This chapter is a joint work with Yohei Yamamoto.

[2] Phillips et al. (2011) and Phillips and Yu (2011) developed methods for a single bubble. Phillips et al. (2015a, 2015b) modified this to account for multiple bubbles. For other testing methods, see Gürkaynak (2008) and Homm and Breitung (2012).

bond prices. Del Negro and Otrok (2007), Stock and Watson (2008) and Moench and Ng (2011) applied the factor model to U.S. house prices.

In the previous chapter, I adopt the framework of principal component estimation of the large dimensional approximate factor model developed by Stock and Watson (2002a), Bai (2003) and Bai and Ng (2002, 2004). Specifically, Bai and Ng (2004) proposed PANIC framework to test for the unit root against stationarity in the common and idiosyncratic components of the factor model. They showed that the standard ADF tests applied to the common and idiosyncratic components have good size and power for the left-tailed version of the ADF tests. However, the previous chapter discovered that the ADF tests in the PANIC framework have very different size and power properties when the right-tailed versions are considered in an explosive environment. First, the test for the common component suffers from serious size distortions when the idiosyncratic components are explosive. Second, the test for the idiosyncratic components exhibits the nonmonotonic power problem, that is, the power can go down to zero when the idiosyncratic component is moderately or strongly explosive. To address these problems, I proposed a method based on cross-sectional regressions to disentangle the common and idiosyncratic components.

In this study, I apply the date-stamping methodology for the origination of explosive behaviors proposed in the seminal work of Phillips et al. (2011) to the large dimensional factor model. To this end, I compare two methods of identifying the common and idiosyncratic components: PANIC and CS. Monte Carlo simulations show that when the explosive behavior lies only in the common component, the origination date is precisely estimated by either the PANIC or CS method. However, when the explosive behaviors exist in the idiosyncratic components, PANIC method loses its power of detection and the origination dates become inaccurate. I also show that these problems are resolved by using the CS method, although some tendency of overdetection is observed when the idiosyncratic components are strongly explosive.

The remainder of this chapter is organized as follows. In section 3.2, I introduce the model. In section 3.3, I explain the date-stamping methodology of Phillips et al. (2011) and two approaches to disentangle the common and idiosyncratic components. In section 3.4, I implement Monte Carlo simulations to assess the empirical probability of correctly or incorrectly detecting explosive behaviors and the accuracy of the date-stamping strategies. Section 3.5 is the conclusion.

## 3.2 Model

I assume that a panel data set of asset prices follow the factor model:

$$X_{i,t} = \mu_i + \lambda_i' F_t + U_{i,t}, \quad \text{for } i = 1, ..., N \text{ and } t = 1, ..., T, \tag{3.2.1}$$

where $X_{i,t}$ is a scalar of the observed asset price with subscripts $i$ and $t$ indicating cross-section and time, respectively. In addition, $\mu_i$ is a cross-section specific intercept, $F_t$ and $\lambda_i$ are $r \times 1$ vectors of the common factors and factor loadings, respectively, and $U_{i,t}$ is a scalar idiosyncratic component. The cross-section and time dimensions $N$ and $T$ are large, while the number of factors $r$ is small indicating that the large dimensional panel data is driven by a small number of factors. For the moment, I consider $r = 1$. I assume that the common factor and the idiosyncratic components follow the first-order AR processes: $F_t = \alpha F_{t-1} + e_t$ and $U_{i,t} = \rho_i U_{i,t-1} + z_{i,t}$, where $\alpha$ and $\rho_i$ are the AR coefficients and $e_t$ and $z_{i,t}$ are stationary disturbances, respectively. I assume $e_t$ and $z_{i,t}$ for any $i$ are mutually independent for all leads and lags.

The fact that the asset price $X_{i,t}$ follows a random walk is consistent with the efficient market hypothesis (Fama, 1970). Therefore, during normal times, both $F_t$ and $U_{i,t}$ follow a random walk ($\alpha = \rho_i = 1$). When asset price $X_{i,t}$ is subject to a speculative bubble, it exhibits explosive behavior, such that $\alpha > 1$ and/or $\rho_i > 1$. Thus, I allow for regimes that switch between the random walk and the explosive process in the common and idiosyncratic components. For simplicity, I consider a single bubble in each component so that

$$F_t = \begin{cases} \alpha F_{t-1} + e_t, & \text{for } T_e^F + 1 \leq t \leq T_f^F, \\ F_{t-1} + e_t, & \text{otherwise,} \end{cases} \tag{3.2.2}$$

where $T_e^F$ and $T_f^F$ are the origination and termination dates of the bubble in the common component. Further,

$$U_{i,t} = \begin{cases} \rho_i U_{i,t-1} + z_{i,t}, & \text{for } T_e^{Ui} + 1 \leq t \leq T_f^{Ui}, \\ U_{i,t-1} + z_{i,t}, & \text{otherwise,} \end{cases} \tag{3.2.3}$$

where $T_e^{Ui}$ and $T_f^{Ui}$ are the origination and termination dates of the bubble in the $i$th idiosyncratic component. As $X_{i,t}$ exhibits bubble when either $F_t$ or $U_{i,t}$ does so, the bubble in $X_{i,t}$ starts at $T_e^{Xi} = \min\{T_e^F, T_e^{Ui}\}$ and ends at $T_f^{Xi} = \max\{T_f^F, T_f^{Ui}\}$. If these do not overlap, multiple bubbles appear in $X_{i,t}$. I let $h = T_f^{X_i} - T_e^{X_i}$ be the length of the explosive period.

When $r > 1$, one can consider the common factors series-by-series to investigate whether the common factor space is explosive. This is adequate because the existence of at least one explosive factor implies that the whole space is explosive. This contrasts with the case of investigating stationarity of common factor space. As Bai and Ng (2004) point out, random walks of individual factors do not necessarily imply a random walk for the common factor space if they have a cointegration relationship and a separate treatment may be needed.

I consider the AR coefficients in the explosive period in the forms of

$$\alpha = 1 + \frac{c}{k_h} \text{ and } \rho_i = 1 + \frac{c_i}{k_h}, \tag{3.2.4}$$

where $c$ and $c_i \geq 0$ for all $i$ are the localizing coefficients. I follow Phillips and Magdalinos (2007) to assume $k_h \to \infty$ and $k_h/h \to 0$ as $h \to \infty$ to justify the use of the methods of Phillips et al. (2011) and the previous chapter.

## 3.3 Date-stamping methodology

### 3.3.1 Univariate time series

Phillips et al. (2011) proposed the following methodology for date-stamping explosive behavior in a univariate time series, say $Y_t$ for $t = 1, ..., T$. I denote the ADF test using the subsample $t = 1, ..., T_0$, where $1 < T_0 < T$ of $Y_t$ by $ADF_Y^{[1,T_0]}$; that is, a $t$-test statistic for the coefficient $\delta_Y$ in the regression $Y_t - Y_{t-1} = \mu_Y + \delta_Y Y_{t-1} + error$. Thus, I construct a sequence of the ADF test statistics, starting from sample $t \in [1, T_0]$ with $T_0$ being the minimal amount of data, and extending forward $[1, T_0 + 1]$, $[1, T_0 + 2]$, ..., $[1, T]$. I identify explosive behaviors when the test statistic exceeds the critical value $(cv_t)$, diverging to infinity as $T \to \infty$, so that the origination date is estimated by:

$$\hat{T}_e^Y = \inf_{T_0 \leq t \leq T} \{t : ADF_Y^{[1,t]} > cv_t\}.$$

In practice, I follow Phillips et al.(2011, 2015a, 2015b) and set the critical value diverging at a rate of double logarithms. I apply this method to the estimated common and idiosyncratic components identified by the following methods.

### 3.3.2 Identifying the common and idiosyncratic components

#### 3.3.2.1 The PANIC method

The first approach to identifying the common and idiosyncratic components is the PANIC method proposed by Bai and Ng (2004). The main concept is simple. I use the principal component method of the first-differenced data ($x_{i,t} = X_{i,t} - X_{i,t-1}$) for the entire sample to estimate the first-differenced common components ($f_t = F_t - F_{t-1}$) and the idiosyncratic components ($u_{i,t} = U_{i,t} - U_{i,t-1}$). Then, the levels of the common and idiosyncratic components are recovered by accumulating their differences. The algorithm is described as follows.

1. Take first-differences of the observed data $x_{i,t} = X_{i,t} - X_{i,t-1}$ for $t = 2, ..., T$.

2. Obtain the principal component estimate of the common components ($\hat{f}_t$, $t = 2, ..., T$) out of $x_{i,t}$ as follows. Let $x$ be a $(T-1) \times N$ matrix, with the $(t, i)$th element being $x_{i,t+1}$ and $\hat{f} = [\hat{f}_2, ..., \hat{f}_T]'$ be a $(T-1) \times r$ matrix of an estimate for the common component. Then, $\hat{f}$ is the $\sqrt{T-1}$ times the eigenvectors of $xx'$ corresponding to the $r$ largest eigenvalues. The factor loadings and the first-differenced idiosyncratic components are estimated by $\hat{\lambda} = x'\hat{f}/(T-1)$, where $\hat{\lambda} = [\hat{\lambda}_1, ..., \hat{\lambda}_N]'$ an $N \times r$ matrix, and $\hat{u}_{i,t} = x_{i,t} - \hat{\lambda}_i'\hat{f}_t$ for $i = 1, ..., N$ and $t = 2, ..., T$, respectively.

3. The levels of the common and idiosyncratic components are obtained by $\hat{F}_t = \sum_{s=2}^{t} \hat{f}_s$ and $\hat{U}_{i,t} = \sum_{s=2}^{t} \hat{u}_{i,s}$, for $i = 1, \ldots, N$ and $t = 2, \ldots, T$, respectively.

Bai and Ng (2004) considered the left-tailed ADF test for the common and idiosyncratic components for the null hypothesis of unit root against an alternative hypothesis of stationarity. They showed that the tests for either component have the same asymptotic distribution as $N, T \to \infty$ and good size and power in finite samples. However, the property of the right-tailed version of the ADF test was not investigated.

#### 3.3.2.2 The CS method

The previous chapter thoroughly investigated the right-tailed version of the ADF tests under the PANIC framework. It showed that the test for the common component suffers from serious size distortions when the idiosyncratic components are explosive. It also showed that the test for the idiosyncratic components exhibits the nonmonotonic power problem such that the power can go down to zero when the idiosyncratic component is moderately or strongly explosive, because they may be misidentified as the common component. To address these problems, it proposed a method based on cross-sectional regressions to disentangle the common and idiosyncratic components. The algorithm is described as follows.

1. Divide the entire sample period ($t = 1, ..., T$) into two: the training sample ($t = 1, ..., T_1$) and the testing sample ($t = T_1 + 1, ..., T$). The training sample must be selected such that no explosive series or only weakly explosive series are included.

2. Use the first-differenced data $x_{i,t}$ in the training sample to estimate the first-differences of the common components and the factor loadings by the principal component method. Denote the first-differenced common components, the level estimates, and the factor loadings by $\hat{f}_t$ and $\hat{F}_t = \sum_{s=2}^{t} \hat{f}_s$ for $t = 2, ..., T_1$ and $\hat{\lambda}_i^*$ for $i = 1, ..., N$. The first differences of the idiosyncratic components are obtained by $\hat{u}_{i,t} = x_{i,t} - \hat{\lambda}_i^{*\prime} \hat{f}_t$ and the level estimates are obtained by $\hat{U}_{i,t} = \sum_{s=2}^{t} \hat{u}_{i,s}$ for $t = 2, ..., T_1$ in the training sample.

3. In the testing sample for $t = T_1 + 1, ..., T$, the common factor is estimated by the cross-sectional regression with $X_{i,t}$ and $\hat{\lambda}_i^*$ as the regressand and regressors for $i = 1, \ldots, N$

$$\hat{F}_t = \left( \sum_{i=1}^{N} \hat{\lambda}_i^* \hat{\lambda}_i^{*\prime} \right)^{-1} \left( \sum_{i=1}^{N} \hat{\lambda}_i^* X_{i,t} \right),$$

and the idiosyncratic components are estimated by $\hat{U}_{i,t} = X_{i,t} - \hat{\lambda}_i^{*\prime} \hat{F}_t$. Their first-differences are $\hat{f}_t = \hat{F}_t - \hat{F}_{t-1}$ and $\hat{u}_{i,t} = \hat{U}_{i,t} - \hat{U}_{i,t-1}$, respectively, for $t = T_1 + 1, ..., T$ .

In practice, this method requires us to select the sample demarcation point ($t = T_1$) in Step 1. Ideally, the training sample should be the longest possible, as far as the factor loading estimates $\hat{\lambda}_i^*$ are not contaminated by the inclusion of explosive series in the sample. Hence, I wish to select the sample $[1, T_1]$ in which no or only weakly explosive series are included in the data $X_{i,t}$. A simple scheme is to apply the aforementioned date-stamping method to the cross-sectional average $\bar{X}_t = N^{-1}\sum_{i=1}^{N} X_{it}$ for $t = 1, ..., T$ and use the estimated origination date as $T_1 + 1$. This criterion would be justified by the fact that, when some series start to be explosive, the average $(\bar{X}_t)$ also exhibits explosive behavior. A more direct criterion is given by the date-stamping method applied to the estimated common component via the full-sample PANIC method. This is because the factor loading estimate will not be contaminated as long as the factor estimate is not contaminated. I will use both criteria in the following section.

## 3.4 Monte Carlo simulations

In this section, I conduct Monte Carlo simulations to investigate the empirical probability of detecting explosive behaviors by the proposed procedures. I also assess the accuracy of the origination date when the explosive behavior is present and detected.

I first generate data using models (3.2.1), (3.2.2), (3.2.3), and (3.2.4), where $r = 1$ and $\mu_i = 0$ for all $i$. I set the following two experiments. In the first experiment, I generate data with explosive behavior only in the common component, such that $c > 0$ and $c_i = 0$ for all $i$. In the second experiment, the data contains explosive behaviors in all the idiosyncratic components but not in the common component so that $c = 0$ and $c_i > 0$ for all $i$. In both experiments, I identify the common and idiosyncratic components and implement the aforementioned date-stamping method for both components, enabling the study of correct and false detections. To report the results of the idiosyncratic components, I particularly select the first cross-section unit. This choice loses no generality as the model is symmetric for cross-section units.

I implement the following steps in each Monte Carlo replication. If the sequence of ADF test statistics exceeds the critical value for more than four consecutive periods, the first date when the test statistic exceeds the critical value is recorded as the origination dates: $\hat{T}_e^F$ for the common component and $\hat{T}_e^{Ui}$ for the idiosyncratic components. The critical values are set at $cv_t = \log(\log(t))/1.5$ following Phillips et al.'s (2011, 2015ab) recommendation for the rate of double logarithms. The scale constant 1.5 is chosen such that the false detection rate becomes approximately 5% in my setting by a separate simulation. Based on 5,000 replications, I compute the empirical probability that the explosive behavior is not detected falsely when there is an explosive behavior in the model. I also compute the empirical probability that the explosive behavior is incorrectly detected when there is no explosive behavior in the model. These are called error rates. Given that the bubble is correctly detected, I compute the averages of $\left(\frac{\hat{T}_e^F - T_e}{T}\right)$ and $\left(\frac{\hat{T}_e^{Ui} - T_e}{T}\right)$ across replications as the bias and the mean squared error (MSE) of $\frac{\hat{T}_e^F}{T}$ and $\frac{\hat{T}_e^{Ui}}{T}$. I present the case of $N = 100$, $T = 200$, $T_e^F = T_e^{Ui} = 40$, and $T_f^F = T_f^{U_i} = 80$ so that the duration of the bubble is $h = 40$. However, reasonable variations regarding the model setting do not affect qualitative results. The minimal amount of data is set at $T_0 = [r_0 T]$, where $r_0 = 0.01 + 1.8/\sqrt{T}$, following Phillips et al. (2015a). Because I am interested in how the results change as the AR coefficients varies, a set of values for $c$ and $c_i = [0.2, 0.4, \ldots, 2.0]$ is considered in the AR coefficients (3.2.4) with $k_h = h^\kappa$ and $\kappa = 0.85$. I compare the error rates for the common and idiosyncratic components, the bias, and the MSE for the PANIC, CS1, and CS2 methods, where CS1 uses $\bar{X}_t$ and CS2 uses $\hat{F}_t$ to select the end of the training sample ($T_1$). I also compute the error rates, the bias and the MSE when the true common and idiosyncratic components are used and these are labeled "Observed". These results are free from the effects of identification scheme and thus serve as a benchmark.

Figure 3.4.1 shows the results of the origination date for the case of explosive behavior in the common component but not in the idiosyncratic components. The error rate for the common component is high when $c$ is small, however, it steadily declines as $c$ increases. This corresponds to the standard power curve of the ADF test against the explosive alternative hypothesis. The error rate for the idiosyncratic components remains low in any methods, however, I observe some tendency for overdetection of CS1 and CS2 when the common component are strongly explosive. This reflects the size-distortions of the CS based test for idiosyncratic components documented by the previous chapter. There is no large differences in the biases and the MSEs across the four methods in this case. Overall, PANIC shows very similar properties to the Observed, which proves the usefulness of PANIC in this case. However, the results of CS1 and CS2 do not differ much from them.

Figure 3.4.2 in turn presents the results of the origination date for the case when the idiosyncratic components have explosive behaviors but the common component does not. Remarkably, the error rate for the common component when using PANIC rapidly increases to one. This corresponds to the size distortion of the ADF test for the common component when the idiosyncratic

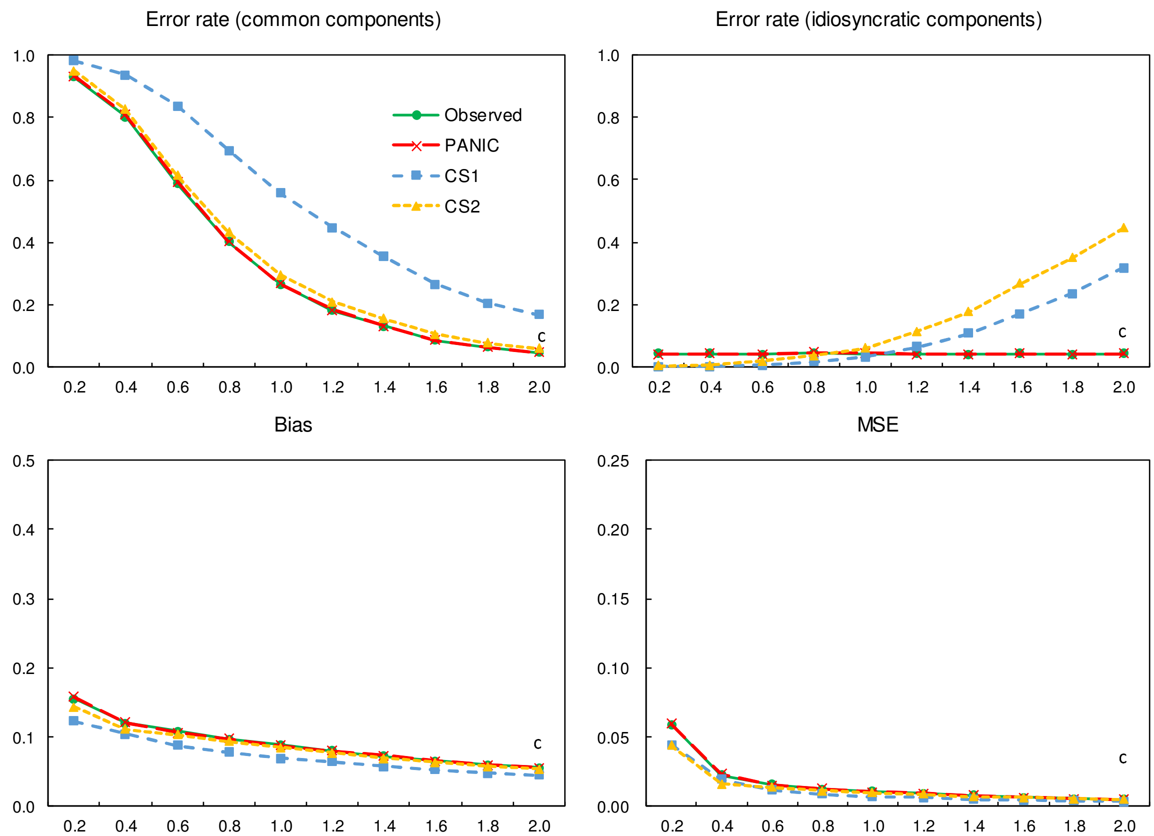


Figure 3.4.1: Bias, MSE and error rates when the explosive behavior is in the common components.

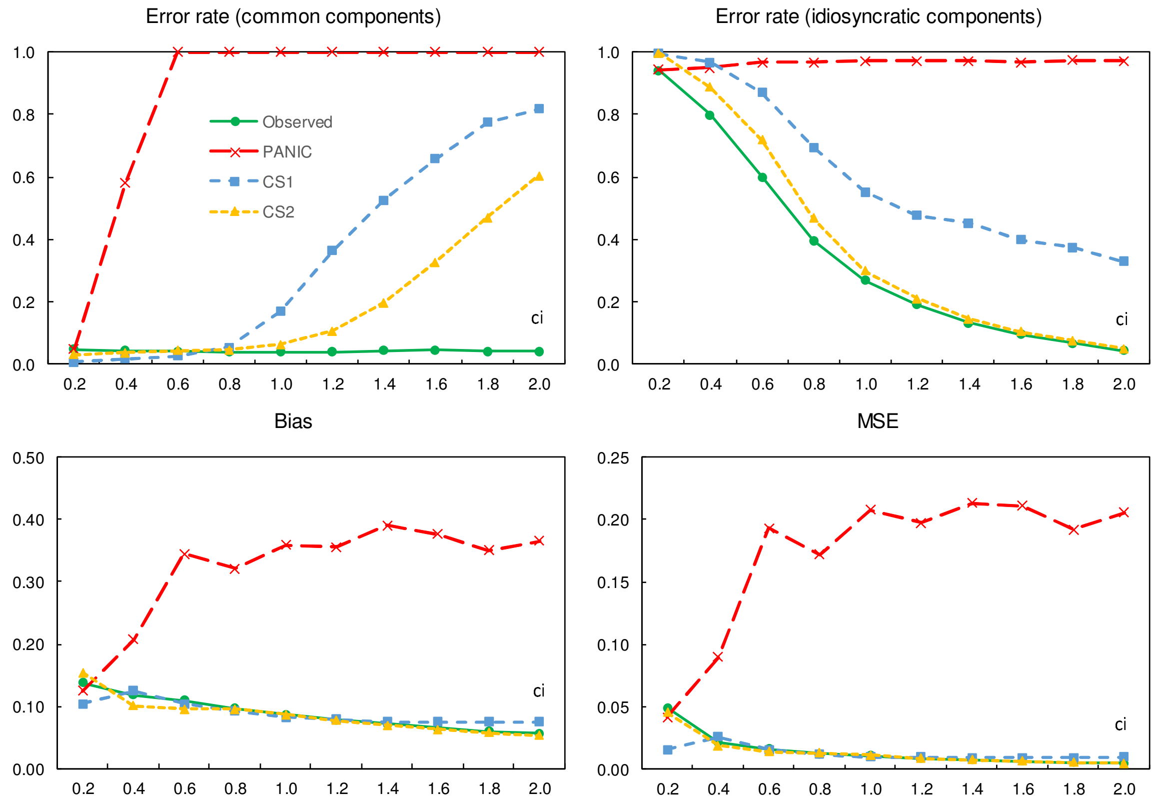


Figure 3.4.2: Bias, MSE and error rates when the explosive behaviors are in the idiosyncratic components.

components are explosive. This occurs because PANIC misidentifies the common and idiosyncratic components and suffers from spurious explosive behavior in the former. More interestingly, the error rate for idiosyncratic components when using PANIC remains very close to one. This is due to the nonmonotonic power problem of the ADF test applied to the idiosyncratic components pointed out in the previous chapter. These imply that the explosive behavior in the common component is falsely detected and the explosive behaviors in the idiosyncratic components are hardly detected by PANIC. In contrast, the error rates of the idiosyncratic components when using CS1 or CS2 show similar patterns to that of the Observed. The error rates of the common component when using CS1 and CS2 are low, although I see some tendency for overdetection when $c_i$ becomes large. The bias and the MSE when using PANIC are high across all values of $c_i$, however, when using CS1 or CS2, the values are very low and similar to those of the Observed. Finally, although the bias and MSE are very similar for CS1 and CS2, the latter tends to provide a lower error rate than the former in these experiments.

In summary, the properties of the right-tailed unit root tests investigated in the previous chapter are passed over to the date-stamping of explosive behavior. When it lies only in the common component, the origination date is precisely estimated by either PANIC or CS. However, when the explosive behaviors exist in the idiosyncratic components, PANIC loses its power of detection and the origination date is inaccurate. These problems are addressed by using CS, although some tendency for overdetection is observed when the idiosyncratic components are strongly explosive.

## 3.5 Conclusions

In this study, I applied the date-stamping methodology for the origination of explosive behaviors proposed in the seminal work of Phillips et al. (2011) to the large dimensional common factor model. To this end, I compared two methods of identifying the common and idiosyncratic components: PANIC and CS. As discovered by the previous chapter, when the PANIC method is used, the ADF test for the common component may suffer from serious size distortions and that for the idiosyncratic components exhibits the nonmonotonic power problem. These features are passed over to the study regarding date-stamping. Monte Carlo simulations show that, when the explosive behavior lies only in the common component, the origination date is precisely estimated by either the PANIC or the CS method. However, when the explosive behaviors exist in the idiosyncratic components, the PANIC method loses its power of detection and the origination dates are inaccurate. These problems are resolved by using the CS method, although some tendency of overdetection is observed when the idiosyncratic components are strongly explosive.

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# Appendix A

# Proofs of Theorems in Chapter 1

## A.1 Preliminaries for Theorem 1.1

Let $V_{NT}$ be an $r \times r$ diagonal matrix of the first $r$ largest eigenvalues of $(NT)^{-1}XX'$ and the $k$th eigenvalue be $v_{NT,k}$. Let $H = (\Lambda'\Lambda/N)(F'\hat{F}/T)V_{NT}^{-1} = (h_1, \ldots, h_r)$ be an $r \times r$ rotation matrix where $h_k$ is the $k$th column. I also use the notation $\delta_{NT} = min(\sqrt{N}, \sqrt{T})$. The following identity equation holds.

$$\hat{f}_{k,t} = h_k' f_t + u_{k,t},$$

where

$$u_{k,t} = v_{NT,k}^{-1}\left(T^{-1}\sum_{s=1}^{T}\hat{f}_{k,s}\gamma_{st} + T^{-1}\sum_{s=1}^{T}\hat{f}_{k,s}\zeta_{st} + T^{-1}\sum_{s=1}^{T}\hat{f}_{k,s}\eta_{st} + T^{-1}\sum_{s=1}^{T}\hat{f}_{k,s}\xi_{st}\right),$$

with $\gamma_{s,t} = N^{-1}\sum_{i=1}^{N}E(e_{i,s}e_{i,t})$, $\zeta_{st} = N^{-1}[e_t'e_s - E(e_t'e_s)]$, $\eta_{st} = N^{-1}f_s'\Lambda' e_t$ and $\xi_{st} = N^{-1}f_t'\Lambda' e_s$.

**Proposition A.1.** *Under Assumptions 1.1–1.4, the following hold.*

1. $T^{-1}\sum_{t=1}^{T}(\hat{f}_{k,t} - h_k' f_t)^2 = O_p(\delta_{NT}^{-2})$,

2. $V_{NT} = O_p(1)$,

3. $h_k = O_p(1)$,

4. $T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}\gamma_{st}^2 \leq m$,

5. $E\left(T^{-1}\sum_{t=1}^{T}\left\|N^{-1/2}\Lambda' e_t\right\|^2\right) \le m$.

*Proof.* First three propositions are shown in Bai (2003). Proof of 1 is Lemma A.1 of Bai (2003). Proof of 2 and 3 are implicitly shown by Lemma A.3 of Bai (2003). The last two propositions are shown in Lemma 1.(i) and Lemma 1.(ii) of Bai and Ng (2002). □

**Lemma A.1.** *Under Assumptions 1.1–1.4, I have*

1. $T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\gamma_{st} = O_p(T^{-\frac{1}{2}}\delta_{NT}^{-1})$,
2. $T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\zeta_{st} = O_p(T^{-\frac{1}{4}}N^{-\frac{1}{2}})$,
3. $T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\eta_{st} = O_p(T^{-\frac{1}{2}}N^{-\frac{1}{2}})$,
4. $T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\xi_{st} = O_p(T^{-\frac{3}{4}}N^{-\frac{1}{2}})$,
5. $T^{-1}\sum_{t=1}^{T}(\hat{f}_{k,t} - h_k' f_t)^3 = O_p(\delta_{NT}^{-3})$,
6. $T^{-1}\sum_{t=1}^{T}(\hat{f}_{k,t} - h_k' f_t)^4 = O_p(\delta_{NT}^{-4})$.

*Proof.* These can be proved almost the same way as Lemma A.2 in Bai (2003).

1. Let us consider

$$T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\gamma_{st} = T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_{k,s} - h_k' f_s\right)\gamma_{st} + T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}h_k' f_s\gamma_{st}.$$

The first term is

$$\left|T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_{k,s} - h_k' f_s\right)\gamma_{st}\right| \le T^{-\frac{3}{2}}\sum_{t=1}^{T}\left[T^{-1}\sum_{s=1}^{T}\left(\hat{f}_{k,s} - h_k' f_s\right)^2\right]^{\frac{1}{2}}\left(\sum_{s=1}^{T}\gamma_{st}^2\right)^{\frac{1}{2}}.$$

Because $T^{-1/2}\sum_{t=1}^{T}|y_t|^{1/2} \le (\sum_{t=1}^{T}|y_t|)^{1/2}$ , I have

$$T^{-\frac{3}{2}}\sum_{t=1}^{T}\left(\sum_{s=1}^{T}\gamma_{st}^2\right)^{\frac{1}{2}} \le T^{-\frac{1}{2}}\left(T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}\gamma_{st}^2\right)^{\frac{1}{2}},$$

by Proposition A.1.4. Therefore, this is $O_p(T^{-\frac{1}{2}}\delta_{NT}^{-1})$ by Proposition A.1.1. The second term without $h_k'$ is $O_p(T^{-1})$ since

$$E\left\|T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}f_s\gamma_{st}\right\| \le \left(\max_{1\le s\le T}E\|f_s\|\right)T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}|\gamma_{st}| \le m,$$

by Assumptions 1.1.1 and 1.3.4. Hence, the lemma holds.

2. Consider

$$T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\zeta_{st}=T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_{k,s}-h_k'f_s\right)\zeta_{st}+T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}h_k'f_s\zeta_{st}.$$

The first term is $O_p(N^{-\frac{1}{2}}\delta_{NT}^{-1})$ by Bai (2003). The second term without $h_k'$ is

$$\begin{aligned}\left\|T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}f_s\zeta_{st}\right\| &\leq T^{-\frac{1}{4}}\left(T^{-\frac{1}{2}}\sum_{s=1}^{T}\|f_s\|^2\right)^{\frac{1}{2}}T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{s=1}^{T}\zeta_{st}^2\right)^{\frac{1}{2}}\\ &= T^{-\frac{1}{4}}O_p(1)\,O_p\left(N^{-\frac{1}{2}}\right).\end{aligned}$$

This is because the first set of parentheses is $O_p(1)$ by Assumption 1.1.3. The second set of parentheses is shown in Bai (2003) and is $O_p(N^{-\frac{1}{2}})$. Hence, the lemma holds.

3. Consider

$$T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\eta_{st}=T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_{k,s}-h_k'f_s\right)\eta_{st}+T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}h_k'f_s\eta_{st}.$$

The first term is

$$\left|T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_{k,s}-h_k'f_s\right)\eta_{st}\right|\leq\left[T^{-1}\sum_{t=1}^{T}\left(\hat{f}_{k,s}-h_k'f_s\right)^2\right]^{\frac{1}{2}}T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{s=1}^{T}\eta_{st}^2\right)^{\frac{1}{2}}.$$

The second set of parentheses is

$$\begin{aligned}T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{s=1}^{T}\eta_{st}^2\right)^{\frac{1}{2}} &\leq T^{-\frac{1}{2}}\left(T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}\eta_{st}^2\right)^{\frac{1}{2}},\\ &= T^{-\frac{1}{2}}\left(T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(N^{-1}f_s'\Lambda'e_t\right)^2\right)^{\frac{1}{2}},\\ &\leq T^{-\frac{1}{4}}N^{-\frac{1}{2}}\left(T^{-1}\sum_{t=1}^{T}\left\|N^{-\frac{1}{2}}\Lambda'e_t\right\|^2T^{-\frac{1}{2}}\sum_{s=1}^{T}\|f_s\|^2\right)^{\frac{1}{2}},\\ &= O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right),\end{aligned}$$

by Assumption 1.1.3 and Proposition A.1.5. Therefore the first term is $O_p(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\delta_{NT}^{-1})$.

The second term without $h_k'$ is

$$\left\|T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T} f_s\eta_{st}\right\| \leq T^{-\frac{1}{4}}\left(T^{-\frac{1}{2}}\sum_{s=1}^{T}\|f_s\|^2\right)^{\frac{1}{2}} T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{s=1}^{T}\eta_{st}^2\right)^{\frac{1}{2}}.$$

Therefore, this is $O_p(T^{-\frac{1}{2}}N^{-\frac{1}{2}})$. Hence, the lemma holds.

4. Consider

$$\begin{aligned} T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\xi_{st} &= T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_{k,s}-h_k' f_s\right)\xi_{st} \\ &\quad + T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T} h_k' f_s \xi_{st}, \\ &= T^{-2}N^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_{k,s}-h_k' f_s\right) e_s'\Lambda f_t \\ &\quad + T^{-2}N^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T} h_k' f_s e_s'\Lambda f_t. \end{aligned}$$

The first term is $O_p(N^{-\frac{1}{2}}\delta_{NT}^{-1})$ by Bai (2003). The second term without $h_k'$ is

$$\begin{aligned} \left\|T^{-2}N^{-1}\sum_{t=1}^{T}\sum_{s=1}^{T} f_s e_s'\Lambda f_t\right\| &\leq T^{-\frac{3}{4}}N^{-\frac{1}{2}}\left\|T^{-\frac{1}{2}}\sum_{t=1}^{T} f_t\right\|\left(T^{-\frac{1}{2}}\sum_{s=1}^{T}\|f_s\|^2\right)^{\frac{1}{2}} \\ &\quad \times\left(T^{-1}\sum_{s=1}^{T}\left\|N^{-\frac{1}{2}}e_s'\Lambda\right\|^2\right)^{\frac{1}{2}}. \end{aligned}$$

The third set of parentheses is bounded by Proposition A.1.5. Therefore, this is $O_p(T^{-\frac{3}{4}}N^{-\frac{1}{2}})$ and the lemma holds.

5. Using Lemma A.1.6,

$$\begin{aligned} \left|T^{-1}\sum_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)^3\right| &\leq \left[T^{-1}\sum_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)^2\right]^{\frac{1}{2}}\left[T^{-1}\sum_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)^4\right]^{\frac{1}{2}} \\ &= O_p\left(\delta_{NT}^{-1}\right)\times O_p\left(\delta_{NT}^{-2}\right) = O_p\left(\delta_{NT}^{-3}\right). \end{aligned}$$

Therefore, the lemma holds.

6. Because Hölder's inequality implies $(x+y+z+u)^4 \leq 4^3(x^4+y^4+z^4+u^4)$ , I obtain

$$\left(\hat{f}_{k,t}-h_k' f_t\right)^4 \leq \hat{v}_{NT,k}^{-4}4^3\left(a_t^2+b_t^2+c_t^2+d_t^2\right),$$

where $a_t = (T^{-1}\sum_{s=1}^{T}\hat{f}_{k,s}\gamma_{st})^2$, $b_t = (T^{-1}\sum_{s=1}^{T}\hat{f}_{k,s}\zeta_{st})^2$, $c_t = (T^{-1}\sum_{s=1}^{T}\hat{f}_{k,s}\eta_{st})^2$ and $d_t = (T^{-1}\sum_{s=1}^{T}\hat{f}_{k,s}\xi_{st})^2$. Because $a_t$ is positive,

$$T^{-1}\sum\nolimits_{t=1}^{T} a_t^2 \le \left(T^{-1}\sum\nolimits_{t=1}^{T} a_t\right)^2.$$

In addition, $T^{-1}\sum_{t=1}^{T} a_t = O_p\left(T^{-1}\right)$ by Bai and Ng (2002). Hence, $T^{-1}\sum_{t=1}^{T} a_t^2 = O_p\left(T^{-2}\right)$. Similarly,

$$T^{-1}\sum\nolimits_{t=1}^{T} b_t^2 = O_p\left(N^{-2}\right), \quad T^{-1}\sum\nolimits_{t=1}^{T} c_t^2 = O_p\left(N^{-2}\right), \quad T^{-1}\sum\nolimits_{t=1}^{T} d_t^2 = O_p\left(N^{-2}\right).$$

Therefore, $T^{-1}\sum_{t=1}^{T}(\hat{f}_{k,t} - h_k' f_t)^4 = O_p(\delta_{NT}^{-4})$.

□

**Lemma A.2.** *Under Assumptions 1.1–1.4,*

1. $\hat{\mu}_{k,1}^f = T^{-1}\sum_{t=1}^{T} h_k' f_t + O_p(T^{-\frac{1}{4}}N^{-\frac{1}{2}})$,
2. $\hat{\mu}_{k,2}^f = T^{-1}\sum_{t=1}^{T}[(f_t - \mu_1^f)^{\otimes 2}]' h_k^{\otimes 2} + O_p(T^{-\frac{1}{4}}N^{-\frac{1}{2}})$,
3. $\hat{\mu}_{k,3}^f = \begin{array}{l} T^{-1}\sum_{t=1}^{T}[(f_t - \mu_1^f)^{\otimes 3}]' h_k^{\otimes 3} - 3T^{-1}\sum_{t=1}^{T}[(f_t - \mu_1^f)^{\otimes 2}]' h_k^{\otimes 2} T^{-1}\sum_{t=1}^{T}(f_t - \mu_1^f)' h_k \\ + O_p(T^{-\frac{1}{4}}N^{-\frac{1}{2}}). \end{array}$

*Proof.*

1. Consider

$$\begin{aligned}
\hat{\mu}_{k,1}^f &= T^{-1}\sum\nolimits_{t=1}^{T}\hat{f}_{k,t} \\
&= T^{-1}\sum\nolimits_{t=1}^{T} h_k' f_t + T^{-1}\sum\nolimits_{t=1}^{T} u_{k,t}.
\end{aligned}$$

The second term is

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T} u_{k,t} &= v_{NT,k}^{-1}\begin{pmatrix} T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\gamma_{st}+T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\zeta_{st} \\ +T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\eta_{st}+T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_{k,s}\xi_{st} \end{pmatrix} \\
&= O_P(1)\begin{bmatrix} O_p\left(T^{-\frac{1}{2}}\delta_{NT}^{-1}\right)+O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right) \\ +O_p\left(T^{-\frac{1}{2}}N^{-\frac{1}{2}}\right)+O_p\left(T^{-\frac{3}{4}}N^{-\frac{1}{2}}\right) \end{bmatrix} \\
&= O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right)
\end{aligned}$$

by Proposition A.1.2 and Lemmas A.1.1 to A.1.4. Hence the lemma holds.

2. Consider

$$\begin{aligned}
\hat{\mu}_{k,2}^{f} &= T^{-1}\sum_{t=1}^{T}\left(\hat{f}_{k,t}-\hat{\mu}_{k,1}^{f}\right)^{2} \\
&= T^{-1}\sum_{t=1}^{T}\begin{pmatrix} \underbrace{\hat{f}_{k,t}-h_k' f_t}_{=A_1}+\underbrace{h_k' f_t-h_k'\mu_1^f}_{=A_2} \\ +\underbrace{h_k'\mu_1^f-T^{-1}\sum_{s=1}^{T}h_k' f_s}_{=A_3}-\underbrace{T^{-1}\sum_{s=1}^{T}u_{k,s}}_{=A_4} \end{pmatrix}^{2}.
\end{aligned}$$

I consider the sums of squares of each term. $T^{-1}\sum_{t=1}^{T}A_1^2$ is

$$T^{-1}\sum_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)^2 = O_P\left(\delta_{NT}^{-2}\right)$$

by Proposition A.1.1. $T^{-1}\sum_{t=1}^{T}A_3^2$ is

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T}\left(h_k'\mu_1^f-T^{-1}\sum_{s=1}^{T}h_k' f_s\right)^2 &= T^{-2}\sum_{t=1}^{T}\left[\left(T^{-\frac{1}{2}}\sum_{s=1}^{T}\left(f_s-\mu_1^f\right)\right)' h_k\right]^2 \\
&= O_p\left(T^{-1}\right)
\end{aligned}$$

by Assumption 1.1.3. $T^{-1}\sum_{t=1}^{T}A_4^2$ is

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{s=1}^{T}u_{k,s}\right)^2 &= \left[O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right)\right]^2 \\
&= O_p\left(T^{-\frac{1}{2}}N^{-1}\right)
\end{aligned}$$

by Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_1 A_2$ is

$$\begin{aligned}\left|T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)\left(h_k' f_t - h_k'\mu_1^f\right)\right| &\leq T^{-1}\sum\nolimits_{t=1}^{T}\left|\hat{f}_{k,t}-h_k' f_t\right|\left\|f_t-\mu_1^f\right\|\|h_k\| \\ &\leq T^{-\frac{1}{4}}\left[T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)^2\right]^{\frac{1}{2}} \\ &\quad\times\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left\|f_t-\mu_1^f\right\|^2\right)^{\frac{1}{2}}\|h_k\| \\ &= O_p\left(T^{-\frac{1}{4}}\delta_{NT}^{-1}\right)\end{aligned}$$

by Assumption 1.1.3 and Proposition A.1.1. $T^{-1}\sum_{t=1}^{T} A_2 A_3$ is

$$\begin{aligned}\left|T^{-1}\sum\nolimits_{t=1}^{T}\left(h_k' f_t - h_k'\mu_1^f\right)\left(h_k'\mu_1^f - T^{-1}\sum\nolimits_{s=1}^{T} h_k' f_s\right)\right| &\leq T^{-1}\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left\|f_t-\mu_1^f\right\|\|h_k\|\right)^2 \\ &= O_p\left(T^{-1}\right)\end{aligned}$$

by Assumption 1.1.3. $T^{-1}\sum_{t=1}^{T} A_3 A_4$ is

$$\begin{aligned}T^{-1}\sum\nolimits_{t=1}^{T}\left(h_k'\mu_1^f - T^{-1}\sum_{s=1}^{T} h_k' f_s\right)\left(T^{-1}\sum\nolimits_{s=1}^{T} u_{k,s}\right) &= O_p\left(T^{-\frac{1}{2}}\right)O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right) \\ &= O_p\left(T^{-\frac{3}{4}}N^{-\frac{1}{2}}\right)\end{aligned}$$

by Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_1 A_4$ is

$$\begin{aligned}T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)\left(T^{-1}\sum\nolimits_{s=1}^{T} u_{k,s}\right) &= \left(T^{-1}\sum\nolimits_{s=1}^{T}\varepsilon_{k,s}\right)^2 \\ &= O_p\left(T^{-\frac{1}{2}}N^{-1}\right)\end{aligned}$$

by Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_1 A_3$ is

$$\begin{aligned}T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)\left(h_k'\mu_1^f - T^{-1}\sum\nolimits_{s=1}^{T} h_k' f_s\right) &= O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right)O_p\left(T^{-\frac{1}{2}}\right) \\ &= O_p\left(T^{-\frac{3}{4}}N^{-\frac{1}{2}}\right)\end{aligned}$$

by Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_2 A_4$ is

$$\begin{aligned} T^{-1}\sum\nolimits_{t=1}^{T}\left(h_k' f_t - h_k'\mu_1^f\right)\left(T^{-1}\sum\nolimits_{s=1}^{T} u_{k,s}\right) &= O_p\left(T^{-\frac{1}{2}}\right) O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right) \\ &= O_p\left(T^{-\frac{3}{4}}N^{-\frac{1}{2}}\right) \end{aligned}$$

by Lemma A.2.1. Therefore,

$$\begin{aligned} T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t} - \hat{\mu}_{k,1}^f\right)^2 &= T^{-1}\sum\nolimits_{t=1}^{T}\left[\left(f_t - \mu_1^f\right)' h_k\right]^2 + O_p\left(T^{-\frac{1}{4}}\delta_{NT}^{-1}\right) \\ &= T^{-1}\sum\nolimits_{t=1}^{T}\left[\left(f_t - \mu_1^f\right)^{\otimes 2}\right]' h_k^{\otimes 2} + O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right). \end{aligned}$$

3. Consider

$$\hat{\mu}_{k,3}^f = T^{-1}\sum\nolimits_{t=1}^{T}\left(A_1 + A_2 + A_3 + A_4\right)^3.$$

$T^{-1}\sum_{t=1}^{T} A_1^3 = O_p\left(\delta_{NT}^{-3}\right)$ by Lemma A.1.5. $T^{-1}\sum_{t=1}^{T} A_3^3$ is

$$\begin{aligned} \left|T^{-1}\sum\nolimits_{t=1}^{T}\left(h_k'\mu_1^f - T^{-1}\sum\nolimits_{s=1}^{T} h_k' f_s\right)^3\right| &\leq T^{-\frac{3}{2}}\left(T^{-\frac{1}{2}}\sum\nolimits_{s=1}^{T}\left\|f_s - \mu_1^f\right\|\|h_k\|\right)^3 \\ &= O_p\left(T^{-\frac{3}{2}}\right) \end{aligned}$$

by Assumption 1.1.3. $T^{-1}\sum_{t=1}^{T} A_4^3$ is

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(T^{-1}\sum\nolimits_{s=1}^{T} u_{k,s}\right)^3 = O_p\left(T^{-\frac{3}{4}}N^{-\frac{3}{2}}\right)$$

by Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_1^2 A_2$ is

$$\begin{aligned} \left|T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t} - h_k' f_t\right)^2\left(h_k' f_t - h_k'\mu_1^f\right)\right| &\leq T^{-\frac{1}{4}}\left[T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t} - h_k' f_t\right)^4\right]^{\frac{1}{2}} \\ &\quad \times\left[T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left\|f_t - \mu_1^f\right\|^2\right]^{\frac{1}{2}}\|h_k\| \\ &= T^{-\frac{1}{4}} O_p\left(\delta_{NT}^{-2}\right) \times O_p(1) = O_p\left(T^{-\frac{1}{4}}\delta_{NT}^{-2}\right) \end{aligned}$$

by Lemma A.1.6 and Assumption 1.1.3. $T^{-1}\sum_{t=1}^{T} A_1^2 A_3$ is

$$T^{-1}\sum_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)^2\left(h_k'\mu_1^f - T^{-1}\sum_{s=1}^{T} h_k' f_s\right) = O_p\left(\delta_{NT}^{-2}\right)\times T^{-\frac{1}{2}} O_p(1) = O_p\left(T^{-\frac{1}{2}}\delta_{NT}^{-2}\right)$$

by Proposition A.1.1. $T^{-1}\sum_{t=1}^{T} A_1^2 A_4$ is

$$\begin{aligned} T^{-1}\sum_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)^2\left(T^{-1}\sum_{s=1}^{T} u_{k,s}\right) &= O_p\left(\delta_{NT}^{-2}\right) O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right) \\ &= O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\delta_{NT}^{-2}\right) \end{aligned}$$

by Proposition A.1.1 and Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_2^2 A_1$ is

$$\begin{aligned} \left|T^{-1}\sum_{t=1}^{T}\left(h_k' f_t - h_k'\mu_1^f\right)^2\left(\hat{f}_{k,t}-h_k' f_t\right)\right| &\leq T^{-1}\sum_{t=1}^{T}\left\|f_t-\mu_1^f\right\|^2\|h_k\|^2\left|\hat{f}_{k,t}-h_k' f_t\right| \\ &\leq T^{-\frac{1}{4}}\left(T^{-\frac{1}{2}}\sum_{t=1}^{T}\left\|f_t-\mu_1^f\right\|^2\right)^{\frac{1}{2}} \\ &\quad\times\left(T^{-1}\sum_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)^2\right)^{\frac{1}{2}}\|h_k\|^2 \\ &= T^{-\frac{1}{4}} O_p(1)\, O_p\left(\delta_{NT}^{-1}\right) O_p(1) \\ &= O_p\left(T^{-\frac{1}{4}}\delta_{NT}^{-1}\right) \end{aligned}$$

by Assumption 1.1.3 and Proposition A.1.1. $T^{-1}\sum_{t=1}^{T} A_2^2 A_4$ is

$$\begin{aligned} \left|T^{-1}\sum_{t=1}^{T}\left(h_k' f_t - h_k'\mu_1^f\right)^2\left(T^{-1}\sum_{s=1}^{T} u_{k,s}\right)\right| &\leq T^{-1}\sum_{t=1}^{T}\left\|f_t-\mu_1^f\right\|^2\|h_k\|^2\left|T^{-1}\sum_{s=1}^{T} u_{k,s}\right| \\ &= O_p\left(T^{-\frac{1}{2}}\right) O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right) \\ &= O_p\left(T^{-\frac{3}{4}}N^{-\frac{1}{2}}\right) \end{aligned}$$

by Assumption 1.1.3 and Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_3^2 A_1$ is

$$\begin{aligned}\left|T^{-1}\sum_{t=1}^{T}\left(h_k'\mu_1^f - T^{-1}\sum_{s=1}^{T} h_k' f_s\right)^2\left(\hat{f}_{k,t} - h_k' f_t\right)\right| &\leq T^{-1}\sum_{t=1}^{T}\left\|f_t - \mu_1^f\right\|^2 \|h_k\|^2 \\ &\quad \times \left|T^{-1}\sum_{t=1}^{T}\left(\hat{f}_{k,t} - h_k' f_t\right)\right| \\ &= O_p\left(T^{-\frac{1}{2}}\right) O_p\left(T^{-\frac{1}{4}} N^{-\frac{1}{2}}\right) \\ &= O_p\left(T^{-\frac{3}{4}} N^{-\frac{1}{2}}\right)\end{aligned}$$

by Assumption 1.1.3 and Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_3^2 A_2$ is

$$T^{-1}\sum_{t=1}^{T}\left(h_k'\mu_1^f - T^{-1}\sum_{s=1}^{T} h_k' f_s\right)^2\left(h_k' f_t - h_k'\mu_1^f\right) = O_p\left(T^{-\frac{3}{2}}\right)$$

by Assumption 1.1.3. $T^{-1}\sum_{t=1}^{T} A_3^2 A_4$ is

$$\begin{aligned}T^{-1}\sum_{t=1}^{T}\left(h_k'\mu_1^f - T^{-1}\sum_{s=1}^{T} h_k' f_s\right)^2\left(T^{-1}\sum_{s=1}^{T} u_{k,s}\right) &= O_p\left(T^{-1}\right) O_p\left(T^{-\frac{1}{4}} N^{-\frac{1}{2}}\right) \\ &= O_p\left(T^{-\frac{5}{4}} N^{-\frac{1}{2}}\right)\end{aligned}$$

by Assumption 1.1.3 and Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_4^2 A_1$ is

$$\begin{aligned}T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{s=1}^{T} u_{k,s}\right)^2\left(\hat{f}_{k,t} - h_k' f_t\right) &= \left(T^{-1}\sum_{s=1}^{T}\varepsilon_{k,s}\right)^3 \\ &= O_p\left(T^{-\frac{3}{4}} N^{-\frac{3}{2}}\right)\end{aligned}$$

by Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_4^2 A_2$ is

$$\begin{aligned}T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{s=1}^{T} u_{k,s}\right)^2\left(h_k' f_t - h_k'\mu_1^f\right) &= O_p\left(T^{-\frac{1}{2}} N^{-1}\right) O_p\left(T^{-\frac{1}{2}}\right) \\ &= O_p\left(T^{-1} N^{-1}\right)\end{aligned}$$

by Assumption 1.1.3 and Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_4^2 A_3$ is

$$\begin{aligned}T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{s=1}^{T} u_{k,s}\right)^2\left(h_k'\mu_1^f - T^{-1}\sum_{s=1}^{T} h_k' f_s\right) &= O_p\left(T^{-\frac{1}{2}} N^{-1}\right) O_p\left(T^{-\frac{1}{2}}\right) \\ &= O_p\left(T^{-1} N^{-1}\right)\end{aligned}$$

by Assumption 1.1.3 and Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_1 A_2 A_3$ is

$$\begin{aligned}
&\left|T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)\left(h_k' f_t-h_k'\mu_1^f\right)\left(h_k'\mu_1^f-T^{-1}\sum\nolimits_{s=1}^{T} h_k' f_s\right)\right| \\
&\quad\leq T^{-\frac{3}{4}}\left[T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)^2\right]^{\frac{1}{2}}\left[T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left\|f_t-\mu_1^f\right\|^2\right]^{\frac{1}{2}}\|h_k\|^2\left(T^{-\frac{1}{2}}\sum\nolimits_{s=1}^{T}\left\|f_s-\mu_1^f\right\|\right) \\
&\quad= T^{-\frac{3}{4}} O_p\left(\delta_{NT}^{-1}\right)\times O_p(1)\times O_p(1) \\
&\quad= O_p\left(T^{-\frac{3}{4}}\delta_{NT}^{-1}\right)
\end{aligned}$$

by Assumption 1.1.3 and Proposition A.1.1. $T^{-1}\sum_{t=1}^{T} A_1 A_2 A_4$ is

$$\begin{aligned}
&\left|T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)\left(h_k' f_t-h_k'\mu_1^f\right)\left(T^{-1}\sum\nolimits_{s=1}^{T} u_{k,s}\right)\right| \\
&\quad\leq T^{-\frac{1}{4}}\left[T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)^2\right]^{\frac{1}{2}}\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left\|f_t-\mu_1^f\right\|^2\right)^{\frac{1}{2}}\|h_k\|\left|T^{-1}\sum\nolimits_{s=1}^{T} u_{k,s}\right| \\
&\quad= T^{-\frac{1}{4}} O_p\left(\delta_{NT}^{-1}\right)\times O_p(1)\times O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right) \\
&\quad= O_p\left(T^{-\frac{1}{2}}N^{-\frac{1}{2}}\delta_{NT}^{-2}\right)
\end{aligned}$$

by Assumption 1.1.3, Proposition A.1.1 and Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_1 A_3 A_4$ is

$$\begin{aligned}
&T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_{k,t}-h_k' f_t\right)\left(h_k'\mu_1^f-T^{-1}\sum\nolimits_{s=1}^{T} h_k' f_s\right)\left(T^{-1}\sum\nolimits_{s=1}^{T} u_{k,s}\right) \\
&\quad= O_p\left(T^{-\frac{1}{2}}N^{-1}\right) O_p\left(T^{-\frac{1}{2}}\right) \\
&\quad= O_p\left(T^{-1}N^{-1}\right)
\end{aligned}$$

by Assumption 1.1.3 and Lemma A.2.1. $T^{-1}\sum_{t=1}^{T} A_2 A_3 A_4$ is

$$\begin{aligned}
&T^{-1}\sum\nolimits_{t=1}^{T}\left(h_k' f_t-h_k'\mu_1^f\right)\left(h_k'\mu_1^f-T^{-1}\sum\nolimits_{s=1}^{T} h_k' f_s\right)\left(T^{-1}\sum\nolimits_{s=1}^{T} u_{k,s}\right) \\
&\quad= O_p\left(T^{-\frac{1}{2}}\right)\times O_p\left(T^{-\frac{1}{2}}\right)\times O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right) \\
&\quad= O_p\left(T^{-\frac{5}{4}}N^{-\frac{1}{2}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\mu}_{k,3} &= T^{-1}\sum\nolimits_{t=1}^{T} A_2^3 + T^{-1}\sum\nolimits_{t=1}^{T} A_2^2 A_3 + O_p\left(T^{-\frac{1}{4}}\delta_{NT}^{-1}\right) \\
&= T^{-1}\sum\nolimits_{t=1}^{T}\left[\left(f_t-\mu_1^f\right)^{\otimes 3}\right]' h_k^{\otimes 3} \\
&\quad -3T^{-1}\sum\nolimits_{t=1}^{T}\left[\left(f_t-\mu_1^f\right)^{\otimes 2}\right]' h_k^{\otimes 2} T^{-1}\sum\nolimits_{t=1}^{T}\left(f_t-\mu_1^f\right)' h_k + O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right).
\end{aligned}$$

Hence, the lemma holds.

□

Let $\alpha_k$ be the probability limit of $\hat{\alpha}_k$ where$\hat{\alpha}_k = (h_k^{\otimes 3\prime}, (-3\mu_2^{f\prime} h_k^{\otimes 2} h_k'))'$.

**Lemma A.3.** *Under Assumptions 1.1–1.4,*

$$T^{-\frac{1}{2}}\sum_{t=1}^{T}\begin{pmatrix}\hat{a}_1'\hat{z}_{1,t}\\ \vdots \\ \hat{a}_r'\hat{z}_{r,t}\end{pmatrix} = \begin{pmatrix}\hat{\alpha}_1'\\ \vdots \\ \hat{\alpha}_r'\end{pmatrix} T^{-\frac{1}{2}}\sum_{t=1}^{T} z_t + O_p\left(T^{\frac{1}{4}}N^{-\frac{1}{2}}\right) \qquad \text{(A.1.1)}$$

*Proof.* It is sufficient to show that (A.1.1) holds for the $k$th element. By using Lemma A.2, the $k$th element of (A.1.1) can expand as follows.

$$\begin{aligned}
T^{-\frac{1}{2}}\sum_{t=1}^{T}\hat{a}_k'\hat{z}_{k,t} &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\left[\left(\hat{f}_{k,t}-\hat{\mu}_{k,1}^f\right)^3 - \hat{\mu}_{k,3}^f\right] - 3\hat{\mu}_{k,2}^f T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(\hat{f}_t-\hat{\mu}_{k,1}^f\right) \\
&= I - 3II.
\end{aligned}$$

Consider $I$.

$$\begin{aligned}
T^{-\frac{1}{2}}\sum_{t=1}^{T}\left[\left(\hat{f}_{k,t}-\hat{\mu}_{k,1}^{f}\right)^{3}-\hat{\mu}_{k,3}^{f}\right] &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\left[\left(f_t-\mu_1^f\right)^{\otimes 3}\right]' h_k^{\otimes 3}-T^{\frac{1}{2}}\mu_3^{f\prime}h_k^{\otimes 3} \\
&\quad -T^{\frac{1}{2}}\left(\hat{\mu}_{k,3}^{f}-\mu_3^{f\prime}h_k^{\otimes 3}\right) \\
&\quad -3T^{-1}\sum_{t=1}^{T}\left[\left(f_t-\mu_1^f\right)^{\otimes 2}\right]' h_k^{\otimes 2}T^{-\frac{1}{2}}\sum_{s=1}^{T}\left(f_t-\mu_1^f\right)' h_k+O_p\left(T^{\frac{1}{4}}N^{-\frac{1}{2}}\right) \\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(\left[\left(f_t-\mu_1^f\right)^{\otimes 3}\right]-\mu_3^f\right)' h_k^{\otimes 3} \\
&\quad +3T^{-1}\sum_{t=1}^{T}\left[\left(f_t-\mu_1^f\right)^{\otimes 2}\right]' h_k^{\otimes 2}T^{-\frac{1}{2}}\sum_{s=1}^{T}\left(f_t-\mu_1^f\right)' h_k-O_p\left(T^{\frac{1}{4}}N^{-\frac{1}{2}}\right) \\
&\quad -3T^{-1}\sum_{t=1}^{T}\left[\left(f_t-\mu_1^f\right)^{\otimes 2}\right]' h_k^{\otimes 2}T^{-\frac{1}{2}}\sum_{s=1}^{T}\left(f_t-\mu_1^f\right)' h_k+O_p\left(T^{\frac{1}{4}}N^{-\frac{1}{2}}\right) \\
&= T^{-1}\sum_{t=1}^{T}\left[\left(f_t-\mu_1^f\right)^{\otimes 3}-\mu_3^f\right]' h_k^{\otimes 3}+O_p\left(T^{\frac{1}{4}}N^{-\frac{1}{2}}\right),
\end{aligned}$$

by Lemma A.2.3. Consider $II$. $\hat{\mu}_{k,2}^{f}=\mu_2^{f\prime}h_k^{\otimes 2}+O_p\left(T^{\frac{1}{4}}N^{-\frac{1}{2}}\right)$ by Lemma A.2.2.

$$\begin{aligned}
T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(\hat{f}_t-\hat{\mu}_{k,1}^{f}\right) &= T^{-\frac{1}{2}}\sum_{t=1}^{T}h_k' f_t+T^{-\frac{1}{2}}\sum_{t=1}^{T}u_{k,t}-T^{\frac{1}{2}}h_k'\mu_1^f-O_p\left(T^{\frac{1}{4}}N^{-\frac{1}{2}}\right) \\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(f_t-\mu_1^f\right)' h_k+O_p\left(T^{\frac{1}{4}}N^{-\frac{1}{2}}\right)
\end{aligned}$$

by A.2.1. Therefore

$$\begin{aligned}
T^{-\frac{1}{2}}\sum_{t=1}^{T}\hat{a}_k'\hat{z}_{k,t} &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(\left[\left(f_t-\mu_1^f\right)^{\otimes 3}\right]-\mu_3^f\right)' h_k^{\otimes 3} \\
&\quad -3\mu_2^{f\prime}h_k^{\otimes 2}T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(f_t-\mu_1^f\right)' h_k+O_p\left(T^{\frac{1}{4}}N^{-\frac{1}{2}}\right) \\
&= T^{-\frac{1}{2}}\sum_{t=1}^{T}\hat{\alpha}_k' z_t+O_p\left(T^{\frac{1}{4}}N^{-\frac{1}{2}}\right)
\end{aligned}$$

with $\tau_k^f=0$. □

**Lemma A.4.** *Under Assumptions 1.1–1.4,*

$$\sqrt{T}\hat{\tau}_k^f = \left(\hat{\mu}_{k,2}^f\right)^{-\frac{3}{2}} \left(T^{-\frac{1}{2}} \sum_{t=1}^{T} \hat{\alpha}_k' z_t\right) + O_p\left(T^{\frac{1}{4}} N^{-\frac{1}{2}}\right)$$

*Proof.* I define $\psi_k^f = \mu_3^{f\prime} h_k^{\otimes 3} / (\mu_2^{f\prime} h_k^{\otimes 2})^{\frac{3}{2}}$ to derive the statement. This is equal to zero when $\tau_k^f = \mu_{k,3}^f (\mu_{k,2}^f)^{-\frac{3}{2}} = 0$ for all $k$.

$$\begin{aligned}
\sqrt{T}\left(\hat{\tau}_k^f - \psi_k^f\right) &= \sqrt{T}\left[\hat{\mu}_{k,3}^f \left(\hat{\mu}_{k,2}^f\right)^{-\frac{3}{2}} - \psi_k^f\right], \\
&= \sqrt{T}\left(\hat{\mu}_{k,2}^f\right)^{-\frac{3}{2}} \left[\hat{\mu}_{k,3}^f - \psi_k^f \left(\hat{\mu}_{k,2}^f\right)^{\frac{3}{2}}\right], \\
&= \left(\hat{\mu}_{k,2}^f\right)^{-\frac{3}{2}} \underbrace{\sqrt{T}\left(\hat{\mu}_{k,3}^f - \mu_3^{f\prime} h_k^{\otimes 3}\right)}_{=I} - \left(\hat{\mu}_{k,2}^f\right)^{-\frac{3}{2}} \psi_k^f \underbrace{\sqrt{T}\left[\left(\hat{\mu}_{k,2}^f\right)^{\frac{3}{2}} - \left(\mu_2^{f\prime} h_k^{\otimes 2}\right)^{\frac{3}{2}}\right]}_{=II}.
\end{aligned}$$

Consider $I$.

$$\begin{aligned}
\sqrt{T}\left(\hat{\mu}_{k,3}^f - \mu_3^{f\prime} h_k^{\otimes 3}\right) \quad &= \quad T^{-\frac{1}{2}} \sum_{t=1}^{T} \left\{\left(f_t - \mu_1^f\right)^{\otimes 3} - \mu_3^f\right\}' h_k^{\otimes 3} \\
&\quad - 3T^{-1} \sum_{t=1}^{T} \left[\left(f_t - \mu_1^f\right)^{\otimes 2}\right]' h_k^{\otimes 2} T^{-\frac{1}{2}} \sum_{s=1}^{T} \left(f_t - \mu_1^f\right)' h_k + O_p\left(T^{\frac{1}{4}} N^{-\frac{1}{2}}\right),
\end{aligned}$$

by Lemma A.2.3. Consider $II$. The delta method implies

$$\left(\hat{\mu}_{k,2}^f\right)^{\frac{3}{2}} - \left(\mu_2^{f\prime} h_k^{\otimes 2}\right)^{\frac{3}{2}} = \frac{3}{2}\left(\mu_2^{f\prime} h_k^{\otimes 2}\right)^{\frac{1}{2}} \left[\hat{\mu}_{k,2}^f - \mu_2^{f\prime} h_k^{\otimes 2}\right] + o_p(1).$$

Therefore,

$$\sqrt{T}\left(\hat{\mu}_{k,2}^f - \mu_2^{f\prime} h_k^{\otimes 2}\right) = T^{-\frac{1}{2}} \sum_{t=1}^{T} \left[\left(f_t - \mu_1^f\right)^{\otimes 2} - \mu_2^f\right]' h_k^{\otimes 2} + O_p\left(T^{\frac{1}{4}} N^{-\frac{1}{2}}\right),$$

by Lemma A.2.2. Now I obtain

$$
\begin{aligned}
\sqrt{T}\left(\hat{\tau}_k^f - \psi_k^f\right) &= \left(\hat{\mu}_{k,2}^f\right)^{-\frac{3}{2}} \sqrt{T}\left(\hat{\mu}_{k,3}^f - \mu_3^{f\prime} h_k^{\otimes 3}\right) - \left(\hat{\mu}_{k,2}^f\right)^{-\frac{3}{2}} \psi_k^f \sqrt{T}\left[\left(\hat{\mu}_{k,2}^f\right)^{\frac{3}{2}} - \left(\mu_2^{f\prime} h_k^{\otimes 2}\right)^{\frac{3}{2}}\right], \\
&= \left(\hat{\mu}_{k,2}^f\right)^{-\frac{3}{2}} \left(\begin{array}{l} T^{-\frac{1}{2}} \sum_{t=1}^T \left[\left(f_t - \mu_1^f\right)^{\otimes 3} - \mu_3^f\right]' h_k^{\otimes 3} \\ -3T^{-1} \sum_{t=1}^T \left[\left(f_t - \mu_1^f\right)^{\otimes 2}\right]' h_k^{\otimes 2} T^{-\frac{1}{2}} \sum_{s=1}^T \left(f_t - \mu_1^f\right)' h_k + O_p\left(T^{\frac{1}{4}} N^{-\frac{1}{2}}\right) \end{array}\right), \\
&\quad - \left(\hat{\mu}_{k,2}\right)^{-\frac{3}{2}} \frac{3}{2} \psi_k^f \left(\mu_2^{f\prime} h_k^{\otimes 2}\right)^{\frac{1}{2}} \left(T^{-\frac{1}{2}} \sum_{t=1}^T \left[\left(f_t - \mu_1^f\right)^{\otimes 2} - \mu_2^f\right]' h_k^{\otimes 2} + O_p\left(T^{\frac{1}{4}} N^{-\frac{1}{2}}\right)\right).
\end{aligned}
$$

Note that

$$
z_t = \left(\left[\left(f_t - \mu_1^f\right)^{\otimes 3} - \mu_3^f\right]', \left(f_t - \mu_1^f\right)'\right)'.
$$

Hence,

$$
\sqrt{T}\hat{\tau}_k = \left(\hat{\mu}_{k,2}\right)^{-\frac{3}{2}} \left(T^{-\frac{1}{2}} \sum_{t=1}^T \hat{\alpha}_k' z_t\right) + O_p\left(T^{\frac{1}{4}} N^{-\frac{1}{2}}\right),
$$

with $\tau_k^f = 0$ for all $k$. □

## A.2 Proof of Theorem 1.1

*Proof.* By using Lemma A.4, I obtain

$$
\begin{aligned}
\sqrt{T}\hat{\tau}^f &= \sqrt{T}\left(\hat{\tau}_1^f, \cdots, \hat{\tau}_r^f\right)', \\
&= \underbrace{\begin{bmatrix} \hat{\mu}_{1,2}^f & & 0 \\ & \ddots & \\ 0 & & \hat{\mu}_{r,2}^f \end{bmatrix}}_{\equiv \hat{M}_2}^{-\frac{3}{2}} T^{-\frac{1}{2}} \sum_{t=1}^T \underbrace{\begin{bmatrix} \hat{\alpha}_1' \\ \vdots \\ \hat{\alpha}_r' \end{bmatrix}}_{\equiv \hat{\alpha}'} z_t + O_p\left(T^{\frac{1}{4}} N^{-\frac{1}{2}}\right), \\
&= \hat{M}_2^{-\frac{3}{2}} \hat{\alpha}' \left(T^{-\frac{1}{2}} \sum_{t=1}^T z_t\right) + O_p\left(T^{\frac{1}{4}} N^{-\frac{1}{2}}\right), \qquad \text{(A.2.1)}
\end{aligned}
$$

where $\hat{\alpha} = (\hat{\alpha}_1, \cdots, \hat{\alpha}_r)'$.

1. By plugging (A.2.1) into the Wald-type test $W_{NT} = T\left[\hat{\tau}^{f\prime}\hat{M}_2^3(\hat{\Gamma}^f)^{-1}\hat{\tau}^f\right]$, where $\hat{\Gamma}^f = T^{-1}\hat{a}'(\sum_{t=1}^T \hat{z}_t\hat{z}_t')\hat{a}$, I obtain

$$\begin{aligned} W_{NT} &= T\left[\hat{\tau}^{f\prime}\hat{M}_2^3(\hat{\Gamma}^f)^{-1}\hat{\tau}^f\right], \\ &= \left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^T z_t\right)'\hat{\alpha}\hat{M}_2^{-\frac{3}{2}}M_2^3\left(\hat{\Gamma}^f\right)^{-1}\hat{M}_2^{-\frac{3}{2}}\hat{\alpha}'\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^T z_t\right) + O_p(T^{\frac{1}{2}}N^{-1}), \\ &= \left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^T z_t'\right)\hat{\alpha}\left(\hat{\Gamma}^f\right)^{-1}\hat{\alpha}'\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^T z_t\right) + O_p(T^{\frac{1}{2}}N^{-1}), \\ &= \left[\left(\hat{\Gamma}^f\right)^{-\frac{1}{2}}\hat{\alpha}'\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^T z_t\right)\right]'\left[\left(\hat{\Gamma}^f\right)^{-\frac{1}{2}}\hat{\alpha}'\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^T z_t\right)\right] + O_p(T^{\frac{1}{2}}N^{-1}), \\ &\Rightarrow \chi_r^2, \end{aligned}$$

as $N, T \to \infty$ and $\sqrt{T}/N \to 0$ by using Assumption 1.13.

2. By plugging (A.2.1) into an $r \times 1$ vector of the elements in $M_{NT}$, that is, $\sqrt{T}\hat{M}_2^{\frac{3}{2}}(\hat{\Gamma}^f)^{-\frac{1}{2}}\hat{\tau}^f$, I obtain

$$\begin{aligned} \sqrt{T}\hat{M}_2^{\frac{3}{2}}\left(\hat{\Gamma}^f\right)^{-\frac{1}{2}}\hat{\tau}^f &= \hat{M}_2^{\frac{3}{2}}\left(\hat{\Gamma}^f\right)^{-\frac{1}{2}}\hat{M}_2^{-\frac{3}{2}}\hat{\alpha}'\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^T z_t\right) + O_p(T^{\frac{1}{4}}N^{-\frac{1}{2}}), \\ &= \left(\hat{\Gamma}^f\right)^{-\frac{1}{2}}\hat{\alpha}'\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^T z_t\right) + O_p(T^{\frac{1}{4}}N^{-\frac{1}{2}}) \\ &\Rightarrow N\left(0_{r\times 1}, I_r\right), \end{aligned}$$

as $N, T \to \infty$ and $\sqrt{T}/N \to 0$ by using Assumption 1.13. The result follows immediately.

□

## A.3 Preliminaries for Theorem 1.2

**Lemma A.5.** *Under Assumptions 1.1–1.4,*

1. $\hat{\lambda}_i - H^{-1}\lambda_i = O_p(T^{-\frac{1}{4}}N^{-\frac{1}{2}})$,

2. $T^{-1}\sum_{t=1}^T(\hat{\lambda}_i'\hat{f}_t - \lambda_i' f_t) = O_p(T^{-\frac{1}{2}}N^{-1})$,

3. $T^{-1}\sum_{t=1}^T(\hat{\lambda}_i'\hat{f}_t - \lambda_i' f_t)^2 = O_p(\delta_{NT}^{-2})$,

4. $T^{-1}\sum_{t=1}^T(\hat{\lambda}_i'\hat{f}_t - \lambda_i' f_t)^3 = O_p(\delta_{NT}^{-3})$,

5. $T^{-1}\sum_{t=1}^{T}(\hat{\lambda}_i'\hat{f}_t - \lambda_i' f_t)^4 = O_p(\delta_{NT}^{-4})$,

*Proof.*

1. According to the proof of Theorem 2 in Bai (2003), I obtain

$$\begin{aligned} \hat{\lambda}_i - H^{-1}\lambda_i &= T^{-1}H'F'e_i - T^{-1}\left(\hat{F}-FH\right)'\left(\hat{F}-FH\right)H^{-1}\lambda_i \\ &\quad - T^{-1}H'F'\left(\hat{F}-FH\right)H^{-1}\lambda_i + T^{-1}\left(\hat{F}-FH\right)' e_i. \end{aligned} \quad \text{(A.3.1)}$$

(a) The first term is $O_p(T^{-1})$ by Assumption 1.4.

(b) The second term is $O_p(\delta_{NT}^{-2})$.

(c) For the third term, I have

$$\begin{aligned} T^{-1}\sum_{t=1}^{T}\left(\hat{f}_t - H'f_t\right)f_t' &= \hat{V}_{NT}^{-1}\begin{pmatrix} T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_s\gamma_{st}f_t' + T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_s\zeta_{st}f_t' \\ +T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_s\eta_{st}f_t' + T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_s\xi_{st}f_t' \end{pmatrix} \\ &= \hat{V}_{NT}^{-1}\left(I + II + III + IV\right). \end{aligned}$$

First, $I = O_p(T^{-\frac{1}{2}}\delta_{NT}^{-1})$ by Bai (2003). For $II$,

$$II = T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_s - H'f_s\right)\zeta_{st}f_t' + T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}H'f_s\zeta_{st}f_t'.$$

For the first term,

$$\begin{aligned} \left\|T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_s - H'f_s\right)\zeta_{st}f_t'\right\| &\leq \left(T^{-1}\sum_{s=1}^{T}\left\|\hat{f}_s - H'f_s\right\|^2\right)^{\frac{1}{2}} \\ &\quad \times \left(T^{-1}\sum_{s=1}^{T}\left\|T^{-1}\sum_{t=1}^{T}\zeta_{st}f_t'\right\|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Then the second set of parentheses is

$$\begin{aligned} T^{-1}\sum_{s=1}^{T}\left\|T^{-1}\sum_{t=1}^{T}\zeta_{st}f_t'\right\|^2 &\leq T^{-\frac{3}{2}}\sum_{s=1}^{T}\left(T^{-\frac{1}{2}}\sum_{t=1}^{T}\|f_t\|^2\right)\left(T^{-1}\sum_{t=1}^{T}\zeta_{st}^2\right) \\ &\leq O_p\left(T^{-\frac{1}{2}}N^{-1}\right). \end{aligned}$$

This is because $T^{-1}\sum_{t=1}^{T}\zeta_{st}^2 = O_p(N^{-1})$ by Bai (2003). Therefore this is $O_p(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\delta_{NT}^{-1})$.

For the second term,

$$\begin{aligned}\left\|T^{-2}\sum\nolimits_{t=1}^{T}\sum\nolimits_{s=1}^{T}f_s\zeta_{st}f_t'\right\| &\leq T^{-\frac{1}{4}}\left(T^{-\frac{1}{2}}\sum\nolimits_{s=1}^{T}\|f_s\|^2\right)^{\frac{1}{2}}\left(T^{-1}\sum\nolimits_{s=1}^{T}\left\|T^{-1}\sum\nolimits_{t=1}^{T}\zeta_{st}f_t'\right\|^2\right)^{\frac{1}{2}}\\ &= O_p\left(T^{-\frac{1}{2}}N^{-\frac{1}{2}}\right).\end{aligned}$$

Thus, $II = O_p(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\delta_{NT}^{-1}) + O_p(T^{-\frac{1}{2}}N^{-\frac{1}{2}})$. For $III$,

$$III = T^{-2}\sum\nolimits_{t=1}^{T}\sum\nolimits_{s=1}^{T}\left(\hat{f}_s - H'f_s\right)\eta_{st}f_t' + T^{-2}\sum\nolimits_{t=1}^{T}\sum\nolimits_{s=1}^{T}H'f_s\eta_{st}f_t'.$$

For the first term,

$$\begin{aligned}\left\|T^{-2}\sum\nolimits_{t=1}^{T}\sum\nolimits_{s=1}^{T}\left(\hat{f}_s - H'f_s\right)\eta_{st}f_t'\right\| &\leq \left(T^{-1}\sum\nolimits_{s=1}^{T}\left\|\hat{f}_s - H'f_s\right\|^2\right)^{\frac{1}{2}}\\ &\quad\times\left(T^{-1}\sum\nolimits_{s=1}^{T}\left\|T^{-1}\sum\nolimits_{t=1}^{T}\eta_{st}f_t'\right\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

For the second set of parentheses,

$$\begin{aligned}T^{-1}\sum\nolimits_{s=1}^{T}\left\|T^{-1}\sum\nolimits_{t=1}^{T}\eta_{st}f_t'\right\|^2 &= N^{-2}T^{-1}\sum\nolimits_{s=1}^{T}\left\|T^{-1}\sum\nolimits_{t=1}^{T}f_s'\Lambda'e_tf_t'\right\|^2\\ &\leq N^{-1}T^{-1}\left(T^{-\frac{1}{2}}\sum\nolimits_{s=1}^{T}\|f_s\|^2\right)\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\|f_t\|^2\right)\\ &\quad\times\left(T^{-1}\sum\nolimits_{t=1}^{T}\left\|N^{-\frac{1}{2}}\Lambda'e_t\right\|^2\right)\\ &= O_p\left(T^{-1}N^{-1}\right).\end{aligned}$$

Therefore this is $O_p(T^{-\frac{1}{2}}N^{-\frac{1}{2}}\delta_{NT}^{-1})$. For the second term,

$$\begin{aligned}\left\|T^{-2}\sum\nolimits_{t=1}^{T}\sum\nolimits_{s=1}^{T}f_s\eta_{st}f_t'\right\| &\leq T^{-\frac{1}{4}}\left(T^{-\frac{1}{2}}\sum\nolimits_{s=1}^{T}\|f_s\|^2\right)^{\frac{1}{2}}\left(T^{-1}\sum\nolimits_{s=1}^{T}\left\|T^{-1}\sum\nolimits_{t=1}^{T}\eta_{st}f_t'\right\|^2\right)^{\frac{1}{2}}.\\ &= O_p\left(T^{-\frac{3}{4}}N^{-\frac{1}{2}}\right).\end{aligned}$$

Thus, $III = O_p(T^{-\frac{1}{2}}N^{-\frac{1}{2}}\delta_{NT}^{-1}) + O_p(T^{-\frac{3}{4}}N^{-\frac{1}{2}})$. The proof of $IV$ is similar to that of

$III$. Thus

$$T^{-1}\sum_{t=1}^{T}\left(\hat{f}_t - H'f_t\right)f_t' = O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\delta_{NT}^{-1}\right) + O_p\left(T^{-\frac{1}{2}}N^{-\frac{1}{2}}\right).$$

(d) For the last term, I have

$$\begin{aligned} T^{-1}\sum_{t=1}^{T}\left(\hat{f}_t - H'f_t\right)e_{i,t} &= \hat{V}_{NT}^{-1}\begin{pmatrix} T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_s\gamma_{st}e_{i,t} + T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_s\zeta_{st}e_{i,t} \\ +T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_s\eta_{st}e_{i,t} + T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{f}_s\xi_{st}e_{i,t} \end{pmatrix}. \\ &= \hat{V}_{NT}^{-1}\left(I + II + III + IV\right). \end{aligned}$$

First, $I = O_p(T^{-\frac{1}{2}}\delta_{NT}^{-1})$ by Bai (2003). For $II$,

$$II = T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_s - H'f_s\right)\zeta_{st}e_{i,t} + T^{-2}H'\sum_{t=1}^{T}\sum_{s=1}^{T}f_s\zeta_{st}e_{i,t}.$$

The first term is

$$\begin{aligned} \left\|T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_s - H'f_s\right)\zeta_{st}e_{i,t}\right\| &\le \left(T^{-1}\sum_{s=1}^{T}\left\|\hat{f}_s - H'f_s\right\|^2\right)^{\frac{1}{2}} \\ &\quad\times\left[T^{-1}\sum_{s=1}^{T}T\left(T^{-1}\sum_{t=1}^{T}\zeta_{st}e_{i,t}\right)^2\right]^{\frac{1}{2}}. \end{aligned}$$

Because $T^{-1}\sum_{t=1}^{T}\zeta_{st}e_{i,t} = O_p(N^{-\frac{1}{2}})$, this is $O_p(N^{-\frac{1}{2}}\delta_{NT}^{-1})$. For the second term,

$$\begin{aligned} \left\|T^{-2}H'\sum_{t=1}^{T}\sum_{s=1}^{T}f_s\zeta_{st}e_{i,t}\right\| &\le \|H\|\,T^{-\frac{1}{4}}\left(T^{-\frac{1}{2}}\sum_{s=1}^{T}\|f_s\|^2\right)^{\frac{1}{2}} \\ &\quad\times\left[T^{-1}\sum_{s=1}^{T}\left(T^{-1}\sum_{t=1}^{T}\zeta_{st}e_{i,t}\right)^2\right]^{\frac{1}{2}}. \end{aligned}$$

Therefore this is $O_p(T^{-\frac{1}{4}}N^{-\frac{1}{2}})$. Thus $II = O_p(N^{-\frac{1}{2}}\delta_{NT}^{-1}) + O_p(T^{-\frac{1}{4}}N^{-\frac{1}{2}})$. For $III$,

$$III = T^{-2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{f}_s - H'f_s\right)\eta_{st}e_{i,t} + T^{-2}H'\sum_{t=1}^{T}\sum_{s=1}^{T}f_s\eta_{st}e_{i,t}.$$

For the first term,

$$\left\| T^{-2} \sum_{t=1}^{T} \sum_{s=1}^{T} \left( \hat{f}_s - H' f_s \right) \eta_{st} e_{i,t} \right\| \leq \left( T^{-1} \sum_{s=1}^{T} \left\| \hat{f}_s - H' f_s \right\|^2 \right)^{\frac{1}{2}} \times \left[ T^{-1} \sum_{s=1}^{T} \left( T^{-1} \sum_{t=1}^{T} \eta_{st} e_{i,t} \right)^2 \right]^{\frac{1}{2}}.$$

For the second set of parentheses,

$$\begin{aligned}
&\left\| T^{-1} \sum_{s=1}^{T} \left( T^{-1} \sum_{t=1}^{T} \eta_{st} e_{i,t} \right)^2 \right\| \\
&\quad = T^{-3} N^{-2} \sum_{s=1}^{T} \left\| f_s' \sum_{t=1}^{T} \left( \Lambda' e_t \right) e_{i,t} \right\|^2 \\
&\quad \leq T^{-1} N^{-2} \sum_{s=1}^{T} \| f_s \|^2 \left\| T^{-1} \sum_{t=1}^{T} \left( \Lambda' e_t \right) e_{i,t} \right\|^2 \\
&\quad \leq N^{-1} T^{-1} \left( T^{-\frac{1}{2}} \sum_{s=1}^{T} \| f_s \|^2 \right) \left( T^{-1} \sum_{t=1}^{T} \left\| N^{-\frac{1}{2}} \Lambda' e_t \right\|^2 \right) \left( T^{-\frac{1}{2}} \sum_{t=1}^{T} e_{i,t}^2 \right) \\
&\quad = O_p \left( T^{-1} N^{-1} \right).
\end{aligned}$$

Therefore this is $O_p(T^{-\frac{1}{2}} N^{-\frac{1}{2}} \delta_{NT}^{-1})$. For the second term,

$$\begin{aligned}
&\left\| T^{-2} H' \sum_{t=1}^{T} \sum_{s=1}^{T} f_s \eta_{st} e_{i,t} \right\| \\
&\quad \leq \| H \| T^{-\frac{1}{4}} \left( T^{-\frac{1}{2}} \sum_{s=1}^{T} \| f_s \|^2 \right)^{\frac{1}{2}} \left[ T^{-1} \sum_{s=1}^{T} \left( T^{-1} \sum_{t=1}^{T} \eta_{st} e_{i,t} \right)^2 \right]^{\frac{1}{2}} \\
&\quad = O_p \left( T^{-\frac{3}{4}} N^{-\frac{1}{2}} \right).
\end{aligned}$$

Thus $III = O_p(T^{-\frac{1}{2}} N^{-\frac{1}{2}} \delta_{NT}^{-1}) + O_p(T^{-\frac{3}{4}} N^{-\frac{1}{2}})$. The proof of $IV$ is similar to that of $III$. Thus,

$$T^{-1} \sum_{t=1}^{T} \left( \hat{f}_t - H' f_t \right) e_{i,t} = O_p \left( T^{-\frac{1}{4}} N^{-\frac{1}{2}} \right).$$

Combining these results yields

$$\begin{aligned}
\hat{\lambda}_i - H^{-1} \lambda_i &= O_p \left( T^{-1} \right) + O_p \left( \delta_{NT}^{-2} \right) + \left( T^{-\frac{1}{4}} N^{-\frac{1}{2}} \delta_{NT}^{-1} \right) + O_p \left( T^{-\frac{1}{2}} N^{-\frac{1}{2}} \right) + O_p \left( T^{-\frac{1}{4}} N^{-\frac{1}{2}} \right) \\
&= O_p \left( T^{-\frac{1}{4}} N^{-\frac{1}{2}} \right).
\end{aligned}$$

2. Consider

$$\begin{aligned} T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right) &= \left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_t-H'f_t\right) \\ &\quad+\left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'H'T^{-1}\sum\nolimits_{t=1}^{T}f_t \\ &\quad+\lambda_i'\left(H^{-1}\right)'T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_t-H'f_t\right), \end{aligned}$$

For the first term,

$$\begin{aligned} \left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_t-H'f_t\right) &= O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right)O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right) \\ &= O_p\left(T^{-\frac{1}{2}}N^{-1}\right). \end{aligned}$$

The second term is

$$\left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'H'T^{-\frac{1}{2}}T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}f_t=O_p\left(T^{-\frac{3}{4}}N^{-\frac{1}{2}}\right).$$

The third term is

$$\lambda_i'\left(H^{-1}\right)'T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{f}_t-H'f_t\right)=O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right).$$

Therefore

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)=O_p\left(T^{-\frac{1}{2}}N^{-1}\right)$$

Thus, the lemma holds.

3. Consider

$$\begin{aligned} T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)^2 &= T^{-1}\sum\nolimits_{t=1}^{T}\left[\begin{array}{l}\left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'\left(\hat{f}_t-H'f_t\right) \\ +\left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'H'f_t+\hat{\lambda}_i'\left(H^{-1}\right)'\left(\hat{f}_t-H'f_t\right)\end{array}\right]^2 \\ &= O_p\left(\delta_{NT}^{-2}\right). \end{aligned}$$

This is because

$$\begin{aligned}
\left|T^{-1}\sum\nolimits_{t=1}^{T}\left[\left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'\left(\hat{f}_t-H'f_t\right)\right]^2\right| &\leq \left\|\hat{\lambda}_i-H^{-1}\lambda_i\right\|^2 T^{-1}\sum\nolimits_{t=1}^{T}\left\|\hat{f}_t-H'f_t\right\|^2 \\
&= O_p\left(T^{-\frac{1}{2}}N^{-1}\delta_{NT}^{-2}\right),
\end{aligned}$$

$$\begin{aligned}
\left|T^{-1}\sum\nolimits_{t=1}^{T}\left[\left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'H'f_t\right]^2\right| &\leq \left\|\hat{\lambda}_i-H^{-1}\lambda_i\right\|^2 T^{-\frac{1}{2}}T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left\|H'f_t\right\|^2 \\
&= O_p\left(T^{-1}N^{-1}\right),
\end{aligned}$$

$$\begin{aligned}
\left|T^{-1}\sum\nolimits_{t=1}^{T}\left[\lambda_i'\left(H^{-1}\right)'\left(\hat{f}_t-H'f_t\right)\right]^2\right| &\leq \left\|\lambda_i'\left(H^{-1}\right)\right\|^2 T^{-1}\sum\nolimits_{t=1}^{T}\left\|\hat{f}_t-H'f_t\right\|^2 \\
&= O_p\left(\delta_{NT}^{-2}\right),
\end{aligned}$$

$$\begin{aligned}
&\left|T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'\left(\hat{f}_t-H'f_t\right)\left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'H'f_t\right| \\
&\quad\leq \left\|\hat{\lambda}_i-H^{-1}\lambda_i\right\|^2 T^{-1}\sum\nolimits_{t=1}^{T}\left\|\hat{f}_t-H'f_t\right\|\left\|H'f_t\right\| \\
&\quad\leq \left\|\hat{\lambda}_i-H^{-1}\lambda_i\right\|^2\left(T^{-1}\sum\nolimits_{t=1}^{T}\left\|\hat{f}_t-H'f_t\right\|^2\right)^{\frac{1}{2}}T^{-\frac{1}{4}}\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left\|H'f_t\right\|^2\right)^{\frac{1}{2}} \\
&\quad= O_p\left(T^{-\frac{1}{2}}N^{-1}\right)O_p\left(\delta_{NT}^{-1}\right)T^{-\frac{1}{4}}, \\
&\quad= O_p\left(T^{-\frac{3}{4}}N^{-1}\delta_{NT}^{-1}\right)
\end{aligned}$$

$$\begin{aligned}
&\left|T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i-H^{-1}\lambda_i\right)'H'f_t\lambda_i'\left(H^{-1}\right)'\left(\hat{f}_t-H'f_t\right)\right| \\
&\quad\leq T^{-1}\sum\nolimits_{t=1}^{T}\left\|\hat{\lambda}_i-H^{-1}\lambda_i\right\|\left\|H'f_t\right\|\left\|\left(H^{-1}\right)\lambda_i\right\|\left\|\hat{f}_t-H'f_t\right\| \\
&\quad\leq \left\|\hat{\lambda}_i-H^{-1}\lambda_i\right\|\left\|\left(H^{-1}\right)\lambda_i\right\|\left(T^{-1}\sum\nolimits_{t=1}^{T}\left\|\hat{f}_t-H'f_t\right\|^2\right)^{\frac{1}{2}}T^{-\frac{1}{4}}\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left\|H'f_t\right\|^2\right)^{\frac{1}{2}} \\
&\quad= O_p\left(T^{-\frac{1}{4}}N^{-\frac{1}{2}}\right)O_p\left(\delta_{NT}^{-1}\right)T^{-\frac{1}{4}}, \\
&\quad= O_p\left(T^{-\frac{1}{2}}N^{-\frac{1}{2}}\delta_{NT}^{-1}\right)
\end{aligned}$$

$$
\begin{aligned}
&\left| T^{-1} \sum\nolimits_{t=1}^{T} \left( \hat{\lambda}_i - H^{-1}\lambda_i \right)' \left( \hat{f}_t - H' f_t \right) \lambda_i' \left( H^{-1} \right)' \left( \hat{f}_t - H' f_t \right) \right| \\
&\quad \leq \left\| \hat{\lambda}_i - H^{-1}\lambda_i \right\| \left\| \left( H^{-1} \right) \lambda_i \right\| T^{-1} \sum\nolimits_{t=1}^{T} \left\| \hat{f}_t - H' f_t \right\|^2 \\
&\quad = O_p \left( T^{-\frac{1}{4}} N^{-\frac{1}{2}} \delta_{NT}^{-2} \right).
\end{aligned}
$$

Thus, the lemma holds. Without loss of generality, I can also prove 4 and 5 in the same way and the lemmas hold.

□

**Lemma A.6.** *Under Assumptions 1.1–1.4,*

1. $\hat{\mu}_{i,1}^e = T^{-1} \sum_{t=1}^{T} e_{i,t} + O_p(T^{-\frac{1}{2}} N^{-1})$.
2. $\hat{\mu}_{i,2}^e = T^{-1} \sum_{t=1}^{T} (e_{i,t} - \mu_{i,1}^e)^2 + O_p(\delta_{NT}^{-2})$.
3. $\hat{\mu}_{i,3}^e = T^{-1} \sum_{t=1}^{T} (e_{i,t} - \mu_{i,1}^e)^3 - 3T^{-1} \sum_{s=1}^{T} (e_{i,t} - \mu_{i,1}^e)^2 T^{-1} \sum_{t=1}^{T} (e_{i,t} - \mu_{i,1}^e) + O_p(\delta_{NT}^{-3})$.

*Proof.*

1.

$$
\begin{aligned}
\lambda_i' f_t + e_{i,t} &= \hat{\lambda}_i' \hat{f}_t + \hat{e}_{i,t}, \\
\Leftrightarrow \quad \hat{e}_{i,t} &= e_{i,t} + \lambda_i' f_t - \hat{\lambda}_i' \hat{f}_t, & \text{(A.3.2)} \\
\Leftrightarrow \quad T^{-1} \sum\nolimits_{t=1}^{T} \hat{e}_{i,t} &= T^{-1} \sum\nolimits_{t=1}^{T} e_{i,t} + T^{-1} \sum\nolimits_{t=1}^{T} \left( \lambda_i' f_t - \hat{\lambda}_i' \hat{f}_t \right). & \text{(A.3.3)}
\end{aligned}
$$

The second term is $O_p(T^{-\frac{1}{2}} N^{-1})$ by Lemma A.5.2. Thus, the lemma holds.

2. Combining (A.3.2) and (A.3.3),

$$
\begin{aligned}
\hat{e}_{i,t} - T^{-1} \sum\nolimits_{t=1}^{T} \hat{e}_{i,t} &= \left( e_{i,t} - \mu_{i,1}^e \right) - \left( T^{-1} \sum\nolimits_{t=1}^{T} e_{i,t} - \mu_{i,1}^e \right) \\
&\quad - \left( \hat{\lambda}_i' \hat{f}_t - \lambda_i' f_t \right) - T^{-1} \sum\nolimits_{t=1}^{T} \left( \hat{\lambda}_i' \hat{f}_t - \lambda_i' f_t \right), \\
&= B_1 - B_2 - B_3 - B_4.
\end{aligned}
$$

Then,

$$
T^{-1} \sum\nolimits_{t=1}^{T} \left( \hat{e}_{i,t} - \hat{\mu}_{i,1}^e \right)^2 = T^{-1} \sum\nolimits_{t=1}^{T} \left( B_1 - B_2 - B_3 - B_4 \right)^2.
$$

I consider the sums of squares of each term. For $T^{-1}\sum_{t=1}^{T} B_2^2$ is

$$T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{t=1}^{T} e_{i,t}-\mu_{i,1}^{e}\right)^2 = O_p\left(T^{-1}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_3^2$,

$$T^{-1}\sum_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i' f_t\right)^2 = O_p\left(\delta_{NT}^{-2}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_4^2$,

$$T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i' f_s\right)\right)^2 = O_p\left(T^{-1}N^{-2}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_1B_2$,

$$T^{-1}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)\left(T^{-1}\sum_{s=1}^{T} e_{i,s}-\mu_{i,1}^{e}\right) = O_p\left(T^{-1}\right)$$

For $T^{-1}\sum_{t=1}^{T} B_2B_3$,

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{t=1}^{T} e_{i,t}-\mu_{i,1}^{e}\right)\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i' f_t\right) &= \left(T^{-1}\sum_{t=1}^{T} e_{i,t}-\mu_{i,1}^{e}\right)\\
&\quad \times T^{-1}\sum_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i' f_t\right),\\
&= O_p\left(T^{-1}N^{-1}\right).
\end{aligned}$$

For $T^{-1}\sum_{t=1}^{T} B_3B_4$,

$$T^{-1}\sum_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i' f_t\right)\left[T^{-1}\sum_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i' f_s\right)\right] = O_p\left(T^{-1}N^{-2}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_1B_4$,

$$T^{-1}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{f}\right)\left[T^{-1}\sum_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i' f_s\right)\right] = O_p\left(T^{-1}N^{-1}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_1 B_3$,

$$\begin{aligned}
&\left|T^{-1}\sum\nolimits_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)\right| \\
&\quad \leq T^{\frac{1}{4}}\left(T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^2\right)^{\frac{1}{2}}\left[T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)^2\right]^{\frac{1}{2}}, \\
&\quad = O_p\left(T^{\frac{1}{4}}\delta_{NT}^{-1}\right).
\end{aligned}$$

Finally, $T^{-1}\sum_{t=1}^{T} B_2 B_4$ is

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(T^{-1}\sum\nolimits_{t=1}^{T} e_{i,t}-\mu_{i,1}^{e}\right)T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)=O_p\left(T^{-1}N^{-1}\right).$$

Thus, Lemma holds.

3. Consider the following equation.

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{e}_{i,t}-\hat{\mu}_{i,1}^{e}\right)^3=T^{-1}\sum\nolimits_{t=1}^{T}\left[B_1-B_2-B_3-B_4\right]^3.$$

Similar to 2, I consider the sums of squares of each term. $T^{-1}\sum_{t=1}^{T} B_2^3 = O_p\left(T^{-\frac{3}{2}}\right)$. For $T^{-1}\sum_{t=1}^{T} B_3^3$,

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)^3=O_p\left(\delta_{NT}^{-3}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_4^3$,

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(T^{-1}\sum\nolimits_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i'f_s\right)\right)^3=O_p\left(T^{-\frac{3}{2}}N^{-3}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_1^2 B_3$,

$$\begin{aligned}
\left|T^{-1}\sum\nolimits_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^2\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)\right| &\leq T^{-\frac{1}{4}}\left[T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^4\right]^{\frac{1}{2}} \\
&\quad\times\left[T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)^2\right]^{\frac{1}{2}}, \\
&= O_p\left(T^{-\frac{1}{4}}\delta_{NT}^{-1}\right)
\end{aligned}$$

For $T^{-1}\sum_{t=1}^{T}B_1^2B_4$,

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^2\left[T^{-1}\sum\nolimits_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i'f_s\right)\right]=O_p\left(T^{-1}N^{-1}\right).$$

For $T^{-1}\sum_{t=1}^{T}B_2^2B_1$,

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(T^{-1}\sum\nolimits_{t=1}^{T}e_{i,t}-\mu_{i,1}^{e}\right)^2\left(e_{i,t}-\mu_{i,1}^{e}\right)=O_p\left(T^{-\frac{3}{2}}\right).$$

For $T^{-1}\sum_{t=1}^{T}B_2^2B_3$,

$$T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum\nolimits_{t=1}^{T}e_{i,t}-\mu_{i,1}^{e}\right)^2\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)=O_p\left(T^{-\frac{3}{2}}N^{-1}\right).$$

For $T^{-1}\sum_{t=1}^{T}B_2^2B_4$,

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(T^{-1}\sum\nolimits_{t=1}^{T}e_{i,t}-\mu_{i,1}^{e}\right)^2T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)=O_p\left(T^{-\frac{3}{2}}N^{-1}\right).$$

For $T^{-1}\sum_{t=1}^{T}B_3^2B_1$,

$$\begin{aligned}\left|T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)^2\left(e_{i,t}-\mu_{i,1}^{e}\right)\right| &\leq T^{-\frac{1}{4}}\left[T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)^4\right]^{\frac{1}{2}}\\ &\quad\times\left[T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^2\right]^{\frac{1}{2}},\\ &= O_p\left(T^{-\frac{1}{4}}\delta_{NT}^{-2}\right).\end{aligned}$$

For $T^{-1}\sum_{t=1}^{T}B_3^2B_2$,

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)^2\left(T^{-1}\sum\nolimits_{t=1}^{T}e_{i,t}-\mu_{i,1}^{e}\right)=O_p\left(T^{-\frac{1}{2}}\delta_{NT}^{-2}\right).$$

For $T^{-1}\sum_{t=1}^{T}B_3^2B_4$,

$$T^{-1}\sum\nolimits_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)^2T^{-1}\sum\nolimits_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i'f_s\right)=O_p\left(T^{-\frac{1}{2}}N^{-1}\delta_{NT}^{-2}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_4^2 B_1$,

$$T^{-1}\sum_{t=1}^{T}\left[T^{-1}\sum_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i'f_s\right)\right]^2\left(e_{i,t}-\mu_{i,1}^e\right)=O_p\left(T^{-\frac{3}{2}}N^{-2}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_4^2 B_2$,

$$T^{-1}\sum_{t=1}^{T}\left[T^{-1}\sum_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i'f_s\right)\right]^2\left(T^{-1}\sum_{t=1}^{T} e_{i,t}-\mu_{i,1}^e\right)=O_p\left(T^{-\frac{3}{2}}N^{-2}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_4^2 B_3$,

$$T^{-1}\sum_{t=1}^{T}\left[T^{-1}\sum_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i'f_s\right)\right]^2\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)=O_p\left(T^{-\frac{3}{2}}N^{-3}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_1 B_2 B_3$,

$$\begin{aligned}&\left|T^{-1}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^e\right)\left(T^{-1}\sum_{t=1}^{T} e_{i,t}-\mu_{i,1}^e\right)\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)\right|\\ &\quad\leq T^{-\frac{3}{4}}\left|T^{-\frac{1}{2}}\sum_{t=1}^{T} e_{i,t}-\mu_{i,1}^e\right|\left[T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^e\right)^2\right]^{\frac{1}{2}}\left[T^{-1}\sum_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)^2\right]^{\frac{1}{2}},\\ &\quad=O_p\left(T^{-\frac{3}{4}}\delta_{NT}^{-1}\right).\end{aligned}$$

For $T^{-1}\sum_{t=1}^{T} B_1 B_2 B_4$,

$$T^{-1}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^e\right)\left(T^{-1}\sum_{t=1}^{T} e_{i,t}-\mu_{i,1}^e\right)T^{-1}\sum_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)=O_p\left(T^{-\frac{3}{2}}N^{-1}\right).$$

For $T^{-1}\sum_{t=1}^{T} B_1 B_3 B_4$,

$$\begin{aligned}&\left|T^{-1}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^e\right)\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)T^{-1}\sum_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i'f_s\right)\right|\\ &\quad\leq\left|T^{-1}\sum_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i'f_s\right)\right|T^{-\frac{1}{4}}\left[T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^e\right)^2\right]^{\frac{1}{2}}\left[T^{-1}\sum_{t=1}^{T}\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i'f_t\right)^2\right]^{\frac{1}{2}},\\ &\quad=O_p\left(T^{-\frac{3}{4}}N^{-1}\delta_{NT}^{-1}\right).\end{aligned}$$

For $T^{-1}\sum_{t=1}^{T} B_2 B_3 B_4$,

$$T^{-1}\sum_{t=1}^{T}\left(T^{-1}\sum_{t=1}^{T} e_{i,t}-\mu_{i,1}^{e}\right)\left(\hat{\lambda}_i'\hat{f}_t-\lambda_i' f_t\right)\left[T^{-1}\sum_{s=1}^{T}\left(\hat{\lambda}_i'\hat{f}_s-\lambda_i' f_s\right)\right]=O_p\left(T^{-\frac{3}{2}}N^{-2}\right).$$

Therefore, the following holds.

$$\begin{aligned}T^{-1}\sum_{t=1}^{T}\left(\hat{e}_{i,t}-\hat{\mu}_{i,1}^{e}\right)^3 &= T^{-1}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^3\\ &\quad -3T^{-1}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^2\left(T^{-1}\sum_{t=1}^{T} e_{i,t}-\mu_{i,1}^{e}\right)+O_p\left(\delta_{NT}^{-3}\right).\end{aligned}$$

□

**Lemma A.7.** *Under Assumptions 1.1–1.4,*

$$T^{-\frac{1}{2}}\sum_{t=1}^{T}\hat{b}_i'\hat{v}_{i,t}=\hat{b}_i' T^{-\frac{1}{2}}\sum_{t=1}^{T} v_{i,t}+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-3}\right) \tag{A.3.4}$$

*Proof.* By using Lemma A.6, I obtain

$$\begin{aligned}T^{-\frac{1}{2}}\sum_{t=1}^{T}\hat{b}_i'\hat{v}_{i,t} &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\left[\left(\hat{e}_{i,t}-\hat{\mu}_{i,1}^{e}\right)^3-\hat{\mu}_{i,3}^{e}\right]-3\hat{\mu}_{i,2}^{e}T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(\hat{e}_t-\hat{\mu}_{i,1}^{e}\right),\\ &= I-3II.\end{aligned}$$

Consider $I$.

$$\begin{aligned}T^{-\frac{1}{2}}\sum_{t=1}^{T}\left[\left(\hat{e}_{i,t}-\hat{\mu}_{i,1}^{e}\right)^3-\hat{\mu}_{i,3}^{e}\right] &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^3-T^{\frac{1}{2}}\mu_{i,3}^{e}\\ &\quad -T^{\frac{1}{2}}\left(\hat{\mu}_{k,3}^{e}-\mu_{i,3}^{e}\right)\\ &\quad -3T^{-1}\sum_{s=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^2 T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-3}\right),\\ &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\left[\left(e_{i,t}-\mu_{i,1}^{e}\right)^3-\mu_{i,3}^{e}\right]\\ &\quad +3T^{-1}\sum_{s=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^2 T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)-O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-3}\right)\\ &\quad -3T^{-1}\sum_{s=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)^2 T^{-\frac{1}{2}}\sum_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^{e}\right)+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-3}\right),\\ &= T^{-\frac{1}{2}}\sum_{t=1}^{T}\left[\left(e_{i,t}-\mu_{i,1}^{e}\right)^3-\mu_{i,3}^{e}\right]+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-3}\right).\end{aligned}$$

by Lemma A.6.3. Consider $II$.

$$
\begin{aligned}
T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left(\hat{e}_t-\hat{\mu}_{i,1}^e\right) &= T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T} e_{i,t}+T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left(\lambda_i' f_t-\hat{\lambda}_i'\hat{f}_t\right)-T^{-1}\sum\nolimits_{t=1}^{T} e_{i,t}-O_p\left(N^{-1}\right), \\
&= T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^e\right)+O_p\left(N^{-1}\right)
\end{aligned}
$$

by A.6.1. Therefore

$$
\begin{aligned}
T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\hat{b}_i'\hat{v}_{i,t} &= T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left[\left(e_{i,t}-\mu_{i,1}^e\right)^3-\mu_{i,3}^e\right]+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-3}\right) \\
&\quad -3\hat{\mu}_{i,2}^e T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left(e_{i,t}-\mu_{i,1}^e\right)+O_p\left(N^{-1}\right), \\
&= T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\hat{b}_i' v_{i,t}+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-3}\right).
\end{aligned}
$$

Therefore, the lemma holds. □

## A.4 Proof of Theorem 1.2

*Proof.* The coefficient of skewness for the $i$th idiosyncratic component is

$$
\sqrt{T}\left(\hat{\tau}_i^e-\tau_i^e\right)=\left(\hat{\mu}_{i,2}^e\right)^{-\frac{3}{2}}\underbrace{\sqrt{T}\left(\hat{\mu}_{i,3}^e-\mu_{i,3}^e\right)}_{=I}-\left(\hat{\mu}_{i,2}^e\right)^{-\frac{3}{2}}\tau_i^e\underbrace{\sqrt{T}\left[\left(\hat{\mu}_{i,2}^e\right)^{\frac{3}{2}}-\left(\mu_{i,2}^e\right)^{\frac{3}{2}}\right]}_{=II}.
$$

For $I$,

$$
\begin{aligned}
\sqrt{T}\left(\hat{\mu}_{i,3}^e-\mu_{i,3}^e\right) &= T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left(\left(e_{i,t}^3-\mu_{i,1}^e\right)^3-\mu_{i,3}^e\right) \\
&\quad -3T^{-1}\sum\nolimits_{s=1}^{T}(e_{i,t}-\mu_{i,1}^e)^2 T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}(e_{i,t}-\mu_{i,1}^e)+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-3}\right),
\end{aligned}
$$

by Lemma A.6.3. For $II$, the delta method implies

$$
\left(\hat{\mu}_{i,2}^e\right)^{\frac{3}{2}}-\left(\mu_{i,2}^e\right)^{\frac{3}{2}}=\frac{3}{2}\left(\mu_{i,2}^e\right)^{\frac{1}{2}}\left(\hat{\mu}_{i,2}^e-\mu_{i,2}^e\right)+o_p\left(1\right).
$$

Therefore,

$$
\sqrt{T}\left(\hat{\mu}_{i,2}^e-\mu_{i,2}^e\right)=T^{-\frac{1}{2}}\sum\nolimits_{t=1}^{T}\left[\left(e_{i,t}-\mu_{i,1}^e\right)-\mu_{i,2}^e\right]+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-2}\right),
$$

by Lemma A.6.2. Now I obtain

$$
\begin{aligned}
\sqrt{T}\left(\hat{\tau}_i^e-\tau_i^e\right) &= \left(\hat{\mu}_{i,2}^e\right)^{-\frac{3}{2}}\sqrt{T}\left(\hat{\mu}_{i,3}^e-\mu_{i,3}^e\right)-\left(\hat{\mu}_{i,2}^e\right)^{-\frac{3}{2}}\tau_i^e\sqrt{T}\left[\left(\hat{\mu}_{i,2}^e\right)^{\frac{3}{2}}-\left(\mu_{i,2}^e\right)^{\frac{3}{2}}\right], \\
&= \left(\hat{\mu}_{i,2}^e\right)^{-\frac{3}{2}}\left[\begin{array}{c} T^{-\frac{1}{2}}\sum_{t=1}^T\left(e_{i,t}^3-\mu_{i,1}^e\right)-\mu_{i,3}^e \\ -3T^{-1}\sum_{s=1}^T(e_{i,t}-\mu_{i,1}^e)^2T^{-\frac{1}{2}}\sum_{t=1}^T(e_{i,t}-\mu_{i,1}^e)\end{array}\right] \\
&\quad -\left(\hat{\mu}_{i,2}^e\right)^{-\frac{3}{2}}\frac{3}{2}\tau_i^e\left(\mu_{i,2}^e\right)^{\frac{1}{2}}T^{-\frac{1}{2}}\sum_{t=1}^T\left(\left(e_{i,t}-\mu_{i,1}^e\right)-\mu_{i,2}^e\right)+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-2}\right).
\end{aligned}
$$

Hence,

$$
\sqrt{T}\hat{\tau}_i^e=\left(\hat{\mu}_{i,2}^e\right)^{-\frac{3}{2}}\left(T^{-\frac{1}{2}}\sum_{t=1}^T\hat{b}_i'v_{i,t}\right)+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-2}\right),
$$

with $\tau_i^e=0$. By plugging this into the test statistic for the $i$th idiosyncratic component $S_{i,NT}=\sqrt{T}(\hat{\mu}_{i,2}^e)^{\frac{3}{2}}(\hat{\Gamma}_i^e)^{-\frac{1}{2}}\hat{\tau}_i^e$, where $\hat{\Gamma}_i^e=T^{-1}\sum_{t=1}^T(\hat{b}_i'\hat{v}_{i,t})^2$, I have

$$
\begin{aligned}
S_{i,NT} &= \sqrt{T}\left(\frac{\hat{\Gamma}_i^e}{\left(\hat{\mu}_{i,2}^e\right)^3}\right)^{-\frac{1}{2}}\hat{\tau}_i^e, \\
&= \left(\frac{\hat{\Gamma}_i^e}{\left(\hat{\mu}_{i,2}^e\right)^3}\right)^{-\frac{1}{2}}\left(\hat{\mu}_{i,2}^e\right)^{-\frac{3}{2}}\left(T^{-\frac{1}{2}}\sum_{t=1}^T\hat{b}_i'v_{i,t}\right)+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-2}\right), \\
&= \left(\hat{\Gamma}_i^e\right)^{-\frac{1}{2}}\left(T^{-\frac{1}{2}}\sum_{t=1}^T\hat{b}_i'v_{i,t}\right)+O_p\left(T^{\frac{1}{2}}\delta_{NT}^{-2}\right), \\
&\Rightarrow N\left(0,1\right),
\end{aligned}
$$

as $N,T\to\infty$ and $\sqrt{T}/N\to 0$ by using Assumption 1.3.3. □

# Appendix B

# Supplementary Material to Chapter 2

## B.1 Results under the LTU Framework

Becheri and van den Akker (2015) and Westerlund (2015) derived the local asymptotic power of the pooled panel unit root tests where the common factors are extracted using the PANIC method. In doing so, the first-order AR coefficients are assumed to shrink to one at a fast rate of $T^{-1}$, that is, the local to unity (LTU) time series rate.[1] Here, I present the local asymptotic power of the common and the idiosyncratic tests of the PANIC and the CS tests by using the LTU framework:

**Assumption B.1.** *The AR coefficients satisfy* $\alpha_T = 1 + c/T$ *and* $\rho_{i,T} = 1 + c_i/T$*, where both* $c$ *and* $c_i$ *are the fixed constants.*

The localizing coefficients $c$ and $c_i$ can be either positive or negative, where $c > 0$ and $c_i > 0$ relate to explosive processes and $c < 0$ and $c_i < 0$ pertain to stationary processes. Therefore, the local asymptotic results are valid against either the explosive alternative hypothesis or the stationary alternative hypothesis. The following theorem is obtained.

**Theorem SA-1.** *Suppose Assumptions 2.1–2.4 and B.1 hold. Let* $W_c(r)$ *and* $W_{c,i}(r)$ *be independent Ornstein and Uhlenbeck processes defined on* $r \in [0, 1]$, $\bar{W}_c(r) = W_c(r) - \int W_c(r)dr$*, and* $\bar{W}_{c,i}(r) = W_{c,i}(r) - \int W_{c,i}(r)dr$*. The following hold as* $N, T \to \infty$*.*

[1] The rate also depends on $N$ because they consider the pooled tests.

*(i-a: The case of no deterministic components, common tests)*

$$t_{\hat{F}} \Rightarrow c\left[\int_0^1 W_c(r)^2 dr\right]^{1/2} + \frac{\int_0^1 W_c(r)dW(r)}{\left[\int_0^1 W_c(r)^2 dr\right]^{1/2}}.$$

*(i-b: The case of no deterministic components, idiosyncratic tests)*

$$t_{\hat{U}}(i) \Rightarrow c_i\left[\int_0^1 W_{c,i}(r)^2 dr\right]^{1/2} + \frac{\int_0^1 W_{c,i}(r)dW_i(r)}{\left[\int_0^1 W_{c,i}(r)^2 dr\right]^{1/2}}.$$

*(ii-a: The case of the intercept, common tests)*

$$\bar{t}_{\hat{F}} \Rightarrow c\left[\int_0^1 \bar{W}_c(r)^2 dr\right]^{1/2} + \frac{\int_0^1 \bar{W}_c(r)dW(r)}{\left[\int_0^1 \bar{W}_c(r)^2 dr\right]^{1/2}}.$$

*(ii-b: The case of the intercept, idiosyncratic tests)*

$$\bar{t}_{\hat{U}}(i) \Rightarrow c_i\left[\int_0^1 \bar{W}_{c,i}(r)^2 dr\right]^{1/2} + \frac{\int_0^1 \bar{W}_{c,i}(r)dW_i(r)}{\left[\int_0^1 \bar{W}_{c,i}(r)^2 dr\right]^{1/2}}.$$

For brevity, I present the proof of Theorem SA-1 under the i.i.d. assumptions $C(L) = 1$ and $D_i(L) = 1$ in Appendix B.2. I can obtain the proof of the tests based on the regression augmented by $p$ lags under $p \to \infty$ and $p^3/\min\{N, T\} \to 0$ by closely following Appendix C of Bai and Ng (2004); hence, it is condensed. This confirms that the factor estimation errors are immaterial in the limit distributions in a locally explosive environment. Note that the independent Ornstein and Uhlenbeck processes reduce to the independent Wiener processes when $c = 0$ in cases (i-a) and (ii-a) and when $c_i = 0$ in cases (i-b) and (ii-b); hence, the results encompass the asymptotic null distributions.

This shows that the size of the common (idiosyncratic) test is robust to the LTU deviations in the idiosyncratic (common and other idiosyncratic) components. As for the power, it ensures that the common (idiosyncratic) test has the standard local power even though the idiosyncratic (common and other idiosyncratic) components have the LTU deviations. Hence, this theorem theoretically confirms Bai and Ng's (2004) Monte Carlo findings in both the left- and right-tailed tests and implies that the PANIC method can disentangle the common and idiosyncratic explosive components.

I next consider the CS tests. Note that the time dimension of the testing sample is now $h$ instead of $T$; hence, I now denote $\alpha_h = 1 + \frac{c}{h}$ and $\rho_{i,h} = 1 + \frac{c_i}{h}$ in Assumption B.1. I present Theorem SA-2 under the LTU framework. For brevity, again, I provide a proof under the i.i.d. assumptions $C(L) = 1$ and $D_i(L) = 1$ in Appendix B.4.

**Theorem SA-2.** *Suppose Assumptions 2.1–2.4 and B.1 hold. The following hold as $N, T, h \to \infty$.*

*(i-a: The case of no deterministic components, common tests)*

$$t^*_{\tilde{F}} \Rightarrow c\left[\int_0^1 W_c(r)^2 dr\right]^{1/2} + \frac{\int_0^1 W_c(r) dW(r)}{\left[\int_0^1 W_c(r)^2 dr\right]^{1/2}}.$$

*(i-b: The case of no deterministic components, idiosyncratic tests)*

$$t^*_{\tilde{U}}(i) \Rightarrow c_i\left[\int_0^1 W_{c,i}(r)^2 dr\right]^{1/2} + \frac{\int_0^1 W_{c,i}(r) dW_i(r)}{\left[\int_0^1 W_{c,i}(r)^2 dr\right]^{1/2}}.$$

*(ii-a: The case of the intercept, common tests)*

$$\bar{t}^*_{\tilde{F}} \Rightarrow c\left[\int_0^1 \bar{W}_c(r)^2 dr\right]^{1/2} + \frac{\int_0^1 \bar{W}_c(r) dW(r)}{\left[\int_0^1 \bar{W}_c(r)^2 dr\right]^{1/2}}.$$

*(ii-b: The case of the intercept, idiosyncratic tests)*

$$\bar{t}^*_{\tilde{U}}(i) \Rightarrow c_i\left[\int_0^1 \bar{W}_{c,i}(r)^2 dr\right]^{1/2} + \frac{\int_0^1 \bar{W}_{c,i}(r) dW_i(r)}{\left[\int_0^1 \bar{W}_{c,i}(r)^2 dr\right]^{1/2}}.$$

Similar to Theorem SA-1, this theorem shows that the common test asymptotically achieves the correct size and is consistent under the LTU framework. The idiosyncratic test also asymptotically attains the correct size and is consistent under the LTU framework.

## B.2 Proof of Theorem SA-1 and Theorem 2.1

Throughout Appendix B.2, I use the notation $\theta = \min\left\{\sqrt{N}, \sqrt{T}\right\}$ and $H = V^{-1}(\hat{f}'f/T)(\Lambda'\Lambda/N)$, where $V$ is the largest eigenvalue of $xx'/(NT)$. I also let $F_{t-1}^c = F_{t-1} - \bar{F}$, where $\bar{F} = T^{-1}\sum_{t=1}^T F_{t-1}$ and $\hat{F}_{t-1}^c = \hat{F}_{t-1} - \overline{\hat{F}}$, where $\overline{\hat{F}} = T^{-1}\sum_{t=1}^T \hat{F}_{t-1}$. Let also $f_t^c = f_t - \bar{f}$, where $\bar{f} = T^{-1}\sum_{t=1}^T f_t$ and $\hat{f}_t^c = \hat{f}_t - \overline{\hat{f}}$, where $\overline{\hat{f}} = T^{-1}\sum_{t=1}^T \hat{f}_t$. In addition, I let $U_{i,t-1}^c = U_{i,t-1} - \bar{U}_i$ where $\bar{U}_i = T^{-1}\sum_{t=1}^T U_{i,t-1}$ and $\hat{U}_{i,t-1}^c = \hat{U}_{i,t-1} - \overline{\hat{U}}_i$, where $\overline{\hat{U}}_i = T^{-1}\sum_{t=1}^T \hat{U}_{i,t-1}$. Let also $u_{i,t}^c = u_{i,t} - \bar{u}_i$, where $\bar{u}_i = T^{-1}\sum_{t=1}^T u_{i,t}$ and $\hat{u}_{i,t}^c = \hat{u}_{i,t} - \overline{\hat{u}}_i$, where $\overline{\hat{u}}_i = T^{-1}\sum_{t=1}^T \hat{u}_{i,t}$. As I explained in the main text, the proofs are presented under $r = 1$, $C(L) = 1$, and $D_i(L) = 1$ for all $i$.

**Lemma B.1.** *Under Assumptions 2.1, 2.3, 2.4, and B.1, the following hold.*

*(a)* $T^{-1/2}F_{[Tr]} \Rightarrow \sigma W_c(r)$,

*(b)* $T^{-3/2}\sum_{t=1}^T F_t \Rightarrow \sigma \int W_c(r)dr$,

*(c)* $T^{-1}\sum_{t=1}^T F_{t-1}e_t \Rightarrow \sigma^2 \int W_c(r)dW(r)$,

*(d)* $T^{-2}\sum_{t=1}^T F_t^2 \Rightarrow \sigma^2 \int W_c(r)^2 dr$,

*(e)* $T^{-1/2}U_{i,[Tr]} \Rightarrow \sigma_i W_{c,i}(r)$,

*(f)* $T^{-3/2}\sum_{t=1}^T U_{i,t} \Rightarrow \sigma_i \int W_{c,i}(r)dr$,

*(g)* $T^{-1}\sum_{t=1}^T U_{i,t-1}z_{i,t} \Rightarrow \sigma_i^2 \int W_{c,i}(r)dW(r)$,

*(h)* $T^{-2}\sum_{t=1}^T U_{i,t}^2 \Rightarrow \sigma_i^2 \int W_{c,i}(r)^2 dr$,

*(i)* $T^{-1}\sum_{t=1}^T F_{t-1}^c e_t \Rightarrow \sigma^2 \int \bar{W}_c(r)dW(r)$,

*(j)* $T^{-2}\sum_{t=1}^T F_{t-1}^{c2} \Rightarrow \sigma^2 \int \bar{W}_c(r)^2 dr$,

*(k)* $T^{-1}\sum_{t=1}^T U_{i,t-1}^c z_{i,t} \Rightarrow \sigma_i^2 \int \bar{W}_{c,i}(r)dW(r)$,

*(l)* $T^{-2}\sum_{t=1}^T U_{i,t-1}^c \Rightarrow \sigma_i^2 \int \bar{W}_{c,i}(r)^2 dr$,

*where* $W_c(r)$ *and* $W_{c,i}(r)$ *are independent Ornstein and Uhlenbeck processes defined on* $r \in [0,1]$ *and* $\bar{W}_c(r) \equiv W_c(r) - \int W_c(r)dr$, $\bar{W}_{c,i}(r) \equiv W_{c,i}(r) - \int W_{c,i}(r)dr$.

**Proof of B.1.** See Phillips (1987) for parts (a)–(h). For parts (i)–(l), the results are directly obtained from them. ■

**Lemma B.2.** *Under Assumptions 2.1, 2.3, 2.4, and B.1, the following hold.*

*(a)* $T^{-1}\sum_{t=1}^T f_t^2 \xrightarrow{p} \Sigma_f$*, a positive constant,*

*(b)* $\mathbb{E}(u_{i,t}) = 0$ *and* $\mathbb{E}\left|u_{i,t}\right|^8 = O(1)$,

*(c)* $\left|\gamma^*_{s,s}\right| = O(1)$ *for all* $s$ *and* $T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\left|\gamma^*_{s,t}\right| = O(1)$*, where* $\gamma^*_{st} = N^{-1}\sum_{i=1}^{N}\mathbb{E}(u_{i,s}u_{i,t})$*,*

*(d)* $\sum_{i=1}^{N}\left|\phi^*_{i,j}\right| = O(1)$ *for all* $j$ *and* $N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\phi^*_{i,j}\right| = O(1)$*, where* $\phi^*_{i,j} = \mathbb{E}(u_{i,t}u_{j,t})$*,*

*(e)* $\zeta^*_{s,t} = O(1)$*, where* $\zeta^*_{s,t} = \mathbb{E}\left|N^{-1/2}\sum_{i=1}^{N}[u_{i,s}u_{i,t} - \mathbb{E}(u_{i,s}u_{i,t})]\right|^4$.

**Proof of Lemma B.2.** (a) I start with

$$f_t = \frac{c}{T}F_{t-1} + e_t.$$

Squaring both sides, summing over $t$, and multiplying both sides by $T^{-1}$ yields

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} f_t^2 &= \frac{c^2}{T^3}\sum_{t=1}^{T} F_{t-1}^2 + 2\tfrac{c}{T^2}\sum_{t=1}^{T} F_{t-1}e_t + \tfrac{1}{T}\sum_{t=1}^{T} e_t^2,\\ &= I + II + T^{-1}\sum_{t=1}^{T} e_t^2 \xrightarrow{p} \sigma^2,\end{aligned}$$

because $I = O_p(T^{-1})$ by using Lemma B.1 (d) and $II = o_p(T^{-1})$ by using Lemma B.1 (c). The convergence of the third term is implied by the weak law of large numbers in Assumption 2.1. Hence, the result follows.

(b) It is straightforward that

$$\mathbb{E}(u_{i,t}) = \mathbb{E}(U_{i,t}) - \mathbb{E}(U_{i,t-1}) = 0,$$

from Assumption 2.3.1. Next,

$$\begin{aligned}\mathbb{E}\left|u_{i,t}\right|^8 &= \mathbb{E}\left|\frac{c}{T}U_{i,t-1} + z_{i,t}\right|^8,\\ &\leq 2^8 \times \max\left\{\frac{c_i^8}{T^8}\mathbb{E}\left|U_{i,t-1}\right|^8, \mathbb{E}\left|z_{i,t}\right|^8\right\},\end{aligned}$$

but

$$\mathbb{E}\left|U_{i,t-1}\right|^8 \leq T^8\rho_{i,T}^{8T}\mathbb{E}\left|z_{i,t}\right|^8,$$

so that

$$\frac{c_i^8}{T^8}\mathbb{E}\left|U_{i,t-1}\right|^8 \leq c_i^8\rho_{i,T}^{8T}\mathbb{E}\left|z_{i,t}\right|^8,$$

where $\rho_{i,T}^{8T} = (1 + \frac{c_i}{T})^{8T} \to \exp(8c_i)$ and $\mathbb{E}\left|z_{i,t}\right|^8 \leq M$ from Assumption 2.3.1. Hence, the result follows.

(c) Without loss of generality, let $s \geq t$. Consider

$$\begin{aligned}
\mathbb{E}(u_{i,s}u_{i,t}) &= \mathbb{E}\left[\left(\frac{c_i}{T}U_{i,s-1} + z_{i,s}\right)\left(\frac{c_i}{T}U_{i,t-1} + z_{i,t}\right)\right], \\
&= \frac{c_i^2}{T^2}\mathbb{E}(U_{i,s-1}U_{i,t-1}) + \frac{c_i}{T}\mathbb{E}\left(U_{i,s-1}z_{i,t}\right) + \frac{c_i}{T}\mathbb{E}\left(U_{i,t-1}z_{i,s}\right) + \mathbb{E}(z_{i,s}z_{i,t}), \\
&= I + II + III + IV.
\end{aligned}$$

However,

$$I \leq \frac{c_i^2}{T}\mathbb{E}(T^{-1}U_{i,s-1}^2) = O(T^{-1}),$$

by using Lemma B.1 (e). For $II$,

$$\begin{aligned}
II &= \frac{c_i}{T}\mathbb{E}\left(U_{i,s-1}z_{i,t}\right) = \frac{c_i}{T}\mathbb{E}[(U_{i,s-1} - U_{i,t})z_{i,t} + u_{i,t}z_{i,t} + U_{i,t-1}z_{i,t}], \\
&= \frac{c_i}{T}\mathbb{E}[(U_{i,s-1} - U_{i,t})z_{i,t}] + \frac{c_i}{T}\mathbb{E}(u_{i,t}z_{i,t}) + \frac{c_i}{T}\mathbb{E}(U_{i,t-1}z_{i,t}), \\
&= IIa + IIb + IIc.
\end{aligned}$$

However, since $U_{i,s-1} = z_{i,s-1} + \rho_{i,T}z_{i,s-2} + \cdots + \rho_{i,T}^{s-t-1}U_{i,t}$,

$$\begin{aligned}
IIa &= \frac{c_i}{T}\mathbb{E}[\left\{z_{i,s-1} + \rho_{i,T}z_{i,s-2} + \rho_{i,T}^2 z_{i,s-3} + \cdots + (\rho_{i,T}^{s-t-1} - 1)U_{i,t}\right\} z_{i,t}], \\
&= \frac{c_i}{T}\mathbb{E}[\left\{z_{i,s-1} + \rho_{i,T}z_{i,s-2} + \rho_{i,T}^2 z_{i,s-3} + \cdots + (\rho_{i,T}^{s-t-1} - 1)z_{i,t} + \rho_{i,T}(\rho_{i,T}^{s-t-1} - 1)U_{i,t-1}\right\} z_{i,t}], \\
&= \frac{c_i}{T}(\rho_{i,T}^{s-t-1} - 1)\mathbb{E}(z_{i,t}^2) = O(T^{-1}),
\end{aligned}$$

from Assumption 2.3.1,

$$\begin{aligned}
IIb &= \frac{c_i}{T}\mathbb{E}[(z_{i,t} + \frac{c_i}{T}z_{i,t-1} + \frac{c_i}{T}\rho_{i,T}z_{i,t-2} + \cdots + \frac{c_i}{T}\rho_{i,T}^t U_{i,0})z_{i,t}], \\
&= \frac{c_i}{T}\mathbb{E}(z_{i,t}^2) = O(T^{-1}),
\end{aligned}$$

from Assumption 2.3.1, and

$$IIc = \frac{c_i}{T^{1/2}} \underbrace{\mathbb{E}(T^{-1/2}U_{i,t-1})}_{=O(1) \text{ by Lemma A1 (e)}} \mathbb{E}(z_{i,t}) = 0,$$

so that $II = O(T^{-1})$. For $III$,

$$III = \frac{c_i}{T}\mathbb{E}\left(U_{i,t-1}z_{i,s}\right) = \frac{c_i}{T}\mathbb{E}\left(U_{i,t-1}\right)\mathbb{E}(z_{i,s}) = 0,$$

since $U_{i,t-1}$ and $z_{i,s}$ are independent as long as $s \geq t$. Therefore,

$$\begin{aligned}\mathbb{E}(u_{i,s}u_{i,t}) &= O(T^{-1}) + O(T^{-1}) + O(T^{-1}) + 0 + \mathbb{E}(z_{i,s}z_{i,t}),\\ &= \begin{cases}\sigma_i^2 + O(T^{-1}) & \text{if } s = t\\ O(T^{-1}) & \text{if } s \neq t\end{cases}.\end{aligned}$$

I now consider

$$\begin{aligned}\gamma_{s,t}^* &= \mathbb{E}\left[N^{-1}\textstyle\sum_{i=1}^N u_{i,s}u_{i,t}\right],\\ &= \begin{cases}N^{-1}\sum_{i=1}^N \sigma_i^2 + O(T^{-1}) & \text{if } s = t\\ O(T^{-1}) & \text{if } s \neq t\end{cases}.\end{aligned}$$

I also have

$$\textstyle\sum_{s=1}^T \left|\gamma_{s,t}^*\right| = N^{-1}\sum_{i=1}^N \sigma_i^2 + O(1),$$

so that

$$T^{-1}\textstyle\sum_{s=1}^T\sum_{t=1}^T \left|\gamma_{s,t}^*\right| = N^{-1}\sum_{i=1}^N \sigma_i^2 + O(1) = O(1).$$

(d) Consider

$$\begin{aligned}\phi_{i,j}^* &= \mathbb{E}(u_{i,t}u_{j,t}) = \mathbb{E}\left[\left(\frac{c_i}{T}U_{i,t-1} + z_{i,t}\right)\left(\frac{c_j}{T}U_{j,t-1} + z_{j,t}\right)\right],\\ &= \frac{c_ic_j}{T^2}\mathbb{E}(U_{i,t-1}U_{j,t-1}) + \frac{c_i}{T}\mathbb{E}(U_{i,t-1}z_{j,t}) + \frac{c_j}{T}\mathbb{E}(U_{j,t-1}z_{i,t}) + \mathbb{E}(z_{i,t}z_{j,t}),\\ &= I + II + III + IV.\end{aligned}$$

For $I$,

$$I = \frac{c_ic_j}{T^2}\mathbb{E}(U_{i,t-1}U_{j,t-1}) = \frac{c_ic_j}{T^2}\phi_{i,j}\left[\textstyle\sum_{l=0}^{t-1}(1+\frac{c_i}{T})^l(1+\frac{c_j}{T})^l\right],$$

and, assuming $c_i \geq c_j$ without loss of generality, I obtain

$$\left[\sum_{l=0}^{t-1}(1+\tfrac{c_i}{T})^{2l}\right] \leq T(1+\frac{c_i}{T})^{2T} = O(T),$$

so that $I = \phi_{i,j} \times O(T^{-1})$. For $II$,

$$II = \frac{c_i}{T^{1/2}}\mathbb{E}(T^{-1/2}U_{i,t-1})\mathbb{E}(z_{j,t}) = 0,$$

from Assumption 2.3.1, and similarly, $III = 0$. $IV = \phi_{i,j}$ by definition. Therefore,

$$\phi_{i,j}^* = \phi_{i,j}[1 + O(T^{-1})],$$

so that

$$\sum_{i=1}^{N}\left|\phi_{i,j}^*\right| = [1+O(T^{-1})]\sum_{i=1}^{N}|\phi_{i,j}| = O(1),$$

from Assumption 2.3.2 and

$$N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|\phi_{i,j}^*\right| = [1+O(T^{-1})]N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}|\phi_{i,j}| = O(1),$$

from Assumption 2.3.2 as well. Hence, the result follows.

(e) Since

$$u_{i,s}u_{i,t} = \frac{c^2}{T^2}U_{i,s-1}U_{i,t-1} + \frac{c}{T}U_{i,s-1}z_{i,t} + \frac{c}{T}U_{i,t-1}z_{i,s} + z_{i,s}z_{i,t},$$

$$\begin{aligned}
\zeta_{s,t}^* &= \mathbb{E}\left|N^{-1/2}\sum_{i=1}^{N}\left[u_{i,s}u_{i,t} - \mathbb{E}(u_{i,s}u_{i,t})\right]\right|^4, \\
&= \mathbb{E}\left|\frac{c^2}{T^2N^{1/2}}\sum_{i=1}^{N}\left[U_{i,s-1}U_{i,t-1} - \mathbb{E}(U_{i,s-1}U_{i,t-1})\right]\right. \\
&\quad + \frac{c}{TN^{1/2}}\sum_{i=1}^{N}\left[U_{i,s-1}z_{i,t} - \mathbb{E}(U_{i,s-1}z_{i,t})\right] \\
&\quad + \frac{c}{TN^{1/2}}\sum_{i=1}^{N}\left[U_{i,t-1}z_{i,s} - \mathbb{E}(U_{i,t-1}z_{i,s})\right] \\
&\quad \left. + \frac{1}{N^{1/2}}\sum_{i=1}^{N}\left[z_{i,s}z_{i,t} - \mathbb{E}(z_{i,s}z_{i,t})\right]\right|^4, \\
&= \mathbb{E}\left|\Phi_1 + \Phi_2 + \Phi_3 + \Phi_4\right|^4, \\
&\leq 4^4 \times \max\left\{\mathbb{E}\left|\Phi_1\right|^4, \mathbb{E}\left|\Phi_2\right|^4, \mathbb{E}\left|\Phi_3\right|^4, \zeta_{s,t}\right\}.
\end{aligned}$$

Consider $\mathbb{E}\left|\Phi_1\right|^4$. Since $U_{i,s-1}=\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}z_{i,l}$ and $U_{i,t-1}=\sum_{m=0}^{t-1}\rho_{i,T}^{t-1-m}z_{i,m}$,

$$\begin{aligned}
\mathbb{E}\left|\Phi_1\right|^4 &= \frac{c_i^8}{T^8}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[U_{i,s-1}U_{i,t-1}-\mathbb{E}(U_{i,s-1}U_{i,t-1})\right]\right|^4, \\
&= \frac{c_i^8}{T^8}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}\sum_{m=0}^{t-1}\rho_{i,T}^{t-1-m}z_{i,l}z_{i,m}\right.\right. \\
&\quad \left.\left.-\textstyle\sum_{l=1}^{s-1}\rho_{i,T}^{s-1-l}\sum_{m=0}^{t-1}\rho_{i,T}^{t-1-m}\mathbb{E}(z_{i,l}z_{i,m})\right]\right|^4, \\
&= \frac{c_i^8}{T^8}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}\sum_{m=0}^{t-1}\rho_{i,T}^{t-1-m}(z_{i,l}z_{i,m}-\mathbb{E}(z_{i,l}z_{i,m}))\right|^4, \\
&\leq \frac{c_i^8}{T^8}\mathbb{E}\left|\textstyle\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}\sum_{m=0}^{t-1}\rho_{i,T}^{t-1-m}\left|N^{-1/2}\sum_{i=1}^N(z_{i,l}z_{i,m}-\mathbb{E}(z_{i,l}z_{i,m}))\right|\right|^4, \\
&\leq \frac{c_i^8}{T^8}T^8\rho_{i,T}^{8T}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N(z_{i,l}z_{i,m}-\mathbb{E}(z_{i,l}z_{i,m}))\right|^4, \\
&= c_i^8\rho_{i,T}^{8T}M=O(1),
\end{aligned}$$

from Assumption 2.3.3. Next,

$$\begin{aligned}
\mathbb{E}\left|\Phi_2\right|^4 &= \frac{c_i^4}{T^4}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[U_{i,s-1}z_{i,t}-\mathbb{E}(U_{i,s-1}z_{i,t})\right]\right|^4, \\
&= \frac{c_i^4}{T^4}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}z_{i,l}z_{i,t}-\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}\mathbb{E}(z_{i,l}z_{i,t})\right]\right|^4, \\
&= \frac{c_i^4}{T^4}\mathbb{E}\left|\textstyle\sum_{l=0}^{s-1}\rho_{i,T}^{s-1-l}N^{-1/2}\sum_{i=1}^N\left[z_{i,l}z_{i,t}-\mathbb{E}(z_{i,l}z_{i,t})\right]\right|^4, \\
&\leq \frac{c_i^4}{T^4}T^4\rho_{i,T}^{4T}\mathbb{E}\left|N^{-1/2}\textstyle\sum_{i=1}^N\left[z_{i,l}z_{i,t}-\mathbb{E}(z_{i,l}z_{i,t})\right]\right|^4, \\
&= c_i^4\rho_{i,T}^{4T}M=O(1),
\end{aligned}$$

and $\mathbb{E}\left|\Phi_3\right|^4=O(1)$ is similarly shown. Therefore,

$$\zeta_{s,t}^*\leq 4^4\times\max\left\{O(1),\zeta_{s,t}\right\}=O(1),$$

from Assumption 2.3.3. Hence, the result follows. ■

**Lemma B.3.** *Under Assumptions 2.1–2.4 and B.1, the following hold.*

*(a)* $T^{-1/2}\sum_{t=1}^T(\hat{f}_t-Hf_t)=O_p(\theta^{-1})$,

*(b)* $T^{-1}\sum_{t=1}^T(\hat{f}_t-Hf_t)^2=O_p(\theta^{-2})$,

*(c)* $T^{-1}\sum_{t=1}^T(\hat{f}_t-Hf_t)u_{i,t}=O_p(\theta^{-2})$,

*(d)* $T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)f_t = O_p(\theta^{-2})$,

*(e)* $T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)\hat{f}_t = O_p(\theta^{-2})$,

*(f)* $\hat{\lambda}_i - H^{-1}\lambda_i = O_p\left(\frac{1}{\min\{N, T^{1/2}\}}\right)$.

**Proof of Lemma B.3.** Part (a) follows Theorem 1 of Bai (2003). For part (b), the proof is straightforward from Theorem 1 of Bai and Ng (2002) if their assumptions are replaced with my Lemma B.2. For parts (c), (d), and (e), the proof follows Lemmas B1, B2, and B3 of Bai (2003), respectively if their assumptions are replaced with my Lemma B.2. For part (f), I have

$$\begin{aligned}
\hat{\lambda}_i - \lambda_i H^{-1} &= T^{-1}H\sum_{t=1}^{T} f_t u_{i,t} \\
&\quad + T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)\hat{f}_t\lambda_i + T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)u_{i,t}, \\
&= T^{-1}H\sum_{t=1}^{T} f_t u_{i,t} + O_p(\theta^{-2}),
\end{aligned}$$

by using Lemma B.3 (e) and (c). Now,

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T} f_t u_{i,t} &= T^{-1}\sum_{t=1}^{T}\left(\tfrac{c}{T}F_{t-1} + e_t\right)\left(\tfrac{c_i}{T}U_{i,t-1} + z_{i,t}\right), \\
&= cc_iT^{-3}\sum_{t=1}^{T} F_{t-1}U_{i,t-1} + cT^{-2}\sum_{t=1}^{T} F_{t-1}z_{i,t} \\
&\quad + c_iT^{-2}\sum_{t=1}^{T} U_{i,t-1}e_t + T^{-1}\sum_{t=1}^{T} e_t z_{i,t}, \\
&= I + II + III + IV.
\end{aligned}$$

However, by using the Cauchy–Schwarz inequality, Lemma B.1 (d) and (h), and Assumptions 2.1 and 2.3.1, I obtain

$$\begin{aligned}
I &\leq cc_iT^{-1}\left(T^{-2}\sum_{t=1}^{T} F_{t-1}^2\right)^{1/2}\left(T^{-2}\sum_{t=1}^{T} U_{i,t-1}^2\right)^{1/2} = O_p(T^{-1}), \\
II &\leq cT^{-1/2}\left(T^{-2}\sum_{t=1}^{T} F_{t-1}^2\right)^{1/2}\left(T^{-1}\sum_{t=1}^{T} z_{i,t-1}^2\right)^{1/2} = O_p(T^{-1/2}), \\
III &\leq c_iT^{-1/2}\left(T^{-2}\sum_{t=1}^{T} U_{i,t-1}^2\right)^{1/2}\left(T^{-1}\sum_{t=1}^{T} e_{t-1}^2\right)^{1/2} = O_p(T^{-1/2}).
\end{aligned}$$

For $IV$, Assumptions 2.1, 2.3.1, and 2.4 imply that $\{e_t z_{i,t}\}_{t=2}^{T}$ is a white noise sequence so that

$IV = O_p(T^{-1/2})$. Therefore,

$$\hat{\lambda}_i - H^{-1}\lambda_i = O_p(T^{-1/2}) + O_p(\theta^{-2}) = O_p\left(\frac{1}{\min\left\{N, T^{1/2}\right\}}\right). \tag{B.2.1}$$

Hence, the result follows. ■

**Lemma B.4.** *Under Assumptions 2.1–2.4 and A, the following hold.*

*(a)* $T^{-1}\sum_{t=1}^{T}\hat{f}_t^2 = T^{-1}H^2\sum_{t=1}^{T}f_t^2 + O_p(\theta^{-2})$,

*(b)* $T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^2 = T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2 + O_p(\theta^{-1})$,

*(c)* $T^{-1}\sum_{t=1}^{T}\hat{F}_{t-1}\hat{f}_t = T^{-1}H^2\sum_{t=1}^{T}F_{t-1}f_t + O_p(\theta^{-1})$,

*(d)* $T^{-1}\sum_{t=1}^{T}\hat{u}_{i,t}^2 = T^{-1}\sum_{t=1}^{T}u_{i,t}^2 + O_p(\theta^{-2})$,

*(e)* $T^{-2}\sum_{t=1}^{T}\hat{U}_{i,t-1}^2 = T^{-2}\sum_{t=1}^{T}U_{i,t-1}^2 + O_p(\theta^{-1})$,

*(f)* $T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t-1}\hat{u}_{i,t} = T^{-1}\sum_{t=1}^{T}U_{i,t-1}u_{i,t} + O_p(\theta^{-1})$,

*(g)* $T^{-1}\sum_{t=1}^{T}\hat{F}_{t-1}^c\hat{f}_t = T^{-1}H^2\sum_{t=1}^{T}F_{t-1}^c f_t + O_p(\theta^{-1})$,

*(h)* $T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^{c2} = T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^{c2} + O_p(\theta^{-1})$,

*(i)* $T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t-1}^c\hat{u}_{i,t} = T^{-1}\sum_{t=1}^{T}U_{i,t-1}^c u_{i,t} + O_p(\theta^{-1})$,

*(j)* $T^{-2}\sum_{t=1}^{T}\hat{U}_{i,t-1}^{c2} = T^{-2}\sum_{t=1}^{T}U_{i,t-1}^{c2} + O_p(\theta^{-1})$.

**Proof of Lemma B.4.** Note that $\hat{F}_0 = 0$ and $\hat{U}_{i,0} = 0$ for all $i$ by definition. (a) I start with the identity

$$\begin{aligned}
T^{-1}\sum_{t=1}^{T}\hat{f}_t^2 &= T^{-1}\sum_{t=1}^{T}\left[Hf_t + (\hat{f}_t - Hf_t)\right]^2, \\
&= T^{-1}H^2\sum_{t=1}^{T}f_t^2 + T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t)^2 \\
&\quad +2T^{-1}H\sum_{t=1}^{T}f_t(\hat{f}_t - Hf_t), \\
&= T^{-1}H^2\sum_{t=1}^{T}f_t^2 + I + II.
\end{aligned}$$

However, $I = O_p(\theta^{-2})$ by using Lemma B.3 (b) and $II = O_p(\theta^{-2})$ by using Lemma B.3 (d). Hence, the result follows.

(b) This part closely follows Bai and Ng's (2004) Lemma B.7. Since

$$\hat{F}_{t-1} = HF_{t-1} + \sum_{s=1}^{t-1}(\hat{f}_s - Hf_s), \tag{B.2.2}$$

squaring both sides, summing over $t$, and multiplying by $T^{-2}$ yields

$$\begin{aligned} T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^{2} &= T^{-2}H^{2}\sum_{t=1}^{T}F_{t-1}^{2}+T^{-1}\sum_{t=1}^{T}\left[T^{-1/2}\sum_{s=1}^{t-1}(\hat{f}_s-Hf_s)\right]^{2} \\ &\quad +2T^{-1}H\sum_{t=1}^{T}F_{t-1}\left[T^{-1/2}\sum_{s=1}^{t-1}(\hat{f}_s-Hf_s)\right], \\ &= T^{-2}H^{2}\sum_{t=1}^{T}F_{t-1}^{2}+I+II. \end{aligned}$$

However, $I=O_p(\theta^{-1})$ by using Lemma B.3 (a). For term $II$, I use the Cauchy–Schwarz inequality to get

$$\begin{aligned} II &\leq 2\left(T^{-2}\sum_{t=1}^{T}F_{t-1}^{2}\right)^{1/2}\left[T^{-1}\sum_{t=1}^{T}\left(T^{-1/2}\sum_{s=1}^{t-1}(\hat{f}_s-Hf_s)\right)^{2}\right]^{1/2}, \\ &= O_p(1)\times O_p(\theta^{-1}), \end{aligned}$$

by using Lemma B.1 (d) for the first term and Lemma B.3 (a) for the second term. Hence, the result follows.

(c) Since $F_t^2=(F_{t-1}+f_t)^2=F_{t-1}^2+f_t^2+2F_{t-1}f_t$ by construction, I obtain

$$F_{t-1}f_t=\frac{1}{2}(F_t^2-F_{t-1}^2-f_t^2).$$

Summing over $t$ and multiplying by $T^{-1}$ yields

$$T^{-1}\sum_{t=1}^{T}F_{t-1}f_t=\tfrac{1}{2}(T^{-1}F_T^2-T^{-1}F_0^2-T^{-1}\sum_{t=1}^{T}f_t^2). \tag{B.2.3}$$

I also have by construction

$$\hat{F}_{t-1}\hat{f}_t=\frac{1}{2}(\hat{F}_t^2-\hat{F}_{t-1}^2-\hat{f}_t^2),$$

so that

$$T^{-1}\sum_{t=1}^{T}\hat{F}_{t-1}\hat{f}_t=\tfrac{1}{2}(T^{-1}\hat{F}_T^2-T^{-1}\hat{F}_0^2-T^{-1}\sum_{t=1}^{T}\hat{f}_t^2). \tag{B.2.4}$$

Subtracting (B.2.3) multiplied by $H^2$ from (B.2.4) yields

$$\begin{aligned} T^{-1}\textstyle\sum_{t=1}^{T}\hat{F}_{t-1}\hat{f}_t &= T^{-1}H^2\textstyle\sum_{t=1}^{T}F_{t-1}f_t + \frac{1}{2T}(\hat{F}_T^2 - H^2F_T^2) - \frac{1}{2T}(\hat{F}_0^2 - H^2F_0^2) \\ &\quad - \left(T^{-1}\textstyle\sum_{t=1}^{T}\hat{f}_t^2 - T^{-1}H^2\textstyle\sum_{t=1}^{T}f_t^2\right), \\ &= T^{-1}H^2\textstyle\sum_{t=1}^{T}F_{t-1}f_t + I + II + III. \end{aligned}$$

For $I$, updating (B.2.2) to the period $T$, squaring both sides, and multiplying by $T^{-1}$ yields

$$\begin{aligned} T^{-1}\hat{F}_T^2 &= T^{-1}H^2F_T^2 + \underbrace{\left[T^{-1/2}\textstyle\sum_{s=1}^{T}(\hat{f}_s - Hf_s)\right]^2}_{=O_p(\theta^{-2}) \text{ by Lemma B.3 (a)}} \\ &\quad +2\underbrace{T^{-1/2}F_T}_{=O_p(1) \text{ by Lemma B.1 (a)}}\underbrace{\left[T^{-1/2}\textstyle\sum_{s=1}^{T}(\hat{f}_s - Hf_s)\right]}_{=O_p(\theta^{-1}) \text{ by Lemma B.3 (a)}}, \end{aligned}$$

so that $I = O_p(\theta^{-1})$. For $II$,

$$\hat{F}_0^2 - H^2F_0^2 = -H^2(\alpha_T^2F_0^2 + e_1^2 + 2\alpha_TF_0e_1),$$

is bounded as $T \to \infty$ so that $II = O_p(T^{-1})$. Term $III$ is $O_p(\theta^{-2})$ by using Lemma B.4 (a). Hence, the result follows.

(d) Since $\hat{u}_{i,t} = x_{i,t} - \hat{\lambda}_i\hat{f}_t$ and $x_{i,t} = u_{i,t} + \lambda_iH^{-1}Hf_t$,

$$\begin{aligned} \hat{u}_{i,t} &= u_{i,t} + \lambda_iH^{-1}Hf_t - \hat{\lambda}_i\hat{f}_t, \\ &= u_{i,t} - \lambda_iH^{-1}(\hat{f}_t - Hf_t) - (\hat{\lambda}_i - \lambda_iH^{-1})\hat{f}_t. \end{aligned} \tag{B.2.5}$$

Squaring both sides, summing over $t$, and multiplying by $T^{-1}$ yields

$$\begin{aligned} T^{-1}\textstyle\sum_{t=1}^{T}\hat{u}_{i,t}^2 &= T^{-1}\textstyle\sum_{t=1}^{T}u_{i,t}^2 + \lambda_i^2H^{-2}T^{-1}\textstyle\sum_{t=1}^{T}(\hat{f}_t - Hf_t)^2 + (\hat{\lambda}_i - \lambda_iH^{-1})^2T^{-1}\textstyle\sum_{t=1}^{T}\hat{f}_t^2 \\ &\quad -2\lambda_iH^{-1}T^{-1}\textstyle\sum_{t=1}^{T}(\hat{f}_t - Hf_t)u_{i,t} - 2(\hat{\lambda}_i - \lambda_iH^{-1})T^{-1}\textstyle\sum_{t=1}^{T}\hat{f}_tu_{i,t} \\ &\quad +2\lambda_i(\hat{\lambda}_i - \lambda_iH^{-1})T^{-1}\textstyle\sum_{t=1}^{T}(\hat{f}_t - Hf_t)\hat{f}_t, \\ &= T^{-1}\textstyle\sum_{t=1}^{T}u_{i,t}^2 + I + II + III + IV + V. \end{aligned}$$

However, $I = O_p(\theta^{-2})$ by using Lemma B.3 (b), $II = O_p(\frac{1}{\min\{T,N^2\}})$ by using Lemma B.3 (f) and $T^{-1}\sum_{t=2}^{T}\hat{f}_t = 1$, and $III = O_p(\theta^{-2})$ by using Lemma B.3 (c). I also have

$$\begin{aligned}
IV &= -2(\hat{\lambda}_i - \lambda_i H^{-1})T^{-1}\textstyle\sum_{t=1}^{T}\hat{f}_t u_{i,t}, \\
&= -2(\hat{\lambda}_i - \lambda_i H^{-1})T^{-1}\textstyle\sum_{t=1}^{T}(\hat{f}_t - Hf_t)u_{i,t} - 2(\hat{\lambda}_i - \lambda_i H^{-1})T^{-1}H\textstyle\sum_{t=1}^{T} f_t u_{i,t}, \\
&= O_p\left(\frac{1}{\min\{T,N^2\}}\right) \times O_p(\theta^{-2}) + O_p\left(\frac{1}{\min\{T,N^2\}}\right) \times O_p(T^{-1/2}),
\end{aligned}$$

by using Lemma B.3 (f) and Lemma B.3 (c) for the first term and by using Lemma B.3 (f) for the second term, and $V = O_p\left(\frac{1}{\min\{T,N^2\}}\right) \times O_p(\theta^{-2})$ by using Lemma B.3 (f) and Lemma B.3 (e). Hence, the result follows.

(e) I have

$$\begin{aligned}
\hat{U}_{i,t} &= \textstyle\sum_{s=1}^{t}\hat{u}_{i,s}, \\
&= \textstyle\sum_{s=1}^{t} u_{i,s} - \lambda_i H^{-1}\sum_{s=1}^{t}(\hat{f}_s - Hf_s) - (\hat{\lambda}_i - \lambda_i H^{-1})\sum_{s=1}^{t}\hat{f}_s, \\
&= U_{i,t} - U_{i,0} - \lambda_i H^{-1}\textstyle\sum_{s=1}^{t}(\hat{f}_s - Hf_s) - (\hat{\lambda}_i - \lambda_i H^{-1})\sum_{s=1}^{t}\hat{f}_s,
\end{aligned}$$

from (B.2.5). Multiplying both sides by $T^{-1/2}$ would yield

$$\begin{aligned}
T^{-1/2}\hat{U}_{i,t} &= T^{-1/2}U_{i,t} - T^{-1/2}U_{i,0} - \lambda_i H^{-1}\left[T^{-1/2}\textstyle\sum_{s=1}^{t}(\hat{f}_s - Hf_s)\right] - (\hat{\lambda}_i - H^{-1}\lambda_i)T^{-1/2}\sum_{s=1}^{t}\hat{f}_s, \\
&= T^{-1/2}U_{i,t} + I + II + III.
\end{aligned}$$

but $I = O_p(T^{-1/2})$ from Assumption 2.3.1, $II = O_p(\theta^{-1})$ by using Lemma B.3 (a), $III = O_p\left(\frac{1}{\min\{N,T^{1/2}\}}\right)$ by using Lemma B.3 (f) and

$$\begin{aligned}
T^{-1/2}\textstyle\sum_{s=1}^{t}\hat{f}_s &= T^{-1/2}\hat{F}_t = T^{-1/2}HF_t + T^{-1/2}\textstyle\sum_{s=1}^{t}(\hat{f}_s - Hf_s), \\
&= O_p(1) + O_p(\theta^{-1}),
\end{aligned}$$

by using Lemma B.1 (b) and Lemma B.3 (a). This results in $T^{-1/2}\hat{U}_{i,t} = T^{-1/2}U_{i,t} + O_p(\theta^{-1})$ so that squaring both sides yields

$$\begin{aligned} T^{-1}\hat{U}_{i,t}^2 &= T^{-1}U_{i,t}^2 + O_p(\theta^{-2}) + O_p(\theta^{-1}) \times T^{-1/2}U_{i,t}, \\ &= T^{-1}U_{i,t}^2 + O_p(\theta^{-1}), \end{aligned} \tag{B.2.6}$$

by using Lemma B.1 (e). Furthermore, summing over $t$ yields

$$T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t}^2 = T^{-1}\sum_{t=1}^{T}U_{i,t}^2 + O_p(\theta^{-1})T^{-1/2}\sum_{t=1}^{T}U_{i,t}.$$

Multiplying both sides by $T^{-1}$ yields

$$\begin{aligned} T^{-2}\sum_{t=1}^{T}\hat{U}_{i,t}^2 &= T^{-2}\sum_{t=1}^{T}U_{i,t}^2 + O_p(\theta^{-1}) \underbrace{T^{-3/2}\sum_{t=1}^{T}U_{i,t}}_{=O_p(1) \text{ by Lemma B.1 (g)}}, \\ &= T^{-2}\sum_{t=1}^{T}U_{i,t}^2 + O_p(\theta^{-1}). \end{aligned}$$

Hence, the result follows.

(f) I use an identity similar to (B.2.4) for $\hat{U}_{i,t}$

$$T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t-1}\hat{u}_{i,t} = \frac{\hat{U}_{i,T}^2}{2T} - \frac{\hat{U}_{i,0}^2}{2T} - \frac{1}{2T}\sum_{t=1}^{T}\hat{u}_{i,t}^2, \tag{B.2.7}$$

and an identity similar to (B.2.3) for $U_{i,t}$

$$T^{-1}\sum_{t=1}^{T}U_{i,t-1}u_{i,t} = \frac{U_{i,T}^2}{2T} - \frac{U_{i,0}^2}{2T} - \frac{1}{2T}\sum_{t=1}^{T}u_{i,t}^2. \tag{B.2.8}$$

Subtracting (B.2.8) from (B.2.7) yields

$$\begin{aligned} & T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t-1}\hat{u}_{i,t} - T^{-1}\sum_{t=1}^{T}U_{i,t-1}u_{i,t} \\ = & \frac{1}{2T}(\hat{U}_{i,T}^2 - U_{i,T}^2) - \frac{1}{2T}(\hat{U}_{i,0}^2 - U_{i,0}^2) - \frac{1}{2T}\left(\sum_{t=1}^{T}\hat{u}_{i,t}^2 - \sum_{t=1}^{T}u_{i,t}^2\right), \\ = & I + II + III. \end{aligned}$$

However, $I$ and $II$ are $O_p(\theta^{-1})$ from (B.2.6) and $III$ is $O_p(\theta^{-2})$ by using Lemma B.4 (d). Hence, the result follows.

(g) Since $F_t^{c2} = (F_{t-1} + f_t - \bar{F})^2 = (F_{t-1}^c + f_t)^2 = F_{t-1}^{c2} + f_t^2 + 2F_{t-1}^c f_t$ by construction, I obtain

$$\begin{aligned} F_{t-1}^c f_t &= \frac{1}{2}(F_t^{c2} - F_{t-1}^{c2} - f_t^2), \\ &= \frac{1}{2}(F_t^2 - F_{t-1}^2 - 2\bar{F} f_t - f_t^2). \end{aligned}$$

Summing over $t$ and multiplying by $T^{-1}$ yields

$$T^{-1} \textstyle\sum_{t=1}^T F_{t-1}^c f_t = \frac{1}{2}(T^{-1}F_T^2 - T^{-1}F_0^2 - 2\bar{F}\bar{f} - T^{-1}\sum_{t=1}^T f_t^2). \tag{B.2.9}$$

I also have by construction

$$\begin{aligned} \hat{F}_{t-1}^c \hat{f}_t &= \frac{1}{2}(\hat{F}_t^{c2} - \hat{F}_{t-1}^{c2} - \hat{f}_t^2), \\ &= \frac{1}{2}(\hat{F}_t^2 - \hat{F}_{t-1}^2 - 2\overline{\hat{F}} \hat{f}_t - \hat{f}_t^2), \end{aligned}$$

so that

$$T^{-1} \textstyle\sum_{t=1}^T \hat{F}_{t-1}^c \hat{f}_t = \frac{1}{2}(T^{-1}\hat{F}_T^2 - T^{-1}\hat{F}_0^2 - 2\overline{\hat{F}}\overline{\hat{f}} - T^{-1}\sum_{t=1}^T \hat{f}_t^2). \tag{B.2.10}$$

Subtracting (B.2.9) multiplied by $H^2$ from (B.2.10) yields

$$\begin{aligned} T^{-1} \textstyle\sum_{t=1}^T \hat{F}_{t-1}^c \hat{f}_t &= T^{-1}H^2 \textstyle\sum_{t=1}^T F_{t-1}^c f_t + \frac{1}{2T}(\hat{F}_T^2 - H^2 F_T^2) - \frac{1}{2T}(\hat{F}_0^2 - H^2 F_0^2) \\ &\quad - \left(T^{-1}\textstyle\sum_{t=1}^T \hat{f}_t^2 - T^{-1}H^2 \sum_{t=1}^T f_t^2\right) - (\overline{\hat{F}}\overline{\hat{f}} - H^2 \bar{F}\bar{f}), \\ &= T^{-1}H^2 \textstyle\sum_{t=1}^T F_{t-1} f_t + I + II + III + IV. \end{aligned}$$

For the terms $I + II + III$, I follow the proof of part (c) to obtain $O_p(\theta^{-1})$. Term $IV$ is

$$\begin{aligned} \overline{\hat{F}}\overline{\hat{f}} - H^2\bar{F}\bar{f} &= (\overline{\hat{F}} - H\bar{F})H\bar{f} + H\bar{F}(\overline{\hat{f}} - H\bar{f}) + (\overline{\hat{F}} - H\bar{F})(\overline{\hat{f}} - H\bar{f}), \\ &= O_p(T^{1/2}\theta^{-1}) \times O_p(T^{-1/2}) \\ &\quad + O_p(T^{1/2}) \times O_p(T^{-1/2}\theta^{-1}) \\ &\quad + O_p(T^{1/2}\theta^{-1}) \times O_p(T^{-1/2}\theta^{-1}), \\ &= O_p(\theta^{-1}), \end{aligned}$$

because $H\bar{f} = O_p(T^{-1/2})$,

$$\begin{aligned}\overline{\hat{F}} &= T^{-1}H\sum_{t=1}^{T} F_{t-1} + T^{-1}\sum_{t=1}^{T}(\hat{F}_{t-1} - HF_{t-1}),\\ &= T^{-1}H\sum_{t=1}^{T} F_{t-1} + T^{-1}\sum_{t=1}^{T}\sum_{s=1}^{t-1}(\hat{f}_s - Hf_s),\\ &= H\bar{F} + O_p(T^{1/2}\theta^{-1}),\end{aligned}$$

and

$$\begin{aligned}\overline{\hat{f}} &= T^{-1}H\sum_{t=1}^{T} f_t + T^{-1}\sum_{t=1}^{T}(\hat{f}_t - Hf_t),\\ &= H\bar{f} + O_p(T^{-1/2}\theta^{-1}).\end{aligned}$$

(h) I have

$$\begin{aligned}T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^{c2} &= T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^{2} - 2\overline{\hat{F}}T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1} + T^{-1}\overline{\hat{F}}^{2},\\ &= T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^{2} - T^{-1}\overline{\hat{F}}^{2} = I + II.\end{aligned}$$

However,

$$I = T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2 + O_p(\theta^{-1}),$$

by using Lemma B.4 (b) and

$$II = T^{-1}H^2\bar{F}^2 + O_p(\theta^{-1}).$$

Hence,

$$\begin{aligned}I + II &= T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2 - T^{-1}H^2\bar{F}^2 + O_p(\theta^{-1}),\\ &= T^{-2}H^2\sum_{t=1}^{T}(F_{t-1} - \bar{F})^2 + O_p(\theta^{-1}).\end{aligned}$$

(i) The proof follows part (g) by using an identity similar to (B.2.10)

$$T^{-1}\sum_{t=1}^{T}\hat{U}_{i,t-1}^{c}\hat{u}_{i,t} = \tfrac{1}{2}(T^{-1}\hat{U}_{i,T}^2 - T^{-1}\hat{U}_{i,0}^2 - 2\overline{\hat{U}}_i\overline{\hat{u}}_i - T^{-1}\sum_{t=1}^{T}\hat{u}_{i,t}^2),$$

and an identity similar to (B.2.9) for $U_{i,t}$

$$T^{-1}\sum_{t=1}^{T}U_{i,t-1}^{c}u_{i,t}=\frac{1}{2}(T^{-1}U_{i,T}^{2}-T^{-1}U_{i,0}^{2}-2\overline{U}_{i}\overline{u}_{i}-T^{-1}\sum_{t=1}^{T}u_{i,t}^{2}).$$

(j) I have

$$\begin{aligned}T^{-2}\sum_{t=1}^{T}(\hat{U}_{i,t-1}-\overline{\hat{U}}_{i})^{2} &= T^{-2}\sum_{t=1}^{T}\hat{U}_{i,t-1}^{2}-2T^{-2}\overline{\hat{U}}_{i}\sum_{t=1}^{T}\hat{U}_{i,t-1}+T^{-2}\sum_{t=1}^{T}\overline{\hat{U}}_{i}^{2},\\ &= T^{-2}\sum_{t=1}^{T}\hat{U}_{i,t-1}^{2}-T^{-1}\overline{\hat{U}}_{i}^{2}=I+II,\end{aligned}$$

but

$$I=T^{-2}\sum_{t=1}^{T}U_{i,t-1}^{2}+O_{p}(\theta^{-1}),$$

by using Lemma B.4 (e) and

$$II=-T^{-1}\bar{U}_{i}^{2}+O_{p}(\theta^{-1}).$$

Hence,

$$\begin{aligned}I+II &= T^{-2}\sum_{t=1}^{T}U_{i,t-1}^{2}-T^{-1}\bar{U}_{i}^{2}+O_{p}(\theta^{-1}),\\ &= T^{-2}\sum_{t=1}^{T}(U_{i,t-1}-\bar{U}_{i})^{2}+O_{p}(\theta^{-1}),\end{aligned}$$

and the result follows. ■

**Proof of Theorem SA-1.** (i-a) The common test is

$$t_{\hat{F}}=\frac{T\hat{\delta}}{\hat{\sigma}\left(T^{-2}\sum_{t=2}^{T}\hat{F}_{t-1}^{2}\right)^{-1/2}}. \tag{B.2.11}$$

Under Assumptions 2.1–2.4 and B.1, I can use Lemma B.4 (b) and (c) so that the numerator becomes

$$\begin{aligned} T\hat{\delta} &= \frac{T^{-1}\sum_{t=1}^{T}\hat{F}_{t-1}\hat{f}_t}{T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^2}, \\ &= \frac{T^{-1}H^2\sum_{t=1}^{T}F_{t-1}f_t + O_p(\theta^{-1})}{T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2 + O_p(\theta^{-1})}, \\ &= \frac{cT^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2 + T^{-1}H^2\sum_{t=1}^{T}F_{t-1}e_t + O_p(\theta^{-1})}{T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2 + O_p(\theta^{-1})}. \end{aligned} \tag{B.2.12}$$

The variance estimate

$$\begin{aligned} \hat{\sigma}^2 &= T^{-1}\sum_{t=1}^{T}\left(\hat{f}_t - \hat{\delta}\hat{F}_{t-1}\right)^2, \\ &= T^{-1}\sum_{t=1}^{T}\left[Hf_t - \hat{\delta}^* HF_{t-1} + \left(\hat{f}_t - Hf_t\right) + \hat{\delta}^* HF_{t-1} - \hat{\delta}\hat{F}_{t-1}\right]^2, \\ &= T^{-1}\sum_{t=1}^{T}\left[He_t - (\hat{\delta}^* - \delta)HF_{t-1} + \left(\hat{f}_t - Hf_t\right) + \hat{\delta}^* HF_{t-1} - \hat{\delta}\hat{F}_{t-1}\right]^2, \\ &= T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{5} D_j\right)^2 \leq T^{-1}\sum_{t=1}^{T} 5\left(\sum_{j=1}^{5} D_j^2\right), \end{aligned}$$

where $\hat{\delta}^* = \frac{T^{-1}H^2\sum_{t=1}^{T}F_{t-1}f_t}{T^{-1}H^2\sum_{t=1}^{T}F_{t-1}^2}$. To ensure this is a consistent estimate, I compute the stochastic orders of the five terms. First, $T^{-1}\sum_{t=1}^{T}D_1^2 = T^{-1}\sum_{t=1}^{T}H^2e_t^2$,

$$T^{-1}\sum_{t=1}^{T}D_2^2 = T^{-1}\frac{\left[T^{-1}H^2\sum_{t=1}^{T}F_{t-1}e_t\right]^2}{T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2} = O_p(T^{-1}),$$

by using Lemma B.1 (c) and (d).

$$T^{-1}\sum_{t=1}^{T}D_3^2 = T^{-1}\sum_{t=1}^{T}\left(\hat{f}_t - Hf_t\right)^2 = O_p(\theta^{-2}),$$

by using Lemma B.3 (b).

$$T^{-1}\sum_{t=1}^{T}D_4^2 = T^{-1}\frac{\left[T^{-1}H^2\sum_{t=1}^{T}F_{t-1}f_t\right]^2}{T^{-2}H^2\sum_{t=1}^{T}F_{t-1}^2} = O_p(T^{-1}),$$

by using Lemma B.1 (c) and (d). Finally,

$$T^{-1}\sum_{t=1}^{T}D_5^2 = T^{-1}\frac{\left[T^{-1}\sum_{t=1}^{T}\hat{F}_{t-1}\hat{f}_t\right]^2}{T^{-2}\sum_{t=1}^{T}\hat{F}_{t-1}^2} = O_p(T^{-1}),$$

by using Lemma B.4 (b) and (c). Therefore, the first term dominates and the variance estimate satisfies $\hat{\sigma}^2 = Q^{-2}\sigma^2 + O_p(\theta^{-2})$ with $Q^{-1} \equiv p\lim H$, for any fixed $c$. Hence, plugging (B.2.12) into (B.2.11) and applying Lemma B.1 (c) and (d), I obtain

$$t_{\hat{F}} \Rightarrow c\left[\int_0^1 W_c(r)^2 dr\right]^{1/2} + \frac{\int_0^1 W_c(r)dW(r)}{\left[\int_0^1 W_c(r)^2 dr\right]^{1/2}}.$$

(i-b) I follow the same steps as above by replacing $\hat{f}_t$ and $\hat{F}_{t-1}$ with $\hat{u}_{i,t}$ and $\hat{U}_{i,t-1}$ and using the corresponding lemmas to show the results. Hence, the proof is condensed.

(ii-a) The common test is

$$\bar{t}_{\hat{F}} = \frac{T\hat{\delta}}{\hat{\sigma}\left[T^{-2}\sum_{t=2}^{T}(\hat{F}_{t-1} - \overline{\hat{F}})^2\right]^{-1/2}}, \tag{B.2.13}$$

where

$$\begin{aligned} T\hat{\delta} &= \frac{T^{-1}\sum_{t=1}^{T}(\hat{F}_{t-1} - \overline{\hat{F}})\hat{f}_t}{T^{-2}\sum_{t=1}^{T}(\hat{F}_{t-1} - \overline{\hat{F}})^2}, \\ &= \frac{T^{-1}H^2\sum_{t=1}^{T}(F_{t-1} - \bar{F})f_t + O_p(\theta^{-1})}{T^{-2}H^2\sum_{t=1}^{T}(F_{t-1} - \bar{F})^2 + O_p(\theta^{-1})}, \\ &= \frac{cT^{-1}H^2\sum_{t=1}^{T}(F_{t-1} - \bar{F})^2 + T^{-1}H^2\sum_{t=1}^{T}(F_{t-1} - \bar{F})e_t + O_p(\theta^{-1})}{T^{-2}H^2\sum_{t=1}^{T}(F_{t-1} - \bar{F})^2 + O_p(\theta^{-1})}, \end{aligned} \tag{B.2.14}$$

by using Lemma B.4 (g) for the numerator and Lemma B.4 (h) for the denominator. For the variance estimate, I can show that $\hat{\sigma}^2 = Q^{-2}\sigma^2 + O_p(\theta^{-2})$ as follows.

$$\begin{aligned} \hat{\sigma}^2 &= T^{-1}\sum_{t=1}^{T}\left[\hat{f}_t - \overline{\hat{f}} - \hat{\delta}\left(\hat{F}_{t-1} - \overline{\hat{F}}\right)\right]^2, \\ &= T^{-1}\sum_{t=1}^{T}\left[Hf_t - H\bar{f} - \hat{\delta}\left(\hat{F}_{t-1} - \overline{\hat{F}}\right) + \left(\hat{f}_t - Hf_t\right) - \left(\overline{\hat{f}} - H\overline{f}\right)\right]^2, \\ &= T^{-1}\sum_{t=1}^{T}\Big[He_t - H\bar{e} + \delta\left(HF_{t-1} - H\bar{F}\right) - \hat{\delta}^*\left(HF_{t-1} - H\bar{F}\right) + \left(\hat{f}_t - Hf_t\right) \\ &\quad - \left(\overline{\hat{f}} - H\overline{f}\right) + \hat{\delta}^*\left(HF_{t-1} - H\bar{F}\right) - \hat{\delta}\left(\hat{F}_{t-1} - H\overline{\hat{F}}\right)\Big]^2, \\ &= T^{-1}\sum_{t=1}^{T}\Big[He_t - H\bar{e} - \left(\hat{\delta}^* - \delta\right)\left(HF_{t-1} - H\bar{F}\right) + \left(\hat{f}_t - Hf_t\right) - \left(\overline{\hat{f}} - H\overline{f}\right) \\ &\quad + \hat{\delta}^*\left(HF_{t-1} - H\bar{F}\right) - \hat{\delta}\left(\hat{F}_{t-1} - \overline{\hat{F}}\right)\Big]^2, \\ &= T^{-1}\sum_{t=1}^{T}\left(\sum_{j=1}^{7} D_j\right)^2 \leq T^{-1}\sum_{t=1}^{T} 7\left(\sum_{j=1}^{7} D_j^2\right), \end{aligned}$$

where $\hat{\delta}^* = \frac{T^{-1}H^2\sum_{t=1}^T(F_{t-1}-\bar{F})f_t}{T^{-1}H^2\sum_{t=1}^T(F_{t-1}-\bar{F})^2}$. However, $T^{-1}\sum_{t=1}^T D_1^2 = T^{-1}H^2\sum_{t=1}^T e_t^2$, $T^{-1}\sum_{t=1}^T D_2^2 = H^2\bar{e}^2 = O_p(T^{-1})$,

$$T^{-1}\sum_{t=1}^T D_3^2 = T^{-1}\frac{\left[T^{-1}H^2\sum_{t=1}^T(F_{t-1}-\bar{F})e_t\right]^2}{T^{-2}H^2\sum_{t=1}^T(F_{t-1}-\bar{F})^2} = O_p(T^{-1}),$$

by using Lemma B.1 (i) and (j),

$$T^{-1}\sum_{t=1}^T D_4^2 = T^{-1}\sum_{t=1}^T\left(\hat{f}_t - Hf_t\right)^2 = O_p(\theta^{-2}),$$

by using Lemma B.3 (b),

$$T^{-1}\sum_{t=1}^T D_5^2 = \left(\bar{\hat{f}} - H\bar{f}\right)^2 = \left[T^{-1}\sum_{t=1}^T\left(\hat{f}_t - Hf_t\right)\right]^2 = O_p(\theta^{-2}),$$

by using Lemma B.3 (a),

$$T^{-1}\sum_{t=1}^T D_6^2 = T^{-1}\frac{\left[T^{-1}H^2\sum_{t=1}^T(F_{t-1}-\bar{F})f_t\right]^2}{T^{-2}H^2\sum_{t=1}^T(F_{t-1}-\bar{F})^2} = O_p(T^{-1}),$$

by using Lemma B.1 (i) and (j),

$$T^{-1}\sum_{t=1}^T D_7^2 = T^{-1}\frac{\left[T^{-1}\sum_{t=1}^T(\hat{F}_{t-1}-\bar{\hat{F}})\hat{f}_t\right]^2}{T^{-2}\sum_{t=1}^T(\hat{F}_{t-1}-\bar{\hat{F}})^2} = O_p(T^{-1}),$$

by using Lemma B.4 (g) and (h). Therefore, by plugging (B.2.14) into (B.2.13) and applying Lemma B.1 (i) and (j), I obtain

$$t_{\hat{F}} \Rightarrow c\left[\int_0^1 \bar{W}_c(r)^2dr\right]^{1/2} + \frac{\int_0^1 \bar{W}_c(r)dW(r)}{\left[\int_0^1 \bar{W}_c(r)^2dr\right]^{1/2}},$$

as $N, T \to \infty$.

(ii-b) I follow the same steps as above by replacing $\hat{f}_t$ and $\hat{F}_{t-1}$ with $\hat{u}_{i,t}$ and $\hat{U}_{i,t-1}$ and using the corresponding lemmas to show the results. Hence, the proof is condensed. ■

**Lemma B.5.** *Under Assumptions 2.1 and 2.5, the following hold.*

*(a)* $k_T^{-1/2}\sum_{t=1}^T \alpha_T^{-t}e_t \Rightarrow N(0,\sigma^2/2c)$,

*(b)* $\sum_{t=1}^T F_t = O_p(k_T T^{1/2}) + O_p(\alpha_T^T k_T^{3/2})$,

*(c)* $\alpha_T^{-T}k_T^{-1}\sum_{t=1}^T F_{t-1}e_t = o_p(1)$,

*(d)* $\alpha_T^{-2T}k_T^{-2}\sum_{t=1}^T F_t^2 = O_p(1)$,

*(e)* $T^{-1}\sum_{t=1}^T f_t^2 = O_p(\alpha_T^{2T}T^{-1}) + O_p(1)$.

*Under Assumptions 2.3.1 and 2.5, the following hold for all i:*

*(f)* $k_T^{-1/2}\sum_{t=1}^T \rho_{i,T}^{-t}z_{i,t} \Rightarrow N(0,\sigma_i^2/2c_i)$,

*(g)* $\sum_{t=1}^T U_{i,t} = O_p(k_T T^{1/2}) + O_p(\rho_{i,T}^T k_T^{3/2})$,

*(h)* ${\rho_{i,T}}^{-T}k_T^{-1}\sum_{t=1}^T U_{i,t-1}z_{i,t} = o_p(1)$,

*(i)* ${\rho_{i,T}}^{-2T}k_T^{-2}\sum_{t=1}^T U_{i,t}^2 = O_p(1)$,

*(j)* $T^{-1}\sum_{t=1}^T u_{i,t}^2 = O_p(\rho_{i,T}^{2T}T^{-1}) + O_p(1)$.

**Proof of Lemma B.5.** Here, I present the proof of only parts (a) to (e). The proofs of parts (f) to (j) are shown in the same way but with $U_{i,t}$ instead of $F_t$ and with Assumption 2.1 replaced by Assumption 2.3.1. I suppress the proofs to conserve space.

(a) See Lemma 4.2 of Phillips and Magdalinos (2007).

(b) I start with the expression

$$\begin{aligned}
\sum_{t=1}^T F_t &= \sum_{t=0}^T \alpha_T^t e_0 + \sum_{t=0}^{T-1}\alpha_T^t e_1 + \sum_{t=0}^{T-2}\alpha_T^t e_2 + \cdots + e_T,\\
&= \frac{1}{1-\alpha_T}\left[(\alpha_T-\alpha_T^{T+1})F_0 + (1-\alpha_T^T)e_1 + (1-\alpha_T^{T-1})e_2 + \cdots + (1-\alpha_T)e_T\right],\\
&= \frac{k_T}{c}\left[\sum_{t=1}^T e_t - \sum_{t=1}^T \alpha_T^{T+1-t}e_t + (\alpha_T-\alpha_T^{T+1})F_0\right],\\
&= \frac{k_T}{c}\sum_{t=1}^T e_t - \frac{\alpha_T^{T+1}k_T}{c}\sum_{t=1}^T \alpha_T^{-t}e_t + \tfrac{k_T}{c}(\alpha_T-\alpha_T^{T+1})F_0,\\
&= I + II + III.
\end{aligned}$$

However, $I = O_p(k_T T^{1/2})$ from Assumption 2.1, $II = O_p(\alpha_T^T k_T^{3/2})$ by using Lemma B.5 (a), and $III = O_p(\alpha_T^T k_T)$ from Assumption 2.1. Hence, the result follows.

(c) I start with the expression for $F_{t-1}$

$$F_{t-1} = e_{t-1} + \alpha_T e_{t-2} + ... + \alpha_T^{t-2}e_1 + \alpha_T^{t-1}F_0 = \alpha_T^{t-1}\sum_{s=1}^{t-1}\alpha_T^{-s}e_s + \alpha_T^{t-1}F_0.$$

Multiplying both sides by $\alpha_T^{-T} k_T^{-1} e_t$ and summing over $t$ yield

$$\begin{aligned}
\alpha_T^{-T} k_T^{-1} \textstyle\sum_{t=1}^{T} F_{t-1} e_t &= \alpha_T^{-T} k_T^{-1} \textstyle\sum_{t=1}^{T} \left( \sum_{s=1}^{t-1} \alpha_T^{t-s-1} e_s \right) e_t + \alpha_T^{-T} k_T^{-1} F_0 \sum_{t=1}^{T} \alpha_T^{t-1} e_t, \\
&= I + II.
\end{aligned}$$

The expected value of this is zero because of Assumption 2.1. To show that this is $o_p(1)$, I confirm that the second moments of both terms are bounded as $T \to \infty$. For $I$, I can simplify the second moment as follows by using Assumption 2.1.

$$\begin{aligned}
& \mathbb{E}\left[ \alpha_T^{-T} k_T^{-1} \textstyle\sum_{t=1}^{T} \left( \sum_{s=0}^{t-1} \alpha_T^{t-s-1} e_s \right) e_t \right]^2 \\
= \;& \alpha_T^{-2T} k_T^{-2} \sigma^4 \textstyle\sum_{t=1}^{T} \sum_{s=0}^{t-1} \alpha_T^{2(t-s-1)}, \\
= \;& \alpha_T^{-2T} \frac{\alpha_T^2 \sigma^4}{k_T(\alpha_T^2 - 1)} \left( \frac{\alpha_T^{2T} - 1}{k_T(\alpha_T^2 - 1)} - \frac{T}{k_T \alpha_T^2} \right), \\
= \;& \frac{\alpha_T^2 \sigma^4}{k_T(\alpha_T^2 - 1)} \left( \frac{1 - \alpha_T^{-2T}}{k_T(\alpha_T^2 - 1)} - \frac{\alpha_T^{-2T} T}{k_T \alpha_T^2} \right).
\end{aligned}$$

However, since $k_T(\alpha_T^2 - 1) \to 2c$, $\alpha_T^2 \to 1$, and $\alpha_T^{-2T} T = o(1)$, this is $O(1)$. For $II$,

$$\begin{aligned}
& \mathbb{E}\left[ \alpha_T^{-T} k_T^{-1} F_0 \textstyle\sum_{t=1}^{T} \alpha_T^{t-1} e_t \right]^2, \\
= \;& \alpha_T^{-2T} k_T^{-1} \frac{\mathbb{E}(F_0^2)\sigma^2}{k_T(\alpha_T^2 - 1)} (\alpha_T^{2T} - 1), \\
= \;& k_T^{-1} \frac{\mathbb{E}(F_0^2)\sigma^2}{k_T(\alpha_T^2 - 1)} (1 - \alpha_T^{-2T}) = O(k_T^{-1}),
\end{aligned}$$

so that the second moment of $II$ diminishes. Therefore, the result follows.

(d) I take squares of both sides of $F_t = \alpha_T F_{t-1} + e_t$ and take summations over $t = 1, ..., T$ to obtain

$$\begin{aligned}
\sum_{t=1}^{T} F_{t-1}^2 &= \frac{1}{\alpha_T^2 - 1}\left\{F_T^2 - F_0^2 - \sum_{t=1}^{T} e_t^2 - 2\alpha_T \sum_{t=1}^{T} F_{t-1}e_t\right\}, \\
\alpha_T^{-2T} k_T^{-2} \sum_{t=1}^{T} F_{t-1}^2 &= \frac{1}{k_T(\alpha_T^2 - 1)}\left\{\frac{\alpha_T^{-2T}}{k_T}(F_T^2 - F_0^2) - \frac{\alpha_h^{-2h}}{k_h}\sum_{t=1}^{T} e_t^2\right. \\
&\quad \left. - \frac{2\alpha_T^{-2T+1}}{k_T}\sum_{t=1}^{T} F_{t-1}e_t\right\}, \\
&= \frac{1}{k_T(\alpha_T^2 - 1)}\left\{I - II - III\right\},
\end{aligned}$$

where $k_T(\alpha_T^2 - 1) \to 2c$. Now I can show that $I = O_p(1)$, because

$$\begin{aligned}
F_T &= \sum_{j=1}^{T} \alpha_T^{T-j} e_j = \alpha_T^T k_T^{1/2}\left(\tfrac{1}{\sqrt{k_T}}\sum_{j=1}^{T} \alpha_T^{-j} e_j\right), \\
F_T^2 &= \alpha_T^{2T} k_T \left(\frac{1}{\sqrt{k_T}}\sum_{j=1}^{T} \alpha_T^{-j} e_j\right)^2 = O_p(\alpha_T^{2T} k_T),
\end{aligned}$$

so that

$$I = \frac{\alpha_T^{-2T}}{k_T} F_T^2 - \frac{\alpha_T^{-2T}}{k_T} F_0^2 = O_p(1) + O_p(\alpha_T^{-2T} k_T^{-1}) = O_p(1).$$

For $II$,

$$II = \left(\frac{\alpha_T^{-2T} T}{k_T}\right)\frac{1}{T}\sum_{t=1}^{T} e_t^2 = o\left(\tfrac{k_T}{T}\right) \times O_p(1) = o_p(1).$$

For $III$ (divided by 2),

$$\begin{aligned}
III &= \frac{\alpha_T^{-2T+1}}{k_T}\sum_{t=1}^{T} F_{t-1}e_t \\
&= \frac{\alpha_T^{-2T+1}}{k_T}\sum_{t=1}^{T}\left(\sum_{j=1}^{t-1} \alpha_T^{t-1-j} e_j\right)e_t + F_0\left(\tfrac{\alpha_T^{-T}}{\sqrt{k_T}}\right)\tfrac{1}{\sqrt{k_T}}\sum_{t=1}^{T} \alpha_T^{-(T-t)} e_t, \\
&= IIIa + IIIb.
\end{aligned}$$

For $IIIa$, because I have

$$\begin{aligned}
\mathbb{E}\left[\frac{\alpha_T^{-2T+1}}{k_T}\sum_{t=1}^{T}\left(\sum_{j=1}^{t-1}\alpha_T^{t-1-j}e_j\right)e_t\right]^2 &= \frac{\sigma^4\alpha_T^{-4T}}{k_T^2}\sum_{t=1}^{T}\sum_{j=1}^{t-1}\alpha_T^{2(t-j-1)}, \\
&= \frac{\sigma^4\alpha_T^{-4T}}{k_T^2(\alpha_T^2-1)}\left[\sum_{t=1}^{T}\alpha_T^{2(t-1)} - T\right] = O(\alpha_T^{-2T}),
\end{aligned}$$

$IIIa = o_p(1)$. For $IIIb$, because $F_0 = O_p(1)$, $\left(\frac{\alpha_T^{-T}}{\sqrt{k_T}}\right) = o\left(\frac{\sqrt{k_T}}{T}\right) = o_p(1)$, and $\left(\frac{1}{\sqrt{k_T}}\sum_{t=1}^{T}\alpha_T^{-(T-t)}e_t\right) = O_p(1)$, $IIIb = o_p(1)$. Hence, $III = o_p(1)$. The result follows. ■

(e) I start with

$$f_t = \frac{c}{k_T}F_{t-1} + e_t.$$

Squaring both sides, summing over $t$, and multiplying by $T^{-1}$ yields

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T} f_t^2 &= \frac{c^2}{Tk_T^2}\sum_{t=1}^{T} F_{t-1}^2 + \tfrac{2c}{Tk_T}\sum_{t=1}^{T} F_{t-1}e_t + \tfrac{1}{T}\sum_{t=1}^{T} e_t^2,\\ &= I + II + III.\end{aligned}$$

However, $I = O_p(\alpha_T^{2T}T^{-1})$ by using Lemma B.5 (d), $II = o_p(\alpha_T^{T}T^{-1})$ by using Lemma B.5 (c), and $III = O_p(1)$ by Assumption 2.1. Hence, the result follows. ■

**Proof of Theorem 2.1**. I start with equation (A.1) of Bai and Ng (2004). Let $u_t = [u_{1,t}\ u_{2,t}\ \cdots\ u_{N,t}]$ be a $1 \times N$ vector of the first differences of the idiosyncratic errors at time $t$.

$$\begin{aligned}\hat{f}_t &= Hf_t + V^{-1}N^{-1}T^{-1}\hat{f}'u\Lambda f_t + V^{-1}N^{-1}T^{-1}\hat{f}'f\Lambda'u_t'\\ &\quad + V^{-1}N^{-1}T^{-1}\hat{f}'uu_t',\end{aligned}$$

or

$$\begin{aligned}\hat{f}_t &= A_1f_t + A_2f_t + N^{-1}\sum_{i=1}^{N} a_{1,i}u_{i,t} + N^{-1}\sum_{i=1}^{N} a_{2,i}u_{i,t},\\ &= Af_t + N^{-1}\sum_{i=1}^{N} a_iu_{i,t},\end{aligned}$$

where $A = A_1 + A_2$ and $a_i = a_{1,i} + a_{2,i}$. I also have $A_1 = V^{-1}N^{-1}T^{-1}\hat{f}'f\Lambda'\Lambda$ from the definition of the $H$ matrix, $A_2 = V^{-1}N^{-1}T^{-1}\hat{f}'u\Lambda$, $a_{1,i} = V^{-1}T^{-1}\hat{f}'f\lambda_i'$, and $a_{2,i} = V^{-1}T^{-1}\hat{f}'u_i$. In the following, because $V^{-1}$ appears in every component $A_1$, $A_2$, $a_{1,i}$ and $a_{2,i}$, I multiply them by $V$ to ease computation and separately derive the bound of $V$.

(i) If $c>0$ and $c_i=0$ for all $i$, then

$$
\begin{aligned}
VA_1 &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1} \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}f_s^2\right|^{1/2}}_{=O_p(\alpha_T^T T^{-1/2})\text{ by Lemma B.5(e)}} \underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}\lambda_i^2\right|}_{=O_p(1)\text{ from Assumption 2.2}} = O_p(\alpha^T T^{-1/2}),\\
VA_2 &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1}\underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}T^{-1}\sum_{s=1}^{T}u_{i,s}^2\right|^{1/2}}_{=O_p(1)}\underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}\lambda_i^2\right|}_{=O_p(1)} = O_p(1),\\
Va_{1,i} &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1}\underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}f_s^2\right|^{1/2}}_{=O_p(\alpha_T^T T^{-1/2})}|\lambda_i| = O_p(\alpha^T T^{-1/2}),\\
Va_{2,i} &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1}\underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}u_{i,s}^2\right|^{1/2}}_{=O_p(1)} = O_p(1).
\end{aligned}
$$

(ii) If $c=0$ and $c_i>0$ for all $i$, then

$$
\begin{aligned}
VA_1 &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1}\underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}f_s^2\right|^{1/2}}_{=O_p(1)}\underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}\lambda_i^2\right|}_{=O_p(1)} = O_p(1),\\
VA_2 &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1}\underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}T^{-1}\sum_{s=1}^{T}u_{i,s}^2\right|^{1/2}}_{=O_p(\rho_{i,T}^T T^{-1/2})\text{ by Lemma B.5 (j)}}\underbrace{\left|N^{-1}\textstyle\sum_{i=1}^{N}\lambda_i^2\right|}_{=O_p(1)} = O_p(\rho_{i,T}^T T^{-1/2}),\\
Va_{1,i} &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1}\underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}f_s^2\right|^{1/2}}_{=O_p(1)}|\lambda_i| = O_p(1),\\
Va_{2,i} &\leq \underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}\hat{f}_s^2\right|^{1/2}}_{=1}\underbrace{\left|T^{-1}\textstyle\sum_{s=1}^{T}u_{i,s}^2\right|^{1/2}}_{=O_p(\rho_{i,T}^T T^{-1/2})} = O_p(\rho_{i,T}^T T^{-1/2}).
\end{aligned}
$$

The largest eigenvalue $V$ of $N^{-1}T^{-1}xx'$ satisfies $V^{1/2}=\left\|N^{-1/2}T^{-1/2}x\right\|$, where $\|\cdot\|$ denotes the Euclidean norm, so that

$$
\begin{aligned}
V &= N^{-1}T^{-1}\|x\|^2,\\
&= N^{-1}T^{-1}\textstyle\sum_{i=1}^{N}\sum_{t=1}^{T}x_{i,t}^2,\\
&= \left\{\begin{array}{ll} O_p(\alpha_T^T T^{-1/2}), & \text{for case (i)}\\ O_p(\rho_{i,T}^T T^{-1/2}), & \text{for case (ii)}\end{array}\right. . \qquad \text{(B.2.15)}
\end{aligned}
$$

Hence, the results follow if the stochastic bound for $V$ is sharp and the expression is asymptotically dominated by the mildly explosive components. ■

## B.3 Proof of Factor Estimation Errors in Theorem 2.1 (i)

In the following, I provide two additional facts pertaining to consistency of factor estimation in Theorem 2.1 (i).

First, I show that the differenced factors can be consistently estimated. From Proof of Theorem 2.1 (i) in Appendix B.2, I have

$$\begin{aligned} \hat{f}_t - (A_1 + A_2) f_t &= N^{-1} \textstyle\sum_{i=1}^N a_{1,i} u_{i,t} + N^{-1} \sum_{i=1}^N a_{2,i} u_{i,t}, \\ &= I + II, \end{aligned} \tag{B.3.1}$$

where $A_1 = V^{-1} N^{-1} T^{-1} \hat{f}' f \Lambda' \Lambda$, $A_2 = V^{-1} N^{-1} T^{-1} \hat{f}' u \Lambda$, $a_{1,i} = V^{-1} T^{-1} \hat{f}' f \lambda_i'$, and $a_{2,i} = V^{-1} T^{-1} \hat{f}' u_i$ as I previously defined. Then,

$$I = V^{-1} (T^{-1} \hat{f}' f) N^{-1} \textstyle\sum_{i=1}^N \lambda_i u_{i,t},$$

with $V = O_p(\alpha_T^T T^{-1/2})$ from (B.2.15) and

$$(T^{-1} \hat{f}' f) \leq \underbrace{\left| T^{-1} \textstyle\sum_{s=1}^T \hat{f}_s^2 \right|^{1/2}}_{=1} \underbrace{\left| T^{-1} \textstyle\sum_{s=1}^T f_s^2 \right|^{1/2}}_{=O_p(\alpha_T^T T^{-1/2}) \text{ by Lemma B.5 (e)}} = O_p(\alpha_T^T T^{-1/2}).$$

Hence,

$$I = O_p(\alpha_T^{-T} T^{1/2}) \times O_p(\alpha_T^T T^{-1/2}) \times o_p(1) = o_p(1).$$

I also have, by using the definition of $a_{2,i}$ in Proof of Theorem 2.1,

$$\begin{aligned} II &= V^{-1} T^{-1} N^{-1} \textstyle\sum_{s=1}^T \hat{f}_s \sum_{i=1}^N u_{i,s} u_{i,t}, \\ &\leq V^{-1} T^{-1} \textstyle\sum_{s=1}^T \hat{f}_s \left| N^{-1} \sum_{i=1}^N u_{i,s}^2 \right|^{1/2} \left| N^{-1} \sum_{i=1}^N u_{i,t}^2 \right|^{1/2}, \\ &\leq V^{-1} \underbrace{\left| T^{-1} \textstyle\sum_{s=1}^T \hat{f}_s^2 \right|}_{=1} \underbrace{\left| T^{-1} \textstyle\sum_{s=1}^T \left| N^{-1} \sum_{i=1}^N u_{i,s}^2 \right| \right|}_{=O_p(1)}^{1/2} \left| N^{-1} \textstyle\sum_{i=1}^N u_{i,t}^2 \right|^{1/2}, \end{aligned}$$

but $V = O_p(\alpha_T^T T^{-1/2})$ from (B.2.15), $T^{-1}\sum_{s=1}^{T}\left|N^{-1}\sum_{i=1}^{N} u_{i,s}^2\right| = O_p(1)$, and $N^{-1}\sum_{i=1}^{N} u_{i,t}^2 = O_p(1)$ under $c_i = 0$. Hence,

$$II = O_p(\alpha_T^{-T} T^{1/2}) \times O_p(1) = o_p(1).$$

I also show that the level factors involve factor estimation errors of order $O_p(T^{1/2}N^{-1/2})$. Using (B.3.1), the level factor estimate is

$$\hat{F}_t = \sum_{s=1}^{t} \hat{f}_s = (A_1 + A_2)\sum_{s=1}^{t} f_s + \sum_{s=1}^{t} N^{-1}\sum_{i=1}^{N} a_{1,i}u_{i,s} + \sum_{s=1}^{t} N^{-1}\sum_{i=1}^{N} a_{2,i}u_{i,s},$$

or

$$\begin{aligned} \hat{F}_t - (A_1 + A_2)F_t &= \sum_{s=1}^{t} N^{-1}\sum_{i=1}^{N} a_{1,i}u_{i,s} + \sum_{s=1}^{t} N^{-1}\sum_{i=1}^{N} a_{2,i}u_{i,s}, \\ &= I + II = O_p(T^{1/2}N^{-1/2}), \end{aligned} \tag{B.3.2}$$

because

$$\begin{aligned} I &= V^{-1}(T^{-1}\hat{f}'f)\sum_{s=1}^{t} N^{-1}\sum_{i=1}^{N} \lambda_i u_{i,t}, \\ &= V^{-1}(T^{-1}\hat{f}'f)T^{1/2}N^{-1/2}(T^{-1/2}N^{-1/2}\sum_{s=1}^{t}\sum_{i=1}^{N} \lambda_i u_{i,t}), \end{aligned}$$

where $V^{-1}(T^{-1}\hat{f}'f) = O_p(1)$ because $V^{-1} = O_p(\alpha_T^{-T}T^{1/2})$ and $T^{-1}\hat{f}'f = O_p(\alpha_T^T T^{-1/2})$ as I showed in term $I$ of the differenced factor. I also have

$$T^{-1/2}N^{-1/2}\sum_{s=1}^{t}\sum_{i=1}^{N} \lambda_i u_{i,t} = O_p(1).$$

Hence,

$$I = O_p(1) \times T^{1/2}N^{-1/2} \times O_p(1) = O_p(T^{1/2}N^{-1/2}).$$

and

$$\begin{aligned} II &= V^{-1}T^{-1}N^{-1}\sum_{l=1}^{T} \hat{f}_l \sum_{i=1}^{N} u_{i,l}\sum_{s=1}^{t} u_{i,s}, \\ &\le V^{-1}\left|T^{-1}\sum_{l=1}^{T} \hat{f}_l^2\right| \quad \left|T^{-1}\sum_{l=1}^{T}\left|N^{-1}\sum_{i=1}^{N} u_{i,l}^2\right|\right|^{1/2}\left|N^{-1}\sum_{i=1}^{N}\sum_{s=1}^{t} u_{i,s}^2\right|^{1/2}, \end{aligned}$$

where $V = O_p(\alpha_T^T T^{-1/2})$ from (B.2.15), $T^{-1}\sum_{l=1}^{T}\hat{f}_l^2 = 1$, $T^{-1}\sum_{l=1}^{T}\left|N^{-1}\sum_{i=1}^{N}u_{i,l}^2\right| = O_p(1)$, and $\left|N^{-1}\sum_{i=1}^{N}\sum_{s=1}^{t}u_{i,s}^2\right|^{1/2} = O_p(T^{1/2})$. Hence, $II$ is dominated by $I$. Therefore, (B.3.1) implies that the factor estimation errors in the differenced factor are $o_p(1)$ and (B.3.2) implies that the factor estimation errors in the level factor are $O_p(T^{1/2}N^{-1/2})$.

## B.4 Proof of Theorem SA-2 and Theorem 2.2

Throughout Appendix B.4, I let $F_{t-1}^c = F_{t-1} - \bar{F}$, where $\bar{F} = h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}$ and $\tilde{F}_{t-1}^c = \tilde{F}_{t-1} - \overline{\overline{F}}$, where $\overline{\overline{F}} = h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}$. Let also $f_t^c = f_t - \bar{f}$, where $\bar{f} = h^{-1}\sum_{t=T+1}^{T+h}f_t$ and $\tilde{f}_t^c = \tilde{f}_t - \overline{\overline{f}}$, where $\overline{\overline{f}} = h^{-1}\sum_{t=T+1}^{T+h}\tilde{f}_t$. In addition, I let $U_{i,t-1}^c = U_{i,t-1} - \bar{U}_i$, where $\bar{U}_i = h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}$ and $\tilde{U}_{i,t-1}^c = \tilde{U}_{i,t-1} - \overline{\overline{U}}_i$, where $\overline{\overline{U}}_i = h^{-1}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}$. Let also $u_{i,t}^c = u_{i,t} - \bar{u}_i$, where $\bar{u}_i = h^{-1}\sum_{t=T+1}^{T+h}u_{i,t-1}$ and $\tilde{u}_{i,t-1}^c = \tilde{u}_{i,t-1} - \overline{\overline{u}}_i$, where $\overline{\overline{u}}_i = h^{-1}\sum_{t=T+1}^{T+h}\tilde{u}_{i,t-1}$. I also let $\rho_h = \max_i \rho_{i,h}$.

**Lemma B.6.** *Under Assumptions 2.1–2.5, the following hold:*

*(a) For $t = T+1, ..., T+h$ uniformly in $t$,*

$$F_t = O_p(\alpha_h^h k_h^{1/2}),$$

*(b) For $t = T+1, ..., T+h$ uniformly in $t$ and all $i$,*

$$U_{i,t} = O_p(\rho_h^h k_h^{1/2}),$$

*(c) For $t = T+1, ..., T+h$ uniformly in $t$,*

$$\tilde{F}_t - HF_t = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(d)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1} - HF_{t-1})^2 = O_p\left(\frac{\alpha_h^{2h}k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h}k_h}{\min\{N, T\}}\right),$$

*(e)*

$$h^{-1}\sum_{t=T+1}^{T+h} F_{t-1}(\tilde{F}_{t-1}-HF_{t-1}) = O_p\left(\frac{\alpha_h^{2h}k_h}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h k_h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(f)*

$$h^{-1}\sum_{t=T+1}^{T+h} \tilde{F}_{t-1}(\tilde{F}_{t-1}-HF_{t-1}) = O_p\left(\frac{\alpha_h^{2h}k_h}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h k_h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right) +O_p\left(\frac{\rho_h^{2h}k_h}{\min\left\{N,T\right\}}\right),$$

*(g)*

$$h^{-1/2}\sum_{t=T+1}^{T+h} (\tilde{F}_{t-1}-HF_{t-1})e_t = O_p\left(\frac{\alpha_h^{h}k_h^{1/2}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(h) For $t=T+1,...,T+h$ uniformly in $t$,*

$$f_t = O_p(\alpha_h^h k_h^{-1/2}),$$

*(i) For $t=T+1,...,T+h$ uniformly in $t$ and all $i$,*

$$u_{i,t} = O_p(\rho_h^h k_h^{-1/2}),$$

*(j) For $t=T+1,...,T+h$ uniformly in $t$,*

$$\tilde{f}_t - Hf_t = O_p\left(\frac{\alpha_h^{h}k_h^{-1/2}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(k)*

$$h^{-1}\sum_{t=T+1}^{T+h} (\tilde{f}_t - Hf_t)^2 = O_p\left(\frac{\alpha_h^{2h}k_h^{-1}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}k_h^{-1}}{\min\{N,T\}}\right),$$

*(l)*

$$h^{-1}\sum_{t=T+1}^{T+h} (\tilde{F}_{t-1}-HF_{t-1})(\tilde{f}_t - Hf_t) = O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right),$$

*(m)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}-HF_{t-1})f_t = O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(n)*

$$h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}(\tilde{f}_t-Hf_t) = O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(o)*

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}(\tilde{f}_t-Hf_t) &= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right) \\ &\quad +O_p\left(\frac{\rho_h^{2h}}{\min\left\{N,T\right\}}\right), \end{aligned}$$

*(p)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{f}_t-Hf_t)e_t = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(q)*

$$\begin{aligned} h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2 &= h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^2+O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right) \\ &\quad +O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\alpha_h^h\rho_h^h h^{-1}k_h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right), \end{aligned}$$

*(r)*

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}e_t &= h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}e_t \\ &\quad +O_p\left(\frac{\alpha_h^h h^{-1/2}k_h^{1/2}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^h h^{-1/2}k_h^{1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right). \end{aligned}$$

**Proof of Lemma B.6.**

(a) For $t=T+j$ with $j=1,...,h$,

$$\begin{aligned} F_t &= e_t+\alpha_h e_{t-1}+...+\alpha_h^{j-1}e_{t+j-1}+\alpha_h^j F_T, \\ &= \alpha_h^j\sum_{s=1}^{j}\alpha_h^{-s}e_{t+j-s}+\alpha_h^j F_T = O_p(\alpha_h^h k_h^{1/2}), \end{aligned}$$

by using Lemma B.5 (a).

(b) The proof is the same as part (a).

(c) I start with

$$\tilde{F}_t = \frac{\sum_{i=1}^N \hat{\lambda}_i^* X_{it}}{\sum_{i=1}^N \hat{\lambda}_i^{*2}} = \frac{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^* \lambda_i}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}} F_t + \frac{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^* U_{i,t}}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}} = I + II.$$

For $I$,

$$\begin{aligned} I &= \frac{N^{-1}\sum_{i=1}^N (H\hat{\lambda}_i^{*2} + \hat{\lambda}_i^* \lambda_i - H\hat{\lambda}_i^{*2})}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}} F_t, \\ &= \left[ H - \frac{HN^{-1}\sum_{i=1}^N \hat{\lambda}_i^* (\hat{\lambda}_i^* - H^{-1}\lambda_i)}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}} \right] F_t, \end{aligned}$$

but

$$\begin{aligned} N^{-1}\sum_{i=1}^N \hat{\lambda}_i^* (\hat{\lambda}_i^* - H^{-1}\lambda_i) &= N^{-1}\sum_{i=1}^N H^{-1}\lambda_i (\hat{\lambda}_i^* - H^{-1}\lambda_i) + N^{-1}\sum_{i=1}^N (\hat{\lambda}_i^* - H^{-1}\lambda_i)^2, \\ &= O_p\left( \frac{1}{\min\left\{N, T^{1/2}\right\}} \right), \end{aligned}$$

by using Lemma B.3 (f). For $II$,

$$II = \frac{N^{-1}\sum_{i=1}^N H^{-1}\lambda_i U_{i,t} + N^{-1}\sum_{i=1}^N (\hat{\lambda}_i^* - H^{-1}\lambda_i) U_{i,t}}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}}.$$

Therefore,

$$\begin{aligned} \tilde{F}_t - HF_t &= -\frac{HN^{-1}\sum_{i=1}^N \hat{\lambda}_i^* (\hat{\lambda}_i^* - H^{-1}\lambda_i)}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}} F_t \\ &\quad + \frac{N^{-1}\sum_{i=1}^N H^{-1}\lambda_i U_{i,t}}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}} + \frac{N^{-1}\sum_{i=1}^N (\hat{\lambda}_i^* - H^{-1}\lambda_i) U_{i,t}}{N^{-1}\sum_{i=1}^N \hat{\lambda}_i^{*2}}, \\ &= O_p\left( \frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h k_h^{1/2}}{N^{1/2}} \right) + O_p\left( \frac{\rho_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}} \right), \\ &= O_p\left( \frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right), \end{aligned}$$

uniformly in $t$ by using (a) and (b). (d) is straightforwardly shown from (c).

(e) Since $F_t = O_p(\alpha_h^h k_h^{1/2})$,

$$\begin{aligned} F_t(\tilde{F}_t - HF_t) &= O_p(\alpha_h^h k_h^{1/2}) \times \left[ O_p\left( \frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right) \right], \\ &= O_p\left( \frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right), \end{aligned}$$

uniformly in $t$ for $t = T+1, \cdots, T+h$ by using (a) and (c). Hence,

$$h^{-1} \textstyle\sum_{t=T+1}^{T+h} F_{t-1}(\tilde{F}_{t-1} - HF_{t-1}) = O_p\left( \frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right).$$

(f) By using (e) and (d),

$$\begin{aligned} \tilde{F}_t(\tilde{F}_t - HF_t) &= HF_t(\tilde{F}_t - HF_t) + (\tilde{F}_t - HF_t)^2, \\ &= O_p\left( \frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right) \\ &\quad + O_p\left( \frac{\alpha_h^{2h} k_h}{\min\left\{N^2, T\right\}} \right) + O_p\left( \frac{\rho_h^{2h} k_h}{\min\left\{N, T\right\}} \right), \\ &= O_p\left( \frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right) \\ &\quad + O_p\left( \frac{\rho_h^{2h} k_h}{\min\left\{N, T\right\}} \right), \end{aligned}$$

uniformly in $t$ for $t = T+1, \cdots, T+h$. Hence,

$$\begin{aligned} h^{-1} \textstyle\sum_{t=T+1}^{T+h} \tilde{F}_{t-1}(\tilde{F}_{t-1} - HF_{t-1}) &= O_p\left( \frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right) \\ &\quad + O_p\left( \frac{\rho_h^{2h} k_h}{\min\left\{N, T\right\}} \right). \end{aligned}$$

(g) Since

$$(\tilde{F}_{t-1} - HF_{t-1})e_t = O_p\left( \frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right),$$

uniformly in $t$ and $(\tilde{F}_{t-1} - HF_{t-1})$ and $e_t$ are independent with $e_t$ being i.i.d. with $\mathbb{E}(e_t) = 0$, I have

$$h^{-1/2}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}-HF_{t-1})e_t=O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right).$$

(h) From (a), I obtain

$$\begin{aligned} f_t &= F_t-F_{t-1}=(1+\frac{c}{k_h})F_{t-1}+e_t-F_{t-1}, \\ &= \frac{c}{k_h}F_{t-1}+e_t=O_p(\alpha_h^h k_h^{-1/2}), \end{aligned}$$

uniformly in $t$. (i) The proof is the same as (h).

(j) I start with

$$\tilde{f}_t=\frac{\sum_{i=1}^N\hat{\lambda}_i^* x_{it}}{\sum_{i=1}^N\hat{\lambda}_i^{*2}}=\frac{N^{-1}\sum_{i=1}^N\hat{\lambda}_i^*\lambda_i}{N^{-1}\sum_{i=1}^N\hat{\lambda}_i^{*2}}f_t+\frac{N^{-1}\sum_{i=1}^N\hat{\lambda}_i^* u_{i,t}}{N^{-1}\sum_{i=1}^N\hat{\lambda}_i^{*2}}=I+II,$$

where

$$\begin{aligned} I &= \left[H-\frac{HN^{-1}\sum_{i=1}^N\hat{\lambda}_i^*(\hat{\lambda}_i^*-\lambda_i H^{-1})}{N^{-1}\sum_{i=1}^N\hat{\lambda}_i^{*2}}\right]f_t, \\ &= Hf_t+O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right), \end{aligned}$$

and $II=O_p\left(\frac{\rho_h^h k_h^{-1/2}}{N^{1/2}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)=O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)$. Hence,

$$\tilde{f}_t-Hf_t=O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

uniformly in $t$. (k) is straightforwardly shown from (j).

(l) (c) and (j) imply that

$$\begin{aligned} (\tilde{F}_{t-1}-HF_{t-1})(\tilde{f}_t-Hf_t) &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right] \\ &\quad\times\left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right), \end{aligned}$$

uniformly in $t$, which yields the result.

(m) Since

$$\begin{aligned}(\tilde{F}_{t-1} - HF_{t-1})f_t &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right] \times O_p(\alpha_h^h k_h^{-1/2}), \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),\end{aligned}$$

uniformly in $t$, which yields the result.

(n) (a) and (j) imply that

$$\begin{aligned}F_{t-1}(\tilde{f}_t - Hf_t) &= O_p(\alpha_h^h k_h^{1/2}) \times \left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),\end{aligned}$$

uniformly in $t$, which yields the result.

(o) By using (n) and (l), I obtain

$$\begin{aligned}\tilde{F}_{t-1}(\tilde{f}_t - Hf_t) &= HF_{t-1}(\tilde{f}_t - Hf_t) + (\tilde{F}_t - HF_t)(\tilde{f}_t - Hf_t), \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N^2, T\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{3/2}, T\right\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N, T\right\}}\right), \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N, T\right\}}\right),\end{aligned}$$

uniformly in $t$, which yields the result.

(p) From part (j), I obtain

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{f}_t - Hf_t)e_t = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right).$$

(q) I have

$$\begin{aligned}
\textstyle\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}-HF_{t-1})^2 &= \textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2+H^2\sum_{t=T+1}^{T+h}F_{t-1}^2-2H\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}F_{t-1}, \\
&= \textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2+H^2\sum_{t=T+1}^{T+h}F_{t-1}^2 \\
&\quad \textstyle -2H\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}-HF_{t-1})F_{t-1}-2H^2\sum_{t=T+1}^{T+h}F_{t-1}^2, \\
&= \textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2-H^2\sum_{t=T+1}^{T+h}F_{t-1}^2 \\
&\quad \textstyle -2H\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}-HF_{t-1})F_{t-1}.
\end{aligned}$$

This yields

$$\begin{aligned}
\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2 &= \textstyle H^2\sum_{t=T+1}^{T+h}F_{t-1}^2+\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}-HF_{t-1})^2 \\
&\quad \textstyle +2H\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}-HF_{t-1})F_{t-1},
\end{aligned}$$

or

$$\begin{aligned}
h^{-2}\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2 &= \textstyle h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^2+h^{-2}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}-HF_{t-1})^2 \\
&\quad \textstyle +2h^{-2}H\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}-HF_{t-1})F_{t-1}, \\
&= \textstyle h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^2 \\
&\quad +O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right) \\
&\quad +O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\alpha_h^{h}\rho_h^{h}h^{-1}k_h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),
\end{aligned}$$

by using (d) and (e).

(r) I have

$$\begin{aligned}
h^{-1}\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}e_t &= \textstyle h^{-1}H\sum_{t=T+1}^{T+h}F_{t-1}e_t+h^{-1}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}-HF_{t-1})e_t, \\
&= \textstyle h^{-1}H\sum_{t=T+1}^{T+h}F_{t-1}e_t \\
&\quad +O_p\left(\frac{\alpha_h^{h}k_h^{1/2}h^{-1/2}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^{h}k_h^{1/2}h^{-1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),
\end{aligned}$$

by using (g). ■

**Lemma B.7.** *Suppose Assumptions 2.1–2.4 and B.1 hold or Assumptions 2.1–2.5 and the following condition hold:*

$$\frac{\alpha_h^h \rho_h^h h^{1/2} k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \to 0.$$

*Then, I have*

$$\hat{\sigma}^2 \xrightarrow{p} Q^{-2}\sigma^2,$$

*as $N, T, h \to \infty$, where*

$$\hat{\sigma}^2 = h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{f}_t - \hat{\delta}^* \tilde{F}_{t-1})^2,$$

*with*

$$\hat{\delta}^* = \frac{\sum_{t=T+1}^{T+h} \tilde{F}_{t-1} \tilde{f}_t}{\sum_{t=T+1}^{T+h} \tilde{F}_{t-1}^2}.$$

**Proof of Lemma B.7.**

I start with the AR coefficient estimator,

$$\hat{\delta}^* = \frac{\sum_{t=T+1}^{T+h} \tilde{F}_{t-1} \tilde{f}_t}{\sum_{t=T+1}^{T+h} \tilde{F}_{t-1}^2}.$$

Since

$$\begin{aligned}
\tilde{f}_t &= \tilde{F}_t - \tilde{F}_{t-1}, \\
&= HF_t - HF_{t-1} + (\tilde{F}_t - HF_t) - (\tilde{F}_{t-1} - HF_{t-1}), \\
&= H\frac{c}{k_h}F_{t-1} + He_t + (\tilde{F}_t - HF_t) - (\tilde{F}_{t-1} - HF_{t-1}), \\
&= \frac{c}{k_h}\tilde{F}_{t-1} + He_t + (\tilde{f}_t - Hf_t) - \frac{c}{k_h}(\tilde{F}_{t-1} - HF_{t-1}),
\end{aligned}$$

I obtain

$$
\begin{aligned}
\hat{\delta}^* &= \frac{c}{k_h} + \frac{H\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}e_t}{\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2} + \frac{\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}(\tilde{f}_t - Hf_t)}{\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2} \\
&\quad + \frac{c}{k_h}\frac{\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}(\tilde{F}_{t-1} - HF_{t-1})}{\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2}, \\
&= \frac{c}{k_h} + \frac{\alpha_h^{-2h}k_h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}e_t}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2} + \frac{\alpha_h^{-2h}k_h^{-2}H\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1} - HF_{t-1})e_t}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2} \\
&\quad + \frac{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}(\tilde{f}_t - Hf_t)}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2} + \frac{c}{k_h}\frac{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}(\tilde{F}_{t-1} - HF_{t-1})}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\alpha_h^{-h}k_h^{-1}\left(\hat{\delta}^* - \frac{c}{k_h}\right) &= \frac{\alpha_h^{-h}k_h^{-1}H^2\sum_{t=T+1}^{T+h}F_{t-1}e_t}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2} + \frac{\alpha_h^{-h}k_h^{-1}H\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1} - HF_{t-1})e_t}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2} \\
&\quad + \frac{\alpha_h^{-h}k_h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}(\tilde{f}_t - Hf_t)}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2} + \frac{c}{k_h}\frac{\alpha_h^{-h}k_h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}(\tilde{F}_{t-1} - HF_{t-1})}{\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2}, \\
&= I + II + III + IV.
\end{aligned}
$$

For the denominator,

$$
\begin{aligned}
\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^2 &= \alpha_h^{-2h}k_h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^2 + O_p\left(\frac{hk_h^{-1}}{\min\{N^2,T\}}\right) + O_p\left(\frac{\alpha_h^{-2h}\rho_h^{2h}hk_h^{-1}}{\min\{N,T\}}\right) \\
&\quad + O_p\left(\frac{hk_h^{-1}}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^{-h}\rho_h^{h}hk_h^{-1}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),
\end{aligned}
$$

by using Lemma B.6 (q) and the four terms of the factor estimation errors are $o_p(1)$ under the stated conditions.

The numerator of $I$ is $o_p(1)$ by using Lemma B.5 (c). For the numerator of $II$,

$$
\alpha_h^{-h}k_h^{-1}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1} - HF_{t-1})e_t = O_p\left(\frac{h^{1/2}k_h^{-1/2}}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^{-h}\rho_h^{h}h^{1/2}k_h^{-1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),
$$

by using Lemma B.6 (g) and it is $o_p(1)$ under the stated conditions.

For the numerator of $III$,

$$\alpha_h^{-h} k_h^{-1} \sum_{t=T+1}^{T+h} \tilde{F}_{t-1}(\tilde{f}_t - Hf_t) = O_p\left(\frac{\alpha_h^h h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^{-h}\rho_h^{2h} h k_h^{-1}}{\min\{N, T\}}\right) = o_p(1),$$

by using Lemma B.6 (o) and it is $o_p(1)$ under the stated conditions.

For the numerator of $IV$,

$$\alpha_h^{-h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{F}_{t-1}(\tilde{F}_{t-1} - HF_{t-1}) = O_p\left(\frac{\alpha_h^h h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h\rho_h^{2h} h k_h^{-1}}{\min\{N, T\}}\right),$$

by using Lemma B.6 (f) and it is $o_p(1)$ under the stated conditions so that I proceed with $\hat{\delta}^* - \frac{c}{k_h} = O_p(\alpha_h^{-h} k_h^{-1})$ and $\hat{\delta}^* = O_p(k_h^{-1})$. Then,

$$\begin{aligned}
\hat{\sigma}^2 &= h^{-1} \sum_{t=T+1}^{T+h} (\tilde{f}_t - \hat{\delta}^* \tilde{F}_{t-1})^2, \\
&= h^{-1} \sum_{t=T+1}^{T+h} [Hf_t - \hat{\delta}^* HF_{t-1} + (\tilde{f}_t - Hf_t) - \hat{\delta}^*(\tilde{F}_{t-1} - HF_{t-1})]^2, \\
&= h^{-1} \sum_{t=T+1}^{T+h} [Hf_t - H\tfrac{c}{k_h}F_{t-1} - (\hat{\delta}^* - \tfrac{c}{k_h})HF_{t-1} + (\tilde{f}_t - Hf_t) - \hat{\delta}^*(\tilde{F}_{t-1} - HF_{t-1})]^2, \\
&= h^{-1} \sum_{t=T+1}^{T+h} [He_t - (\hat{\delta}^* - \tfrac{c}{k_h})HF_{t-1} + (\tilde{f}_t - Hf_t) - \hat{\delta}^*(\tilde{F}_{t-1} - HF_{t-1})]^2, \\
&= h^{-1} \sum_{t=T+1}^{T+h} [H^2 e_t^2 + (\hat{\delta}^* - \tfrac{c}{k_h})^2 H^2 F_{t-1}^2 + (\tilde{f}_t - Hf_t)^2 + \hat{\delta}^{*2}(\tilde{F}_{t-1} - HF_{t-1})^2 \\
&\quad + 2H^2(\hat{\delta}^* - \frac{c}{k_h})F_{t-1}e_t + 2H(\tilde{f}_t - Hf_t)e_t - 2\hat{\delta}^*(\tilde{F}_{t-1} - HF_{t-1})He_t \\
&\quad + 2(\hat{\delta}^* - \frac{c}{k_h})HF_{t-1}(\tilde{f}_t - Hf_t) - 2(\hat{\delta}^* - \frac{c}{k_h})\hat{\delta}^* HF_{t-1}(\tilde{F}_{t-1} - HF_{t-1}) \\
&\quad + 2\hat{\delta}^*(\tilde{f}_t - Hf_t)(\tilde{F}_{t-1} - HF_{t-1})], \\
&= h^{-1} \sum_{t=T+1}^{T+h} H^2 e_t^2 + \sum_{k=1}^{9} D_k,
\end{aligned}$$

has nine terms of the factor estimation errors. I now show that they are all $o_p(1)$ under the stated conditions. For $D_1$,

$$\begin{aligned}
D_1 &= (\hat{\delta}^* - \frac{c}{k_h})^2 H^2 h^{-1} \textstyle\sum_{t=T+1}^{T+h} F_{t-1}^2, \\
&= O_p(\alpha_h^{-2h} k_h^{-2}) \times O_p\left(\alpha_h^{2h} h^{-1} k_h^2\right), \\
&= O_p(h^{-1}) = o_p(1).
\end{aligned}$$

For $D_2$,

$$\begin{aligned}
D_2 &= h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{f}_t - H f_t)^2, \\
&= O_p\left(\frac{\alpha_h^{2h} k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h^{-1}}{\min\{N, T\}}\right),
\end{aligned}$$

by using Lemma B.6 (k) and it is $o_p(1)$ under the stated conditions. For $D_3$,

$$\begin{aligned}
D_3 &= \hat{\delta}^{*2} h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{F}_{t-1} - H F_{t-1})^2, \\
&= O_p(k_h^{-2}) \times \left[O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^{2h} k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h^{-1}}{\min\{N, T\}}\right),
\end{aligned}$$

by using Lemma B.6 (d) and it is $o_p(1)$ under the stated conditions. For $D_4$,

$$\begin{aligned}
D_4 &= 2H^2(\hat{\delta}^* - \frac{c}{k_h}) h^{-1} \textstyle\sum_{t=T+1}^{T+h} F_{t-1} e_t, \\
&= O_p(\alpha_h^{-h} k_h^{-1}) \times o_p\left(\alpha_h^{h} h^{-1} k_h\right), \\
&= o_p(h^{-1}) = o_p(1),
\end{aligned}$$

by using Lemma B.5 (c). For $D_5$,

$$\begin{aligned}
D_5 &= 2H h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{f}_t - H f_t) e_t, \\
&= O_p\left(\frac{\alpha_h^{h} k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^{h} k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),
\end{aligned}$$

by using Lemma B.6 (p) and it is $o_p(1)$ under the stated conditions. For $D_6$,

$$
\begin{aligned}
D_6 &= 2\hat{\delta}^* H h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{F}_{t-1} - HF_{t-1}) e_t, \\
&= O_p(k_h^{-1}) \times \left[ O_p\left( \frac{\alpha_h^h h^{1/2} k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h h^{1/2} k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right) \right], \\
&= O_p\left( \frac{\alpha_h^h h^{1/2} k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h h^{1/2} k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right),
\end{aligned}
$$

by using Lemma B.6 (g) and it is $o_p(1)$ under the stated conditions. For $D_7$,

$$
\begin{aligned}
D_7 &= 2(\hat{\delta}^* - \frac{c}{k_h}) H h^{-1} \textstyle\sum_{t=T+1}^{T+h} F_{t-1} (\tilde{f}_t - Hf_t), \\
&= O_p(\alpha_h^{-h} k_h^{-1}) \times \left[ O_p\left( \frac{\alpha_h^{2h}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right) \right], \\
&= O_p\left( \frac{\alpha_h^h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right),
\end{aligned}
$$

by using Lemma B.6 (o) and it is $o_p(1)$ under the stated conditions. For $D_8$,

$$
\begin{aligned}
D_8 &= 2(\hat{\delta}^* - \frac{c}{k_h}) \hat{\delta}^* H h^{-1} \textstyle\sum_{t=T+1}^{T+h} F_{t-1} (\tilde{F}_{t-1} - HF_{t-1}), \\
&= O_p(\alpha_h^{-h} k_h^{-1}) \times O_p(k_h^{-1}) \times \left[ O_p\left( \frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right) \right], \\
&= O_p\left( \frac{\alpha_h^h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right),
\end{aligned}
$$

by using Lemma B.6 (e) and it is $o_p(1)$ under the stated conditions. For $D_9$,

$$
\begin{aligned}
D_9 &= 2\hat{\delta}^* h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{f}_t - Hf_t)(\tilde{F}_{t-1} - HF_{t-1}), \\
&= O_p\left(k_h^{-1}\right) \times \left[ O_p\left( \frac{\alpha_h^{2h}}{\min\left\{N^2, T\right\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h}{\min\left\{N^{3/2}, T\right\}} \right) + O_p\left( \frac{\rho_h^{2h}}{\min\left\{N, T\right\}} \right) \right], \\
&= O_p\left( \frac{\alpha_h^{2h} k_h^{-1}}{\min\left\{N^2, T\right\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h k_h^{-1}}{\min\left\{N^{3/2}, T\right\}} \right) + O_p\left( \frac{\rho_h^{2h} k_h^{-1}}{\min\left\{N, T\right\}} \right),
\end{aligned}
$$

by using Lemma B.6 (l) and it is $o_p(1)$ under the stated conditions. Therefore,

$$
\hat{\sigma}^2 = h^{-1} \textstyle\sum_{t=T+1}^{T+h} H^2 e_t^2 + o_p(1),
$$

under the stated conditions, which yields the result. Note that carefully investigating all 17 terms of the factor estimation errors gives the dominating terms $O_p\left(\frac{\alpha_h^h h k_h^{-1}}{\min\{N,T^{1/2}\}}\right)$, and $O_p\left(\frac{\rho_h^h h k_h^{-1}}{\min\{N^{1/2},T^{1/2}\}}\right)$ that appear in the numerator of $IV$. ■

**Lemma B.8.** *Under Assumptions 2.1–2.5, the following hold:*

*(a) For* $t = T+1, ..., T+h$ *uniformly in* $t$,

$$\tilde{U}_{i,t} - U_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(b)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1} - U_{i,t-1})^2 = O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2,T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N,T\}}\right),$$

*(c)*

$$h^{-1}\sum_{t=T+1}^{T+h} U_{i,t-1}(\tilde{U}_{i,t-1} - U_{i,t-1}) = O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N,T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N^{1/2},T^{1/2}\}}\right),$$

*(d)*

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}(\tilde{U}_{i,t-1} - U_{i,t-1}) &= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\left\{N^2, T\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N, T^{1/2}\right\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h} k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right), \end{aligned}$$

*(e)*

$$h^{-1/2}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1} - U_{i,t-1}) z_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N,T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

*(f) For* $t = T+1, ..., T+h$ *uniformly in* $t$,

$$\tilde{u}_{i,t} - u_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(g)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{u}_{i,t} - u_{i,t})^2 = O_p\left(\frac{\alpha_h^{2h} k_h^{-1}}{\min\{N^2,T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h^{-1}}{\min\{N,T\}}\right),$$

*(h)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}-U_{i,t-1})(\tilde{u}_{i,t}-u_{i,t}) = O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N^{3/2},T\}}\right) +O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right),$$

*(i)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}-U_{i,t-1})u_{i,t}=O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

*(j)*

$$h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}(\tilde{u}_{i,t}-u_{i,t})=O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

*(k)*

$$h^{-1}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}(\tilde{u}_{i,t}-u_{i,t}) = O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\{N^{1/2},T^{1/2}\}}\right) +O_p\left(\frac{\alpha_h^{h}\rho_h^{h}}{\min\{N,T^{1/2}\}}\right),$$

*(l)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{u}_{i,t}-u_{i,t})z_{i,t}=O_p\left(\frac{\alpha_h^{h}k_h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{h}k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

*(m)*

$$h^{-2}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}^2 = h^{-2}H^2\sum_{t=T+1}^{T+h}U_{i,t-1}^2+O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right) +O_p\left(\frac{\alpha_h^{h}\rho_h^{h}h^{-1}k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N^{1/2},T^{1/2}\}}\right),$$

*(n)*

$$h^{-1}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}z_{i,t} = h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}z_{i,t} +O_p\left(\frac{\alpha_h^{h}h^{-1/2}k_h^{1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{h}h^{-1/2}k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right).$$

**Proof of Lemma B.8.**

(a) Since

$$\begin{aligned}
\tilde{U}_{i,t} - U_{i,t} &= (\hat{\lambda}_i^* - H^{-1}\lambda_i)HF_t + H^{-1}\lambda_i^*(\tilde{F}_t - HF_t) \\
&\quad +(\hat{\lambda}_i^* - H^{-1}\lambda_i)(\tilde{F}_t - HF_t), \\
&= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) \\
&\quad +O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\
&\quad +O_p\left(\frac{1}{\min\left\{N, T^{1/2}\right\}}\right) \\
&\quad \times\left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),
\end{aligned}$$

by using Lemma B.6 (c) for $t = T+1, \cdots, T+h$ uniformly in $t$. (b) The result is straightforwardly shown from (a).

(c) Since

$$\begin{aligned}
U_{i,t}(\tilde{U}_{i,t} - U_{i,t}) &= O_p\left(\rho_h^h k_h^{1/2}\right) \times \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),
\end{aligned}$$

uniformly in $t$, the result follows.

(d) By using (b) and (c),

$$\begin{aligned}
\tilde{U}_{i,t}(\tilde{U}_{i,t} - U_{i,t}) &= U_{i,t}(\tilde{U}_{i,t} - U_{i,t}) + (\tilde{U}_{i,t} - U_{i,t})^2, \\
&= O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\
&\quad +O_p\left(\frac{\alpha_h^{2h} k_h}{\min\left\{N^2, T\right\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\left\{N, T\right\}}\right), \\
&= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\left\{N^2, T\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),
\end{aligned}$$

uniformly in $t$ for $t = T+1, \cdots, T+h$. Hence,

$$\begin{aligned} h^{-1} \textstyle\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}(\tilde{U}_{i,t-1} - U_{i,t-1}) &= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\{N, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N^{1/2}, T^{1/2}\}}\right). \end{aligned}$$

(e) Since

$$(\tilde{U}_{i,t-1} - U_{i,t-1}) z_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right),$$

and $(\tilde{U}_{i,t-1} - U_{i,t-1})$ and $z_{i,t}$ are independent with $z_{i,t}$ being i.i.d. with $\mathbb{E}(z_{i,t}) = 0$, I have

$$h^{-1/2} \textstyle\sum_{t=T+1}^{T+h} (\tilde{U}_{i,t-1} - U_{i,t-1}) z_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right).$$

(f) This is because

$$\begin{aligned} \tilde{u}_{i,t} - u_{i,t} &= (\hat{\lambda}_i^* - H^{-1}\lambda_i) H f_t + H^{-1}\lambda_i (\tilde{f}_t - H f_t) \\ &\quad + (\hat{\lambda}_i^* - H^{-1}\lambda_i)(\tilde{f}_t - H f_t), \\ &= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right), \end{aligned}$$

uniformly in $t$ by using Lemma B.6 (j).

(g) is straightforwardly shown from (f).

(h) (a) and (f) imply that

$$\begin{aligned} (\tilde{U}_{i,t-1} - U_{i,t-1})(\tilde{u}_{i,t} - u_{i,t}) &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right)\right] \\ &\quad \times \left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\{N^{1/2}, T^{1/2}\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\{N, T\}}\right), \end{aligned}$$

uniformly in $t$, which yields the result.

(i) Since

$$
\begin{aligned}
(\tilde{U}_{i,t-1} - U_{i,t-1})u_{i,t} &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right] \times O_p(\rho_h^h k_h^{-1/2}), \\
&= O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),
\end{aligned}
$$

uniformly in $t$, which yields the result.

(j) Lemma B.6 (b) and Lemma B.8 (f) imply that

$$
\begin{aligned}
U_{i,t-1}(\tilde{u}_{i,t} - u_{i,t}) &= O_p(\rho_h^h k_h^{1/2}) \times \left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),
\end{aligned}
$$

uniformly in $t$, which yields the result.

(k) By using (j) and (h), I obtain

$$
\begin{aligned}
\tilde{U}_{i,t-1}(\tilde{u}_{i,t} - u_{i,t}) &= U_{i,t-1}(\tilde{u}_{i,t} - u_{i,t}) + (\tilde{U}_{i,t-1} - U_{i,t-1})(\tilde{u}_{i,t} - u_{i,t}), \\
&= O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\
&\quad + O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N^2, T\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{3/2}, T\right\}}\right) \\
&\quad + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N, T\right\}}\right), \\
&= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N^2, T\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N, T^{1/2}\right\}}\right) \\
&\quad + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),
\end{aligned}
$$

uniformly in $t$, which yields the result.

(l) From (f), I obtain

$$
h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{u}_{i,t} - u_{i,t}) z_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right).
$$

(m) I have

$$\begin{aligned}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}-U_{i,t-1})^2 &= \sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}^2+\sum_{t=T+1}^{T+h}U_{i,t-1}^2-2\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}U_{i,t-1},\\ &= \sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}^2+\sum_{t=T+1}^{T+h}U_{i,t-1}^2\\ &\quad -2\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}-U_{i,t-1})U_{i,t-1}-2\sum_{t=T+1}^{T+h}U_{i,t-1}^2,\\ &= \sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}^2-\sum_{t=T+1}^{T+h}U_{i,t-1}^2-2\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}-U_{i,t-1})U_{i,t-1}.\end{aligned}$$

Solving the first term on the right-hand side gives

$$\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}^2=\sum_{t=T+1}^{T+h}U_{i,t-1}^2+\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}-U_{i,t-1})^2+2\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}-U_{i,t-1})U_{i,t-1},$$

so that

$$\begin{aligned}h^{-2}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}^2 &= h^{-2}\sum_{t=T+1}^{T+h}U_{i,t-1}^2+h^{-2}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}-U_{i,t-1})^2\\ &\quad +2h^{-2}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}-U_{i,t-1})U_{i,t-1},\\ &= h^{-2}\sum_{t=T+1}^{T+h}U_{i,t-1}^2\\ &\quad +O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right)\\ &\quad +O_p\left(\frac{\alpha_h^h\rho_h^h h^{-1}k_h}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),\end{aligned}$$

by using (b) and (c).

(n) I have

$$\begin{aligned}h^{-1}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}z_{i,t} &= h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}z_{i,t}+h^{-1}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}-U_{i,t-1})z_{i,t},\\ &= h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}z_{i,t}\\ &\quad +O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),\end{aligned}$$

by using (e). ■

**Lemma B.9.** *Suppose Assumptions 2.1–2.4 and B.1 hold or Assumptions 2.1–2.5 and the following* *conditions hold:*

$$\frac{\alpha_h^h h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}} \to 0 \text{ and } \frac{\rho_h^h h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \to 0.$$

*Then, I have*

$$\hat{\sigma}_i^2 \xrightarrow{p} \sigma_i^2,$$

*for any $i$ as $N, T, h \to \infty$, where*

$$\hat{\sigma}_i^2 = h^{-1} \sum_{t=T+1}^{T+h} (\tilde{u}_{i,t} - \hat{\delta}_i^* \tilde{U}_{i,t-1})^2,$$

*with*

$$\hat{\delta}_i^* = \frac{\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1} \tilde{u}_{i,t}}{\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2}.$$

**Proof of Lemma B.9.**

I start with the AR coefficient estimator,

$$\hat{\delta}_i^* = \frac{\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1} \tilde{u}_{i,t}}{\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2}.$$

Since

$$\begin{aligned}
\tilde{u}_{i,t} &= \tilde{U}_{i,t} - \tilde{U}_{i,t-1}, \\
&= U_{i,t} - U_{i,t-1} + (\tilde{U}_{i,t} - U_{i,t}) - (\tilde{U}_{i,t-1} - U_{i,t-1}), \\
&= \frac{c_i}{k_h} U_{i,t-1} + z_{i,t} + (\tilde{U}_{i,t} - U_{i,t}) - (\tilde{U}_{i,t-1} - U_{i,t-1}), \\
&= \frac{c_i}{k_h} \tilde{U}_{i,t-1} + z_{i,t} + (\tilde{u}_{i,t} - u_{i,t}) - \frac{c_i}{k_h} (\tilde{U}_{i,t-1} - U_{i,t-1}),
\end{aligned}$$

I obtain

$$\begin{aligned}
\hat{\delta}_i^* &= \frac{c_i}{k_h} + \frac{\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1} z_{i,t}}{\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2} + \frac{\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1} (\tilde{u}_{i,t} - u_{i,t})}{\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2} \\
&\quad + \frac{c_i}{k_h} \frac{\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1} (\tilde{U}_{i,t-1} - U_{i,t-1})}{\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2}, \\
&= \frac{c_i}{k_h} + \frac{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1} z_{i,t}}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2} + \frac{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} (\tilde{U}_{i,t-1} - U_{i,t-1}) z_{i,t}}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2} \\
&\quad + \frac{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1} (\tilde{u}_{i,t} - u_{i,t})}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2} + \frac{c_i}{k_h} \frac{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1} (\tilde{U}_{i,t-1} - U_{i,t-1})}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2}.
\end{aligned}$$

This yields

$$\begin{aligned}
\rho_h^{-h} k_h^{-1} \left( \hat{\delta}_i^* - \frac{c_i}{k_h} \right) &= \frac{\rho_h^{-h} k_h^{-1} \sum_{t=T+1}^{T+h} U_{i,t-1} z_{i,t}}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2} + \frac{\rho_h^{-h} k_h^{-1} \sum_{t=T+1}^{T+h} (\tilde{U}_{i,t-1} - U_{i,t-1}) z_{i,t}}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2} \\
&\quad + \frac{\rho_h^{-h} k_h^{-1} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1} (\tilde{u}_{i,t} - u_{i,t})}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2} + \frac{c_i}{k_h} \frac{\rho_h^{-h} k_h^{-1} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1} (\tilde{U}_{i,t-1} - U_{i,t-1})}{\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2}, \\
&= I + II + III + IV.
\end{aligned}$$

For the denominator,

$$\begin{aligned}
\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^2 &= \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^2 + O_p\left( \frac{\alpha_h^{2h} \rho_h^{-2h} h k_h^{-1}}{\min\{N^2, T\}} \right) + O_p\left( \frac{h k_h^{-1}}{\min\{N, T\}} \right) \\
&\quad + O_p\left( \frac{\alpha_h^h \rho_h^{-h} h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right),
\end{aligned}$$

by using Lemma B.8 (m) and the four terms of the factor estimation errors are $o_p(1)$ under the stated conditions.

The numerator of $I$ is $o_p(1)$ by using Lemma B.5 (h). For the numerator of $II$,

$$\rho_h^{-h} k_h^{-1} \sum_{t=T+1}^{T+h} (\tilde{U}_{i,t-1} - U_{i,t-1}) z_{i,t} = O_p\left( \frac{\alpha_h^h \rho_h^{-h} h^{1/2} k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{h^{1/2} k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right),$$

by using Lemma B.8 (e) and it is $o_p(1)$ under the stated conditions. For the numerator of $III$,

$$\rho_h^{-h}k_h^{-1}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}(\tilde{u}_{i,t}-u_{i,t}) = O_p\left(\frac{\alpha_h^{2h}\rho_h^{-h}hk_h^{-1}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{h}hk_h^{-1}}{\min\{N^{1/2},T^{1/2}\}}\right) +O_p\left(\frac{\alpha_h^{h}hk_h^{-1}}{\min\{N,T^{1/2}\}}\right),$$

by using Lemma B.8 (k) and it is $o_p(1)$ under the stated conditions. For the numerator of $IV$,

$$\rho_h^{-h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}(\tilde{U}_{i,t-1}-U_{i,t-1}) = O_p\left(\frac{\alpha_h^{2h}\rho_h^{-h}hk_h^{-1}}{\min\{N^2,T\}}\right)+O_p\left(\frac{\alpha_h^{h}hk_h^{-1}}{\min\{N,T^{1/2}\}}\right) +O_p\left(\frac{\rho_h^{h}hk_h^{-1}}{\min\{N^{1/2},T^{1/2}\}}\right),$$

by using Lemma B.8 (d) and it is $o_p(1)$ under the stated conditions. Therefore, I proceed with $\hat{\delta}_i^*-\frac{c_i}{k_h}=O_p(\rho_h^{-h}k_h^{-1})$ and $\hat{\delta}_i^*=O_p(k_h^{-1})$. I now consider

$$\begin{aligned}
\hat{\sigma}_i^2 &= h^{-1}\sum_{t=T+1}^{T+h}(\tilde{u}_{i,t}-\hat{\delta}_i^*\tilde{U}_{i,t-1})^2,\\
&= h^{-1}\sum_{t=T+1}^{T+h}[u_{i,t}-\hat{\delta}_i^*U_{i,t-1}+(\tilde{u}_{i,t}-u_{i,t})-\hat{\delta}_i^*(\tilde{U}_{i,t-1}-U_{i,t-1})]^2,\\
&= h^{-1}\sum_{t=T+1}^{T+h}[u_{i,t}-\tfrac{c_i}{k_h}U_{i,t-1}-(\hat{\delta}_i^*-\tfrac{c_i}{k_h})U_{i,t-1}+(\tilde{u}_{i,t}-u_{i,t})-\hat{\delta}_i^*(\tilde{U}_{i,t-1}-U_{i,t-1})]^2,\\
&= h^{-1}\sum_{t=T+1}^{T+h}[z_{i,t}-(\hat{\delta}_i^*-\tfrac{c_i}{k_h})U_{i,t-1}+(\tilde{u}_{i,t}-u_{i,t})-\hat{\delta}_i^*(\tilde{U}_{i,t-1}-U_{i,t-1})]^2,\\
&= h^{-1}\sum_{t=T+1}^{T+h}[z_{i,t}^2+(\hat{\delta}_i^*-\tfrac{c_i}{k_h})^2U_{i,t-1}^2+(\tilde{u}_{i,t}-u_{i,t})^2+\hat{\delta}_i^{*2}(\tilde{U}_{i,t-1}-U_{i,t-1})^2\\
&\quad +2(\hat{\delta}_i^*-\frac{c_i}{k_h})U_{i,t-1}z_{i,t}+2(\tilde{u}_{i,t}-u_{i,t})z_{i,t}-2\hat{\delta}_i^*(\tilde{U}_{i,t-1}-U_{i,t-1})z_{i,t}\\
&\quad +2(\hat{\delta}_i^*-\frac{c_i}{k_h})U_{i,t-1}(\tilde{u}_{i,t}-u_{i,t})-2(\hat{\delta}_i^*-\frac{c_i}{k_h})\hat{\delta}_i^*U_{i,t-1}(\tilde{U}_{i,t-1}-U_{i,t-1})\\
&\quad +2\hat{\delta}_i^*(\tilde{u}_{i,t}-u_{i,t})(\tilde{U}_{i,t-1}-U_{i,t-1})],\\
&= h^{-1}\sum_{t=T+1}^{T+h}z_{i,t}^2+\sum_{k=1}^{9}D_k,
\end{aligned}$$

has nine terms of the factor estimation errors. I now show that they are all $o_p(1)$ under the stated conditions. For $D_1$,

$$\begin{aligned} D_1 &= (\hat{\delta}_i^* - \frac{c_i}{k_h})^2 h^{-1} \textstyle\sum_{t=T+1}^{T+h} U_{i,t-1}^2, \\ &= O_p(\rho_h^{-2h} k_h^{-2}) \times O_p\left(\rho_h^{2h} h^{-1} k_h^2\right), \\ &= O_p(h^{-1}) = o_p(1). \end{aligned}$$

For $D_2$,

$$\begin{aligned} D_2 &= h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{u}_{i,t} - u_{i,t})^2, \\ &= O_p\left(\frac{\alpha_h^{2h} k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h^{-1}}{\min\{N, T\}}\right), \end{aligned}$$

by using Lemma B.8 (g) and it is $o_p(1)$ under the stated conditions. For $D_3$,

$$\begin{aligned} D_3 &= \hat{\delta}^{*2} h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{U}_{i,t-1} - U_{i,t-1})^2, \\ &= O_p(k_h^{-2}) \times \left[O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^{2h} k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h^{-1}}{\min\{N, T\}}\right), \end{aligned}$$

by using Lemma B.8 (b) and it is $o_p(1)$ under the stated conditions. For $D_4$,

$$\begin{aligned} D_4 &= 2(\hat{\delta}_i^* - \frac{c_i}{k_h}) h^{-1} \textstyle\sum_{t=T+1}^{T+h} U_{i,t-1} z_{i,t}, \\ &= O_p(\rho_h^{-h} k_h^{-1}) \times o_p\left(\rho_h^{h} k_h h^{-1}\right) = o_p(1). \end{aligned}$$

For $D_5$,

$$\begin{aligned} D_5 &= 2h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{u}_{i,t} - u_{i,t}) z_{i,t}, \\ &= O_p\left(\frac{\alpha_h^{h} k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^{h} k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right), \end{aligned}$$

by using Lemma B.8 (l) and it is $o_p(1)$ under the stated conditions. For $D_6$,

$$\begin{aligned} D_6 &= 2\hat{\delta}_i^* h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{U}_{i,t-1} - U_{i,t-1}) z_{i,t}, \\ &= O_p(k_h^{-1}) \times \left[ O_p\left( \frac{\alpha_h^h h^{1/2} k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h h^{1/2} k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right) \right], \\ &= O_p\left( \frac{\alpha_h^h h^{1/2} k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h h^{1/2} k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right), \end{aligned}$$

by using Lemma B.8 (e) and it is $o_p(1)$ under the stated conditions. For $D_7$,

$$\begin{aligned} D_7 &= 2(\hat{\delta}_i^* - \frac{c_i}{k_h}) h^{-1} \textstyle\sum_{t=T+1}^{T+h} U_{i,t-1} (\tilde{u}_{i,t} - u_{i,t}), \\ &= O_p(\rho_h^{-h} k_h^{-1}) \times \left[ O_p\left( \frac{\alpha_h^h \rho_h^h}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^{2h}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right) \right], \\ &= O_p\left( \frac{\alpha_h^h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right), \end{aligned}$$

by using Lemma B.8 (j) and it is $o_p(1)$ under the stated conditions. For $D_8$,

$$\begin{aligned} D_8 &= 2(\hat{\delta}_i^* - \frac{c_i}{k_h}) \hat{\delta}_i^* h^{-1} \textstyle\sum_{t=T+1}^{T+h} U_{i,t-1} (\tilde{U}_{i,t-1} - U_{i,t-1}), \\ &= O_p(\rho_h^{-h} k_h^{-1}) \times \left[ O_p\left( \frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^{2h} k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right) \right], \\ &= O_p\left( \frac{\alpha_h^h}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{\rho_h^h}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right), \end{aligned}$$

by using Lemma B.8 (c) and it is $o_p(1)$ under the stated conditions. For $D_9$,

$$\begin{aligned} D_9 &= 2\hat{\delta}_i^* h^{-1} \textstyle\sum_{t=T+1}^{T+h} (\tilde{u}_{i,t} - u_{i,t})(\tilde{U}_{i,t-1} - U_{i,t-1}), \\ &= O_p\left(k_h^{-1}\right) \times \left[ O_p\left( \frac{\alpha_h^{2h}}{\min\left\{N^2, T\right\}} \right) + O_p\left( \frac{\rho_h^{2h}}{\min\left\{N, T\right\}} \right) \right], \\ &= O_p\left( \frac{\alpha_h^{2h} k_h^{-1}}{\min\left\{N^2, T\right\}} \right) + O_p\left( \frac{\rho_h^{2h} k_h^{-1}}{\min\left\{N, T\right\}} \right), \end{aligned}$$

by using Lemma B.8 (h) and it is $o_p(1)$ under the stated conditions. Therefore,

$$\hat{\sigma}_i^2 = h^{-1} \textstyle\sum_{t=T+1}^{T+h} z_{i,t}^2 + o_p(1),$$

under the stated conditions. Note that the two conditions

$$\frac{\alpha_h^h h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}} \to 0 \text{ and } \frac{\rho_h^h h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \to 0,$$

are obtained by carefully investigating all 17 terms of the factor estimation errors. ■

**Lemma B.10.** *Under Assumptions 2.1–2.5, the following hold:*

*(a)*

$$\bar{\tilde{F}} - H\bar{F} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(b)*

$$\bar{\tilde{f}} - H\bar{f} = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right).$$

**Proof of Lemma B.10.** (a) I have

$$\bar{\tilde{F}} - H\bar{F} = h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1} - HF_{t-1}) = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

by using Lemma B.6 (c). (b) I have

$$\bar{\tilde{f}} - H\bar{f} = h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\tilde{f}_t - Hf_t) = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

by using Lemma B.6 (j). ■

**Lemma B.11.** *Under Assumptions 2.1–2.5, the following hold:*

*(a) For $t = T+1, ..., T+h$ uniformly in $t$,*

$$F_t^c = O_p(\alpha_h^h k_h^{1/2}),$$

*(b) For $t = T+1, ..., T+h$ uniformly in $t$ and all $i$,*

$$U_{i,t}^c = O_p(\rho_h^h k_h^{1/2}),$$

*(c) For $t = T+1, ..., T+h$ uniformly in $t$,*

$$\tilde{F}_t^c - HF_t^c = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(d)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^c - HF_{t-1}^c)^2 = O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right),$$

*(e)*

$$h^{-1}\sum_{t=T+1}^{T+h} F_{t-1}^c(\tilde{F}_{t-1}^c - HF_{t-1}^c) = O_p\left(\frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(f)*

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h} \tilde{F}_{t-1}^c(\tilde{F}_{t-1}^c - HF_{t-1}^c) &= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h} k_h}{\min\left\{N, T\right\}}\right), \end{aligned}$$

*(g)*

$$h^{-1/2}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^c - HF_{t-1}^c)e_t = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(h) For $t = T+1, ..., T+h$ uniformly in $t$,*

$$f_t^c = O_p(\alpha_h^h k_h^{-1/2}),$$

*(i) For $t = T+1, ..., T+h$ uniformly in $t$ and all $i$,*

$$u_{i,t}^c = O_p(\rho_h^h k_h^{-1/2}),$$

*(j) For $t = T+1, ..., T+h$ uniformly in $t$,*

$$\tilde{f}_t^c - Hf_t^c = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(k)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{f}_t^c - Hf_t^c)^2 = O_p\left(\frac{\alpha_h^{2h}k_h^{-1}}{\min\{N^2,T\}}\right) + O_p\left(\frac{\rho_h^{2h}k_h^{-1}}{\min\{N,T\}}\right),$$

*(l)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^c - HF_{t-1}^c)(\tilde{f}_t^c - Hf_t^c) = O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right),$$

*(m)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^c - HF_{t-1}^c)f_t^c = O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(n)*

$$h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}^c(\tilde{f}_t^c - Hf_t^c) = O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(o)*

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^c(\tilde{f}_t^c - Hf_t^c) &= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N,T\right\}}\right), \end{aligned}$$

*(p)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{f}_t^c - Hf_t^c)e_t = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(q)*

$$\begin{aligned} h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2} &= h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^{c2} + O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right) + O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h\rho_h^h h^{-1}k_h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right), \end{aligned}$$

*(r)*

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^c e_t &= h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}^c e_t \\ &\quad + O_p\left(\frac{\alpha_h^h h^{-1/2}k_h^{1/2}}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h h^{-1/2}k_h^{1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right). \end{aligned}$$

**Proof of Lemma B.11.**

(a) I have

$$\begin{aligned} F_{t-1}^{c} &= F_{t-1} - h^{-1}\textstyle\sum_{t=T+1}^{T+h} F_{t-1}, \\ &= O_p(\alpha_h^h k_h^{1/2}), \end{aligned}$$

by using Lemma B.6 (a).

(b) The proof is the same as (a).

(c) I have

$$\begin{aligned} \tilde{F}_{t-1}^{c} - HF_{t-1}^{c} &= (\tilde{F}_{t-1} - HF_{t-1}) - (\overline{\tilde{F}} - H\overline{F}), \\ &= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right), \end{aligned}$$

uniformly in $t$, for $t = T+1, \cdots T+h$, by using Lemmas B.6 (c) and B.10 (a). (d) is straightforwardly shown from (c).

(e) I have

$$\begin{aligned} F_{t-1}^{c}(\tilde{F}_{t-1}^{c} - HF_{t-1}^{c}) &= O_p(\alpha_h^h k_h^{1/2}) \times \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right), \end{aligned}$$

uniformly in $t$, for $t = T+1, \cdots T+h$, by using (a) and (c). The result follows.

(f) I have

$$\begin{aligned} \tilde{F}_{t-1}^{c}(\tilde{F}_{t-1}^{c} - HF_{t-1}^{c}) &= HF_{t-1}^{c}(\tilde{F}_{t-1}^{c} - HF_{t-1}^{c}) + (\tilde{F}_{t-1}^{c} - HF_{t-1}^{c})^2, \\ &= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^{2h} k_h}{\min\left\{N^2, T\right\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\left\{N, T\right\}}\right), \\ &= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h} k_h}{\min\left\{N, T\right\}}\right), \end{aligned}$$

uniformly in $t$, for $t = T+1, \cdots T+h$, by using results derived in (e) and (d). The result follows.

(g) I have

$$(\tilde{F}_{t-1}^c - HF_{t-1}^c)e_t = (\tilde{F}_{t-1} - HF_{t-1})e_t - (\overline{\tilde{F}} - H\overline{F})e_t,$$

so that

$$\begin{aligned} h^{-1/2}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^c - HF_{t-1}^c)e_t &= h^{-1/2}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1} - HF_{t-1})e_t + (\overline{\tilde{F}} - H\overline{F})h^{-1/2}\sum_{t=T+1}^{T+h} e_t, \\ &= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right), \\ &= O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right), \end{aligned}$$

by using Lemma B.6 (g) and Lemma B.10 (a).

(h) I have

$$f_t^c = f_t - h^{-1}\sum_{t=T+1}^{T+h} f_t = O_p(\alpha_h^h k_h^{-1/2}).$$

(i) can be shown same as (h).

(j) I have

$$\begin{aligned} \tilde{f}_t^c - Hf_t^c &= (\tilde{f}_t - Hf_t) - (\overline{\tilde{f}} - H\bar{f}), \\ &= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right), \end{aligned}$$

uniformly in $t$, for $t = T+1, \cdots T+h$, by using Lemma B.6 (j) and Lemma B.10 (b). (k) is straightforwardly shown from part (j).

(l) I have

$$\begin{aligned} (\tilde{F}_{t-1}^c - HF_{t-1}^c)(\tilde{f}_t^c - Hf_t^c) &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right] \\ &\quad \times \left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right], \\ &= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N^2, T\right\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N, T\right\}}\right), \end{aligned}$$

uniformly in $t$, for $t = T+1, \cdots T+h$, by using the results obtained in (c) and (j). This yields the result.

(m) I have

$$
\begin{aligned}
(\tilde{F}_{t-1}^{c} - HF_{t-1}^{c})f_t^c &= \left[O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right] \times O_p(\alpha_h^h k_h^{-1/2}), \\
&= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),
\end{aligned}
$$

uniformly in $t$, for $t = T+1, \cdots T+h$, by using the result obtained in (c). This yields the result.

(n) (a) and (j) imply that

$$
\begin{aligned}
F_{t-1}^{c}(\tilde{f}_t^c - Hf_t^c) &= O_p(\alpha_h^h k_h^{1/2}) \times \left[O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right)\right], \\
&= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),
\end{aligned}
$$

uniformly in $t$, for $t = T+1, \cdots T+h$, which yields the result.

(o) I have

$$
\begin{aligned}
\tilde{F}_{t-1}^{c}(\tilde{f}_t^c - Hf_t^c) &= HF_{t-1}^{c}(\tilde{f}_t^c - Hf_t^c) + (\tilde{F}_t^c - HF_t^c)(\tilde{f}_t^c - Hf_t^c), \\
&= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\
&\quad + O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N^2, T\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{3/2}, T\right\}}\right) \\
&\quad + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N, T\right\}}\right), \\
&= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\
&\quad + O_p\left(\frac{\rho_h^{2h}}{\min\left\{N, T\right\}}\right),
\end{aligned}
$$

uniformly in $t$, for $t = T+1, \cdots T+h$, by using the results obtained in (n) and (l). This yields the result.

(p) I have

$$
\begin{aligned}
h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\tilde{f}_t^c - Hf_t^c)e_t &= h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\tilde{f}_t - Hf_t)e_t + (\overline{\tilde{f}} - H\bar{f})h^{-1}\textstyle\sum_{t=T+1}^{T+h} e_t, \\
&= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\
&\quad + O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right), \\
&= O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),
\end{aligned}
$$

by using Lemma B.6 (p) and Lemma B.10 (b). Hence,

$$
h^{-1}\textstyle\sum_{t=T+1}^{T+h}(\tilde{f}_t^c - Hf_t^c)e_t = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right).
$$

(q) I have

$$
\begin{aligned}
\textstyle\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^c - HF_{t-1}^c)^2 &= \textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2} + H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2} - 2H\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^c F_{t-1}^c, \\
&= \textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2} + H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2} \\
&\quad -2H\textstyle\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^c - HF_{t-1}^c)F_{t-1} - 2H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2}, \\
&= \textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2} - H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2} \\
&\quad -2H\textstyle\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^c - HF_{t-1}^c)F_{t-1}^c.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2} &= H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2} + \textstyle\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^c - HF_{t-1}^c)^2 \\
&\quad +2H\textstyle\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^c - HF_{t-1}^c)F_{t-1}^c,
\end{aligned}
$$

or

$$
\begin{aligned}
h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2} &= h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^{c2}+h^{-2}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^{c}-HF_{t-1}^{c})^2 \\
&\quad +2h^{-2}H\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^{c}-HF_{t-1}^{c})F_{t-1}^{c}, \\
&= h^{-2}H^2\sum_{t=T+1}^{T+h}F_{t-1}^{c2} \\
&\quad +O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right) \\
&\quad +O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\alpha_h^{h}\rho_h^{h}h^{-1}k_h}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}
$$

by using (d) and (e).

(r) I have

$$
\begin{aligned}
h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c}e_t &= h^{-1}H\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_t+h^{-1}\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^{c}-HF_{t-1}^{c})e_t, \\
&= h^{-1}H\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_t \\
&\quad +O_p\left(\frac{\alpha_h^{h}k_h^{1/2}h^{-1/2}}{\min\{N,T^{1/2}\}}\right)+O_p\left(\frac{\rho_h^{h}k_h^{1/2}h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}
$$

by using (g). ■

**Lemma B.12.** *Suppose Assumptions 2.1–2.4 and B.1 hold or Assumptions 2.1–2.5 and the following condition hold:*

$$\frac{\alpha_h^{h}\rho_h^{h}h^{1/2}k_h^{-1/2}}{\min\{N^{1/2},T^{1/2}\}}\to 0.$$

*Then, I have*

$$\hat{\sigma}^2 \xrightarrow{p} Q^{-2}\sigma^2,$$

*as $N,T,h\to\infty$, where*

$$\hat{\sigma}^2=h^{-1}\sum_{t=T+1}^{T+h}(\tilde{f}_t^{c}-\hat{\delta}^*\tilde{F}_{t-1}^{c})^2.$$

**Proof of Lemma B.12.** The proof follows that of Lemma B.7 by replacing Lemma B.6 with Lemma B.11. Thus, it is not repeated.

**Lemma B.13.** *Under Assumptions 2.1–2.5, the following hold:*

*(a) For* $t = T+1, ..., T+h$ *uniformly in* $t$,

$$\tilde{U}_{i,t}^{c} - U_{i,t}^{c} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(b)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}^{c} - U_{i,t-1}^{c})^2 = O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\{N, T\}}\right),$$

*(c)*

$$h^{-1}\sum_{t=T+1}^{T+h} U_{i,t-1}^{c}(\tilde{U}_{i,t-1}^{c} - U_{i,t-1}^{c}) = O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(d)*

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^{c}(\tilde{U}_{i,t-1}^{c} - U_{i,t-1}^{c}) &= O_p\left(\frac{\alpha_h^{2h} k_h}{\min\{N^2, T\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h k_h}{\min\left\{N, T^{1/2}\right\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h} k_h}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right), \end{aligned}$$

*(e)*

$$h^{-1/2}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}^{c} - U_{i,t-1}^{c}) z_{i,t} = O_p\left(\frac{\alpha_h^h k_h^{1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(f) For* $t = T+1, ..., T+h$ *uniformly in* $t$,

$$\tilde{u}_{i,t}^{c} - u_{i,t}^{c} = O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),$$

*(g)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{u}_{i,t}^{c} - u_{i,t}^{c})^2 = O_p\left(\frac{\alpha_h^{2h} k_h^{-1}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\rho_h^{2h} k_h^{-1}}{\min\{N, T\}}\right),$$

*(h)*

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}^{c} - U_{i,t-1}^{c})(\tilde{u}_{i,t}^{c} - u_{i,t}^{c}) &= O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2, T\}}\right) + O_p\left(\frac{\alpha_h^h \rho_h^h}{\min\left\{N^{3/2}, T\right\}}\right) \\ &\quad + O_p\left(\frac{\rho_h^{2h}}{\min\{N, T\}}\right), \end{aligned}$$

*(i)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1}^{c}-U_{i,t-1}^{c})u_{i,t}^{c}=O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(j)*

$$h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}^{c}(\tilde{u}_{i,t}^{c}-u_{i,t}^{c})=O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(k)*

$$\begin{aligned}h^{-1}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}^{c}(\tilde{u}_{i,t}^{c}-u_{i,t}^{c}) &= O_p\left(\frac{\alpha_h^{2h}}{\min\left\{N^2,T\right\}}\right)+O_p\left(\frac{\rho_h^{2h}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right)\\ &\quad+O_p\left(\frac{\alpha_h^h\rho_h^h}{\min\left\{N,T^{1/2}\right\}}\right),\end{aligned}$$

*(l)*

$$h^{-1}\sum_{t=T+1}^{T+h}(\tilde{u}_{i,t}^{c}-u_{i,t}^{c})z_{i,t}=O_p\left(\frac{\alpha_h^h k_h^{-1/2}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^h k_h^{-1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),$$

*(m)*

$$\begin{aligned}h^{-2}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}^{c2} &= h^{-2}H^2\sum_{t=T+1}^{T+h}U_{i,t-1}^{c2}+O_p\left(\frac{\alpha_h^{2h}h^{-1}k_h}{\min\{N^2,T\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right)\\ &\quad+O_p\left(\frac{\alpha_h^h\rho_h^h h^{-1}k_h}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right),\end{aligned}$$

*(n)*

$$\begin{aligned}h^{-1}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}^{c}z_{i,t} &= h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}^{c}z_{i,t}\\ &\quad+O_p\left(\frac{\alpha_h^h h^{-1/2}k_h^{1/2}}{\min\left\{N,T^{1/2}\right\}}\right)+O_p\left(\frac{\rho_h^h h^{-1/2}k_h^{1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}}\right).\end{aligned}$$

**Proof of Lemma B.13.** They can be shown straightforwardly from Lemma B.8 as I did in Lemma B.11 from Lemma B.6. Thus, the proof is not repeated. ■

**Lemma B.14.** *Suppose Assumptions 2.1–2.4 and B.1 hold or Assumptions 2.1–2.5 and the following conditions hold:*

$$\frac{\alpha_h^h h k_h^{-1}}{\min\left\{N,T^{1/2}\right\}}\to 0 \text{ and } \frac{\rho_h^h h k_h^{-1}}{\min\left\{N^{1/2},T^{1/2}\right\}}\to 0.$$

*Then, I have*

$$\hat{\sigma}_i^2 \xrightarrow{p} \sigma_i^2,$$

*for any $i$ as $N, T, h \to \infty$, where*

$$\hat{\sigma}_i^2 = h^{-1} \sum_{t=T+1}^{T+h} (\tilde{u}_{i,t}^c - \hat{\delta}_i^* \tilde{U}_{i,t-1}^c)^2.$$

**Proof of Lemma B.14.** The proof follows that of Lemma B.9 by replacing Lemma B.8 with Lemma B.13. Thus, it is not repeated. ■

**Lemma B.15.** *Let $\Theta \sim N(0, \sigma^2/2c)$ and $\Theta_i \sim N(0, \sigma_i^2/2c_i)$. Under Assumptions 2.1–2.5, the following hold as $T$ and $h \to \infty$.*

*(a) Suppose $c > 0$.*

*If $T/k_h \to 0$, then*

$$\alpha_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} F_{t-1}^2 \approx \tfrac{1}{2c} \Theta^2.$$

*If $T/k_h \to \pi$ $(0 < \pi < \infty)$, then*

$$\alpha_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} F_{t-1}^2 \approx \tfrac{1}{2c} \left( \tfrac{F_T}{\sqrt{T}} \sqrt{\pi} + \Theta \right)^2.$$

*If $T/k_h \to \infty$, then*

$$\alpha_h^{-2h} k_h^{-1} T^{-1} \sum_{t=T+1}^{T+h} F_{t-1}^2 \approx \tfrac{1}{2c} \left( \tfrac{F_T}{\sqrt{T}} \right)^2.$$

*(b) Suppose $c_i > 0$.*

*If $T/k_h \to 0$, then*

$$\rho_{i,h}^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^2 \approx \tfrac{1}{2c_i} \Theta_i^2.$$

*If $T/k_h \to \pi$ $(0 < \pi < \infty)$, then*

$$\rho_{i,h}^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^2 \approx \tfrac{1}{2c_i} \left( \tfrac{U_{i,T}}{\sqrt{T}} \sqrt{\pi} + \Theta_i \right)^2.$$

*If $T/k_h \to \infty$, then*

$$\rho_{i,h}^{-2h} k_h^{-1} T^{-1} \sum_{t=T+1}^{T+h} U_{i,t-1}^2 \approx \tfrac{1}{2c_i} \left( \tfrac{U_{i,T}}{\sqrt{T}} \right)^2.$$

**Proof of Lemma B.15.**

(a) I take squares of both sides of $F_t = \alpha_T F_{t-1} + e_t$ to obtain

$$\begin{aligned} F_t^2 &= \alpha_h^2 F_{t-1}^2 + 2\alpha_h F_{t-1} e_t + e_t^2, \\ (\alpha_h^2 - 1) F_{t-1}^2 &= F_t^2 - F_{t-1}^2 - 2\alpha_h F_{t-1} e_t - e_t^2. \end{aligned}$$

I then take summations over $t = T+1, ..., T+h$ to obtain

$$\begin{aligned} (\alpha_h^2 - 1) \textstyle\sum_{t=T+1}^{T+h} F_{t-1}^2 &= F_{T+h}^2 - F_T^2 - \textstyle\sum_{t=T+1}^{T+h} e_t^2 - 2\alpha_h \sum_{t=T+1}^{T+h} F_{t-1} e_t, \\ \textstyle\sum_{t=T+1}^{T+h} F_{t-1}^2 &= \frac{1}{\alpha_h^2 - 1} \left\{ F_{T+h}^2 - F_T^2 - \textstyle\sum_{t=T+1}^{T+h} e_t^2 - 2\alpha_h \sum_{t=T+1}^{T+h} F_{t-1} e_t \right\}, \\ \alpha_h^{-2h} k_h^{-2} \textstyle\sum_{t=T+1}^{T+h} F_{t-1}^2 &= \frac{1}{k_h^2(\alpha_h^2 - 1)} \left\{ \alpha_h^{-2h}(F_{T+h}^2 - F_T^2) - \alpha_h^{-2h} \textstyle\sum_{t=T+1}^{T+h} e_t^2 \right. \\ &\quad \left. -2\alpha_h^{-2h+1} \textstyle\sum_{t=T+1}^{T+h} F_{t-1} e_t \right\}, \\ &= \frac{1}{k_h(\alpha_h^2 - 1)} \left\{ \frac{\alpha_h^{-2h}}{k_h}(F_{T+h}^2 - F_T^2) - \frac{\alpha_h^{-2h}}{k_h} \textstyle\sum_{t=T+1}^{T+h} e_t^2 \right. \\ &\quad \left. - \frac{2\alpha_h^{-2h+1}}{k_h} \textstyle\sum_{t=T+1}^{T+h} F_{t-1} e_t \right\}, \\ &= \frac{1}{k_h(\alpha_h^2 - 1)} \left\{ I - II - III \right\}. \end{aligned}$$

I now consider terms $II$, $III$, and $I$ in order. For $II$,

$$\frac{\alpha_h^{-2h}}{k_h} \textstyle\sum_{t=T+1}^{T+h} e_t^2 = \left( \alpha_h^{-2h} \frac{h}{k_h} \right) \left( \frac{1}{h} \sum_{t=T+1}^{T+h} e_t^2 \right) = o(1) \times O_p(1) = o_p(1),$$

by using Proposition A.1 of Phillips and Magdalinos (2007) $\alpha_h^{-2h} = o(k_h^2 h^{-2})$ for the first component and the weak law of large numbers for the second component.

For $III$, by plugging

$$F_{t-1} = \alpha_h^{t-T-1} F_T + \textstyle\sum_{j=1}^{t-T-j-1} \alpha_h^{t-T-j-1} e_{T+j},$$

in $III$ (divided by 2) yields

$$\begin{aligned}
\frac{\alpha_h^{-2h+1}}{k_h}\sum_{t=T+1}^{T+h}F_{t-1}e_t &= \frac{\alpha_h^{-2h+1}}{k_h}\sum_{t=T+1}^{T+h}\left(\sum_{j=1}^{t-T-1}\alpha_h^{t-T-j-1}e_{T+j}\right)e_t \\
&\quad +\frac{1}{k_h}F_T\alpha_h^{-2h+1}\sum_{t=T+1}^{T+h}\alpha_h^{t-T-1}e_t, \\
&= \frac{\alpha_h^{-2h}}{k_h}\sum_{t=T+1}^{T+h}\left(\sum_{j=1}^{t-T-1}\alpha_h^{t-T-j}e_{T+j}\right)e_t+\tfrac{1}{k_h}F_T\alpha_h^{-h}\sum_{t=T+1}^{T+h}\alpha_h^{t-T-h}e_t, \\
&= \frac{\alpha_h^{-2h}}{k_h}\sum_{t=T+1}^{T+h}\left(\sum_{j=1}^{t-T-1}\alpha_h^{t-T-j}e_{T+j}\right)e_t \\
&\quad +\underbrace{\left(\frac{F_T}{\sqrt{T}}\right)}_{=O_p(1)}\underbrace{\left(\sqrt{\frac{T}{k_h}}\alpha_h^{-h}\right)}_{=o(T^{1/2}h^{-1/2})}\underbrace{\left(\frac{1}{\sqrt{k_h}}\sum_{t=T+1}^{T+h}\alpha_h^{t-T-h}e_t\right)}_{=O_p(1)}, \\
&= \frac{\alpha_h^{-2h}}{k_h}\sum_{t=T+1}^{T+h}\left(\sum_{j=1}^{t-T-1}\alpha_T^{t-T-j}e_{T+j}\right)e_t+o_p\left(T^{1/2}h^{-1/2}\right),
\end{aligned}$$

because $\sqrt{\frac{T}{k_h}}\alpha_h^{-h}=\sqrt{\frac{T}{k_h}}\times o\left(k_hh^{-1}\right)=o\left(T^{1/2}k_h^{1/2}h^{-1}\right)=o\left(T^{1/2}h^{-1/2}\right)$. For $\frac{1}{\sqrt{k_h}}\sum_{t=T+1}^{T+h}\alpha_h^{t-T-h}e_t=O_p(1)$, I used Lemma 4.2 of Phillips and Magdalinos (2007). In addition, I can show

$$\frac{\alpha_h^{-2h}}{k_h}\sum_{t=T+1}^{T+h}\left(\sum_{j=1}^{t-T-h}\alpha_h^{t-T-j}e_{T+j}\right)e_t=O_p(\alpha_h^{-h})=o_p(1),$$

by following Phillips and Magdalinos (2007).

Finally, I consider term $I$ as follows.

$$\begin{aligned}
\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}F_{t-1}^2 &= \frac{1}{k_h(\alpha_h^2-1)}\left\{\frac{\alpha_h^{-2h}}{k_h}(F_{T+h}^2-F_T^2)+o_p\left(T^{1/2}h^{-1/2}\right)\right\}, \\
&= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{\alpha_h^{-2h}}{k_h}\right)F_{T+h}^2 \\
&\quad -\frac{1}{k_h(\alpha_h^2-1)}\left(\alpha_h^{-2h}\frac{T}{k_h}\right)\frac{F_T^2}{T}+o_p\left(T^{1/2}h^{-1/2}\right),
\end{aligned}$$

because $k_h(\alpha_h^2-1)\to 2c$. But the second term is $o_p(Th^{-1})$ because $\alpha_h^{-2h}=o(k_hh^{-1})$ so that

$\alpha_h^{-2h}(T/k_h) = o\left(Th^{-1}\right)$ and $F_T^2/T = O_p(1)$. Furthermore,

$$\begin{aligned}
\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^2 &= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{\alpha_h^{-2h}}{k_h}\right)F_{T+h}^2 + o_p\left(Th^{-1}\right),\\
&= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{\alpha_h^{-2h}}{k_h}\right)\left(\alpha_h^h F_T + \textstyle\sum_{j=1}^h \alpha_h^{h-j}e_{T+j}\right)^2 + o_p\left(Th^{-1}\right),\\
&= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{1}{k_h}\right)\left(F_T + \textstyle\sum_{j=1}^h \alpha_h^{-j}e_{T+j}\right)^2 + o_p\left(Th^{-1}\right),\\
&= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{F_T}{\sqrt{T}}\sqrt{\frac{T}{k_h}} + \frac{1}{\sqrt{k_h}}\textstyle\sum_{j=1}^h \alpha_h^{-j}e_{T+j}\right)^2 + o_p\left(Th^{-1}\right).
\end{aligned}$$

Therefore, if $T/k_h \to 0$, then $T/h \to 0$ by Assumption 2.5 and

$$\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^2 \approx \frac{1}{2c}\left(\frac{1}{\sqrt{k_n}}\textstyle\sum_{j=1}^h \alpha_h^{-j}e_{T+j}\right)^2 \Rightarrow \frac{1}{2c}\Theta^2,$$

by Lemma B.5 (a). If $T/k_h \to \pi$ $(0 < \pi < \infty)$, then $T/h \to 0$ and

$$\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^2 \approx \frac{1}{2c}\left(\frac{F_T}{\sqrt{T}}\sqrt{\pi} + \Theta\right)^2.$$

If $T/k_h \to \infty$, then

$$\begin{aligned}
\alpha_h^{-2h}k_h^{-1}T^{-1}\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^2 &= \frac{1}{k_h(\alpha_h^2-1)}\frac{k_h}{T}\left(\frac{F_T}{\sqrt{T}}\sqrt{\frac{T}{k_h}} + \frac{1}{\sqrt{k_h}}\textstyle\sum_{j=1}^h \alpha_h^{-j}e_{T+j}\right)^2 + o_p\left(k_h h^{-1}\right),\\
&= \frac{1}{k_h(\alpha_h^2-1)}\left(\frac{F_T}{\sqrt{T}} + \underbrace{\sqrt{\frac{k_h}{T}}}_{=o(1)}\underbrace{\frac{1}{\sqrt{k_h}}\textstyle\sum_{j=1}^h \alpha_h^{-j}e_{T+j}}_{=O_p(1)\text{ by Lemma A5(a)}}\right)^2 + o_p\left(k_h h^{-1}\right),\\
&\approx \frac{1}{2c}\left(\frac{F_T}{\sqrt{T}}\right)^2.
\end{aligned}$$

(b) I follow the same steps as above by replacing $F_t^c$ and $F_t$ with $U_{i,t}^c$ and $U_{i,t}$ to show the results. Hence, the proof is condensed. ■

I now provide a proof for the asymptotic properties of the CS tests under the LTU framework (Theorem SA-2) provided in Appendix B.1 and under the MLTU framework (Theorem 2.2) presented in Section 2.4.

**Proof of Theorem SA-2.**

(i-a) The $t$ test statistic is

$$t^*_{\tilde{F}} = \frac{h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}\tilde{f}_t}{\hat{\sigma}\left(h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}^2_{t-1}\right)^{1/2}}.$$

The numerator is

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}\tilde{f}_t &= \frac{c}{h^2}\sum_{t=T+1}^{T+h}\tilde{F}^2_{t-1} + h^{-1}H^2\sum_{t=T+1}^{T+h}F_{t-1}e_t \\ &\quad -\frac{c}{h^2}\sum_{t=T+1}^{T+h}(\tilde{F}^2_{t-1} - H^2F^2_{t-1}) \\ &\quad +h^{-1}H\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1} - HF_{t-1})f_t \\ &\quad +h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}(\tilde{f}_t - Hf_t), \\ &= \frac{c}{h^2}\sum_{t=T+1}^{T+h}\tilde{F}^2_{t-1} + I + II + III + IV, \end{aligned}$$

but $I = o_p(1)$. Further, $II$, $III$, and $IV$ are shown to be $o_p(1)$ by using Lemma B.6 (q), (m), and (o) because $\alpha^h_h, \rho^h_h = O(1)$ when $k_h = h$. For the denominator,

$$\begin{aligned} h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}^2_{t-1} &= h^{-2}H^2\sum_{t=T+1}^{T+h}F^2_{t-1} + O_p\left(\frac{1}{\min\{N^2,T\}}\right) + O_p\left(\frac{\rho^{2h}_h}{\min\{N,T\}}\right) \\ &\quad +O_p\left(\frac{1}{\min\left\{N,T^{1/2}\right\}}\right) + O_p\left(\frac{\rho^h_h}{\min\left\{N^{1/2},T^{1/2}\right\}}\right) = o_p(1), \end{aligned}$$

by using Lemma B.6 (q). The consistency of $\hat{\sigma}$ is shown in Lemma B.7 because under $k_h = h$, $\frac{\alpha^h_h\rho^h_h h^{1/2}k_h^{-1/2}}{\min\left\{N^{1/2},T^{1/2}\right\}} = o(1)$. Therefore,

$$t^*_{\tilde{F}} = c\left(h^{-2}\sum_{t=T+1}^{T+h}F^2_{t-1}\right)^{1/2} + \frac{h^{-1}\sum_{t=T+1}^{T+h}F_{t-1}e_t}{\sigma\left(h^{-2}\sum_{t=T+1}^{T+h}F^2_{t-1}\right)^{1/2}} + o_p(1),$$

which leads to the result.

(i-b) The $t$ test statistic is

$$t^*_{\tilde{U}}(i) = \frac{h^{-1}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}\tilde{u}_{i,t}}{\hat{\sigma}_i\left(h^{-2}\sum_{t=T+1}^{T+h}\tilde{U}^2_{i,t-1}\right)^{1/2}}.$$

The numerator is

$$\begin{aligned} h^{-1}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}\tilde{u}_{i,t} &= \frac{c_i}{h^2}\sum_{t=T+1}^{T+h}\tilde{U}^2_{i,t-1} + h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}z_{i,t} - \frac{c_i}{h^2}\sum_{t=T+1}^{T+h}(\tilde{U}^2_{i,t-1} - U^2_{i,t-1}) \\ &\quad + h^{-1}\sum_{t=T+1}^{T+h}(\tilde{U}_{i,t-1} - U_{i,t-1})u_{i,t} + h^{-1}\sum_{t=T+1}^{T+h}\tilde{U}_{i,t-1}(\tilde{u}_{i,t} - u_{i,t}), \\ &= \frac{c}{h^2}\sum_{t=T+1}^{T+h}\tilde{U}^2_{i,t-1} + I + II + III + IV, \end{aligned}$$

but $I = o_p(1)$. Further, $II$, $III$, and $IV$ are shown to be $o_p(1)$ by using Lemma B.8 (m), (i), and (k). For the denominator, under $k_h = h$

$$\begin{aligned} h^{-2}\sum_{t=T+1}^{T+h}\tilde{U}^2_{i,t-1} &= h^{-2}\sum_{t=T+1}^{T+h}U^2_{i,t-1} + O_p\left(\frac{\alpha_h^{2h}}{\min\{N^2,T\}}\right) \\ &\quad + O_p\left(\frac{1}{\min\{N,T\}}\right) + O_p\left(\frac{\alpha_h^h}{\min\left\{N,T^{1/2}\right\}}\right) \\ &\quad + O_p\left(\frac{1}{\min\left\{N^{1/2},T^{1/2}\right\}}\right), \end{aligned}$$

by using Lemma B.8 (m) and the four terms of the factor estimation errors are $o_p(1)$. The consistency of $\hat{\sigma}_i$ is shown in Lemma B.9 because under $k_h = h$, $\frac{\alpha_h^h h k_h^{-1}}{\min\left\{N,T^{1/2}\right\}} = o_p(1)$ and $\frac{\rho_h^h h k_h^{-1}}{\min\left\{N^{1/2},T^{1/2}\right\}} = o_p(1)$. Therefore,

$$t^*_{\tilde{U}}(i) = c_i\left(h^{-2}\sum_{t=T+1}^{T+h}U^2_{i,t-1}\right)^{1/2} + \frac{h^{-1}\sum_{t=T+1}^{T+h}U_{i,t-1}z_{i,t}}{\sigma_i\left(h^{-2}\sum_{t=T+1}^{T+h}U^2_{i,t-1}\right)^{1/2}} + o_p(1),$$

which leads to the result.

(ii-a) The result is directly shown from (i-a) by using Lemmas B.11 and B.12 instead of Lemmas B.6 and B.7.

(ii-b) The result is directly shown from (i-b) by using Lemmas B.13 and B.14 instead of Lemmas B.8 and B.9. ■

**Proof of Theorem 2.2.**

(a) When $c = 0$, the $t$ test statistic is

$$\bar{t}^*_{\tilde{F}} = \frac{h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}^c_{t-1}\tilde{f}^c_t}{\hat{\sigma}\left(h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}^{c2}_{t-1}\right)^{1/2}}.$$

The numerator is

$$\begin{aligned}
h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}^c_{t-1}\tilde{f}^c_t &= h^{-1}H^2\sum_{t=T+1}^{T+h}F^c_{t-1}e_t + h^{-1}H\sum_{t=T+1}^{T+h}(\tilde{F}^c_{t-1} - HF^c_{t-1})e_t \\
&\quad + h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}^c_{t-1}(\tilde{f}^c_t - Hf^c_t), \\
&= h^{-1}H^2\sum_{t=T+1}^{T+h}F^c_{t-1}e_t + O_p\left(\frac{h^{-1/2}k_h^{1/2}}{\min\{N,T^{1/2}\}}\right) \\
&\quad + O_p\left(\frac{\rho_h^h h^{-1/2}k_h^{1/2}}{\min\{N^{1/2},T^{1/2}\}}\right) + O_p\left(\frac{1}{\min\{N,T^{1/2}\}}\right) \\
&\quad + O_p\left(\frac{\rho_h^h}{\min\{N^{1/2},T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^{2h}}{\min\{N,T\}}\right),
\end{aligned}$$

by using Lemma B.11 (g) and (o) and the five terms of the factor estimation errors are $o_p(1)$ under the stated condition. For the denominator,

$$\begin{aligned}
h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}^{c2}_{t-1} &= h^{-2}H^2\sum_{t=T+1}^{T+h}F^{c2}_{t-1} + O_p\left(\frac{h^{-1}k_h}{\min\{N^2,T\}}\right) + O_p\left(\frac{\rho_h^{2h}h^{-1}k_h}{\min\{N,T\}}\right) \\
&\quad + O_p\left(\frac{h^{-1}k_h}{\min\{N,T^{1/2}\}}\right) + O_p\left(\frac{\rho_h^h h^{-1}k_h}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}$$

by using Lemma B.11 (q) and the four terms of the factor estimation errors are $o_p(1)$ under the stated condition. The consistency of $\hat{\sigma}$ is shown in Lemma B.12 under the same condition. Therefore,

$$\bar{t}^*_{\tilde{F}} = \frac{h^{-1}\sum_{t=T+1}^{T+h}F^c_{t-1}e_t}{\sigma\left(h^{-2}\sum_{t=T+1}^{T+h}F^{c2}_{t-1}\right)^{1/2}} + o_p(1),$$

which leads to the result.

When $c > 0$, the $t$ test statistic is

$$t^*_{\tilde{F}} = \frac{k_h^{-1}\sum_{t=T+1}^{T+h}\tilde{F}^c_{t-1}\tilde{f}^c_t}{\hat{\sigma}\left(k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}^{c2}_{t-1}\right)^{1/2}}.$$

The numerator is

$$
\begin{aligned}
k_h^{-1}\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c}\tilde{f}_t^{c} &= \frac{c}{k_h^2}\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2}+k_h^{-1}H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_t \\
&\quad -\frac{c}{k_h^2}\textstyle\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^{c2}-H^2F_{t-1}^{c2}) \\
&\quad +k_h^{-1}H\textstyle\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^{c}-HF_{t-1}^{c})f_t \\
&\quad +k_h^{-1}\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c}(\tilde{f}_t^{c}-Hf_t^{c}), \\
&= \frac{c}{k_h^2}\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2}+I+II+III+IV.
\end{aligned}
$$

Therefore, if I scale the $t$ test by $\alpha_h^{-h}$

$$
\begin{aligned}
\alpha_h^{-h}t_{\tilde{F}}^{*} &= \frac{c}{\hat{\sigma}}\left(\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2}\right)^{1/2} \\
&\quad +\frac{\alpha_h^{-2h}}{\hat{\sigma}\left(\alpha_h^{-2h}k_h^{-2}\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2}\right)^{1/2}}(I+II+III+IV).
\end{aligned}
$$

I now show that the first term is asymptotically equal to a positive value or diverges to positive infinity and the second term disappears. The first term is

$$
\begin{aligned}
\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h}\tilde{F}_{t-1}^{c2} &= \alpha_h^{-2h}k_h^{-2}H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2}+\alpha_h^{-2h}k_h^{-2}\textstyle\sum_{t=T+1}^{T+h}(\tilde{F}_{t-1}^{c2}-H^2F_{t-1}^{c2}), \\
&= \alpha_h^{-2h}k_h^{-2}H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c2}+O_p\left(\frac{hk_h^{-1}}{\min\{N^2,T\}}\right) \\
&\quad +O_p\left(\frac{\alpha_h^{-2h}\rho_h^{2h}hk_h^{-1}}{\min\{N,T\}}\right)+O_p\left(\frac{hk_h^{-1}}{\min\{N,T^{1/2}\}}\right) \\
&\quad +O_p\left(\frac{\alpha_h^{-h}\rho_h^{h}hk_h^{-1}}{\min\{N^{1/2},T^{1/2}\}}\right),
\end{aligned}
$$

by using Lemma B.11 (q) and the four terms of the factor estimation errors are $o_p(1)$ under the stated condition. I next consider the second term.

$$
\alpha_h^{-2h}\times I=\alpha_h^{-2h}k_h^{-1}H^2\textstyle\sum_{t=T+1}^{T+h}F_{t-1}^{c}e_t=o_p(1),
$$

by using Lemma B.5 (c),

$$\alpha_h^{-2h} \times II = \frac{c\alpha_h^{-2h}}{k_h^2} \sum_{t=T+1}^{T+h} (\tilde{F}_{t-1}^{c2} - H^2 F_{t-1}^{c2}) = o_p(1),$$

as shown in the first term,

$$\begin{aligned} \alpha_h^{-2h} \times III &= \alpha_h^{-2h} k_h^{-1} H \sum_{t=T+1}^{T+h} (\tilde{F}_{t-1}^c - H F_{t-1}^c) f_t^c, \\ &= O_p\left(\frac{h k_h^{-1}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^{-h} \rho_h^h h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}}\right), \end{aligned}$$

by using Lemma B.11 (m) and it is $o_p(1)$ under the stated condition, and

$$\begin{aligned} \alpha_h^{-2h} \times IV &= \alpha_h^{-2h} k_h^{-1} \sum_{t=T+1}^{T+h} \tilde{F}_{t-1}^c (\tilde{f}_t^c - H f_t^c), \\ &= O_p\left(\frac{h k_h^{-1}}{\min\{N, T^{1/2}\}}\right) + O_p\left(\frac{\alpha_h^{-h} \rho_h^h h k_h^{-1}}{\min\{N^{1/2}, T^{1/2}\}}\right) \\ &\quad + O_p\left(\frac{\alpha_h^{-2h} \rho_h^{2h} h k_h^{-1}}{\min\{N, T\}}\right), \end{aligned}$$

by using Lemma B.11 (o) and it is $o_p(1)$ under the stated condition. The consistency of $\hat{\sigma}$ is shown in Lemma B.12 under the stated condition. Therefore,

$$\alpha_h^{-h} \bar{t}_{\tilde{F}}^* = \frac{c}{\sigma} \left(\alpha_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} F_{t-1}^{c2}\right)^{1/2} + o_p(1),$$

under the stated conditions.

Finally,

$$\begin{aligned} \alpha_h^{-h} \bar{t}_{\tilde{F}}^* &= \frac{c}{\sigma} \left(\alpha_h^{-2h} k_h^{-2} H^2 \sum_{t=T+1}^{T+h} F_{t-1}^{c2}\right)^{1/2} + o_p(1), \\ &= \frac{c}{\sigma} \left(\alpha_h^{-2h} k_h^{-2} H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2 - \alpha_h^{-2h} k_h^{-2} h H^2 \bar{F}^2\right)^{1/2} + o_p(1), \end{aligned} \quad \text{(B.4.1)}$$

but because $\bar{F}^2 = O_p(k_h^2 h^{-1}) + O_p(\alpha_h^{2h} k_h^3 h^{-2})$ from Lemma B.5 (b) and $k_h h^{-1} = o_p(1)$,

$$\alpha_h^{-2h} k_h^{-2} h H^2 \bar{F}^2 = O_p(\alpha_h^{-2h}) + O_p(k_h h^{-1}) = o_p(1).$$

By using Lemma B.15 (a), $\alpha_h^{-2h} k_h^{-2} H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2$ or $\alpha_h^{-2h} k_h^{-1} T^{-1} H^2 \sum_{t=T+1}^{T+h} F_{t-1}^2$ is asymptotically equal to the stated values. Plugging these into (B.4.1) yields the final results.

(b) When $c_i = 0$, the $t$ test statistic is

$$\bar{t}_{\tilde{U}}^*(i) = \frac{h^{-1} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^c \tilde{u}_{i,t}^c}{\hat{\sigma}_i \left( h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^{c2} \right)^{1/2}}.$$

The numerator is

$$\begin{aligned} h^{-1} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^c \tilde{u}_{i,t}^c &= h^{-1} \sum_{t=T+1}^{T+h} U_{i,t-1}^c z_{i,t} + h^{-1} \sum_{t=T+1}^{T+h} (\tilde{U}_{i,t-1}^c - U_{i,t-1}^c) z_{i,t} \\ &\quad + h^{-1} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^c (\tilde{u}_{i,t}^c - u_{i,t}^c), \\ &= h^{-1} \sum_{t=T+1}^{T+h} U_{i,t-1}^c z_{i,t} + O_p\left( \frac{\alpha_h^h h^{-1/2} k_h^{1/2}}{\min\{N, T^{1/2}\}} \right) \\ &\quad + O_p\left( \frac{\rho_h^h h^{-1/2} k_h^{1/2}}{\min\{N^{1/2}, T^{1/2}\}} \right) + O_p\left( \frac{\alpha_h^{2h}}{\min\{N^2, T\}} \right) \\ &\quad + O_p\left( \frac{\rho_h^{2h}}{\min\{N^{1/2}, T^{1/2}\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h}{\min\{N, T^{1/2}\}} \right), \end{aligned}$$

by using Lemma B.13 (e) and (k) and the five terms of the factor estimation errors are $o_p(1)$ under the stated condition. For the denominator,

$$\begin{aligned} h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^{c2} &= h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^{c2} + O_p\left( \frac{\alpha_h^{2h} h^{-1} k_h}{\min\{N^2, T\}} \right) \\ &\quad + O_p\left( \frac{\rho_h^{2h} h^{-1} k_h}{\min\{N, T\}} \right) + O_p\left( \frac{\alpha_h^h \rho_h^h h^{-1} k_h}{\min\{N, T^{1/2}\}} \right) \\ &\quad + O_p\left( \frac{\rho_h^{2h} h^{-1} k_h}{\min\{N^{1/2}, T^{1/2}\}} \right), \end{aligned}$$

by using Lemma B.13 (m) and the four terms of the factor estimation errors are $o_p(1)$ under the stated condition. The consistency of $\hat{\sigma}_i$ is shown in Lemma B.14 under the same condition. Therefore,

$$\bar{t}_{\tilde{U}}^*(i) = \frac{h^{-1} \sum_{t=T+1}^{T+h} U_{i,t-1}^c z_{i,t}}{\sigma_i \left( h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^{c2} \right)^{1/2}} + o_p(1),$$

which leads to the result.

When $c_i > 0$, the $t$ test statistic is

$$\bar{t}^*_{\tilde{U}}(i) = \frac{k_h^{-1} \sum_{t=T+1}^{T+h} \tilde{U}^c_{i,t-1} \tilde{u}^c_{i,t}}{\hat{\sigma}_i \left( k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}^{c2}_{i,t-1} \right)^{1/2}}.$$

The numerator is

$$\begin{aligned} k_h^{-1} \sum_{t=T+1}^{T+h} \tilde{U}^c_{i,t-1} \tilde{u}^c_{i,t} &= \frac{c_i}{k_h^2} \sum_{t=T+1}^{T+h} \tilde{U}^{c2}_{i,t-1} + k_h^{-1} \sum_{t=T+1}^{T+h} U^c_{i,t-1} z_{i,t} - \frac{c_i}{k_h^2} \sum_{t=T+1}^{T+h} (\tilde{U}^{c2}_{i,t-1} - U^{c2}_{i,t-1}) \\ &\quad + k_h^{-1} \sum_{t=T+1}^{T+h} (\tilde{U}^c_{i,t-1} - U^c_{i,t-1}) u_{i,t} + k_h^{-1} \sum_{t=T+1}^{T+h} \tilde{U}^c_{i,t-1} (\tilde{u}^c_{i,t} - u^c_{i,t}), \\ &= \frac{c_i}{k_h^2} \sum_{t=T+1}^{T+h} \tilde{U}^{c2}_{i,t-1} + I + II + III + IV. \end{aligned}$$

Therefore, if I scale the $t$ test by $\rho_h^{-h}$, then

$$\begin{aligned} \rho_h^{-h} \bar{t}^*_{\tilde{U}}(i) &= \frac{c_i}{\hat{\sigma}_i} \left( \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}^{c2}_{i,t-1} \right)^{1/2} \\ &\quad + \frac{\rho_h^{-2h}}{\hat{\sigma}_i \left( \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}^{c2}_{i,t-1} \right)^{1/2}} (I + II + III + IV). \end{aligned}$$

I now show that the first term is equal to a positive value or diverges to positive infinity and the second term disappears. First,

$$\begin{aligned} \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} \tilde{U}^2_{i,t-1} &= \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U^2_{i,t-1} + \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} (\tilde{U}^2_{i,t-1} - U^2_{i,t-1}), \\ &= \rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U^2_{i,t-1} + O_p\left( \frac{\alpha_h^{2h} \rho_h^{-2h} h k_h^{-1}}{\min\{N^2, T\}} \right) + O_p\left( \frac{h k_h^{-1}}{\min\{N, T\}} \right) \\ &\quad + O_p\left( \frac{\alpha_h^h \rho_h^{-h} h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}} \right) + O_p\left( \frac{h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}} \right), \end{aligned}$$

by using Lemma B.13 (m) and the last four terms of the factor estimation errors are $o_p(1)$ under $\frac{\alpha_h^h h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}} \to 0$. I next consider the second term.

$$\rho_h^{-2h} \times I = \rho_h^{-2h} k_h^{-1} \sum_{t=T+1}^{T+h} U^c_{i,t-1} z_{i,t} = o_p(1),$$

by using Lemma B.5 (h),

$$\rho_h^{-2h} \times II = \frac{c_i \rho_h^{-2h}}{k_h^2} \sum_{t=T+1}^{T+h} (\tilde{U}_{i,t-1}^{c2} - U_{i,t-1}^{c2}) = o_p(1),$$

as shown in the first term,

$$\begin{aligned}
\rho_h^{-2h} \times III &= \rho_h^{-2h} k_h^{-1} \sum_{t=T+1}^{T+h} (\tilde{U}_{i,t-1}^{c} - U_{i,t-1}^{c}) u_{i,t}, \\
&= O_p\left(\frac{\alpha_h^h \rho_h^{-h} h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}}\right) + O_p\left(\frac{h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right),
\end{aligned}$$

by using Lemma B.13 (i) and it is $o_p(1)$ under the stated condition, and

$$\begin{aligned}
\rho_h^{-2h} \times IV &= \rho_h^{-2h} k_h^{-1} \sum_{t=T+1}^{T+h} \tilde{U}_{i,t-1}^{c} (\tilde{u}_{i,t} - u_{i,t}), \\
&= O_p\left(\frac{\alpha_h^{2h} \rho_h^{-2h} h k_h^{-1}}{\min\left\{N^2, T\right\}}\right) + O_p\left(\frac{h k_h^{-1}}{\min\left\{N^{1/2}, T^{1/2}\right\}}\right) \\
&\quad + O_p\left(\frac{\alpha_h^h \rho_h^{-h} h k_h^{-1}}{\min\left\{N, T^{1/2}\right\}}\right),
\end{aligned}$$

by using Lemma B.13 (k) and it is $o_p(1)$ under the stated condition. The consistency of $\hat{\sigma}_i$ is shown in Lemma B.14 under the stated condition. Therefore,

$$\rho_h^{-h} \bar{t}_{\tilde{U}}^*(i) = \frac{c_i}{\sigma_i} \left(\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^{c2}\right)^{1/2} + o_p(1).$$

Finally,

$$\begin{aligned}
\rho_h^{-h} \bar{t}_{\tilde{U}}^*(i) &= \frac{c_i}{\sigma_i} \left(\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{i,t-1}^{c2}\right)^{1/2} + o_p(1), \\
&= \frac{c_i}{\sigma_i} \left(\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{t-1}^{2} - \rho_h^{-2h} k_h^{-2} h \bar{U}^2\right)^{1/2} + o_p(1), \qquad \text{(B.4.2)}
\end{aligned}$$

but because $\bar{U}^2 = O_p(k_h^2 h^{-1}) + O_p(\alpha_h^{2h} k_h^3 h^{-2})$ from Lemma B.5 (g) and $k_h h^{-1} = o_p(1)$,

$$\rho_h^{-2h} k_h^{-2} h \bar{U}^2 = O_p(\alpha_h^{-2h}) + O_p(k_h h^{-1}) = o_p(1).$$

By using Lemma B.15 (b), $\rho_h^{-2h} k_h^{-2} \sum_{t=T+1}^{T+h} U_{t-1}^2$ or $\rho_h^{-2h} k_h^{-1} T^{-1} \sum_{t=T+1}^{T+h} U_{t-1}^2$ is asymptotically equal to the stated values. Plugging these results into (B.4.2) yields the final results. ■