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A MINIMAX ANALYSIS OF MERTON'S PROBLEM

YOICHI KUWANA

Graduate School of Economics, Hitotsubashi University
Kunitachi, Tokyo 186-8601, Japan
kuwana@stat.hit-u.ac.jp

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Abstract

A minimax solution to Merton's optimal investment/consumption decision problem will be derived. The result holds under general assumptions on the unknown drift process and the utility functions.

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I. *Introduction*

Various generalizations have been made to the problem of optimal investment/consumption decisions since the first introduction by Merton (1971). One of such ramifications is the topic of partial observations where drift of the uncertain asset's price process is assumed to be unobservable. Bayesian approach and filtering theory have been the most frequently employed tools for analyzing the problem. The literature in this context contains Detemple (1986, 1991), Dothan and Feldman (1986), Feldman (1989, 1992), Gennotte (1986), and Kuwana (1993, 1995).

In this article, we take an alternative approach to the problem based on a minimax formulation. Unobservable drift is allowed to be more general than is commonly assumed in Bayesian situations. Also, arbitrariness from the assumption of prior distribution can be avoided.

We start with a set Θ of càdlàg functions on $\mathbb{R}_+$, Θ is viewed as the parameter space for our minimax analysis. Various information structures can be implemented by restricting the parameter space Θ. For instance, the parameter space $\{\mu \in \Theta : \mu_t = c, \forall t \in [0, \infty)\}$ corresponds to the constant unknown drift problem considered in Kuwana (1999). Another example is a simple regime switching or a change point problem where the restricted parameter space can be written as $\{\mu \in \Theta : \mu_s = \alpha, s \in [0, T), \mu_t = \beta, t \in [T, \infty), T \in \mathbb{R}_+\}$. By specifying probability measures on these restricted parameter space, decision problems fit into the Bayesian framework.

II. *The Setup*

Let $\{Z_t\}_{t\geq 0}$ be a standard Brownian motion on some probability space $(\Omega, \{\mathfrak{F}_t\}_{t\geq 0}, P_0)$. Also, let $\{\mu_t\}_{t\geq 0}$ be adapted càdlàg stochastic process on $(\Omega, \{\mathfrak{F}_t\}, P_0)$ which takes values in Θ. Assume that $P_0(\int_0^T \mu_s^2 ds < \infty) = 1\}$. For a given càdlàg function $\mu_s \in \Theta$ that is a path of the process $\{\mu_s\}$, there is a probability measure P_μ such that

$$W_t = Z_t - \int_0^t \mu_s ds$$

is a standard Brownian motion on $(\Omega, \{\mathfrak{F}_t\}_{t\geq 0}, P_\mu)$. It should be noted again that μ_s in above integral is not stochastic.

Consider a financial market where two assets are traded. One is a bond which pays constant rate of return r over time without uncertainty. The other is a stock which has uncertain price fluctuation. To be more specific, let p_s be the price of the stock at time s. Then p_s is assumed to satisfy a stochastic differential equation given by

$$\frac{dp_s}{p_s} = dZ_s = \mu_s ds + dW_s. \tag{1}$$

Here we assumed that the volatility parameter is constant over time and equals to 1. At time t, an investor/consumer is endowed with x units of fortune. Then he continuously invests a proportion π_s of his wealth in the stock and a proportion $1 - \pi_s$ in the bond until a finite horizon T. Simultaneously, a proportion c_s of his wealth is lost due to consumption. It is assumed that his trading volume is so small that his strategy does not affect the price of assets in the market. For a given path $\mu_s \in \Theta$ and $\{\mathfrak{F}_s\}$-adapted decision processes $\{\pi_s\}_{t\leq s\leq T}$ and $\{c_s\}_{t\leq s\leq T}$, his wealth can be expressed as

$$\begin{aligned} dX_s^{\mu,\pi,c} &= \{(1-\pi_s)r - c_s\}X_s^{\mu,\pi,c}ds + \pi_s X_s^{\mu,\pi,c}dZ_s \\ &= \{(\mu_s - r)\pi_s + r - c_s\}X_s^{\mu,\pi,c}ds + \pi_s X_s^{\mu,\pi,c}dW_s. \end{aligned} \tag{2}$$

Define a loss function as

$$L(\mu, \{X_s^{\mu,\pi,c}\}_{t\leq s\leq T}) = -\int_t^T e^{-\delta s}U_1(c_s X_s^{\mu,\pi,c}ds) - e^{-\delta T}U_2(X_T^{\mu,\pi,c}), \tag{3}$$

where δ is a known positive constant discount rate and $U_i : \mathbb{R}_+ \to \mathbb{R}, i = 1,2$ are twice continuously differentiable, strictly concave and strictly increasing utility functions with $U_i(0) > -\infty, i = 1,2$. Denote $\mathcal{A}$ by the set of Markov strategies (π, c). For a given càdlàg function $\mu \in \Theta$ and $(\pi, c) \in \mathcal{A}$, the risk function associated with the loss function (3) is given by

$$R^{(t,x)}(\mu, \pi, c) = E_\mu^{(t,x)}\left[L(\mu, \{X_t^{\mu,\pi,c}\}_{t\leq s\leq T})\right], \tag{4}$$

where $E_\mu^{(t,x)}[\cdot]$ denote the P_μ expectation with the process $\{X_s^{\mu,\pi,c}\}_{t\leq s\leq T}$ starting at $X_t^{\mu,\pi,c} = x$.

When the path $\mu \in \Theta$ is known, the agent's objective is just to minimize (4), i.e. to maximize sum of expected cumulative utility from consumption from time t to T and the utility of his expected terminal wealth at time T. If the agent has prior knowledge about μ with a prior probability measure Λ, he will minimized the Bayes risk:

$$\int_{\Theta} R^{(t,x)}(\mu,\pi,c)\Lambda(d\mu).$$

We implement a simple information structure of μ by restricting Θ. Let I be a closed interval in $\mathbb{R}$. Define a parameter subspace

$$\Theta_I = \{\mu \in \Theta : \mu_s \in I, \forall s \in [0,T]\}.$$

For example, $\mu \in \Theta_{[r+\varepsilon,\infty)}, \varepsilon > 0$ represents a knowledge that μ_s stays at least ε higher than the riskless rate. Also, $\mu \in \Theta_{\mathbb{R}}$ contains no information about μ.

Our problem is to find the minimax strategy for the agent, i.e. to find a strategy (π^*, c^*) that satisfies

$$\sup_{\mu\in\Theta_I} R^{(t,x)}(\mu,\pi^*,c^*) = \inf_{(\pi,c)\in\mathcal{A}} \sup_{\mu\in\Theta_I} R^{(t,x)}(\mu,\pi,c) \tag{5}$$

for given t and x.

III. *The Minimax Solution*

In order to obtain the minimax solution for the problem (5), we need a representation for the difference of risks.

Lemma 1. *Suppose $\mu_{0,s} = \mu_0$ is constant over time. Then, for any $\mu_1 \in \Theta$ and Markov strategy (π, c), we have*

$$\begin{aligned} &R^{(t,x)}(\mu_1,\pi,c) - R^{(t,x)}(\mu_0,\pi,c) \\ =&E^{(t,x)}_{\mu_1}\left[\int_t^T (\mu_{1,s}-\mu_0)\pi_s X_s^{\mu_1,\pi,c} R_x^{(s,X_s^{\mu_1,\pi,c})}(\mu_0,\pi,c)ds\right]. \end{aligned} \tag{6}$$

Proof. Write

$$R_i(t,x) = R^{(t,x)}(\mu_i,\pi,c)$$

and

$$X_{i,s} = X_s^{\mu_i,\pi,c}$$

for $i = 0, 1$. Then $R_0(t,x)$ satisfies a Cauchy problem:

$$\begin{cases} \dfrac{\partial R_0}{\partial t} - e^{-\delta t}U_1(cx) + \{r - c + (\mu_0 - r)\pi\}x\dfrac{\partial R_0}{\partial x} + \dfrac{1}{2}\pi^2x^2\dfrac{\partial^2 R_0}{\partial x^2} = 0 \\ R_0(T,x) = -\dfrac{e^{-\delta T}}{\delta}U_2(x). \end{cases} \tag{7}$$

By applying Ito's lemma to $R_0(s, X_{1,s})$ and using above equation, we have

$$\begin{aligned} d(R_0(s, X_{1,s})) &= \frac{\partial R_0(s, X_{1,s})}{\partial s} ds + R_{0x}(s, X_{1,s}) dX_{1.s} + \frac{1}{2} R_{0xx}(s, X_{1,s}) \pi^2 X_{1,s}^2 ds \\ &= e^{-\delta s} U_1(c_s X_{1,s}) ds + (\mu_{1.s} - \mu_0) \pi_s X_{1,s} R_{0x}(s, X_{1,s}) ds \\ &\qquad + \pi_s X_{1,s} R_{0x}(s, X_{1,s}) dW_s. \end{aligned}$$

By integrating from t to T and taking expectations of both sides, we have

$$\begin{aligned} E_{\mu_1}^{(t,x)}[R_0(T, X_{1,T}) - R_0(t, X_{1,t})] = E_{\mu_1}^{(t,x)} \Bigg[\int_t^T & e^{-\delta s} U_1(c_s X_{1,s}) ds \\ & + \int_t^T (\mu_{1,s} - \mu_0) \pi_s X_{1,s} R_{0x}(s, X_{1,s}) ds \Bigg]. \end{aligned}$$

Note that

$$E_{\mu_1}^{(t,x)} \left[R_0(T, X_{1,T}) - R_0(t, X_{1,t}) - \int_t^T e^{-\delta s} U_1(c_s X_{1,s}) ds \right] = R_1(t, x) - R_0(t, x).$$

This completes the proof. $\square$

An expression of the optimal strategy when μ is a known constant was given by Karatzas, Lehoczky and Shreve (1987). In this situation, the optimal Markov investment strategy $\pi^*(s, x)$ is explicitly given by

$$\pi^*(s, x) = -\frac{\mu - r}{x} \cdot \left. \frac{\partial \mathcal{H}(s, y)}{\partial y} \right|_{y = \mathcal{K}(s,x)} \tag{8}$$

where

$$\mathcal{H}(t, y) = \int_{-\infty}^{\infty} \left\{ \int_0^{T-t} \alpha_1(z, \theta; y) d\theta + \alpha_2(z, T-t; y) \right\} dz,$$

$$\alpha_i(z, \theta; y) = e^{rz - (r + r^2/2)\theta} I_i \left(e^{y - (\mu - r)z + (\delta - r - r^2/2 + \mu^2/2)\theta} \right) \cdot \frac{1}{\sqrt{2\pi\theta}} \exp\left[-\frac{z^2}{2\theta} \right], \quad i = 1, 2,$$

and $I_i, i = 1, 2$ are inverse functions of $U_i, i = 1, 2$ respectively. Also, $\mathcal{K}(s, x)$ is defined as the inverse function of $\mathcal{H}(s, y)$ w.r.t. y, i.e. $\mathcal{K}$ satisfies

$$\mathcal{H}(s, \mathcal{K}(s, x)) = x, \quad x > 0.$$

It is not hard to see that $\frac{\partial \mathcal{H}(s,y)}{\partial y}$ is negative since $I_i, i = 1, 2$ are strictly decreasing. Now we present the main result.

Proposition 2. *For a given closed interval I in $\mathbb{R}$, let $\mu^* = \arg\inf_{m \in I} |m - r|$. Let (π^*, c^*) be the optimal strategy for a constant drift μ^*. Then (π^*, c^*) is the unique minimax strategy for the parameter space Θ_I. In other words, for any $(\pi, c) \in \mathcal{A}$ such that $(\pi, c) \neq (\pi^*, c^*)$, we have*

$$\sup_{\mu \in \Theta_I} R^{(t,x)}(\mu, \pi^*, c^*) < \sup_{\mu \in \Theta_I} R^{(t,x)}(\mu, \pi, c).$$

Proof. When $\mu^* = r$, it follows from the expression (8) that $\pi^* = 0$. In this case, the optimal wealth process $\{X_s^{\mu,\pi^*,c^*}\}$ does not depend on μ. Thus $R^{(t,x)}(\mu, \pi^*, c^*)$ is constant over all $\mu \in \Theta$. (π^*, c^*) is clearly the unique Bayes decision for unit mass prior at r. Hence it is the unique minimax decision.

Suppose $\mu^* > r$. Then by Lemma 1, for any $\mu \in \Theta_I$,

$$\begin{aligned} &R^{(t,x)}(\mu, \pi^*, c^*) - R^{(t,x)}(\mu^*, \pi^*, c^*) \\ =&E_\mu^{(t,x)} \left[\int_t^T (\mu_s - \mu^*) \pi^*(s, X_s^{\mu^*,\pi^*,c^*}) X_s^{\mu,\pi^*,c^*} R_x^{(s, X_s^{\mu,\pi^*,c^*})}(\mu^*, \pi^*, c^*) ds \right]. \end{aligned}$$

Without loss of generality, we may assume that $U_i(x) > 0, i = 1, 2$ by offsetting $-U_i(0)$ when $U_i(0) < 0$. Therefore, $\pi^* x R_x^{(t,x)}(\mu^*, \pi^*, c^*)$ is nonpositive. Since $\mu_s - \mu^* \geq 0$, we have

$$R^{(t,x)}(\mu, \pi^*, c^*) \leq R^{(t,x)}(\mu^*, \pi^*, c^*)$$

for all $\mu \in \Theta_I$. Here the equality is achieved only if $\mu = \mu^*$ a.e. Since (π^*, c^*) is the unique Bayes for a unit mass prior at μ^*, for any $(\pi, c) \neq (\pi^*, c^*)$ we have

$$\sup_{\mu \in \Theta_I} R^{(t,x)}(\mu, \pi^*, c^*) \leq R^{(t,x)}(\mu^*, \pi^*, c^*) < R^{(t,x)}(\mu^*, \pi, c) \leq \sup_{\mu \in \Theta_I} R^{(t,x)}(\mu^*, \pi, c).$$

Hence (π^*, c^*) is the unique minimax strategy. The proof when $\mu^* < r$ is similar. □

Proposition 2 asserts that if no information is available, a minimax investor never invests to the uncertain asset no matter how less risk aversive he might be. This result seems somewhat conservative. It is conjectured that this phenomenon will go away by taking the agent's regret into consideration.

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