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The general equilibrium effects of localised technological progress: A Classical approach

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Abstract

We study the general equilibrium effects of localised technical progress à la Atkinson-Stiglitz on income distribution in economies in which capital is a vector of reproducible and heterogeneous goods. We show that there is no obvious relation between ex-ante profitable innovations and the income distribution that actually emerges in equilibrium. Unlike in the standard macroeconomic literature, localised technical progress may lead to indeterminacy in equilibrium factor prices, and individually rational choices of technique do not necessarily lead to optimal outcomes. Innovations may even cause the disappearance of *all* equilibria.

JEL: O33; D33; D51.

Keywords: localised technical progress, income distribution, general equilibrium.

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1 Introduction

The relation between technical change and the distribution of income and wealth, and in particular the effect of innovation on the dynamics of profitability, are central in a number of current debates, both in academia and in the popular press (such as those on widening inequalities and secular stagnation sparked, respectively, by Piketty [26] and Summers [33]). Yet, standard macroeconomic models provide only rather partial insights on the effects of innovations on profits, and on distribution, more generally.

As Acemoglu ([1], p.443) has forcefully argued, much of the literature has typically assumed, and still largely assumes, that "technological improvements could be viewed as increasing productivity at all factor proportions (in particular, at all combinations of capital and labour)". This implies ignoring that technical change is typically *localised*, in that it "improves the productivity of the techniques (or 'activities') currently being used and perhaps some similar techniques with neighbouring capital-labour ratios" (ibid.), and also *biased*, in that it has direct distributional implications – indeed, innovations are often introduced in order to economise on productive factors that become relatively scarce.

While there is now a substantial literature addressing biased, or directed, technical progress (see Acemoglu [1] and the contributions therein), localised technical change has received less attention. As Atkinson and Stiglitz ([2], p.573) noted in a classic paper, standard macroeconomics seems to "have forgotten the origins of the neo-classical production function: as the number of production processes increases (in an activity analysis model), the production possibilities can be more and more closely approximated by a smooth, differentiable curve. But the different points on the curve still represent different processes of production, and associated with each of these processes there will be certain technical knowledge specific to that technique ... [I]f one brings about a technological improvement in one of the blue-prints this may have little or no effect on the other blue-prints" (ibid.).

In this paper, we follow Atkinson and Stiglitz ([2]) and focus on localised technical change in an activity analysis model. Building on the classical framework developed by von Neumann [23] and Sraffa [32], and later extended by Roemer [27, 28, 29, 30], we explore the effect of localised innovations on profitability and distribution in a fully disaggregated dynamic general equilibrium model. Unlike in the standard macroeconomic literature (including the multi-sectoral extensions of the Ramsey growth model),[1] where capital is conceived as a single commodity, we model capital as a bundle of reproducible and heterogeneous commodities. Apart from being empirically more realistic, this assumption raises a number of interesting and complex issues, as is well known in capital theory (Sraffa [32]).

To be specific, we set up a dynamic general equilibrium model in which, in every period, agents exchange goods and services on a number of interrelated markets. Propertyless agents simply supply labour, while capitalists activate optimal production activities and adopt production techniques yielding the maximum rate of profit. Production takes time. Hence, capital and labour are traded at the beginning of each production period, while consumption goods are exchanged after production has taken place.

We adopt a classical view of the functioning of capitalism and assume the economy to be driven by an accumulation motive. A production technique, in this framework, is a blueprint describing how to combine a vector of produced inputs with labour in order

[1] See, among the many others, Kongsamut et al. [17], Ngai and Pissarides [24], and Boppart [5]. For a critical survey see Kurose [18].

to produce outputs. At the beginning of each production period, the production set consists of a set of known blueprints which agents can choose from in order to activate production. When innovations do emerge, they expand the production set by generating new blueprints that may be used at the beginning of the next production period.[2] The general equilibrium effects of this type of localised innovations is far from obvious.

Suppose the economy starts out on a balanced growth path with an equilibrium price vector and an optimally chosen technique that, absent any perturbations, would remain unchanged over time. In standard macroeconomic models, where technical progress is not localised, innovations lead to an inward shift of the isoquants and, under relatively mild conditions, the economy moves to a (unique) equilibrium in which the new technology is universally adopted, productive factors are fully employed, and a well defined functional distribution of income emerges, which depends on the bias of technical change.

A localised innovation, in contrast, is defined by the discovery of a new blueprint, or activity, namely a single vector of physical and labour inputs placed outside of the existing input-requirement set. Therefore, even if the original technology was smooth, localised technical change introduces a kink in the new input-requirement set. Further, even if they expand the production possibilities set, not all localised innovations are necessarily cost-reducing.

We define localised innovations as profitable, when they are cost-reducing at given equilibrium prices, and prove that *if* (i) a new equilibrium exists in which (ii) a new profitable technique is adopted and (iii) the wage rate remains unchanged, then profits tend to increase (Theorem 2).[3] Once assumptions (i)-(iii) are relaxed, however, our findings are much more nuanced and perhaps surprising.

First, if a new equilibrium with full employment of productive factors is reached, then the effect of innovation on distribution is a priori unclear, as there exist (infinitely) many profit rates and wage rates that can be supported in equilibrium. It is even possible for localised technical change not to be Pareto improving, as there are equilibria in which either the wage rate or the profit rate *decrease* compared to the equilibrium with the old technology (Theorem 3). The actual distributional outcome depends on the equilibrium selection mechanism.

This result is reminiscent of the well-known indeterminacy of the functional distribution of income in Sraffa's [32] system of production prices (for recent analyses of indeterminacy in Sraffian models, see Mandler [20] and Yoshihara and Kwak [37]). Nonetheless, the indeterminacy in Theorem 3 is quite different: it is the result of the equilibrium transition triggered by innovations, and it obtains under a more general equilibrium notion.

Second, more generally, the distributive implications of technical progress depend on the general equilibrium effects of technical change. Localised innovations may contribute to maintain labour unemployed, or even – as conjectured by Acemoglu [1] – *create* techno-

[2] As in Atkinson and Stiglitz ([2]), we assume that localised innovations are the result of learning by doing, and do not explicitly consider R&D activities and the process of generating innovations. This allows us to focus on the general equilibrium effects of innovation on both prices and income distribution.

[3] Theorem 2 is a generalisation of the so-called 'Okishio theorem'. Okishio [25] proved that if the real wage rate is fixed at the (historically and culturally determined) subsistence level, then any cost-reducing innovation will increase the equilibrium profit rate. This results has been interpreted as proving that the Marxian theory of the falling rate of profit is invalid and sparked substantial controversy and a vast literature (see, for example, Roemer [27, 28] and Franke [11], and the references therein). In this literature, cost-reducing innovations are assumed to be automatically and universally adopted, and their general equilibrium effects are largely ignored.

logical unemployment in equilibrium. In these cases, technical change leads to an increase in the equilibrium profit rate (Theorem 4).

This conclusion does not hold in general, though, as cost-reducing localised innovations may lead the profit rate to *fall*. If the new technique increases labour productivity while it makes the present capital stock abundant relative to the population, then its introduction drives the equilibrium profit rate to zero (Theorems 5 and 7). In line with Karl Marx's [22] famous intuition, an innovation that is profitable for individual capitalists at current prices yields, after it is universally adopted, a change in the equilibrium price vector – and consequently in individual behaviour – eventually leading the equilibrium profit rate to decrease. Indeed, the equilibrium profit rate falls (albeit not necessarily to zero) even if the new technique worsens labour productivity, though in this case the mechanism is subtler and less intuitive, as the innovation is *not* adopted: it has a pure general equilibrium effect leading to capitalists to opt for an older technique (Theorem 6).

Third, innovations can be highly disruptive and the process of 'creative destruction' is anything but smooth, as Schumpeter [31] emphasised. For there exist cost-reducing, capital-saving and labour-using innovations that destroy the existing equilibrium and yet are *not* adopted in the new equilibrium (Corollary 1). Innovations may paradoxically lead *older* techniques to become profitable again, due to changes in equilibrium prices (Theorem 6). Indeed, innovations may lead to the disappearance of equilibrium altogether: the process of creative destruction entails disequilibrium dynamics (Section 5.3).[4]

The rest of the paper is structured as follows. Sections 2 and 3 present the economy and the equilibrium concept. Section 4 introduces a taxonomy of innovations. Section 5 highlights some general equilibrium implications of localised technical progress. Section 6 characterises the conditions under which technical change leads to a falling profit rate. Section 7 concludes.

2 The economy

Consider a closed economy with n produced goods. We focus on process innovations and assume the set of commodities to be constant over time.

2.1 Technology, innovation, and knowledge

At the beginning of each production period $t = 1, 2, \ldots$, there is a finite set $\mathcal{B}_t$ of Leontief production techniques (A_t, L_t), where A_t is a $n \times n$ nonnegative, productive, and indecomposable matrix of material input coefficients, whose i-th column is denoted A_{it}, and $L_t = (L_{1t}, \ldots, L_{it}, \ldots, L_{nt}) > \mathbf{0} \equiv (0, \ldots, 0)$ is a $1 \times n$ vector of labour coefficients.[5] The set $\mathcal{B}_t$ contains the blueprints that can be used at t to produce the n goods and the set

[4]In his analysis of the choice of technique in linear economies with joint production, Bidard [3] defines an algorithm that ensures the convergence to an optimal technique in a given class of production sets. This suggests that, as in our model, outside of that class the algorithm may not converge leading to disequilibrium dynamics. The mechanism underlying the non-existence of equilibrium is, however, different: in our model joint production is ruled out and non-convergence derives instead from the interaction between individually optimal choice of technique and the general equilibrium effects of innovations.

[5]Vector inequalities: for all x, $y \in \mathbb{R}^n$, $x \geqq y$ if and only if $x_i \geqq y_i$ $(i = 1, \ldots, n)$; $x \geq y$ if and only if $x \geqq y$ and $x \neq y$; $x > y$ if and only if $x_i > y_i$ $(i = 1, \ldots, n)$.

of all conceivable production techniques at t, $\mathcal{P}_t$, is the convex hull of $\mathcal{B}_t$.[6]

The stock of knowledge does not depreciate: once a production technique is discovered, it remains available for agents to use. But knowledge can be accumulated. Formally, $\mathcal{B}_{t-1} \subseteq \mathcal{B}_t$ holds in general, and *technical progress* takes place between $t-1$ and t if and only if $\mathcal{B}_{t-1} \subset \mathcal{B}_t$ and $(A_t^*, L_t^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$: (A_t^*, L_t^*) is an *innovation*, which is available in t.[7] Because we are interested in the effects of innovation on profitability in competitive market economies, we suppose that information both about $\mathcal{B}_{t-1}$ and about any new technique $(A_t^*, L_t^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ is available to all agents in the economy in t.

At all t, for any $(A_t, L_t) \in \mathcal{B}_t$, let $\delta(A_t) = \frac{1}{1+\Pi(A_t)}$ denote the Frobenius eigenvalue of A_t. By the assumptions on $\mathcal{B}_t$, $\Pi(A_t) > 0$ and $\delta(A_t) < 1$.

2.2 Agents

We study some fundamental dynamic laws of capitalist economies characterised by a drive to accumulate, and assume that agents aim to maximise their wealth subject to reaching a minimum consumption standard.[8]

Let $\mathcal{N}_t = \{1, \ldots, N_t\}$ be the set of agents at t with generic element ν. At the beginning of t, each $\nu \in \mathcal{N}_t$ is endowed with a (possibly zero) vector of produced goods, $\omega_{t-1}^{\nu} \in \mathbb{R}_+^n$ and one unit of labour.[9]

We follow Roemer [29, 30] in making the time structure of production explicit: "Time is essential in production, in the sense that capitalists must pay today for inputs before revenues are received tomorrow." (Roemer [29], p.506) Moreover, "there is no financial capital market: capitalists are limited in the extent of their production by the level of internal finance. [Because] allowing capitalists to borrow from each other ... does not change the results reported here." (Roemer [29], p.506) Within each period of production, market exchanges take place at two points in time: at the beginning of the period, productive inputs are traded at prices $p_t^b \in \mathbb{R}_+^n$ and labour contracts are signed; at the end of the period, outputs are traded at prices $p_t \in \mathbb{R}_+^n$ and workers are paid the nominal wage $w_t \geq 0$.

At the beginning of every t, each $\nu \in \mathcal{N}_t$ must form expectations $(p_t^{e\nu}, w_t^{e\nu})$ about (p_t, w_t). Because agents have the same preferences and possess the same information, we shall assume them to have identical expectations and drop the superscript ν for simplicity: $(p_t^{e\nu}, w_t^{e\nu}) = (p_t^e, w_t^e)$ for all $\nu \in \mathcal{N}_t$.

Given $\left(p_t^b, p_t^e, w_t^e\right)$, at the beginning of each t, every agent $\nu \in \mathcal{N}_t$ chooses her labour supply, l_t^{ν}, and uses wealth, $W_t^{\nu} = p_t^b \omega_{t-1}^{\nu}$, either to buy goods δ_t^{ν} (spending $p_t^b \delta_t^{\nu}$) for sale at the end of the period or to finance production. In the latter case, each agent chooses a production technique, $(A_t^{\nu}, L_t^{\nu}) \in \mathcal{P}_t$, which is activated at level x_t^{ν} by investing (part of) W_t^{ν} to finance the operating costs of the activities she activates, $p_t^b A_t^{\nu} x_t^{\nu}$, and by hiring

[6] We follow the literature and focus on circulating capital but our analysis can be extended to classical models with fixed capital. See, for example, Roemer [28], Bidard [4], and Flaschel et al [10], and Kiedrowski [16].

[7] This is a generalisation of Jones's [15] model of 'ideas'. We assume that innovations are discovered in a given period, after production has begun, and can only be used in the following period. The assumption that only one new technique can emerge in a period is for simplicity and yields no loss of generality.

[8] The model is a dynamic extension of Roemer's [29, 30] accumulating economy with a labour market.

[9] In every period t, we take the distribution of endowments as exogenously given and abstract from all issues related to bequests and the endowment of newly born agents. This is without any loss of generality and none of our results depends on it.

workers $L_t^\nu x_t^\nu$, which are paid ex post the expected amount $w_t^e L_t^\nu x_t^\nu$.[10] Thus, expected gross revenue at the end of t is $p_t^e x_t^\nu + w_t^e l_t^\nu + p_t^e \delta_t^\nu$, which is used to pay wages and finance accumulation, $p_t^e \omega_t^\nu$, subject to purchasing a consumption bundle $b \in \mathbb{R}_+^n$, $b > \mathbf{0}$, per unit of labour performed.[11]

Let $\triangle \equiv \left\{p \in \mathbb{R}_+^n \mid pb = 1\right\}$. In every t, given $\left(p_t^b, p_t^e, w_t^e\right) \in \triangle^2 \times \mathbb{R}_+$, agents are assumed to choose (A_t^ν, L_t^ν), $\xi_t^\nu \equiv (x_t^\nu, l_t^\nu, \delta_t^\nu)$, and ω_t^ν to solve:[12]

$$MP_t^\nu : \max_{(A_t^\nu, L_t^\nu); \xi_t^\nu; \omega_t^\nu} p_t^e \omega_t^\nu$$

subject to

$$\begin{aligned} \left[p_t^e - w_t^e L_t^\nu\right] x_t^\nu + w_t^e l_t^\nu + p_t^e \delta_t^\nu &= p_t^e b l_t^\nu + p_t^e \omega_t^\nu & (1) \\ p_t^b A_t^\nu x_t^\nu + p_t^b \delta_t^\nu &= p_t^b \omega_{t-1}^\nu, & (2) \\ 0 &\leqq l_t^\nu \leqq 1, & (3) \\ (A_t^\nu, L_t^\nu) &\in \mathcal{P}_t; & (4) \\ x_t^\nu, \delta_t^\nu, \omega_t^\nu &\in \mathbb{R}_+^n. & (5) \end{aligned}$$

In other words, we focus on a temporary resource allocation problem whereby agents choose an optimal plan in each production period. The analysis of the transition process sparked by technical progress is developed in section 4 below, where we explicitly consider the change in production techniques occurring after the emergence of an innovation.

Finally, let $v_t \equiv L_t(I - A_t)^{-1}$ denote the standard vector of employment multipliers. In the rest of the paper, we assume that for all $(A_t, L_t) \in \mathcal{B}_t$, $1 > v_t b$ holds: this is a basic condition for the productiveness of the economy.

3 Equilibrium

An *accumulation economy at* t is described by the tuple $\mathcal{N}_t$, $\mathcal{B}_t$, b, and $\Omega_{t-1} \equiv \left(\omega_{t-1}^\nu\right)_{\nu \in \mathcal{N}_t} \in \mathbb{R}_+^{nN_t}$ and is denoted as $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Let $x_t \equiv \sum_{\nu \in \mathcal{N}_t} x_t^\nu$, and let a similar notation hold for δ_t, ω_t, and l_t. Similar to Roemer [29, 30], the equilibrium notion of this economy can be defined.

Definition 1 *A competitive equilibrium (CE) for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ *is a vector* $\left(p_t^b, p_t, w_t\right) \in \triangle^2 \times \mathbb{R}_+$ *and associated* $((A_t^\nu, L_t^\nu); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}$ *such that:*

(a) $((A_t^\nu, L_t^\nu); \xi_t^\nu; \omega_t^\nu)$ *solves* MP_t^ν*, for all* $\nu \in \mathcal{N}_t$ *(individual optimality);*

(b) $\sum_{\nu \in \mathcal{N}_t} A_t^\nu x_t^\nu + \delta_t \leqq \omega_{t-1}$ *(social feasibility of production);*

(c) $\sum_{\nu \in \mathcal{N}_t} L_t^\nu x_t^\nu = l_t$ *(labour market);*

(d) $x_t + \delta_t \geqq \sum_{\nu \in \mathcal{N}_t} b L_t^\nu x_t^\nu + \omega_t$ *with* $x_t > \mathbf{0}$ *(commodity markets);*

(e) $p_t = p_t^e = p_t^b$ *and* $w_t = w_t^e$ *(realised expectations).*

[10] The model can be extended to allow agents to operate production activities with their own capital as self-employed producers. Given the convexity of the optimisation programme MP_t^ν below, this makes no difference to our results.

[11] Given our analytical focus on the relation between technical change and profitability, we do not explicitly analyse the agents' consumption choices and treat b as a parameter. This is without significant loss of generality.

[12] Constraints (1) and (2) are written as equalities without loss of generality.

In other words, at a CE, (a) all agents optimise; (b) aggregate capital is sufficient for production plans; (c) the labour market is in equilibrium; (d) the total supply of commodities is sufficient for consumption and accumulation plans; and (e) agents' expectations are realised ex post, $(p^e_t, w^e_t) = (p_t, w_t)$. For the sake of notational simplicity, because at a CE expectations are realised and $p_t = p^b_t = p^e_t$, we shall write the price vector as $(p_t, w_t) \in \triangle \times \mathbb{R}_+$.

Several points should be noted about Definition 1. First, the concept of CE is a temporary equilibrium notion which focuses on each period in isolation. The dynamic evolution of the economy can thus be conceived of as a sequence of temporary equilibria. This is a natural choice given our focus on the general equilibrium effects of localised technical progress on distribution.[13] Second, it focuses on non-trivial allocations with a positive gross output vector, $x_t > \mathbf{0}$. This is without loss of generality because agents will optimally activate all sectors if the profit rate is positive; and even if the profit rate is zero, $x_t > \mathbf{0}$ can always be the product of optimal choices, consistent with such a CE.

Third, following Roemer [29, 30], we assume that agents have *stationary expectations* and suppose that beginning-of-period commodity prices will also rule at the end of the period, $p^e_t = p^b_t$. This is "a standard assumption of the temporary equilibrium literature" (Roemer 1980, p. 529) which can also be interpreted as imposing not overly demanding conditions on agents' rationality and foresight in expectation formation, consistent with a large literature on bounded rationality and behavioural economics.

It is now possible to derive some preliminary results. First of all, for all $(p_t, w_t) \in \triangle \times \mathbb{R}_+$ and $(A, L) \in \mathcal{P}_t$, let $\pi^{(p_t,w_t)}_{it}(A, L) \equiv \frac{p_{it} - p_t A_i - w_t L_i}{p_t A_i}$, $i = 1, \ldots, n$; and let $\pi^{(p_t,w_t)}_t(A, L) \equiv \max_{i=1,\ldots,n} \pi^{(p_t,w_t)}_{it}(A, L)$. It is immediate to prove that for any $(p_t, w_t) \in \triangle \times \mathbb{R}_+$, if $((A^\nu_t, L^\nu_t); \xi^\nu_t; \omega^\nu_t)$ solves MP^ν_t then (A^ν_t, L^ν_t) must yield the maximum profit rate: $\pi^{(p_t,w_t)}_t(A^\nu_t, L^\nu_t) = \pi^{\max}_t \equiv \max_{(A,L)\in\mathcal{P}_t} \pi^{(p_t,w_t)}_t(A, L)$.[14] It is immediate to show that for all $(p_t, w_t) \in \triangle \times \mathbb{R}_+$ such that $w_t \leqq p_t b$, $1 > v_t b$ implies $\pi^{(p_t,w_t)}_t(A_t, L_t) > 0$ and, a fortiori, $\pi^{\max}_t > 0$.

In principle, in equilibrium different production techniques may be in use. However, as they all yield the same (maximum) profit rate, we shall assume without loss of generality that all agents who activate some production process opt for the same (A_t, L_t), and drop the superscript ν.

Next, constraints (1)-(2) imply that at a CE the following equation holds:

$$p_t \omega^\nu_t = [p_t - p_t A_t - w_t L_t]\, x^\nu_t + (w_t - p_t b)\, l^\nu_t + p_t \omega^\nu_{t-1}, \forall \nu \in \mathcal{N}_t. \tag{6}$$

Then, Lemma 1 derives some properties of the optimal solution to MP^ν_t.[15]

Lemma 1 *Let $\left((p_t, w_t), ((A_t, L_t); \xi^\nu_t; \omega^\nu_t)_{\nu\in\mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. For all $\nu \in \mathcal{N}_t$: if $\pi^{\max}_t > 0$, then $p_t A_t x^\nu_t = p_t \omega^\nu_{t-1}$ and if $w_t > p_t b$, then $l^\nu_t = 1$.*

Lemma 2 proves that equilibrium prices are strictly positive and competition leads to the equalisation of sectoral profit rates in equilibrium.

Lemma 2 *Let $\left((p_t, w_t), ((A_t, L_t); \xi^\nu_t; \omega^\nu_t)_{\nu\in\mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Then $\pi^{\max}_t \geqq 0$, $p_t = (1 + \pi^{\max}_t)\, p_t A_t + w_t L_t$, $p_t > \mathbf{0}$, and $w_t \geqq p_t b$.*

[13] For an analysis of intertemporal general equilibrium from a classical perspective, see Dana et al [8], Veneziani [35], Freni et. al. [12], Galanis et al [13], and Takahashi [34].

[14] The maximum profit rate $\pi^{\max}_t$ is well defined as $\mathcal{B}_t$ is finite.

[15] The results in this section follow rather straightforwardly from MP^ν_t and Definition 1 and their proofs are therefore omitted. (See the Addendum.)

Lastly, Theorem 1 derives some key properties of competitive equilibria.[16]

Theorem 1 *Let* $\left((p_t, w_t), ((A_t, L_t); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ *be a CE for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.
(i) If $\pi_t^{\max} > 0$ *and* $w_t > p_t b$, *then* $N_t = l_t = L_t A_t^{-1} \omega_{t-1}$.
(ii) If $N_t > L_t A_t^{-1} \omega_{t-1}$, *then* $w_t = p_t b$;
(iii) If $N_t < L_t A_t^{-1} \omega_{t-1}$, *then* $\pi_t^{\max} = 0$.

Given an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with aggregate capital stocks $\omega_{t-1} \geq \mathbf{0}$, we define the set of activities (A, L) such that, given ω_{t-1}, all agents can reach subsistence b:

$$\mathcal{B}_t(\omega_{t-1}; b) \equiv \left\{(A, L) \in \mathcal{B}_t \mid A^{-1}\omega_{t-1} > \mathbf{0} \text{ and } (I - bL) A^{-1} \omega_{t-1} \geqq \mathbf{0}\right\}.$$

In other words, if $(A, L) \in \mathcal{B}_t(\omega_{t-1}; b)$ is adopted, then there exists a profile of actions $(x_t^\nu;\ l_t^\nu; \delta_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}$ satisfying Definition 1(b)-(d).

4 Technical progress and technical change

We are interested in the effects of innovations on distribution. To be precise, we focus on an *inter-period change of technique* from (A_{t-1}, L_{t-1}), which is chosen in period $t-1$, to a new technique (A^*, L^*) available in $\mathcal{B}_t \setminus \mathcal{B}_{t-1}$ due to technical progress, and its effect on the equilibrium distribution of income. We denote such inter-period change of technique as $(A_{t-1}, L_{t-1}) \mapsto (A^*, L^*)$.

In order to abstract from other factors – such as those related to the dynamics of productive inputs – we consider a subset of equilibria such that, *absent any innovation*, equilibrium prices would be invariant across two periods. Formally:

Definition 2 *Let* $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \xi_{t-1}^\nu; \omega_{t-1}^\nu\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ *be a CE for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. *Then, the CE is persistent (PCE) if and only if there exists a profile* $(\xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}$, *such that* $\left((p_{t-1}, w_{t-1}), ((A_{t-1}, L_{t-1}); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ *is a CE for* $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-1})$ *with* $\omega_{t-1} = \sum_{\nu \in \mathcal{N}_{t-1}} \omega_{t-1}^\nu$.

In other words, if a CE for the period $t-1$ economy is persistent, then neither equilibrium prices (p_{t-1}, w_{t-1}) nor the production technique (A_{t-1}, L_{t-1}) need to vary in period t, as long as no technical progress takes place between the two periods. Therefore, the notion of PCE is primarily an analytical device to examine the effect of technical progress in isolation, and it describes a possibly counterfactual allocation that would emerge at t if the economy had the same production set as at $t-1$, $\mathcal{B}_t = \mathcal{B}_{t-1}$.

Absent technical progress, the conditions for the persistence of a CE are not particularly strong, as they basically require capital accumulation to appropriately adjust to changes in demographic conditions. Formally:[17]

Proposition 1 *Let* $\left((p_{t-1}, w_{t-1}), \left((A_{t-1}, L_{t-1}); \xi_{t-1}^\nu; \omega_{t-1}^\nu\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ *be a CE for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. *Then,*

[16] Theorem 1 only provides necessary conditions for the existence of a CE: this is all we need for our analysis of technical change. A complete characterisation of the necessary and sufficient conditions for the existence of equilibrium can be found in the Addendum.

[17] Proposition 1 follows immediately from Theorem 1 and its proof is therefore omitted.

(i) if $\pi_{t-1}^{\max} > 0$ *and* $w_{t-1} > p_{t-1}b$*, then this CE is persistent if and only if* $A_{t-1}^{-1}\omega_{t-1} > \mathbf{0}$ *and* $N_t = L_{t-1}A_{t-1}^{-1}\omega_{t-1}$*;*
(ii) if $\pi_{t-1}^{\max} > 0$ *and* $w_{t-1} = p_{t-1}b$*, then this CE is persistent if and only if* $A_{t-1}^{-1}\omega_{t-1} > \mathbf{0}$ *and* $N_t \geqq L_{t-1}A_{t-1}^{-1}\omega_{t-1}$*;*
(iii) if $\pi_{t-1}^{\max} = 0$ *and* $w_{t-1} > p_{t-1}b$*, then this CE is persistent if and only if there exists* $\delta \geqq \mathbf{0}$ *such that* $A_{t-1}^{-1}(\omega_{t-1} - \delta) > \mathbf{0}$ *and* $N_t = L_{t-1}A_{t-1}^{-1}(\omega_{t-1} - \delta)$*.*

The concept of PCE allows us to analyse what may be thought of as Schumpeterian innovations: new techniques that create unforeseen profit opportunities, disrupt existing production processes, and cause fundamental shifts in the distribution of income.

To see this, suppose that the economy is at a PCE in period $t-1$ and technical progress occurs before productive inputs are bought and production starts in period t. Not all innovations $(A^*, L^*) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ alter incentives and lead agents to deviate from the PCE. If $\pi_t^{(p_{t-1},w_{t-1})}(A^*, L^*) \leqq \pi_{t-1}^{\max} = \pi_{t-1}^{(p_{t-1},w_{t-1})}(A_{t-1}, L_{t-1})$ holds, then (p_{t-1}, w_{t-1}) and (A_{t-1}, L_{t-1}) would still constitute a CE in period t. This motivates our focus on innovations that are profitable in the following sense:

Definition 3 *Let* $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu};\ \omega_{t-1}^{\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ *be a PCE for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$*.* $(A, L) \mapsto (A^*, L^*)$ *is profitable if and only if:*

$$\left(1 + \pi_{t-1}^{\max}\right) p_{t-1}A^* + w_{t-1}L^* \leq \left(1 + \pi_{t-1}^{\max}\right) p_{t-1}A + w_{t-1}L.$$

In other words, a new technique is profitable if at prices (p_{t-1}, w_{t-1}), a producer can expect *extra profits* by switching to it. Thus, Definition 3 characterises a necessary condition for the PCE to disappear.

Two points are worth noting about Definition 3. First, the premise that the economy is at a PCE in period $t-1$ is crucial. If the CE were not persistent, then Definition 3 would not capture a relevant condition for innovations to disrupt behaviour, as the economy may move to a different equilibrium because of demographic factors and/or due to capital accumulation. Similarly, the fact that the new technique would have been profitable at last period's prices would be immaterial for today's decisions.

Second, and most important for our analysis, Definition 3 does not tell us anything, a priori, about the effect of technical progress on distribution and profits. For, on the one hand, the condition in Definition 3 is not sufficient to guarantee that the new technique will be adopted at the new, generically different, equilibrium prices (p_t^*, w_t^*) in period t. On the other hand, even if the new technique (A^*, L^*) is indeed optimal at (p_t^*, w_t^*), $\pi_t^{*\max} = \pi_t^{(p_t^*,w_t^*)}(A^*, L^*)$ may be higher or lower than $\pi_{t-1}^{\max} = \pi_t^{(p_{t-1},w_{t-1})}(A_{t-1}, L_{t-1})$. Therefore it is unclear whether technical progress has a positive effect on profitability, as Schumpeter suggested, or rather it may drive the equilibrium profit rate to fall, as Marx argued.

In a seminal contribution, Okishio [25] proved that if the wage rate is fixed at the subsistence level, then the equilibrium profit rate always increases, thus casting doubts on the law of the falling rate of profit. Theorem 2 generalises the Okishio Theorem (OT).[18]

[18] Theorem 2 is more general than standard versions of OT in that we adopt a general equilibrium concept which implies, but does not reduce to, the equalisation of sectoral profit rates.

Theorem 2 *Let* $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ *be a PCE for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. *Let* $(A, L) \mapsto (A^*, L^*)$ *be profitable. If* $\left((p^*_t, w^*_t), ((A^*, L^*); \xi^{\nu}_t; \omega^{\nu}_t)_{\nu \in \mathcal{N}_t}\right)$ *is a CE for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ *with* $w^*_t = w_{t-1}$, *then* $\pi^{*\max}_t > \pi^{\max}_{t-1}$.

Proof: By Lemma 2, $p_{t-1} = \left(1 + \pi^{\max}_{t-1}\right) p_{t-1} A + w_{t-1} L$ and $p^*_t = (1 + \pi^{*\max}_t) p^*_t A^* + w_{t-1} L^*$. Since the change of technique is profitable, $p_{t-1} \geq \left(1 + \pi^{\max}_{t-1}\right) p_{t-1} A^* + w_{t-1} L^*$ holds. Then:

$$p_{t-1} = w_{t-1} L \left[I - \left(1 + \pi^{\max}_{t-1}\right) A\right]^{-1} \geq w_{t-1} L^* \left[I - \left(1 + \pi^{\max}_{t-1}\right) A^*\right]^{-1}. \tag{7}$$

where $\left[I - \left(1 + \pi^{\max}_{t-1}\right) A^*\right]^{-1} > \mathbf{0}$. Post-multiplying both sides of equation (7) by $b > \mathbf{0}$ and recalling that $p_{t-1} \in \Delta$, we obtain

$$1 = w_{t-1} L \left[I - \left(1 + \pi^{\max}_{t-1}\right) A\right]^{-1} b > w_{t-1} L^* \left[I - \left(1 + \pi^{\max}_{t-1}\right) A^*\right]^{-1} b.$$

Since $p^*_t = w^*_t L^* \left[I - (1 + \pi^{*\max}_t) A^*\right]^{-1}$, $w^*_t = w_{t-1}$, and $p^*_t \in \Delta$, we also have

$$1 = w_{t-1} L^* \left[I - (1 + \pi^{*\max}_t) A^*\right]^{-1} b > w_{t-1} L^* \left[I - \left(1 + \pi^{\max}_{t-1}\right) A^*\right]^{-1} b.$$

The result then follows noting that $L^* \left[I - (1 + \pi) A^*\right]^{-1}$ is increasing in π. ■

Theorem 2 is far from obvious. As Roemer ([27], p.409) put it: "Clearly if a capitalist introduces a cost-reducing technical change his short-run profit rate rises. This, however, produces a disequilibrium; what the theorem says is that after prices have readjusted to equilibrate the profit rate again, the new profit rate will be higher than the old rate."

Nonetheless, Theorem 2 holds under rather restrictive assumptions. It proves that *if* (i) a new equilibrium exists in which (ii) the new technique is adopted and (iii) the wage rate remains unchanged, then OT continues to hold. As noted in the Introduction, the implications of localised technical change are significantly less clear-cut once conditions (i)-(iii) are relaxed.

Consider a simple economy with two inputs, capital and labour. In the standard approach, technical progress implies an inward shift of the isoquants, technical change is always cost-reducing, and innovations are adopted in equilibrium. Assuming differentiability, the slope of the new isoquant at the point corresponding to the economy's capital and labour endowments represents the new, unique, equilibrium factor prices associated with increased production and the full employment of both inputs.

In contrast, a localised innovation $(A^*_t, L^*_t) \in \mathcal{B}_t \backslash \mathcal{B}_{t-1}$ is a single new activity – a vector of material and labour inputs – placed outside of the existing input-requirement set. The new input-requirement set is the convex hull of these two components, and therefore has a kink corresponding to the new technique, even if the original input-requirement set had a smooth boundary.

As a result, the interaction of localised innovations, labour market conditions and maximising behaviour portrays a more complex, and arguably more realistic picture of technical progress. First, even if the new technique is profitable in terms of Definition 3, it is not necessarily adopted in equilibrium. Further, it is not necessarily compatible with the full employment of capital and labour, which in turn makes the equilibrium transition more complicated. Finally, even if the universal adoption of the new technique was compatible with the full employment of capital and labour, it is not obvious what the new equilibrium price vector would be.

5 Technical progress and general equilibrium

In order to examine the relation between cost-reducing technical change and productivity in an activity analysis model, we follow Roemer [27] and define various types of technical changes that are relevant to analyse localised innovations.[19]

Definition 4 $(A, L) \mapsto (A^*, L^*)$ *is:*

(i) capital-using labour-saving (CU-LS) if and only if $A^* \geq A$ *and* $L^* \leq L$*, with* $A_i^* \geq A_i$ *and* $L_i^* < L_i$ *some* i*; and capital-saving labour-using (CS-LU) if and only if* $A^* \leq A$ *and* $L^* \geq L$*, with* $A_i^* \leq A_i$ *and* $L_i^* > L_i$ *some* i*;*
(ii) progressive if and only if $v^* < v$*; neutral if and only if* $v^* = v$*; and regressive if and only if* $v^* > v$*.*

Two features of Definition 4 are worth stressing. First, in part (i) innovations are defined in physical, rather than monetary terms in order to abstract from the general equilibrium effects of technical change on prices. Only technical changes that are weakly monotonic in all produced inputs are considered. Although this may seem restrictive in an n-good space, it is in line with the definitions used in the literature (and in policy debates), and with intuitive notions of the mechanisation process that has characterised much of capitalist development.

Second, part (ii) provides a link between innovations and productivity: the adoption of a new technique is progressive if it leads to a uniform *decrease* in employment multipliers, and therefore to an *increase* in labour productivity. As Flaschel et al [10] show these innovations expand the economy's production possibility frontier. Regressive technical changes have the opposite effect.[20]

In order to derive the next results, we impose more structure on technical progress and focus on technical changes whose main effect is on labour, rather than on capital inputs.

Definition 5 *Let* $(A, L) \mapsto (A^*, L^*)$ *take place in sector* i *such that* $A_i \neq A_i^*$*. Then, the change of technique is* labour inelastic *if and only if* $|L_i - L_i^*| > |LA^{-1}(A_i - A_i^*)|$.

The intuition is straightforward in a one-good economy: a change of technique is labour inelastic if the percentage change in produced input is smaller than the percentage change in labour input. In an n-good economy, $(A_i - A_i^*)$ is the change in the *vector* of commodity inputs necessary to produce one unit of good i. Definition 5 uses the linear operator LA^{-1} to transform the units of physical goods into labour: $LA^{-1}(A_i - A_i^*)$ represents the amount of direct labour demand necessary for the operation of the variational commodity inputs. Then, Definition 5 states that a change of technique is labour inelastic if and only if the change in the profile of commodity inputs measured in labour units is smaller than the change of direct labour input necessary to produce one unit of good i.

For each $(A, L) \in \mathcal{B}_t$, let

$$F(\pi; (A, L)) = \begin{cases} \frac{1}{L[I-(1+\pi)A]^{-1}b} & \text{if } \pi \in [0, \Pi(A)), \\ 0 & \text{if } \pi = \Pi(A). \end{cases}$$

[19] See also Flaschel et al. [10].

[20] Part (ii) focuses on changes of technique that modify all employment multipliers in the same direction. As Roemer ([27], p.410) notes, this is without any loss of generality if one considers changes of technique of the type described in part (i).

Then, the *wage-profit curve associated with* (A, L) can be defined as follows:

$$\pi w(A, L) \equiv \left\{(\pi, w) \in \mathbb{R}_+^2 \mid w = F(\pi; (A, L)) \text{ for } \pi \in [0, \Pi(A)]\right\}.$$

The *wage-profit frontier associated with* $\mathcal{B}_t$ is the envelope of the various wage-profit curves and can be defined as follows:

$$\begin{aligned} \pi w(\mathcal{B}_t) \equiv & \left\{(\pi, w) \in \mathbb{R}_+^2 \mid \exists (A, L) \in \mathcal{B}_t : (\pi, w) \in \pi w(A, L) \right. \\ & \left. \& \ \forall (A', L') \in \mathcal{B}_t, \forall (\pi', w') \in \pi w(A', L') : w' = w \Rightarrow \pi' \leqq \pi \right\}. \end{aligned}$$

The concepts of the wage-profit curve and wage-profit frontier provide the analytical tools to examine the optimal choice of technique and the interaction between technical progress and distribution. For in equilibrium only techniques that lie on $\pi w(\mathcal{B}_t)$ will be adopted. Formally:

Lemma 3 *A technique* $(A, L) \in \mathcal{B}_t$ *with* $p = (1 + \pi) pA + wL$ *for some* $(p, w) \in \triangle \times \mathbb{R}_+, w > 0$ *is such that* $p \leqq (1 + \pi) pA' + wL'$ *for all* $(A', L') \in \mathcal{B}_t$ *if and only if* $(\pi, w) \in \pi w(A, L) \cap \pi w(\mathcal{B}_t)$.

Proof: See Kurz and Salvadori ([19]; Theorem 5.1). ■

In other words, a technique (A, L) is cost minimising, and is therefore adopted, if no other technique in $\mathcal{B}_t$ allows for a wage rate higher than w at π.

For each (A, L), the intercepts of $\pi w(A, L)$ on the vertical axis (w) and on the horizontal axis (π) are, respectively, the points $\left(0, \frac{1}{vb}\right)$ and $(\Pi(A), 0)$. Therefore, for any $(A, L), (A', L') \in \mathcal{B}_t$, if $v > v'$ and $A \leq A'$, then $\left(0, \frac{1}{vb}\right) \leq \left(0, \frac{1}{v'b}\right)$ and $(\Pi(A), 0) \geq (\Pi(A'), 0)$ hold. Moreover, $\pi w(A, L)$ and $\pi w(A', L')$ intersect at least once, and quite possibly more than once. Finally, given a wage-profit curve $\pi w(A, L)$ and given $(\pi, w) \in \mathbb{R}_+^2$, let $\pi w(A, L; (\pi, w)) \equiv \{(\pi', w') \in \pi w(A, L) \mid (\pi', w') \geq (\pi, w)\}$.

The wage-profit frontier is conceptually equivalent to the factor price frontier used in standard microeconomic models, but there are some key differences. For example, some well-known paradoxes in capital theory, such as reswitching and capital reversing can only be analysed focusing on the wage profit frontier (Sraffa [32], Kurz and Salvadori [19]). Perhaps more importantly for our analysis, in the standard approach technical progress is conceived of as yielding an outward shift of the *whole* factor prices frontier (and possibly a change in its shape). Thus, for any (π, w) in the original frontier, there is (π', w') in the new frontier such that $(\pi', w') \geq (\pi, w)$. Instead, when it is profitable in the sense of Definition 3, localised technical change has an analogous effect only in a *neighbourhood* of the original equilibrium distribution (π, w), and there exists a non-empty set $\pi w(A^*, L^*; (\pi, w))$ such that for any $(\pi', w') \in \pi w(A^*, L^*; (\pi, w))$, $(\pi', w') \geq (\pi, w)$.

5.1 Indeterminacy

In the analysis of the interaction between technical progress and the equilibrium income distribution, it is natural to start from innovations which allow for the full employment of all factors of production. Theorem 3 shows that even in this special case, the distributive effects of technical change are difficult to predict and may not be Pareto-improving.

Theorem 3 *Let* $\left((p_{t-1}, w_{t-1}), ((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu})_{\nu \in \mathcal{N}_{t-1}}\right)$ *be a PCE for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. *Let* $(A, L) \mapsto (A^*, L^*)$ *be profitable with* $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$ *and* $N_t = L^* A^{*-1} \omega_{t-1}$. *Then, there exists* $\pi w\left(A^*, L^*; (\pi_{t-1}^{\max}, w_{t-1})\right) \neq \varnothing$ *such that for any* $(\pi', w') \in \pi w\left(A^*, L^*; (\pi_{t-1}^{\max}, w_{t-1})\right)$, *there exists a CE* $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{\nu}; \omega_t^{\nu})_{\nu \in \mathcal{N}_t}\right)$ *for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ *with* $w_t^* = w'$ *and* $\pi_t^{*\max} = \pi'$. *Furthermore, if* $\pi_{t-1}^{\max} > 0$, *and* $w_{t-1} > p_{t-1} b$, *then there exist CEs with either* $\pi_t^{*\max} < \pi_{t-1}^{\max}$ *or* $w_t^* < w_{t-1}$.

Proof: 1. Because $\left((p_{t-1}, w_{t-1}), ((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu})_{\nu \in \mathcal{N}_{t-1}}\right)$ is a PCE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$, $\pi_{t-1}(A, L) = \pi_{t-1}^{\max}$. Then, by Lemma 3, $(\pi_{t-1}^{\max}, w_{t-1}) \in \pi w(\mathcal{B}_{t-1})$.

Because $(A, L) \mapsto (A^*, L^*)$ is profitable, $(1 + \pi_{t-1}^{\max}) p_{t-1} A^* + w_{t-1} L^* \leq (1 + \pi_{t-1}^{\max}) p_{t-1} A + w_{t-1} L = p_{t-1}$ implies $p_{t-1}\left[I - (1 + \pi_{t-1}^{\max}) A^*\right] > \mathbf{0}$ which in turn implies $\left[I - (1 + \pi_{t-1}^{\max}) A^*\right]$ is invertible with $\left[I - (1 + \pi_{t-1}^{\max}) A^*\right]^{-1} > \mathbf{0}$, where the indecomposability of A^* ensures the strict sign of the last inequality. Then, profitability implies $p_{t-1} \geq w_{t-1} L^* \left[I - (1 + \pi_{t-1}^{\max}) A^*\right]^{-1}$, which in turn implies

$$1 = p_{t-1} b > w_{t-1} L^* \left[I - (1 + \pi_{t-1}^{\max}) A^*\right]^{-1} b = \frac{w_{t-1}}{\widehat{w}} \text{ for } \widehat{w} \equiv \frac{1}{L^* \left[I - (1 + \pi_{t-1}^{\max}) A^*\right]^{-1} b}.$$

Then, $\widehat{w} > w_{t-1}$. Moreover, let $\widehat{p} \equiv \widehat{w} L^* \left[I - (1 + \pi_{t-1}^{\max}) A^*\right]^{-1} > \mathbf{0}$. Then, $\widehat{p} \in \Delta$.

Clearly, $(\pi_{t-1}^{\max}, \widehat{w}) \in \pi w(\mathcal{B}_t) \setminus \pi w(\mathcal{B}_{t-1})$. Because $F(\pi; (A', L'))$ is strictly decreasing and continuous for each $(A', L') \in \mathcal{B}_t$, it follows that for all $w^* \in [w_{t-1}, \widehat{w}]$, there exists $(\pi^*, w^*) \in \pi w(\mathcal{B}_t) \setminus \pi w(\mathcal{B}_{t-1})$ with $(\pi^*, w^*) \geq (\pi_{t-1}^{\max}, w_{t-1})$.

Insert Figure 1 around here.

2. Consider any $(\pi', w') \in \pi w(A^*, L^*) \cap \pi w(\mathcal{B}_t)$ such that $(\pi', w') \geq (\pi_{t-1}^{\max}, w_{t-1})$. By Lemma 3, there is a $p' \in \triangle$ such that $p' = w' L^* [I - (1 + \pi') A^*]^{-1} > \mathbf{0}$ and (A^*, L^*) is optimal at (p', w'). Hence, since $N_t = L^* A^{*-1} \omega_{t-1}$ and $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, it follows that (A^*, L^*), $x_t^{\nu} = \frac{p' \omega_{t-1}^{\nu}}{p' \omega_{t-1}} A^{*-1} \omega_{t-1}$, $l_t^{\nu} = 1$, $\delta_t^{\nu} = \mathbf{0}$, and $\omega_t^{\nu} = \frac{p' x_t^{\nu} - w' L^* x_t^{\nu} + w' - 1}{(p' - L^*) A^{*-1} \omega_{t-1}} (I - b L^*) A^{*-1} \omega_{t-1}$ solve MP_t^{ν} for all $\nu \in \mathcal{N}_t$ and Definition 1(b)-(e) are satisfied. Then, $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{\nu};\ \omega_t^{\nu})_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $p_t^* = p'$, $w_t^* = w'$, and $\pi_t^{*\max} = \pi'$.

3. If $\pi_{t-1}^{\max} > 0$, and $w_{t-1} > p_{t-1} b$ then by the continuity of $F(\pi; (A', L'))$ for all $(A', L') \in \mathcal{B}_t$, a straightforward modification of the argument in step 1 can be used to show that there exists $(\pi^*, w^*) \in \pi w(A^*, L^*) \cap \pi w(\mathcal{B}_t)$ such that *either* $0 < \pi_{t-1}^{\max} - \pi^* < \varepsilon$ with $\pi^* \geqq 0$, *or* $0 < w_{t-1} - w^* < \varepsilon$ with $w^* \geqq 1$, for some $\varepsilon > 0$. The existence of a CE can then be proved as in step 2. ■

Theorem 3 suggests that when a profitable change of technique guarantees the full employment of labour and capital, a new equilibrium emerges at t in which the new technique is indeed adopted. The effect of innovation on distribution is not clear a priori, however, because of the (infinitely) many profit rates and wage rates that can be supported in equilibrium. Interestingly, technical progress may even make either capitalists or workers *strictly worse off* as there exist equilibria at t with either $\pi_t^{*\max} < \pi_{t-1}^{\max}$ or $w_t^* < w_{t-1}$. The distributional outcome will depend on the actual equilibrium selection mechanism.[21]

[21] One possible solution to this indeterminacy is to consider some form of bargaining over distributions as an equilibrium selection mechanism. See, e.g., Cogliano et al. [7] and Yoshihara and Kaneko [36].

It is worth emphasising, again, that this indeterminacy is due to the localised nature of innovations and does not occur in standard macroeconomic models. It would arise even if the original base set $\mathcal{B}_{t-1}$ contained a (possibly uncountably) infinite number of techniques, and we allowed aggregate consumption demands to vary with prices.

5.2 Technological unemployment

In the previous subsection, we analysed equilibria in which a new technique is adopted and both capital and labour are fully employed. Yet this is by no means guaranteed in the case of localised technical change, which may lead to technological unemployment – as conjectured by Acemoglu [1]. For example, even if the economy was originally at a CE characterised by full employment, CU-LS technical change may lead labour to become relatively abundant *if the new technique is adopted.* As we have already noted, however, unlike in the standard macroeconomic literature, nothing guarantees that the new technique will indeed be adopted: while the new technique is profitable at the equilibrium prices ruling at $t-1$, the very introduction of the new technique is likely to cause disequilibrium in commodity markets and in the labour market, which in turn would cause prices to change. Even though the wage profit curve associated with the new, profitable technique will be part of the wage profit frontier in a neighbourhood of the original equilibrium, this is not necessarily true when it is sufficiently far away from the original equilibrium.

Theorem 4 derives the conditions under which profitable, CU-LS technical change leads to a new CE in which the newly discovered technique is adopted:

Theorem 4 *Let* $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ *be a PCE with* $\pi^{\max}_{t-1} > 0$ *and sufficiently small* $w_{t-1} - p_{t-1}b \geqq 0$ *for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. *Let* $(A, L) \mapsto (A^*, L^*)$ *be profitable, CU-LS, and labour inelastic with* $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$. *Then there exists a CE* $\left((p^*_t, w^*_t), \left((A^*, L^*); \xi^{*\nu}_t; \omega^{*\nu}_t\right)_{\nu \in \mathcal{N}_t}\right)$ *for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. *Moreover, for any CE in which* (A^*, L^*) *is adopted for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$, $w^*_t = p^*_t b$ *and* $\pi^{*\max}_t > \pi^{\max}_{t-1}$ *hold.*

Proof: 1. Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a PCE with $\pi^{\max}_{t-1} > 0$ and sufficiently small $w_{t-1} - p_{t-1}b \geqq 0$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. By Proposition 1, $N_t \geqq LA^{-1}\omega_{t-1}$. Since $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, let $x^*_t \equiv A^{*-1}\omega_{t-1} > \mathbf{0}$. Because $(A, L) \mapsto (A^*, L^*)$ is CU-LS and labour inelastic, the same argument as in the proofs of Theorem A1(i) and Theorem A2 for the case with $A^* x^*_t = \omega_{t-1}$ can be used to prove that $N_t > L^* A^{*-1}\omega_{t-1}$ holds.

2. Because $(A, L) \mapsto (A^*, L^*)$ is profitable, as in step 1 of the proof of Theorem 3, there exists $\left(\pi^{\max}_{t-1}, \widehat{w}\right) \in \pi w(\mathcal{B}_t) \setminus \pi w(\mathcal{B}_{t-1})$ and for all $w^* \in [w_{t-1}, \widehat{w}]$ there exists $(\pi^*, w^*) \in \pi w(\mathcal{B}_t) \setminus \pi w(\mathcal{B}_{t-1})$ with $(\pi^*, w^*) \geq \left(\pi^{\max}_{t-1}, w_{t-1}\right)$.

Because $F(\pi; (A', L'))$ is strictly decreasing and continuous for each $(A', L') \in \mathcal{B}_t$, there exists some $\epsilon > 0$ such that for any w' with $\epsilon \geqq w_{t-1} - w' \geqq 0$, there exists π' such that $(\pi', w') \in \pi w(\mathcal{B}_t) \setminus \pi w(\mathcal{B}_{t-1})$. By assumption $w_{t-1} - p_{t-1}b \geqq 0$ is sufficiently small, and therefore we can consider $w' = 1 = p_{t-1}b$. By Lemma 3, there is a price vector $p' \in \triangle$ such that $p' = w'L^*\left[I - (1 + \pi')A^*\right]^{-1} > \mathbf{0}$ and (A^*, L^*) is cost minimising at (p', w').

3. Noting that $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, it follows that (A^*, L^*), $x^{*\nu}_t = \frac{p'\omega^{\nu}_{t-1}}{p'\omega_{t-1}} A^{*-1}\omega_{t-1}$, $l^{*\nu}_t = \frac{L^* A^{*-1}\omega_{t-1}}{N_t}$, $\delta^{*\nu}_t = \mathbf{0}$, and $\omega^{*\nu}_t = \frac{p' x^{*\nu}_t - w' L^* x^{*\nu}_t + (w'-1) l^{*\nu}_t}{(p' - L^*) A^{*-1}\omega_{t-1}} (I - bL^*) A^{*-1}\omega_{t-1}$ solve MP^{ν}_t

for all $\nu \in \mathcal{N}_t$ and Definition 1(b)-(e) are satisfied. Thus, $\left(\left(p_t^*, w_t^*\right), \left(\left(A^*, L^*\right); \xi_t^{*\nu}; \omega_t^{*\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ is a CE with $p_t^* = p'$ and $w_t^* = w' = p_t^* b = 1$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Moreover, as $N_t > L^* A^{*-1} \omega_{t-1}$ holds, Theorem 1(ii) implies that there is no other CE $\left(\left(p_t'^*, w_t'^*\right), \left(\left(A^*, L^*\right); \xi_t'^{*\nu}; \omega_t'^{*\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ such that $w_t'^* \neq p_t'^* b = 1$.

4. By Theorem 2, $w_t^* \leqq w_{t-1}$ implies $\pi_t^{*\max} > \pi_{t-1}^{\max}$. ■

Theorem 4 shows that profitable, CU-LS innovations may indeed be adopted in equilibrium at t, provided w_{t-1} is sufficiently low. If this is not the case, however, new techniques may not be adopted. To see this, suppose $(A, L) \mapsto (A^*, L^*)$ is CU-LS, labour inelastic, and profitable at (p_{t-1}, w_{t-1}) of a PCE, and $N_{t-1} = LA^{-1}\omega_{t-2}$ holds. Then, $N_t > L^* A^{*-1} \omega_{t-1}$, as per the proof of Theorem A1(i). Thus, if $\left(\left(p_t^*, w_t^*\right), \left(\left(A^*, L^*\right); \xi_t^{*\nu}; \omega_t^{*\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ is a CE, then it must be $w_t^* = p_t^* b = 1$ by Theorem 1(ii).

Insert Figure 2 around here.

Yet, while (A^*, L^*) yields higher profits than (A, L) in a neighbourhood of (p_{t-1}, w_{t-1}), it does not necessarily maximise the profit rate at (p_t^*, w_t^*) if $w_t^* = 1$ is much lower than w_{t-1}. In this case, it is possible for (A, L) to be optimal at (p_t^*, w_t^*), and there may be a CE with prices (p_t^*, w_t^*) and actions $\left((A, L); (x_t^\nu; 1; \mathbf{0}); \omega_t^\nu\right)_{\nu \in \mathcal{N}_t}$ with $Ax_t = \omega_{t-1}$. Figure 3 describes this situation.

Insert Figure 3 around here.

The above argument can be summarised by the following corollary:

Corollary 1 *Let* $\left(\left(p_{t-1}, w_{t-1}\right), \left(\left(A, L\right); \xi_{t-1}^\nu; \omega_{t-1}^\nu\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ *be a PCE with* $\pi_{t-1}^{\max} > 0$ *for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. *Let* $(A, L) \mapsto (A^*, L^*)$ *be profitable, CU-LS, and labour inelastic with* $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$. *Then, there exists a CE* $\left(\left(p_t^*, w_t^*\right), \left(\left(A^*, L^*\right); \xi_t^{*\nu}; \omega_t^{*\nu}\right)_{\nu \in \mathcal{N}_t}\right)$ *for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ *if and only if* $(\pi_t^{*\max}, w_t^* = 1) \in \pi w(\mathcal{B}_t) \cap \pi w(A^*, L^*)$.

5.3 Non-existence of equilibrium

Corollary 1 characterises the conditions under which what may be deemed a market failure occurs: if the condition in Corollary 1 is violated, there exists no equilibrium in which a new technique is adopted *even if* it is profitable and increases labour productivity.

Indeed, in this case localised innovations may cause an even deeper failure and disrupt the functioning of capitalist economies in a more surprising and counterintuitive way: technical progress may cause the economy to reach *no* equilibrium at t. To see this, consider the following example, which builds on that in the previous subsection.

Example: Let $N_t \equiv 6$, $\omega_{t-1} \equiv \begin{pmatrix} 1 \\ 2.5 \end{pmatrix}$, $b \equiv \begin{pmatrix} 0.001 \\ 0.09 \end{pmatrix}$, $\mathcal{B}_{t-1} \equiv \{(A, L), (A^{**}, L^{**})\}$ and $\mathcal{B}_t \equiv \{(A, L), (A^*, L^*), (A^{**}, L^{**})\}$, where

$$\begin{aligned}
(A, L) &\equiv \left(\begin{bmatrix} 0.085 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, \ (0.56, \ 0.4) \right), \\
(A^*, L^*) &\equiv \left(\begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}, \ (0.4, \ 0.4) \right), \text{ and} \\
(A^{**}, L^{**}) &\equiv \left(\begin{bmatrix} 0.05 & 0.05 \\ 0.2 & 0.3 \end{bmatrix}, \ (0.4, \ 0.45) \right).
\end{aligned}$$

All techniques in $\mathcal{B}_t$ are productive and indecomposable. Furthermore, $(A, L), (A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$ and $(A^{**}, L^{**}) \in \mathcal{B}_{t-1} \backslash \mathcal{B}_t(\omega_{t-1}, b)$ with $LA^{-1}\omega_{t-1} = N_t$, $L^*A^{*-1}\omega_{t-1} < N_t$, $L^{**}A^{**-1}\omega_{t-1} > N_t$, and $A^{**-1}\omega_{t-1} = (35, -15) \ngeq \mathbf{0}$. Finally, $(A, L) \mapsto (A^*, L^*)$ is CU-LS, $v^{**}b > vb > v^*b$, and $\Pi(A^{**}) > \Pi(A) > \Pi(A^*)$.

In this economy, the wage-profit frontiers at $t-1$ and t are depicted as follows:

Insert Figure 4 around here.

Observe that $\pi w(A, L)$ is part of $\pi w(\mathcal{B}_{t-1})$ (around the combination of the highest wage rate and zero profit rate, as $\frac{1}{v^*b} > \frac{1}{vb} > \frac{1}{v^{**}b}$) but not of $\pi w(\mathcal{B}_t)$. Moreover, $\pi w(A^{**}, L^{**})$ is part of $\pi w(\mathcal{B}_t)$ at $w = 1$ and for wage rates sufficiently close to one.

It is straightforward to prove the existence of a CE in which (A, L) is activated at $t-1$ provided (p_{t-1}, w_{t-1}) is such that w_{t-1} is sufficiently close to $\frac{1}{vb}$. Then, because $\pi w(A^*, L^*)$ is part of $\pi w(\mathcal{B}_t)$ for wage levels close to $\frac{1}{vb}$, $(A, L) \mapsto (A^*, L^*)$ is profitable.

Suppose that (A^*, L^*) is activated at a CE (p_t^*, w_t^*) at period t. Then, by Theorem 1(ii), $w_t^* = 1$ must hold as $L^*A^{*-1}\omega_{t-1} < N_t$. However, (A^{**}, L^{**}) is uniquely optimal at $(p_t^*, w_t^* = 1)$, yielding a contradiction.

Suppose that (A^{**}, L^{**}) is activated at a CE (p_t^*, w_t^*) at period t. Then, by Theorem 1(iii), $\pi_t^{*\max} = 0$ must hold as $L^{**}A^{**-1}\omega_{t-1} > N_t$. However, (A^*, L^*) is uniquely optimal at the prices (p_t^*, w_t^*) with $\pi_t^{*\max} = 0$, as $\frac{1}{v^{**}b} < \frac{1}{vb} < \frac{1}{v^*b}$, yielding a contradiction.

Therefore, if a CE exists at period t, then in equilibrium agents must activate an activity that is a convex combination of (A^*, L^*) and (A^{**}, L^{**}), and agents must be indifferent between the two techniques. Hence, in equilibrium $(\pi_t^{c\max}, w_t^c)$ must correspond to a point of intersection of $\pi w(A^*, L^*)$ and $\pi w(A^{**}, L^{**})$ on $\pi w(\mathcal{B}_t)$ in Figure 4. In this case, $\pi_t^{c\max} > 0$ and $w_t^c > 1$ hold. Therefore, by Theorem 1(i), it follows that if (p_t^c, w_t^c) with $(\pi_t^{c\max}, w_t^c) > (0, 1)$ is a CE, then there must exist $x^*, x^{**} \in \mathbb{R}_+^n$ such that $A^*x^* + A^{**}x^{**} = \omega_{t-1}$, $L^*x^* + L^{**}x^{**} = N_t$, and $x^c = x^* + x^{**} \geqq N_t b > \mathbf{0}$.

We show that such x^*, x^{**} do not exist. First, as $L^*A^{*-1}\omega_{t-1} < N_t$, there is no $x^* \in \mathbb{R}_+^n$ such that $A^*x^* = \omega_{t-1}$ and $L^*x^* = N_t$. Then, by the Minkowski-Farkas Lemma (Gale, [14], p. 44, Theorem 2.6), there exists $(p'', w'') \in \mathbb{R}^{2+1}$ such that $p''A^* + w''L^* \geqq \mathbf{0}$ and $p''\omega_{t-1} + w''N_t < 0$. Indeed, if $(p'', w'') \equiv ((0.5, 0.5), -0.3)$, then

$$p''A^* + w''L^* = (0.5, 0.5) \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} - 0.3 \cdot (0.4, 0.4) = (0.03, 0.08) > \mathbf{0},$$

$$p''\omega_{t-1} + w''N_t = (0.5, 0.5) \begin{pmatrix} 1 \\ 2.5 \end{pmatrix} - 0.3 \cdot 6 = 1.75 - 1.8 < 0.$$

Moreover, we have:

$$p''A^{**} + w''L^{**} = (0.5, 0.5) \begin{bmatrix} 0.05 & 0.05 \\ 0.2 & 0.3 \end{bmatrix} - 0.3 \cdot (0.4, 0.45) = (0.005, 0.04) > \mathbf{0}.$$

Therefore, in summary, there exists $(p'', w'') \in \mathbb{R}^{2+1}$ such that $p''A^* + w''L^* \geqq \mathbf{0}$, $p''A^{**} + w''L^{**} \geqq \mathbf{0}$, and $p''\omega_{t-1} + w''N_t < 0$ for this economy. Then, by Minkowski-Farkas Lemma (Gale, [14], p. 44, Theorem 2.6), there is no $x^*, x^{**} \in \mathbb{R}_+^2$ such that

$$\begin{bmatrix} A^* & A^{**} \\ L^* & L^{**} \end{bmatrix} \begin{pmatrix} x^* \\ x^{**} \end{pmatrix} = \begin{pmatrix} \omega_{t-1} \\ N_t \end{pmatrix}.$$

Hence, no convex combination of (A^*, L^*) and (A^{**}, L^{**}) can be activated in equilibrium at t.

As a result, the economy with $\mathcal{B}_t = \{(A, L), (A^*, L^*), (A^{**}, L^{**})\}$ reaches *no* equilibrium at t after localised technical progress takes place. ■

6 The falling profit rate

In the previous section, we have shown that – once the general equilibrium effects of technical progress are taken into account – the distributive effects of innovations are not obvious. Absent a significant shift in bargaining power towards workers, however, innovations – and especially labour saving technical progress – tend to increase equilibrium profits. These results would seem to confirm the main intuition of OT and provide yet another obituary for Marx's theory of the falling profit rate. In this section, we show that, at a general level, this conclusion would be unwarranted – or would at least need to be qualified – and there are indeed some ex-ante profitable innovations that may lead to a decrease in the equilibrium profit rate.

Our first result characterises the conditions under which profitable CS-LU change of technique leads to a falling profit rate.

Theorem 5 *Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a PCE with $\pi^{\max}_{t-1} > 0$ and $w_{t-1} > p_{t-1}b$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A, L) \mapsto (A^*, L^*)$ be profitable, CS-LU, and labour inelastic with $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$. Then, the following statements (1), (2), and (3) are equivalent: (1) there exists a CE $\left((p^*_t, w^*_t), \left((A^*, L^*); \xi^{*\nu}_t; \omega^{*\nu}_t\right)_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$; (2) $\pi^{*\max}_t = 0$ for any CE in which (A^*, L^*) is adopted for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$; and (3) $(A^*, L^*) \in \arg\min_{(A', L') \in \mathcal{B}_t} L'(I - A')^{-1} b$.*

Proof: ((3) ⇒ (1)) Let $(A^*, L^*) \in \arg\min_{(A', L') \in \mathcal{B}_t} L'(I - A')^{-1} b$. Then $\frac{1}{v'b} \leqq \frac{1}{v^*b}$ holds for all $(A', L') \in \mathcal{B}_t$. Therefore by Lemma 3 it follows that (A^*, L^*) is optimal at $p^*_t = w^*_t v^* > \mathbf{0}$ and $w^*_t = \frac{1}{v^*b}$.

Since $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b)$, let $x^*_t \equiv A^{*-1}\omega_{t-1} > \mathbf{0}$. The argument used in the proof of Theorem A1(ii) can be adapted to prove that $N_t = LA^{-1}\omega_{t-1} < L^*A^{*-1}\omega_{t-1}$. Then, let $k \equiv \frac{N_t}{L^*A^{*-1}\omega_{t-1}} < 1$. For all $\nu \in \mathcal{N}_t$, let $x^{*\nu}_t = \frac{p^*_t \omega^{\nu}_{t-1}}{p^*_t \omega_{t-1}} k A^{*-1}\omega_{t-1} > \mathbf{0}$, $\delta^{*\nu}_t = \frac{p^*_t \omega^{\nu}_{t-1}}{p^*_t \omega_{t-1}}(1 - k)\omega_{t-1} \geq \mathbf{0}$, $l^{*\nu}_t = 1$, and

$$\omega^{*\nu}_t = \frac{p^*_t x^{*\nu}_t - w^*_t L^* x^{*\nu}_t + w^*_t - 1 + p^*_t \delta^{*\nu}_t}{\left[k\left(p^*_t - L^*\right) A^{*-1}\omega_{t-1} + (1 - k)\, p^*_t \omega_{t-1}\right]} \left[k\left(I - bL^*\right) A^{*-1}\omega_{t-1} + (1 - k)\,\omega_{t-1}\right] \geqq \mathbf{0}.$$

It is not difficult to check that $((A^*, L^*), (x^{*\nu}_t, 1; \delta^{*\nu}_t; \omega^{*\nu}_t))$ solve MP^{ν}_t for all $\nu \in \mathcal{N}_t$ and Definition 1(b)-(e) are satisfied.

((1) ⇒ (2)) It directly follows from Theorem A1(ii).

((2) ⇒ (3)) Let $\left((p^*_t, w^*_t), \left((A^*, L^*); \xi^{*\nu}_t; \omega^{*\nu}_t\right)_{\nu \in \mathcal{N}_t}\right)$ be a CE with $\pi^{*\max}_t = 0$. By construction, $p^*_t = w^*_t v^*$ and $w^*_t = \frac{1}{v^*b}$. Suppose, contrary to the statement, that for some $(A', L') \in \mathcal{B}_t$, $\frac{1}{v'b} > \frac{1}{v^*b}$. Then, $\left(\pi^{*\max}_t = 0, w^*_t = \frac{1}{v^*b}\right) \notin \pi w(\mathcal{B}_t)$ and by Lemma 3, (A^*, L^*) is not optimal, a contradiction. ■

Theorem 5 shows the existence of localised innovations that are profitable from the viewpoint of an individual capitalist but which, if adopted universally, lead the equilibrium profit rate to fall. From a broad theoretical perspective, this result contradicts OT and may therefore be dubbed the *Anti-Okishio Theorem*. It identifies a scenario in which individually rational actions lead to collectively suboptimal outcomes, an intuition which is at the core of Marx's theory of technical change.[22]

How robust is the insight of Theorem 5? Does the equilibrium profit rate fall as a result of profitable, CS-LU technical change if condition (3) in Theorem 5 is not satisfied, or – more strongly – if technical change is regressive? This is not obvious. It can be shown that if technical change is CS-LU and regressive, then the new technique will not be adopted in equilibrium, even if it is profitable (see Theorem A1(ii) in the appendix). In this case, either an equilibrium emerges in which an old technique is adopted, or no equilibrium exists – as in the case discussed in section 5.3. Theorem 6 addresses the first scenario.

Theorem 6 *Let* $\left((p_{t-1}, w_{t-1}), ((A, L); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1})_{\nu \in \mathcal{N}_{t-1}}\right)$ *be a PCE with* $\pi^{\max}_{t-1} > 0$ *and* $w_{t-1} > p_{t-1}b$ *for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. *Let* $(A, L) \mapsto (A^*, L^*)$ *be profitable, CS-LU and regressive. Let* $\{(A^{**}, L^{**})\} = \left[\arg\min_{(A', L') \in \mathcal{B}_{t-1}} L' (I - A')^{-1} b\right]$ *be such that* $(A^{**}, L^{**}) \in \mathcal{B}_{t-1}(\omega_{t-1}, b)$. *Then, there exists a CE* $\left((p^{**}_t, w^{**}_t), ((A^{**}, L^{**}); \xi^{**\nu}_t; \omega^{**\nu}_t)_{\nu \in \mathcal{N}_t}\right)$ *with* $\pi^{**\max}_t < \pi^{\max}_{t-1}$ *for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ *if and only if* $N_t \leqq L^{**} A^{**-1} \omega_{t-1}$.

Proof: ($\Leftarrow$) Because $(A, L) \mapsto (A^*, L^*)$ is regressive, it follows that $vb < v^*b$. Therefore, by the definition of (A^{**}, L^{**}), $v^{**}b < v^*b$ and $\left(0, \frac{1}{v^{**}b}\right) \in \pi w(A^{**}, L^{**}) \cap \pi w(\mathcal{B}_t)$. Because $F(\pi; (A^{**}, L^{**}))$ is strictly decreasing and continuous, there exists $\varepsilon > 0$ such that $\frac{1}{v^{**}b} - \varepsilon > w_{t-1}$ and for all $w' \in \left[\frac{1}{v^{**}b} - \varepsilon, \frac{1}{v^{**}b}\right]$ there exists $\pi' \geqq 0$ such that $(\pi', w') \in \pi w(A^{**}, L^{**}) \cap \pi w(\mathcal{B}_t)$. By Lemma 3, for all such (π', w'), there exists $p' \in \Delta$ with $p' = w' L^{**} [I - (1 + \pi') A^{**}]^{-1} > \mathbf{0}$, such that (A^{**}, L^{**}) is optimal at (p', w'). By construction, $w' > w_{t-1}$ implies $\pi' < \pi^{max}_{t-1}$.

Suppose $N_t = L^{**} A^{**-1} \omega_{t-1}$. Since $(A^{**}, L^{**}) \in \mathcal{B}_{t-1}(\omega_{t-1}, b)$, then as shown in the proof of Theorem 3, there is a CE $\left((p^{**}_t, w^{**}_t), ((A^{**}, L^{**}); \xi^{**\nu}_t; \omega^{**\nu}_t)_{\nu \in \mathcal{N}_t}\right)$ with $(p^{**}_t, w^{**}_t) = (p', w')$ and $\pi^{**\max}_t = \pi' < \pi^{\max}_{t-1}$ for all of the above mentioned $(\pi', w') \in \pi w(A^{**}, L^{**}) \cap \pi w(\mathcal{B}_t)$.

Suppose $N_t < L^{**} A^{**-1} \omega_{t-1}$. Since $(A^{**}, L^{**}) \in \mathcal{B}_{t-1}(\omega_{t-1}, b)$, then as shown in the proof of Theorem 5, there is a CE $\left((p^{**}_t, w^{**}_t), ((A^{**}, L^{**}); \xi^{**\nu}_t; \omega^{**\nu}_t)_{\nu \in \mathcal{N}_t}\right)$ with $(p^{**}_t, w^{**}_t) = \left(\frac{1}{v^{**}b} v^{**}, \frac{1}{v^{**}b}\right)$ and $\pi^{**\max}_t = 0 < \pi^{\max}_{t-1}$.

($\Rightarrow$) Let $\left((p^{**}_t, w^{**}_t), ((A^{**}, L^{**}); \xi^{**\nu}_t; \omega^{**\nu}_t)_{\nu \in \mathcal{N}_t}\right)$ be a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with $\pi^{**\max}_t < \pi^{\max}_{t-1}$. If $N_t > L^{**} A^{**-1} \omega_{t-1}$, then at the CE it would be $w^{**}_t = 1 < w_{t-1}$, which contradicts the assumption that $\pi^{**\max}_t < \pi^{\max}_{t-1}$. ■

Theorem 6 suggests that the insight of Theorem 5 is indeed robust: there exist a range of scenarios in which the emergence of individually profitable innovations leads to a decline in the equilibrium profit rate.[23] The mechanism highlighted in Theorem 6,

[22] Setting aside the empirically less relevant case of innovations that shift the *whole* wage-profit frontier out, it can be shown that the conditions in Theorem 5 describe a scenario characterised by so-called *re-switching* and *reverse capital deepening*; see Kurz and Salvadori [19]. This suggests that there may be some interesting and perhaps surprising connections between the theory of the falling profit rate and some central insights of classical capital theory. (For a discussion, see the Addendum.)

[23] Observe that because $(A, L) \mapsto (A^*, L^*)$ is regressive, there always exists $(A^{**}, L^{**}) \neq (A^*, L^*)$ such that $(A^{**}, L^{**}) \in \arg\min_{(A', L') \in \mathcal{B}_t} L' (I - A')^{-1} b$.

however, is rather different and the result provides an original perspective on the debates on the falling rate of profit. For it shows that technical progress may indeed lead to a decline in profitability because of the general equilibrium effects of localised innovations even though, unlike in Theorem 5, *the new technique is not adopted in equilibrium.*

The main effect of localised innovations, in Theorem 6, is to disrupt consolidated production activities. The appearance of the new, profitable technique (A^*, L^*) leads agents to abandon old production methods, moving the economy away from equilibrium. The new technique is not optimal at any CE, however, because it implies a CS-LU and regressive type of technical change, and therefore is not adopted. One may imagine an equilibrating process of trial and error in which the economy deviates from the original price system (p_{t-1}, w_{t-1}) and eventually settles on another equilibrium in which a previously suboptimal technique, (A^{**}, L^{**}), is adopted.[24] If capital becomes relatively abundant and $N_t < L^{**}A^{**-1}\omega_{t-1}$, then the profit rate falls to zero. However, and perhaps more surprisingly, Theorem 6 proves that there is a decrease in the equilibrium profit rate even if the economy moves to an equilibrium with full employment of labour and capital, $N_t = L^{**}A^{**-1}\omega_{t-1}$, although in this case the new equilibrium profit rate is positive.

Two additional comments are in order. First, if the condition in Theorem 6 is violated, and there is an excess supply of labour with (A^{**}, L^{**}), then using a similar argument as in section 5.3 it can be shown that there may be no CE in the economy. When technical progress is localised, the non-existence of equilibrium may be a pervasive problem.

Second, Theorems 5 and 6 hold under the assumption of full employment of labour at the PCE in $t-1$. What if, instead, there is a sufficiently big industrial reserve army of the unemployed? It can be shown that if $N_{t-1} > LA^{-1}\omega_{t-2}$ in $t-1$, then a profitable CS-LU change of technique will be adopted in equilibrium, and lead to an increase in the profit rate, provided it is gradual.[25] This scenario could obtain, for example, in a developing economy in which aggregate capital is still low relative to the labour force.

Theorems 5 and 6 prove that CS-LU changes of technique may cause the profit rate to fall. Is this the *only* scenario that may lead to a decrease in the profit rate? Not really. Theorem 7 proves that if general equilibrium effects are considered, then the profit rate may fall even in the standard case of CU-LS change of technique.[26]

Theorem 7 *Let* $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ *be a PCE with* $\pi^{\max}_{t-1} > 0$ *for* $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. *Let* $(A, L) \mapsto (A^*, L^*)$ *be profitable and CU-LS with* $(A^*, L^*) \notin \mathcal{B}_t(\omega_{t-1}, b)$. *Then, the following statements (1), (2), and (3) are equivalent: (1) there exists a CE* $\left((p^*_t, w^*_t), \left((A^*, L^*); \xi^{*\nu}_t; \omega^{*\nu}_t\right)_{\nu \in \mathcal{N}_t}\right)$ *for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$*; (2)* $\pi^{*\max}_t = 0$ *for any CE in which* (A^*, L^*) *is adopted for* $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$*; and (3) there exists* $x^* > \mathbf{0}$ *such that* $(I - bL^*)x^* \geqq A^*x^* - \omega_{t-1}$ *with* $A^*x^* \leq \omega_{t-1}$ *and* $L^*x^* = N_t$*, and moreover,* $(A^*, L^*) \in \arg\min_{(A', L') \in \mathcal{B}_t} L'(I - A')^{-1} b$.

Indeed, any profitable CU-LS changes of technique will *always* lead the profit rate to fall to zero in equilibrium, under condition (3) of Theorem 7 with $(A^*, L^*) \notin \mathcal{B}_t(\omega_{t-1}, b)$. It is not difficult to find an economy in which this condition is non-vacuous.

[24]Interestingly, although $(A, L) \mapsto (A^*, L^*)$ is CS-LU, the production technique that is actually adopted in equilibrium is *more* capital intensive than the original technique (A, L), where the value of capital is evaluated using the price vector corresponding to the switching point of these techniques on $\pi w(\mathcal{B}_{t-1})$.

[25]For a formal statement, see the Addendum.

[26]The proof of Theorem 7 is similar to that of Theorem 5 – noting that $(A^*, L^*) \notin \mathcal{B}_t(\omega_{t-1}, b)$ leads to $\pi^{*\max}_t = 0$ as shown in Theorem A1(i) – and is therefore omitted. (See the Addendum.)

7 Conclusions

Our results paint a much more complex picture of the effects of innovations than in standard macroeconomic models. There is no obvious relation between ex-ante profitable innovations and the (functional) distribution of income that emerges in equilibrium after localised technical change is implemented. If technical change leads to an equilibrium with full employment of productive factors, the distribution of income is undetermined, and it is even possible for either the profit rate or the wage rate to decrease. But with localised innovations there is no guarantee that the equilibrium will be characterised by full employment. Furthermore, a localised innovation that is profitable for individual capitalists at current prices does not necessarily yield an increase in profitability: after it is universally adopted, a change in the equilibrium price vector – and consequently in individual behaviour – may occur eventually leading the equilibrium profit rate to decrease.

Methodologically, our analysis suggests that the distributive effects of technical progress cannot be fully understood in models that do not capture the dialectic between individual choices and aggregate outcomes, and the complex network of effects induced by localised technical change. A general equilibrium approach to technical change allows us to model some aspects of the Schumpeterian process of creative destruction. Even though they affect only the production techniques currently in use – unlike in the standard analysis of technical progress – localised innovations disrupt consolidated production practices and move an economy away from its original equilibrium. Indeed, we have shown that they may even cause the disappearance of *all* equilibria and lead the economy to a persistent disequilibrium dynamics.

This methodological insight is, we believe, robust and our theoretical approach provides a rich framework for the analysis of innovations. In closing, we briefly mention three possible extensions of our analysis. First, we have focused only on *process* innovations – new ways of combining inputs in the production of a given set of goods. It would be interesting to investigate the distributive effects of *product* innovations – the invention of new goods and services. Second, given our focus on the effect of the appearance of innovations on the functional distribution of income, we have not explicitly modelled the process of discovering new techniques. Yet, from the general equilibrium perspective adopted in our paper, it would be interesting to endogenise R&D activities and investment, and then examine how the decisions of R&D investors interact with the choices of capitalists in productive sectors in driving changes in the equilibrium income distribution (for a preliminary analysis in an one-good model, see Cogliano et al. [7]). Finally, we have followed the classical literature on localised innovations by focusing on economies with homogeneous labour. It would be worth extending our analysis to more complex models with heterogeneous labour inputs:[27] in addition to allowing for a richer picture of production processes, and of innovations, this would also provide a more nuanced analysis of the distributive effects of innovations which goes beyond the stark two-class framework of the canonical classical model by including cleavages within the working class (for example, between high-skilled and low-skilled workers).

[27] For a discussion of classical models with multiple non-reproducible factors, see, e.g., Kurz and Salvadori [19] and Ekeland and Guesnerie [9].

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A Appendix

The results in sections 5 and 6 explicitly tackle the issue of the existence of equilibrium and characterise the distribution of income at a new equilibrium induced by technical change. Thus, they hold under specific assumptions concerning, for example, technology and endowments. In this appendix, we relax these assumptions and analyse the effects of innovations on income distribution, under the assumption that the economy moves to a new equilibrium in which the new technique is actually adopted.[28]

Theorem A1 analyses the distributive effect of technical change in an economy which, lacking any innovations, has settled onto a steady state growth path with full employment of labour ($w_{t-1} > p_{t-1}b$) and capital ($\pi_{t-1}^{\max} > 0$).

Theorem A1: Let $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu};\ \omega_{t-1}^{\nu}\right)_{\nu\in\mathcal{N}_{t-1}}\right)$ be a PCE with $\pi_{t-1}^{\max} > 0$ and $w_{t-1} > p_{t-1}b$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A, L) \mapsto (A^*, L^*)$ be profitable and labour inelastic. Suppose it results in a CE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu\in\mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. Then:

(i) if $(A, L) \mapsto (A^*, L^*)$ is CU-LS, then $w_t^* = p_t^* b$ and $\pi_t^{*\max} > \pi_{t-1}^{\max}$ whenever $A^* x_t^* = \omega_{t-1}$; otherwise, $\pi_t^{*\max} = 0$;

(ii) if $(A, L) \mapsto (A^*, L^*)$ is CS-LU, then $\pi_t^{*\max} = 0$ and the change of technique cannot be regressive.

Proof: As $\left((p_{t-1}, w_{t-1}), \left((A, L); \xi_{t-1}^{\nu}; \omega_{t-1}^{\nu}\right)_{\nu\in\mathcal{N}_{t-1}}\right)$ is a PCE with $\pi_{t-1}^{\max} > 0$ and $w_{t-1} > p_{t-1}b$, Proposition 1 implies that $N_t = LA^{-1}\omega_{t-1}$, and there exist $(\xi_t^{\nu})_{\nu\in\mathcal{N}_t} = (x_t^{\nu}; 1; \mathbf{0})_{\nu\in\mathcal{N}_t}$ and $(\omega_t^{\nu})_{\nu\in\mathcal{N}_t}$ such that $x_t > \mathbf{0}$ with $Ax_t = \omega_{t-1}$, and $\left((p_{t-1}, w_{t-1}), ((A, L); \xi_t^{\nu};\ \omega_t^{\nu})_{\nu\in\mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-1})$.

Part (i). By Proposition 8 in Roemer [27], if $(A, L) \mapsto (A^*, L^*)$ is profitable and CU-LS then $v^* < v$.

Suppose first that $A^* x_t^* \leq \omega_{t-1}$ holds. By Lemma 2, $p_t^* > \mathbf{0}$, and therefore $p_t^* A^* x_t^* < p_t^* \omega_{t-1}$. Then, by Lemma 1, $\pi_t^{*\max} = 0$ holds.

Next, suppose that $A^* x_t^* = \omega_{t-1}$ holds. Because this change of technique is CU-LS, $A^* x_t \geq Ax_t = \omega_{t-1}$ and $L^* x_t < Lx_t = N_t$. Therefore, since $A^* x_t^* = \omega_{t-1}$, we obtain $A^*(x_t - x_t^*) \geq \mathbf{0}$. We consider two cases.

Case 1: $0 < x_t^* \leqq x_t$.

Clearly, $L^* x_t^* < Lx_t = N_t$, so that $w_t^* = p_t^* b$ follows from Lemma 1 and Definition 1(c).

Let $\pi' \in [0, \Pi(A^*))$ be such that $\rho(\pi') \equiv w_{t-1} L^* [I - (1 + \pi') A^*]^{-1} b = 1$. To see that such π' exists, observe that $\rho(\pi')$ is a continuous function and $lim_{\pi'\to\Pi(A^*)}\rho(\pi') = \infty$, while $\rho(0) = w_{t-1} v^* b < w_{t-1} v b \leqq p_{t-1} b = 1$.

Setting $p' \equiv w_{t-1} L^* [I - (1 + \pi') A^*]^{-1}$, we have $(p', w_{t-1}) \in \triangle \times \mathbb{R}_+$ with $p' = (1 + \pi') p' A^* + w_{t-1} L^* > \mathbf{0}$. Using the same argument as in the proof of Theorem 2, it can be proved that $\pi' > \pi_{t-1}^{\max}$. Then, since $p_t^* = (1 + \pi_t^{*\max}) p_t^* A^* + w_t^* L^*$ and $w_t^* = p_t^* b < w_{t-1}$, we obtain $\pi_t^{*\max} > \pi'$.

Case 2: $x_t^* \nleqq x_t$.

[28] In Theorems A1 and A2, we focus on persistent CEs such that at the beginning of t, if the period $t-1$ optimal technique (A, L) was adopted, then $N_t \geqq LA^{-1}\omega_{t-1}$ would hold. If $N_t < LA^{-1}\omega_{t-1}$ then by Theorem 1 the effect of innovations on our primary variable of interest, the equilibrium profit rate, is not particularly interesting.

We only need to show $L^*x_t^* < Lx_t$. The rest of the proof then follows as in case 1. Suppose, by way of contradiction, that $L^*x_t^* \geqq Lx_t = N_t$. By Definition 1(c), this implies $L^*x_t^* = N_t$. Given $L^* \leq L$ and $x_t^* > \mathbf{0}$, this implies $Lx_t^* > L^*x_t^* = N_t$. Next, $A^*x_t^* = \omega_{t-1}$, and $Ax_t = \omega_{t-1}$ imply $N_t = LA^{-1}Ax_t = LA^{-1}A^*x_t^*$. Therefore $LA^{-1}(A^* - A)x_t^* < 0$.

Because technical change is labour inelastic, it follows that $(L^* - L)x_t^* < LA^{-1}(A^* - A)x_t^* = LA^{-1}A^*x_t^* - Lx_t^*$, which implies $L^*x_t^* < LA^{-1}A^*x_t^* = N_t$ which yields the desired contradiction.

Part (ii). Suppose, ad absurdum, that $\pi_t^{*\max} > 0$. By Lemma 1, $p_t^*A^*x_t^{*\nu} = p_t^*\omega_{t-1}^\nu$, all $\nu \in \mathcal{N}_t$ and by Lemma 2, $p_t^* > \mathbf{0}$. Therefore by Definition 1(b), $A^*x_t^* = \omega_{t-1}$ and, noting that $Lx_t = LA^{-1}\omega_{t-1} = N_t$ it follows that $LA^{-1}A^*x_t^* = N_t$. By Definition 1(c), and noting that $L^* \geq L$ and $x_t^* > \mathbf{0}$, it follows that $N_t \geqq L^*x_t^* > Lx_t^*$. Therefore $LA^{-1}(A^* - A)x_t^* > 0$. Because technical change is labour inelastic, $(L^* - L)x_t^* > LA^{-1}(A^* - A)x_t^* = N_t - Lx_t^*$, which implies $L^*x_t^* > N_t$, in contradiction with Definition 1(c).

To see that $(A, L) \mapsto (A^*, L^*)$ cannot be regressive, observe that if $\pi_t^{*\max} = 0$ at the CE, then $(p_t^*, w_t^*) = \left(\frac{v^*}{v^*b}, \frac{1}{v^*b}\right)$. As (A^*, L^*) is optimal at prices (p_t^*, w_t^*), it follows that $v^* \leqq v^*A + L$. Thus, $v^* \leqq v$ holds, and technical change cannot be regressive. ■

Suppose the economy is on a growth path with full employment of productive factors, but a new technique (A^*, L^*) emerges, at the end of period $t-1$, and technical change is profitable. If (A^*, L^*) is adopted in equilibrium, then by Theorem A1(i) two things can happen: if technical change is CU-LS, and it leads to the emergence of an excess supply of labour and unemployment, then the profit rate increases and the wage rate falls to the subsistence level. This is the Marxian "industrial reserve army of the unemployed". Together, Theorem 1, Proposition 1, and Theorem A1 may be interpreted as illustrating Marx's [21] general law of capitalist accumulation. If, however, technical change is CS-LU, or more generally the shift to the new technique makes aggregate capital abundant relative to the labour force, then the equilibrium profit rate falls to zero.[29]

Theorem A2 characterises equilibria with a new technique when the aggregate capital stock at t is *not* sufficient to allow for the full employment of labour using the old production technique ($N_t > LA^{-1}\omega_{t-1}$):

Theorem A2: Let $\left((p_{t-1}, w_{t-1}), ((A, L); \xi_{t-1}^\nu; \omega_{t-1}^\nu)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a PCE with $Ax_{t-1} = \omega_{t-2}$ and $Lx_{t-1} < N_{t-1}$ for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$. Let $(A, L) \mapsto (A^*, L^*)$ be profitable and labour inelastic. Suppose it results in a CE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. If $(A, L) \mapsto (A^*, L^*)$ is either CU-LS or CS-LU with sufficiently small $(A - A^*, L^* - L)$, then $w_t^* = p_t^*b$ and $\pi_t^{*\max} > \pi_{t-1}^{\max}$ whenever $A^*x_t^* = \omega_{t-1}$; otherwise, $\pi_t^{*\max} = 0$.

Proof: 1. As the CE in period $t-1$ is persistent, Proposition 1 implies that $N_t \geqq LA^{-1}\omega_{t-1}$, and there exist $(\xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}$ such that $x_t > \mathbf{0}$ with $Ax_t = \omega_{t-1}$ and $\left((p_{t-1}, w_{t-1}), ((A, L); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ is a CE for $E(\mathcal{N}_t; \mathcal{B}_{t-1}; b; \Omega_{t-1})$.
2. If $A^*x_t^* \leq \omega_{t-1} = Ax_t$, then the result follows as in the proof of Theorem A1(i). Therefore consider the case with $A^*x_t^* = \omega_{t-1}$.
3. Let $(A, L) \mapsto (A^*, L^*)$ be CU-LS. Suppose $L^*x_t^* \geqq Lx_t$. Because $L^* \leq L$ and $x_t^* > \mathbf{0}$, it follows that $Lx_t^* > L^*x_t^*$. Noting that $Lx_t = LA^{-1}\omega_{t-1}$, we obtain $LA^{-1}(A^* - A)x_t^* < 0$.

[29] If technical change is CS-LU and regressive, then the new technique will not be adopted in equilibrium consistent with Theorems 5 and 6.

Since technical change is labour inelastic, it follows that $(L^* - L) x_t^* < LA^{-1}(A^* - A) x_t^* < 0$, which in turn implies $L^* x_t^* < Lx_t$, yielding the desired contradiction. Thus, $L^* x_t^* < Lx_t \leqq N_t$. Therefore, Theorem 1(ii) implies $w_t^* = p_t^* b$, which in turn implies $\pi_t^{*\max} > \pi_{t-1}^{\max}$ by Theorem 2.

4. Let $(A, L) \mapsto (A^*, L^*)$ be CS-LU. Suppose that $N_t = LA^{-1}\omega_{t-1}$. As $A^* x_t^* = \omega_{t-1}$, $N_t = LA^{-1}A^* x_t^*$ holds. As in the proof of Theorem A1(ii), it can be shown that $N_t \geqq L^* x_t^* > Lx_t^*$ holds, and so $LA^{-1}(A^* - A) x_t^* > 0$. Then, as in the proof of Theorem A1(ii), the labour inelasticity of the technical change implies that $L^* x_t^* > N_t$, in contradiction with Definition 1(c). Therefore, given that $N_t \geqq LA^{-1}\omega_{t-1}$ holds, we cannot but conclude that $LA^{-1}\omega_{t-1} < N_t$. Since $(A - A^*, L^* - L)$ is sufficiently small, $L^* A^{*-1}\omega_{t-1}$ is sufficiently close to $LA^{-1}\omega_{t-1}$, which implies that $N_t > L^* A^{*-1}\omega_{t-1}$ holds. Then, $w_t^* = p_t^* b$ follows from Theorem 1(ii), and by Theorem 2, $\pi_t^{*\max} > \pi_{t-1}^{\max}$. ■

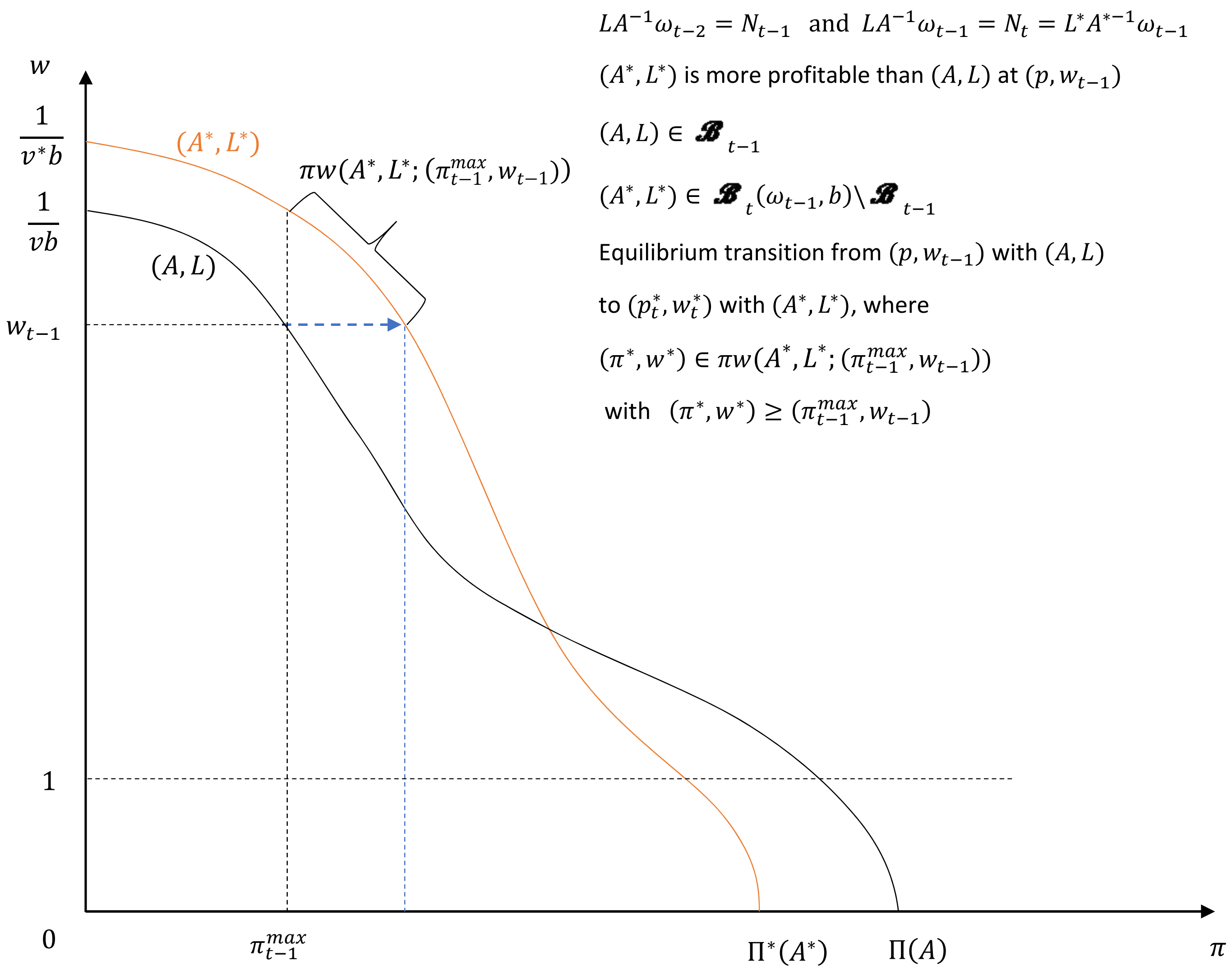

$LA^{-1}\omega_{t-2} = N_{t-1}$ and $LA^{-1}\omega_{t-1} = N_t = L^*A^{*-1}\omega_{t-1}$
(A^*, L^*) is more profitable than (A, L) at (p, w_{t-1})
$(A, L) \in \mathcal{B}_{t-1}$
$(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \setminus \mathcal{B}_{t-1}$
Equilibrium transition from (p, w_{t-1}) with (A, L)
to (p_t^*, w_t^*) with (A^*, L^*), where
$(\pi^*, w^*) \in \pi w(A^*, L^*; (\pi_{t-1}^{max}, w_{t-1}))$
with $(\pi^*, w^*) \geq (\pi_{t-1}^{max}, w_{t-1})$
w
$\frac{1}{v^*b}$
$\frac{1}{vb}$
w_{t-1}
1
0
(A^*, L^*)
$\pi w(A^*, L^*; (\pi_{t-1}^{max}, w_{t-1}))$
(A, L)
π_{t-1}^{max}
$\Pi^*(A^*)$
$\Pi(A)$
π

Figure 1

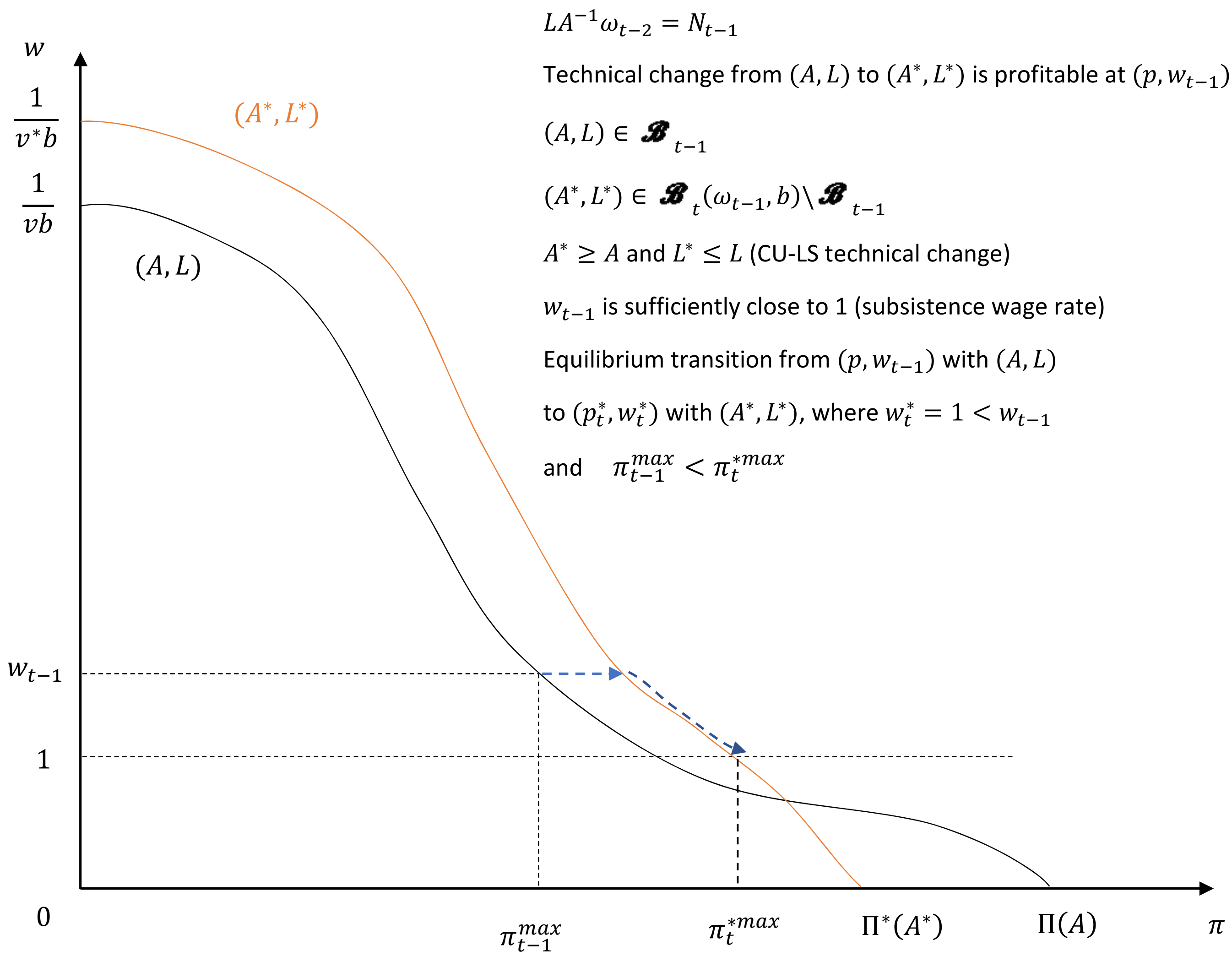
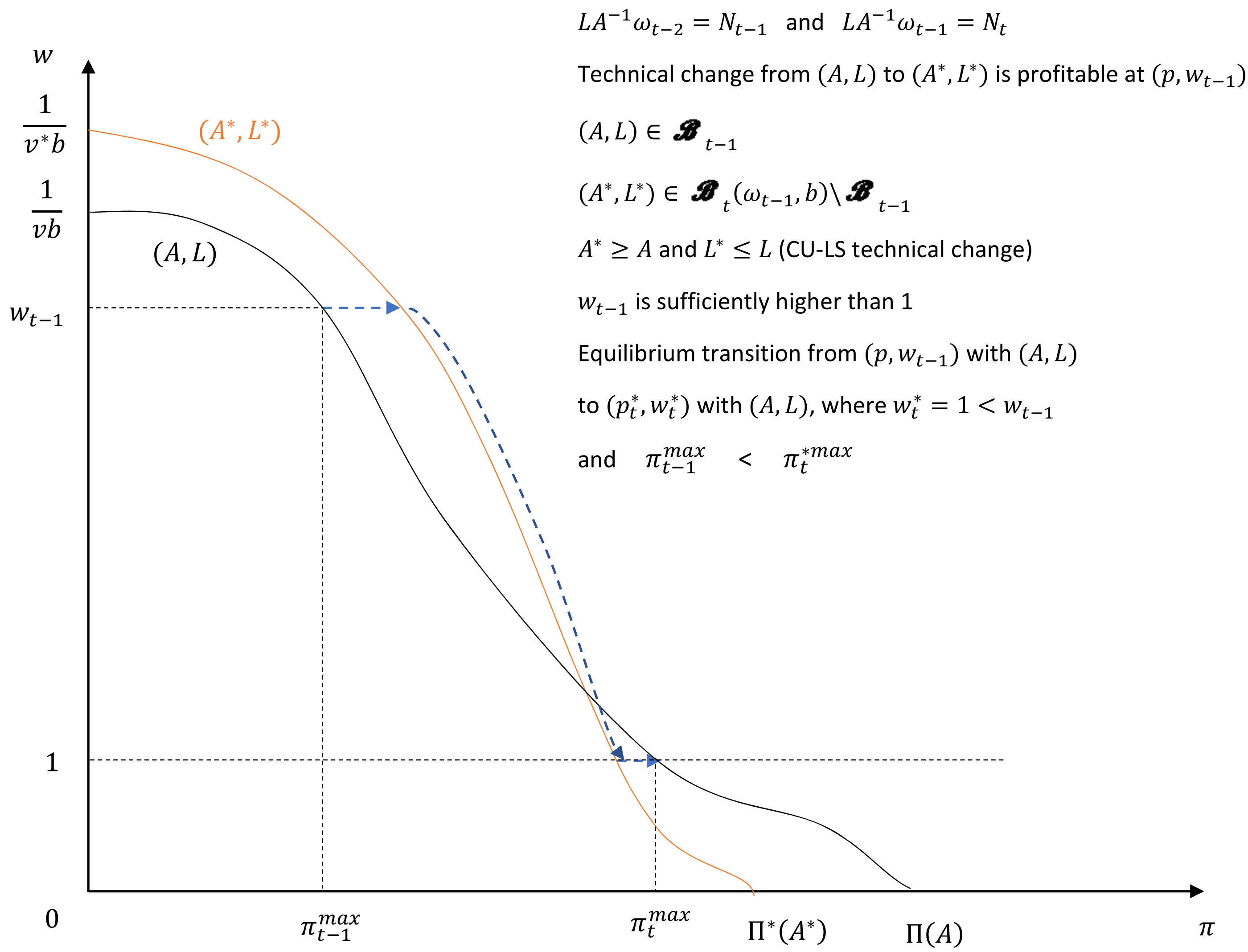

$LA^{-1}\omega_{t-2} = N_{t-1}$
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L) \in \mathscr{B}_{t-1}$
$(A^*, L^*) \in \mathscr{B}_t(\omega_{t-1}, b) \backslash \mathscr{B}_{t-1}$
$A^* \geq A$ and $L^* \leq L$ (CU-LS technical change)
w_{t-1} is sufficiently close to 1 (subsistence wage rate)
Equilibrium transition from (p, w_{t-1}) with (A, L)
to (p_t^*, w_t^*) with (A^*, L^*), where $w_t^* = 1 < w_{t-1}$
and $\pi_{t-1}^{max} < \pi_t^{*max}$
w
$\frac{1}{v^*b}$
$\frac{1}{vb}$
(A^*, L^*)
(A, L)
w_{t-1}
1
0
π_{t-1}^{max}
π_t^{*max}
$\Pi^*(A^*)$
$\Pi(A)$
π

Figure 2



$LA^{-1}\omega_{t-2} = N_{t-1}$ and $LA^{-1}\omega_{t-1} = N_t$
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L) \in \mathscr{B}_{t-1}$
$(A^*, L^*) \in \mathscr{B}_t(\omega_{t-1}, b) \backslash \mathscr{B}_{t-1}$
$A^* \geq A$ and $L^* \leq L$ (CU-LS technical change)
w_{t-1} is sufficiently higher than 1
Equilibrium transition from (p, w_{t-1}) with (A, L)
to (p_t^*, w_t^*) with (A, L), where $w_t^* = 1 < w_{t-1}$
and π_{t-1}^{max} < π_t^{*max}
w
$\frac{1}{v^* b}$
$\frac{1}{vb}$
(A^*, L^*)
(A, L)
w_{t-1}
1
0
π_{t-1}^{max}
π_t^{max}
$\Pi^*(A^*)$
$\Pi(A)$
π

Figure 3

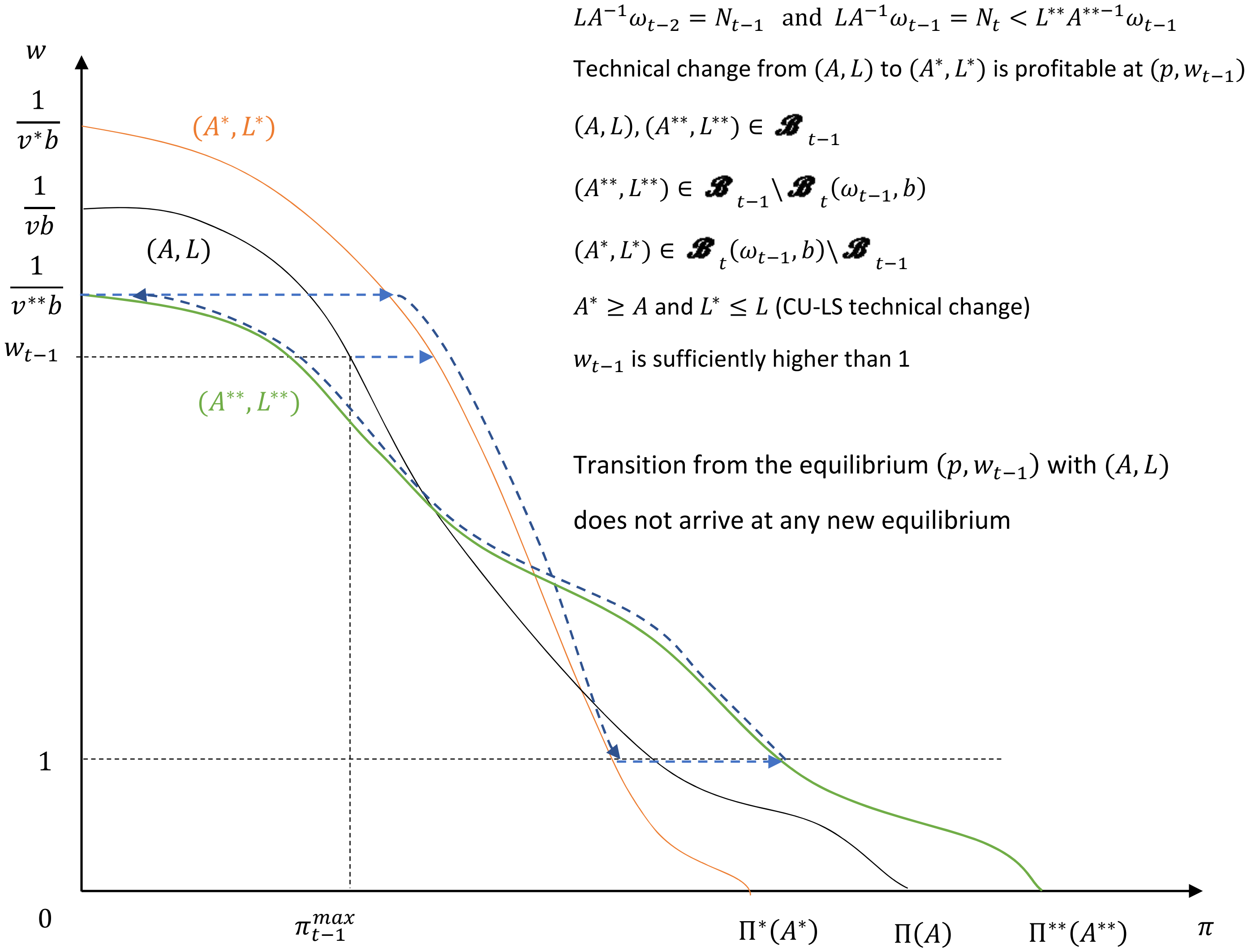

$LA^{-1}\omega_{t-2} = N_{t-1}$ and $LA^{-1}\omega_{t-1} = N_t < L^{**}A^{**-1}\omega_{t-1}$
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L), (A^{**}, L^{**}) \in \mathscr{B}_{t-1}$
$(A^{**}, L^{**}) \in \mathscr{B}_{t-1} \backslash \mathscr{B}_t(\omega_{t-1}, b)$
$(A^*, L^*) \in \mathscr{B}_t(\omega_{t-1}, b) \backslash \mathscr{B}_{t-1}$
$A^* \geq A$ and $L^* \leq L$ (CU-LS technical change)
w_{t-1} is sufficiently higher than 1
Transition from the equilibrium (p, w_{t-1}) with (A, L)
does not arrive at any new equilibrium
w
$\frac{1}{v^* b}$
$\frac{1}{vb}$
$\frac{1}{v^{**} b}$
w_{t-1}
(A^*, L^*)
(A, L)
(A^{**}, L^{**})
1
0
π_{t-1}^{max}
$\Pi^*(A^*)$
$\Pi(A)$
$\Pi^{**}(A^{**})$
π

Figure 4

Addendum for the paper: "*The general equilibrium effects of localised technological progress: A Classical approach*"

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Abstract

Section 1 contains the proofs omitted in the paper. Section 2 provides a discussion of the connections between the falling rate of profit and capital theory mentioned in section 6 of the paper. Section 3 provides a formal statement of the distributive implications of technical progress in developing economies. Section 4 provides the proof of existence of equilibrium.

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1 Proofs

Proof of Lemma 1: The result follows immediately from equation (6). ■

Proof of Lemma 2: At a CE, as $x_t > \mathbf{0}$ holds, $\pi_t^{\max} \geqq 0$ must hold. Indeed, if $\pi_t^{\max} < 0$, then $x_t^\nu = \mathbf{0}$ constitutes an optimal action at the CE for every ν. Moreover, since only sectors yielding the maximum profit rate are activated at the solution to MP_t^ν, $x_t > \mathbf{0}$ implies that $p_t = (1 + \pi_t^{\max}) p_t A_t + w_t L_t$ holds. Then, by the productiveness and the indecomposability of A_t, $p_t = \pi_t^{\max} p_t A_t (I - A_t)^{-1} + w_t L_t (I - A_t)^{-1} > \mathbf{0}$ holds. Finally, $x_t > \mathbf{0}$ implies that $L_t x_t > 0$ and therefore by Definition 1(c) and (6), it must be $w_t \geqq p_t b$. ■

Proof of Theorem 1: *Part (i).* By Lemma 1, $p_t A_t x_t^\nu = p_t \omega_{t-1}^\nu$ and $l_t^\nu = 1$ for all $\nu \in \mathcal{N}_t$. Then, $p_t A_t x_t = p_t \omega_{t-1}$ holds, and by Definition 1(c), $L_t x_t = l_t = N_t$. By Lemma 2, $p_t > \mathbf{0}$. Therefore, $p_t A_t x_t = p_t \omega_{t-1}$ and Definition 1(b) imply $A_t x_t = \omega_{t-1}$. Since $x_t = A_t^{-1} \omega_{t-1}$, then $N_t = L_t A_t^{-1} \omega_{t-1}$ holds.

Part (ii). Suppose, contrary to the statement, that $w_t > p_t b$. Then, by Lemma 1, $l_t^\nu = 1$, all $\nu \in \mathcal{N}_t$. But, then noting that $A_t x_t \leqq \omega_{t-1}$ by Definition 1(b), $N_t > L_t A_t^{-1} \omega_{t-1}$, implies that $L_t x_t < l_t$ holds, contradicting Definition 1(c).

Part (iii). Suppose, contrary to the statement, that $\pi_t^{\max} > 0$. Then, by Lemma 1, $p_t A_t x_t^\nu = p_t \omega_{t-1}^\nu$, for all $\nu \in \mathcal{N}_t$, and so $p_t A_t x_t = p_t \omega_{t-1}$. Therefore since by Lemma 2 $p_t > \mathbf{0}$ holds, Definition 1(b) implies $A_t x_t = \omega_{t-1}$. But, since $N_t < L_t A_t^{-1} \omega_{t-1}$, $L_t x_t > l_t$ holds, contradicting Definition 1(c). ■

Proof of Theorem 7: ((2) ⇒ (3)) Let $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ be a CE with $\pi_t^{*\max} = 0$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. As $\pi_t^{*\max} = 0$, it follows that $(p_t^*, w_t^*) = \left(\frac{1}{v^* b} v^*, \frac{1}{v^* b}\right)$ and $(A^*, L^*) \in \arg\min_{(A', L') \in \mathcal{B}_t} L' (I - A')^{-1} b$. Moreover, as $w_t^* > p_t^* b$, Definition 1(c)-(d) together with Lemma 1 imply $x_t^* \equiv \sum_{\nu \in \mathcal{N}_t} x_t^{*\nu} > \mathbf{0}$ and $L^* x_t^* = N$. Suppose, by way of contradiction, that $A^* x_t^* = \omega_{t-1}$. Because $(A^*, L^*) \notin \mathcal{B}_t (\omega_{t-1}, b)$, then $(I - bL^*) x_t^* \ngeqq \mathbf{0}$. However, by Definition 1(d), $(I - bL^*) x_t^* + \delta_t^* \geqq \mathbf{0}$ must hold, which implies that $\delta_t^* \geq \mathbf{0}$. Then, $A^* x_t^* + \delta_t^* \geq \omega_{t-1}$, which contradicts Definition 1(b). Therefore, $A^* x_t^* \leq \omega_{t-1}$ should hold, and $\delta_t^* = \omega_{t-1} - A^* x_t^*$. Thus, by Definition 1(d), $(I - bL^*) x_t^* \geqq A^* x_t^* - \omega_{t-1}$.

((1) ⇒ (2)) We show that at any CE in which (A^*, L^*) is adopted it must be $\pi_t^{*\max} = 0$. Suppose, ad absurdum, that there exists a CE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^{*\nu}; \omega_t^{*\nu})_{\nu \in \mathcal{N}_t}\right)$ with $\pi_t^{*\max} > 0$. Then, by Lemma 1 and

Lemma 2, $A^* x_t^* = \omega_{t-1}$. Then, as $(A^*, L^*) \notin \mathcal{B}_t(\omega_{t-1}, b)$, it implies that $(I - bL^*) x_t^* \ngeqq \mathbf{0}$. Then, as shown in the first part of the proof of Theorem 7, we derive a contradiction from Definition 1(b)-(d).

((3) $\Rightarrow$ (1)) Suppose there exist $x^* > \mathbf{0}$ such that $(I - bL^*) x^* \geqq A^* x^* - \omega_{t-1}$ with $A^* x^* \leq \omega_{t-1}$ and $L^* x^* = N_t$; and $(A^*, L^*) \in \arg\min_{(A', L') \in \mathcal{B}_t} L' (I - A')^{-1} b$. Then, (A^*, L^*) is optimal at $(p_t^*, w_t^*) \equiv \left(\frac{1}{v^* b} v^*, \frac{1}{v^* b}\right)$. Further, let $\delta^* \equiv \omega_{t-1} - A^* x^* \geq \mathbf{0}$. Then, for all $\nu \in \mathcal{N}_t$, let $x_t^{*\nu} = \frac{p_t^* \omega_{t-1}^{*\nu}}{p_t^* \omega_{t-1}^*} x^* \geqq \mathbf{0}$, $\delta_t^{*\nu} = \frac{p_t^* \omega_{t-1}^{*\nu}}{p_t^* \omega_{t-1}^*} \delta^* \geqq \mathbf{0}$, $l_t^{*\nu} = 1$, and

$$\omega_t^{*\nu} = \frac{(p_t^* - w_t^* L^*) x_t^{*\nu} + w_t^* - p_t^* b + p_t^* \delta_t^{*\nu}}{p_t^* [(I - bL^*) x^* + \omega_{t-1} - A^* x^*]} [(I - bL^*) x^* + \omega_{t-1} - A^* x^*] \geqq \mathbf{0}.$$

It is immediate to prove that $((A^*, L^*), (x_t^{*\nu}, 1, \delta_t^{*\nu}); \omega_t^{*\nu})$ solves MP_t^ν for all $\nu \in \mathcal{N}_t$. Furthermore, by construction, Definition 1(b)-(e) are satisfied and $\pi_t^{*\max} = 0$ holds. ■

2 The falling rate of profit and capital theory

As mentioned in section 6 of the paper, Theorem 5 shows some interesting and perhaps surprising connections between the theory of the falling profit rate and some central insights of classical capital theory.

As an illustration, and without any loss of generality, consider the simplest possible case of technical change, whereby only one technique is known in period $t-1$, so that $\mathcal{B}_{t-1} = \{(A, L)\}$ and $\mathcal{B}_t = \{(A, L), (A^*, L^*)\}$. Under the conditions of Theorem 5, the wage-profit curve of the new technique, $\pi w(A^*, L^*)$, dominates the wage-profit curve of (A, L), $\pi w(A, L)$, at least in a neighbourhood of points $\left(0, \frac{1}{v^* b}\right)$ and $(\Pi(A^*), 0)$,[1] as well as in the non-empty subset $\pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$.[2]

[1] The former follows noting that if the condition in Theorem 5 holds, then $\frac{1}{v' b} \leqq \frac{1}{v^* b}$ for all $(A', L') \in \mathcal{B}_t$ and $\pi w(A^*, L^*)$ coincides with the wage-profit frontier $\pi w(\mathcal{B}_t)$ in a neighbourhood of $(\pi_t^{*\max}, w_t^*) = \left(0, \frac{1}{v^* b}\right)$. The latter follows noting that $A^* \leq A$ implies $\Pi(A^*) > \Pi(A)$.

[2] Because technical change is profitable, an argument similar to that used for Theorem 3 shows that the set $\pi w\left(A^*, L^*; \left(\pi_{t-1}^{\max}, w_{t-1}\right)\right)$ is non-empty and coincides with $\pi w(\mathcal{B}_t)$.

Then, there are two scenarios in which the profit rate will fall. In the first, $\pi w(A^*, L^*)$ completely dominates $\pi w(A, L)$ as shown in Figure A1.

Insert Figure A1 around here.

In this case, technical change is profitable at *any* prices and yet, according to Theorem 5 the adoption of (A^*, L^*) leads the equilibrium profit rate to drop to zero. This is quite a strong – and perhaps surprising – result from a theoretical viewpoint, but it is possibly of limited empirical relevance, because innovations that are profitable at any prices are rare.

Alternatively, if $\pi w(A^*, L^*)$ does not completely dominate $\pi w(A, L)$, and given that the former dominates the latter in at least three regions, the two curves must intersect at least twice, as shown in Figure A2.

Insert Figure A2 around here.

Figure A2 describes a situation in which a *reswitching of techniques* (Kurz and Salvadori [2], p.148) occurs: because $\frac{1}{vb} < \frac{1}{v^*b}$, close to the vertical axis the wage-profit frontier coincides with the wage-profit curve of the technique (A^*, L^*), which is therefore optimal for a sufficiently small (or zero) profit rate. Further, as (A^*, L^*) is the optimal technique at $\pi^* = 0$, the corresponding wage rate, $w^* = \frac{1}{v^*b}$ is higher than the wage rate, $w = \frac{1}{vb}$, associated with $\pi = 0$ under (A, L). In this case, as well-known in the literature on the Cambridge capital controversy, the capital-labour ratio of (A^*, L^*) is higher than that of (A, L) when the values of capital are measured by means of the commodity price vectors corresponding to each of the two switching points, and so (A^*, L^*) is a more capital-intensive technique than (A, L).

As the profit rate increases, a switching point arrives after which the frontier coincides with $\pi w(A, L)$ and the more labour-intensive technique (A, L) becomes optimal. However, since $\Pi(A^*) > \Pi(A)$ another switching point exists after which, as the profit rate *increases* further, the *capital intensive* technique (A^*, L^*) becomes optimal again – a phenomenon known in the literature as *capital reversing* (Kurz and Salvadori [2], pp.447-451).

In other words, setting aside the empirically less relevant case of an innovation unambiguously dominating older techniques, the above arguments show that there exists an interesting relation between capital theory – and the phenomena known as reswitching of techniques and capital reversing, – and the theory of the falling profit rate.

3 CS-LU Technical Change in Developing Economies

Consider a developing economy in which the social endowments of capital stocks accumulated in the past are still very low relative to the size of population. In this case, it is natural to assume that a persistent CE is characterised by $N_{t-1} > LA^{-1}\omega_{t-2}$ and ask whether the premise of Theorem A2 can be satisfied. This is particularly relevant if a CS-LU change of technique is considered, as in the next result.

Theorem 8: Let $\left((p, w_{t-1}), \left((A, L); \xi^{\nu}_{t-1}; \omega^{\nu}_{t-1}\right)_{\nu \in \mathcal{N}_{t-1}}\right)$ be a persistent CE for $E(\mathcal{N}_{t-1}; \mathcal{B}_{t-1}; b; \Omega_{t-2})$ associated with $N_{t-1} > LA^{-1}\omega_{t-2}$. Let $(A^*, L^*) \in \mathcal{B}_t(\omega_{t-1}, b) \backslash \mathcal{B}_{t-1}$ be a new technique. Let $(A, L) \mapsto (A^*, L^*)$ be profitable, labour inelastic, and CS-LU with sufficiently small $(A - A^*, L^* - L)$. Then, there exists a CE $\left((p^*, w^*_t), \left((A^*, L^*); \xi^{*\nu}_t; \omega^{*\nu}_t\right)_{\nu \in \mathcal{N}_t}\right)$ with $w^*_t = 1$ and $\pi^{*\max}_t > \pi^{\max}_{t-1}$ for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$.

Proof: Following the proof of Theorem A2, we can see that $N_t > L^* A^{*-1}\omega_{t-1}$ holds. Then, noting that $N_{t-1} > LA^{-1}\omega_{t-2}$ implies $w_{t-1} = 1$, it follows that (A^*, L^*), $x^{*\nu}_t = \frac{p^*\omega^{\nu}_{t-1}}{p^*\omega_{t-1}} A^{*-1}\omega_{t-1}$, $l^{*\nu}_t = \frac{L^* A^{*-1}\omega_{t-1}}{N_t}$, $\delta^{*\nu}_t = \mathbf{0}$, and $\omega^{*\nu}_t = \frac{p^* x^{*\nu}_t - w^* L^* x^{*\nu}_t + (w^* - 1) l^{*\nu}_t}{(p^* - L^*) A^{*-1}\omega_{t-1}} (I - bL^*) A^{*-1}\omega_{t-1}$ solve MP^{ν}_t for all $\nu \in \mathcal{N}_t$ and Definition 1(b)-(e) are satisfied for $(\pi^{*\max}_t, w^*_t = 1) \in \pi w(\mathcal{B}_t) \cap \pi w(A^*, L^*)$ with $p^* = w^*_t L^* \left[I - (1 + \pi^{*\max}_t) A^*\right]^{-1} > \mathbf{0}$, as in the proof of Theorem 4. Thus, $\left((p^*, w^*_t), \left((A^*, L^*); \xi^{*\nu}_t; \omega^{*\nu}_t\right)_{\nu \in \mathcal{N}_t}\right)$ constitutes a CE for $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$. By Theorem A2, $\pi^{*\max}_t > \pi^{\max}_{t-1}$ holds. ■

Theorem 8 shows that the premise of Theorem A2 is satisfied: if there is a sufficiently big industrial reserve army of the unemployed, then a profitable, gradual, CS-LU change of technique will indeed be adopted in equilibrium, and lead to an increase in the profit rate, even if this change of technique is regressive.

Both the assumption $N_{t-1} > LA^{-1}\omega_{t-2}$, and the characteristics of the new equilibrium described in Theorem 8 are quite realistic in developing economies, in which aggregate labour is abundant relative to the level of accumulated capital stock. These economies may wish to import the advanced technology (a more capital-intensive technique) from advanced economies, but their aggregate capital endowments are often insufficient to adopt capital-intensive techniques. In this case, developing economies may modify such advanced technology into a slightly more labour-intensive one, as in the case of the Japanese economy just after the Meiji Revolution around the mid 19th century (see, e.g., Allen [1]).

4 The existence of a PCE

In this appendix, we analyse the existence of PCEs for an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ with a set of agents $\mathcal{N}_t$, a set of base techniques $\mathcal{B}_t$, a subsistence vector b, and a distribution of physical endowments Ω_{t-1}, in period t. Recall that for each $(A, L) \in \mathcal{B}_t$, $\pi w(A, L)$ is the wage-profit curve associated with (A, L) and $\pi w(\mathcal{B}_t)$ is the wage-profit frontier associated with the set $\mathcal{B}_t$. Then, let

$$\overline{\mathcal{B}}_t \equiv \{(A, L) \in \mathcal{B}_t \mid \exists (\pi, w) \in \pi w(A, L) \cap \pi w(\mathcal{B}_t) \text{ s.t. } (\pi, w) \geq (0, 1)\}.$$

4.1 The existence of PCEs with full employment

Given an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ at period t and given $\mathcal{N}_{t+1}$, we define the following set of productive endowments:

$$C_t^* \equiv \left\{\omega \in \mathbb{R}_+^n \mid \exists x^* > \mathbf{0} \ \& \ (A, L) \in \overline{\mathcal{B}}_t \text{: } \begin{array}{c} Ax^* = \omega, \ (I - bL)x^* \geq \mathbf{0}, \ Lx^* = N_t, \\ A^{-1}(x^* - N_t b) > \mathbf{0}, \ LA^{-1}(x^* - N_t b) = N_{t+1} \end{array}\right\}.$$

We will prove that $\omega_{t-1} \in C_t^*$ is the necessary and sufficient condition for the existence of a PCE with full employment of all productive factors. As a first step, we prove that the set C_t^* is well-defined. To show it, let us define

$$\overline{\mathcal{B}}_t(N_t, N_{t+1}) \equiv \left\{(A, L) \in \overline{\mathcal{B}}_t \mid L\left[I - \frac{N_{t+1}}{N_t}A\right]^{-1} b = 1\right\}.$$

Then:

Theorem A.3: Consider an economy $E(\mathcal{N}_t; \mathcal{P}_t; b; \Omega_{t-1})$ at period t and a set $\mathcal{N}_{t+1}$. Then, if $\overline{\mathcal{B}}_t(N_t, N_{t+1}) \neq \varnothing$, then $C_t^* \neq \varnothing$.

Proof. As $\overline{\mathcal{B}}_t(N_t, N_{t+1}) \neq \varnothing$, let $(A, L) \in \overline{\mathcal{B}}_t(N_t, N_{t+1})$. Let $(1 + g) \equiv \frac{N_{t+1}}{N_t}$. Then, $L[I - (1 + g)A]^{-1} b = 1$ holds. The last equation implies that there exists $p \in \triangle$ such that $p \equiv L\left[I-(1+g)A\right]^{-1} > \mathbf{0}$. Therefore, $p = p\left[(1+g)A + bL\right]$ holds, which implies that the Frobenius eigenvalue of the matrix $\left[(1+g)A + bL\right]$ is equal to 1 and is associated with the unique Frobenius eigenvector $p > \mathbf{0}$. Then, there exists the Frobenius eigenvector $x^* > \mathbf{0}$ such that $x^* = \left[(1+g)A + bL\right]x^*$ unique up to $Lx^* = N_t$. Then, $(1+g)Ax^* = x^* - N_t b$ holds. As $(1+g)Ax^* > \mathbf{0}$ by the indecomposability of A and $x^* > \mathbf{0}$, $x^* - N_t b = (I - bL)x^* > \mathbf{0}$. Moreover, $A^{-1}(x^* - N_t b) = A^{-1}(1+g)Ax^* = (1+g)x^* > \mathbf{0}$. Finally,

$$LA^{-1}(1+g)Ax^* = LA^{-1}(x^* - N_t b)$$

which is equivalent to

$$(1+g)Lx^* = LA^{-1}(x^* - N_t b) \Leftrightarrow \frac{N_{t+1}}{N_t}N_t = LA^{-1}(x^* - N_t b).$$

Thus, by $\omega \equiv Ax^*$, we can see that $\omega \in C_t^*$. ■

For each $(A, L) \in \mathcal{B}_t$, by the intermediate value theorem there exists $\pi \in (0, \Pi(A))$ such that $L\left[I-(1+\pi)A\right]^{-1}b = 1$ holds. Then, if $\overline{\mathcal{B}}_t$ contains sufficiently many different Leontief techniques, it is likely that a technique $(A, L) \in \overline{\mathcal{B}}_t$ will exist such that for some $\pi = \frac{N_{t+1}}{N_t} - 1$, $L\left[I-(1+\pi)A\right]^{-1}b = 1$ holds. In that case, the set C_t^* is likely to be non-empty.

Theorem A.4: Consider an economy $E(\mathcal{N}_t; \mathcal{P}_t; b; \Omega_{t-1})$ at period t and a set $\mathcal{N}_{t+1}$. Then, there exists a PCE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ such that $A^*x^* = \omega_{t-1}$ and $L^*x^* = N_t$ if and only if $\omega_{t-1} \in C_t^*$.

Proof. $(\Leftarrow)$ Let $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ be a PCE such that $A^*x_t^* = \omega_{t-1}$ and $L^*x_t^* = N_t$. As $(\pi_t^{*\max}, w_t^*) \geq (0,1)$ holds in this case, $(A^*, L^*) \in \overline{\mathcal{B}}_t$ holds. At this CE, $\omega_t = x_t^* - N_t b$ holds. As the CE is persistent, $\omega_t \geq \mathbf{0}$ holds, which implies that $x_t^* - N_t b = (I - bL)x_t^* \geq \mathbf{0}$. Moreover, by Proposition 1(i), $A^{*-1}\omega_t > \mathbf{0}$ holds, which implies that $A^{*-1}(x_t^* - N_t b) > \mathbf{0}$. Finally, by Proposition 1(i), $L^*A^{*-1}\omega_t = N_{t+1}$. Thus, $L^*A^{*-1}(x_t^* - N_t b) = N_{t+1}$ holds. In conclusion, $\omega_{t-1} \in C_t^*$ holds.

$(\Rightarrow)$ Suppose $\omega_{t-1} \in C_t^*$. Then, there exist $x^* > \mathbf{0}$ and $(A, L) \in \overline{\mathcal{B}}_t$ such that $Ax^* = \omega_{t-1}$, $(I - bL)x^* \geq \mathbf{0}$, $Lx_t^* = N_t$, $A^{-1}(x^* - N_t b) > \mathbf{0}$, and $LA^{-1}(x^* - N_t b) = N_{t+1}$. As $(A, L) \in \overline{\mathcal{B}}_t$, there exists $(\pi_t^{*\max}, w_t^*) \geq (0,1)$ such that for $p_t^* \equiv w_t^* L\left[I - (1+\pi_t^{*\max})A\right]^{-1} > \mathbf{0}$,

$$p_t^* = (1+\pi_t^{*\max})p_t^*A + w_t^*L \leqq (1+\pi_t^{*\max})p_t^*A' + w_t^*L' \text{ for any } (A', L') \in \mathcal{B}_t.$$

Then, for each $\nu \in \mathcal{N}_t$, a suitable optimal action profile $(\xi_t^\nu; \omega_t^\nu)$ with $\xi_t^\nu = (x_t^{\nu *}, 1, 0)$ can be specified so as to satisfy $\sum_{\nu \in \mathcal{N}_t} x_t^{\nu *} = x^*$ and $\sum_{\nu \in \mathcal{N}_t} \omega_t^\nu = \omega_t \equiv x^* - N_t b \geq \mathbf{0}$. Thus, $\left((p_t^*, w_t^*), ((A, L); \xi_t^\nu; \omega_t^\nu)_{\nu \in \mathcal{N}_t}\right)$ is a CE with the full employment of all productive factors. Finally, $A^{-1}\omega_t > \mathbf{0}$ and $LA^{-1}\omega_t = N_{t+1}$ follow from $A^{-1}(x^* - N_t b) > \mathbf{0}$ and $LA^{-1}(x^* - N_t b) = N_{t+1}$ respectively, and $\omega_t = x^* - N_t b$. Thus, Proposition 1 implies that this CE is persistent. ■

4.2 The existence of PCEs with unemployment of labour

Given an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$, let

$$\overline{\mathcal{B}}_t^* \equiv \{(A, L) \in \mathcal{B}_t \mid \exists (\pi, w) \in \pi w (A, L) \cap \pi w (\mathcal{B}_t) \text{ s.t. } \pi > 0 \ \& \ w = 1\}.$$

Given an economy $E(\mathcal{N}_t; \mathcal{B}_t; b; \Omega_{t-1})$ at period t and a set $\mathcal{N}_{t+1}$, a set of endowments of produced inputs is defined as follows:

$$C_t^{**} \equiv \left\{ \omega \in \mathbb{R}_+^n \mid \exists x^* > \mathbf{0} \ \& \ (A, L) \in \overline{\mathcal{B}}_t^*: \begin{array}{c} Ax^* = \omega, \ (I - bL)x^* \geq \mathbf{0}, \ Lx^* < N_t, \\ A^{-1}(I - bL)x^* > \mathbf{0}, \ LA^{-1}(I - bL)x^* < N_{t+1} \end{array} \right\}.$$

We will show that $\omega_{t-1} \in C_t^{**}$ is the necessary and sufficient condition for the existence of a PCE with unemployment of labour.

First, we show that C_t^{**} is well-defined:

Theorem A.5: Consider an economy $E(\mathcal{N}_t; \mathcal{P}_t; b; \Omega_{t-1})$ at period t and a set $\mathcal{N}_{t+1}$. Then, $C_t^{**} \neq \varnothing$.

Proof. Let $(A, L) \in \overline{\mathcal{B}}_t^*$. Then, by the intermediate value theorem, there exists $g_{(A,L)} > 0$ such that $L\left[I - \left(1 + g_{(A,L)}\right)A\right]^{-1} b = 1$. The last equation implies that there exists $p \in \triangle$ such that $p \equiv L\left[I - \left(1 + g_{(A,L)}\right)A\right]^{-1} > \mathbf{0}$. Therefore, $p = p\left[\left(1 + g_{(A,L)}\right)A + bL\right]$ holds. The last equations imply that the Frobenius eigenvalue of the matrix $\left[\left(1 + g_{(A,L)}\right)A + bL\right]$ is 1 associated with the unique Frobenius eigenvector $p > \mathbf{0}$. Then, there exists the Frobenius eigenvector $x^* > \mathbf{0}$ such that $x^* = \left[\left(1 + g_{(A,L)}\right)A + bL\right]x^*$ with $Lx^* < \frac{N_{t+1}}{1 + g_{(A,L)}}$. Then, $\left(1 + g_{(A,L)}\right)Ax^* = x^* - bLx^* > \mathbf{0}$ holds by $x^* > \mathbf{0}$ and the indecomposability of A. Moreover, $A^{-1}(I - bL)x^* = A^{-1}\left(1 + g_{(A,L)}\right)Ax^* = \left(1 + g_{(A,L)}\right)x^* > \mathbf{0}$ holds. Finally, we have

$$LA^{-1}\left(1 + g_{(A,L)}\right)Ax^* = LA^{-1}(x^* - bLx^*)$$

which is equivalent to $\left(1+g_{(A,L)}\right) Lx^* = LA^{-1}\left(x^* - bLx^*\right)$. Then, $\left(1+g_{(A,L)}\right) Lx^* < N_{t+1}$ as $Lx^* < \frac{N_{t+1}}{1+g_{(A,L)}}$. Thus, $LA^{-1}\left(x^* - bLx^*\right) < N_{t+1}$. Then, by $\omega \equiv Ax^*$, we can see that $\omega \in C_t^{**}$. ■

Theorem A.6: Consider an economy $E(\mathcal{N}_t; \mathcal{P}_t; b; \Omega_{t-1})$ at period t and a set $\mathcal{N}_{t+1}$. Then, there exists a PCE $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ such that $A^*x^* = \omega_{t-1}$ and $L^*x^* < N_t$ if and only if $\omega_{t-1} \in C_t^{**}$.

Proof. ($\Leftarrow$) Let $\left((p_t^*, w_t^*), ((A^*, L^*); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ be a PCE such that $A^*x_t^* = \omega_{t-1}$ and $L^*x_t^* < N_t$. As $\pi_t^{*\max} > 0$ and $w_t^* = 1$ hold in this case, $(A^*, L^*) \in \overline{\mathcal{B}}_t^*$ holds. In this CE, $\omega_t = x_t^* - bL^*x_t^*$ holds. As the CE is persistent, $\omega_t \geq \mathbf{0}$ holds, which implies that $(I - bL)\, x_t^* \geq \mathbf{0}$. Moreover, by Proposition 1(ii), $A^{*-1}\omega_t > \mathbf{0}$ holds, which implies that $A^{*-1}\left(x_t^* - bL^*x_t^*\right) > \mathbf{0}$. Finally, as this CE is persistent with unemployment of labour, it follows from Proposition 1(ii) that $L^*A^{*-1}\omega_t < N_{t+1}$. Thus, $L^*A^{*-1}\left(x_t^* - bL^*x_t^*\right) < N_{t+1}$ holds. In conclusion, $\omega_{t-1} \in C_t^{**}$ holds.

($\Rightarrow$) Let $\omega_{t-1} \in C_t^{**}$ hold. Then, there exist $x^* > \mathbf{0}$ and $(A, L) \in \overline{\mathcal{B}}_t^*$ such that $Ax^* = \omega_{t-1}$, $(I - bL)\, x^* \geq \mathbf{0}$, $Lx^* < N_t$, $A^{-1}(I - bL)\, x^* > \mathbf{0}$, and $LA^{-1}(I - bL)\, x^* < N_{t+1}$. As $(A, L) \in \overline{\mathcal{B}}_t^*$, there exist $\pi_t^{*\max} > 0$ and $w_t^* = 1$ such that for $p_t^* \equiv w_t^* L\left[I - (1 + \pi_t^{*\max})\, A\right]^{-1} > \mathbf{0}$,

$$p_t^* = (1 + \pi_t^{*\max})\, p_t^* A + w_t^* L \leqq (1 + \pi_t^{*\max})\, p_t^* A' + w_t^* L' \text{ for any } (A', L') \in \mathcal{B}_t.$$

Then, for each $\nu \in \mathcal{N}_t$, a suitable optimal action profile $(\xi_t^\nu; \omega_t^\nu)$ with $\xi_t^\nu = \left(x_t^{\nu*}, \frac{Lx^*}{N_t}, 0\right)$ can be specified so as to satisfy $\sum_{\nu\in\mathcal{N}_t} x_t^{\nu*} = x^*$ and $\sum_{\nu\in\mathcal{N}_t} \omega_t^\nu = \omega_t \equiv x^* - bLx^*$. Thus, $\left((p_t^*, w_t^*), ((A, L); \xi_t^\nu; \omega_t^\nu)_{\nu\in\mathcal{N}_t}\right)$ is a CE with unemployment of labour. Finally, $A^{-1}\omega_t > \mathbf{0}$ and $LA^{-1}\omega_t < N_{t+1}$ follow from $A^{-1}(I - bL)\, x^* > \mathbf{0}$ and $LA^{-1}\left(x^* - bLx^*\right) < N_{t+1}$ respectively, noting that $\omega_t = x^* - bLx^*$. Thus, Proposition 1(ii) implies that this CE is persistent. ■

References

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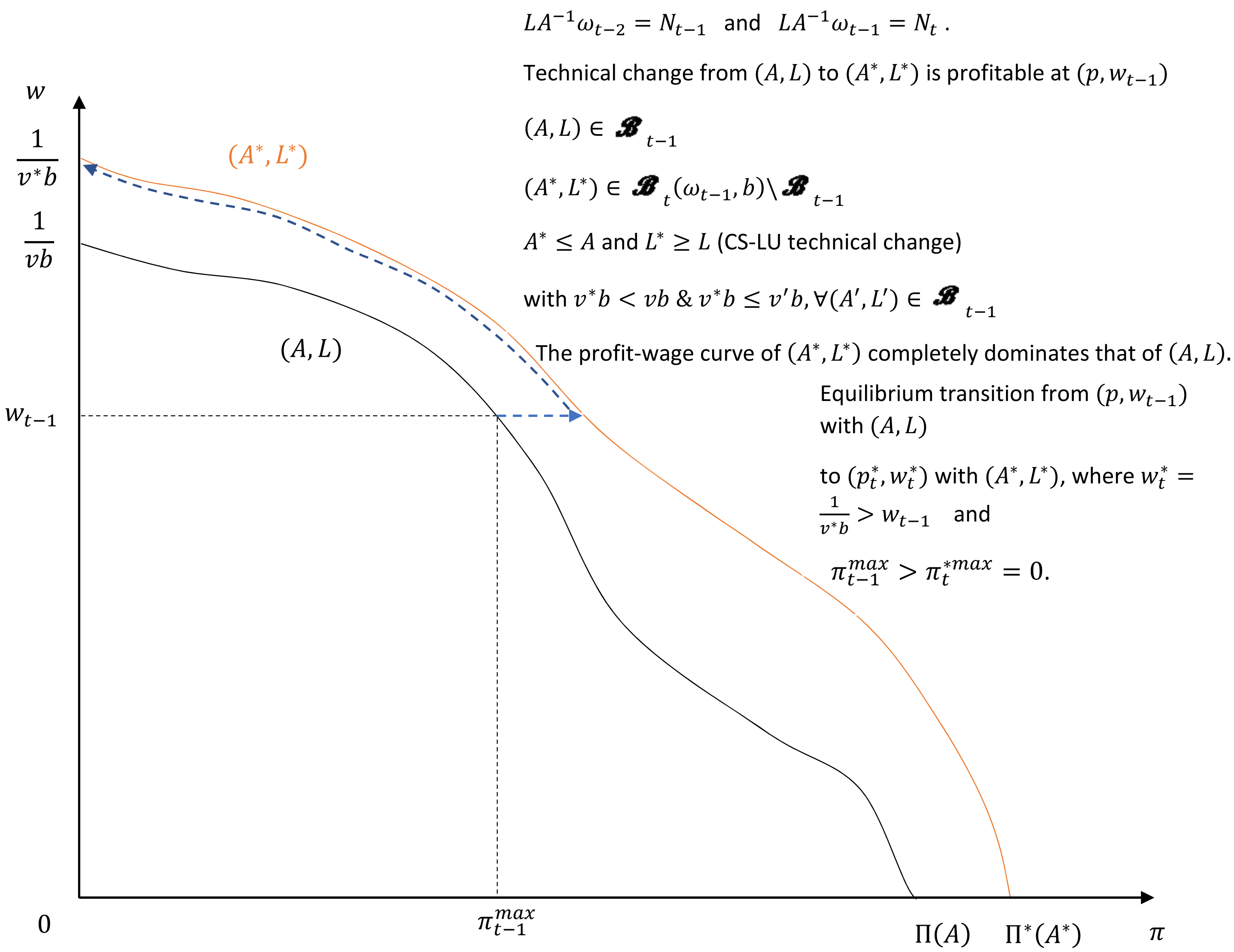
[2] Kurz, D.K., Salvadori, N., 1995. *Theory of Production: A Long-Period Analysis.* Cambridge University Press, Cambridge.



$LA^{-1}\omega_{t-2} = N_{t-1}$ and $LA^{-1}\omega_{t-1} = N_t$.
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L) \in \mathscr{B}_{t-1}$
$(A^*, L^*) \in \mathscr{B}_t(\omega_{t-1}, b) \backslash \mathscr{B}_{t-1}$
$A^* \leq A$ and $L^* \geq L$ (CS-LU technical change)
with $v^*b < vb$ & $v^*b \leq v'b, \forall(A', L') \in \mathscr{B}_{t-1}$
The profit-wage curve of (A^*, L^*) completely dominates that of (A, L).
Equilibrium transition from (p, w_{t-1}) with (A, L)
to (p_t^*, w_t^*) with (A^*, L^*), where $w_t^* = \frac{1}{v^*b} > w_{t-1}$ and
$\pi_{t-1}^{max} > \pi_t^{*max} = 0.$
w
$\frac{1}{v^*b}$
$\frac{1}{vb}$
w_{t-1}
(A^*, L^*)
(A, L)
0
π_{t-1}^{max}
$\Pi(A)$
$\Pi^*(A^*)$
π

Figure A1

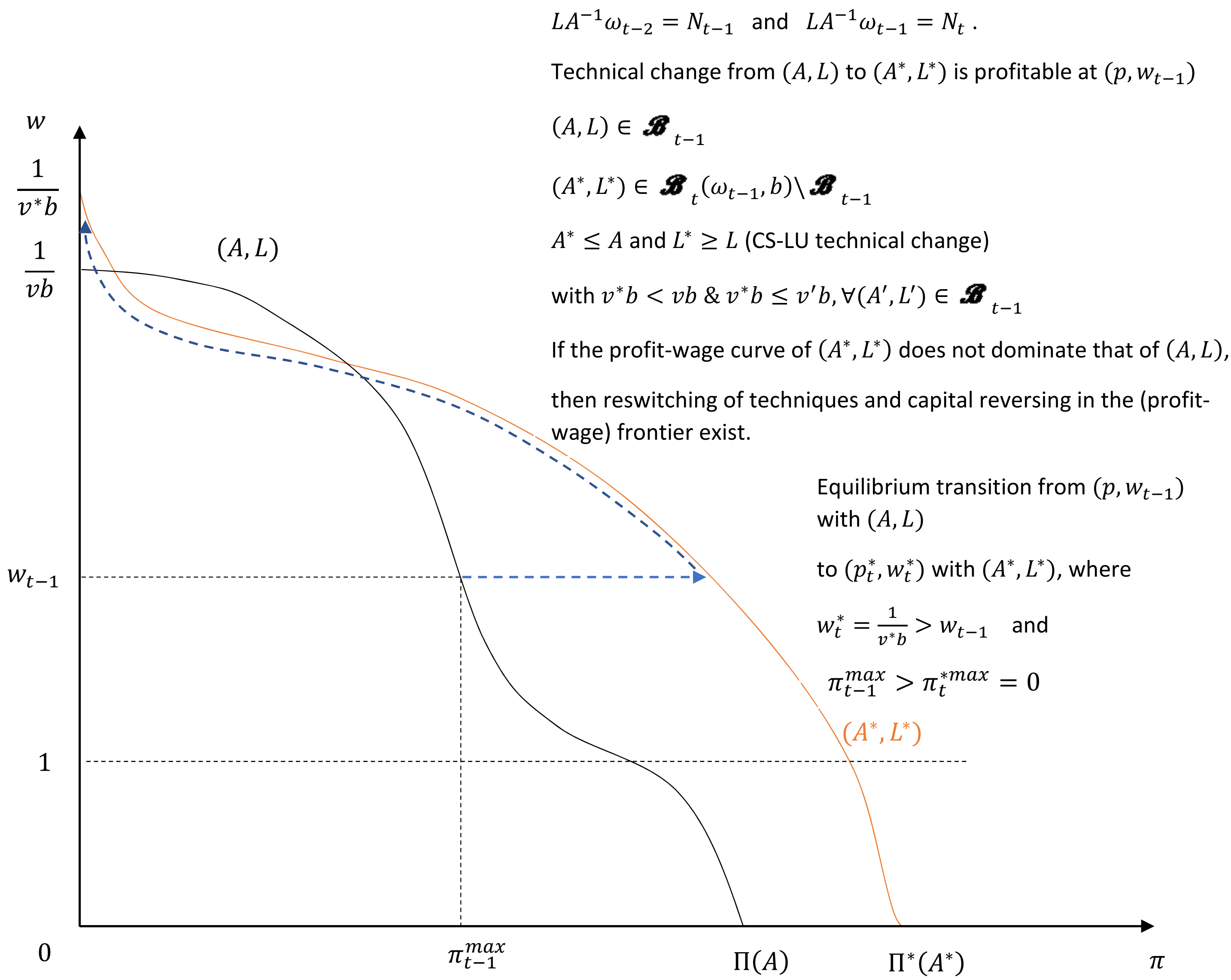

$LA^{-1}\omega_{t-2} = N_{t-1}$ and $LA^{-1}\omega_{t-1} = N_t$.
Technical change from (A, L) to (A^*, L^*) is profitable at (p, w_{t-1})
$(A, L) \in \mathscr{B}_{t-1}$
$(A^*, L^*) \in \mathscr{B}_t(\omega_{t-1}, b) \backslash \mathscr{B}_{t-1}$
$A^* \leq A$ and $L^* \geq L$ (CS-LU technical change)
with $v^*b < vb$ & $v^*b \leq v'b, \forall (A', L') \in \mathscr{B}_{t-1}$
If the profit-wage curve of (A^*, L^*) does not dominate that of (A, L),
then reswitching of techniques and capital reversing in the (profit-wage) frontier exist.
Equilibrium transition from (p, w_{t-1}) with (A, L)
to (p_t^*, w_t^*) with (A^*, L^*), where
$w_t^* = \frac{1}{v^*b} > w_{t-1}$ and
$\pi_{t-1}^{max} > \pi_t^{*max} = 0$
(A^*, L^*)
(A, L)
w
$\frac{1}{v^*b}$
$\frac{1}{vb}$
w_{t-1}
1
0
π_{t-1}^{max}
$\Pi(A)$
$\Pi^*(A^*)$
π

Figure A2