Characterizing the Nash bargaining solution with continuity and almost no individual rationality*

Kensei Nakamura†

This version: April 22, 2023

Graduate School of Economics Discussion Paper No.2023-02

Abstract

In the classical bargaining problems, we characterize the Nash solution (Nash, 1950) by a very mild condition of individual rationality called *Possibility of Utility Gain* and continuity with respect to feasible sets or disagreement points together with Nash's axioms except *Weak Pareto Optimality*. We also provide alternative and unified proofs for other efficiency-free characterizations of the Nash solution.

1 Introduction

Nash (1950) formulated the bargaining problems and characterized a bargaining solution satisfying the axioms of *Scale Invariance*, *Symmetry*, *Contraction Independence*, and *Weak Pareto Optimality*. This solution is called the Nash

*This paper is based on Chapter 2 of my B.A. thesis submitted to Hitotsubashi University. I am grateful to Koichi Tadenuma for very helpful comments. I am responsible for any remaining errors.

†Graduate School of Economics, Hitotsubashi University, Kunitachi, Tokyo 186-8601, Japan. E-mail : kensei.nakamura.econ@gmail.com

solution. Since then, many researchers have investigated the properties of this solution and provided other characterizations.

In the real world, people sometimes reach inefficient agreements. Even when the resulting outcome is physically efficient, they often spend considerable time resolving conflicts. When the timing of agreements affects their utility levels, conflicts may be resolved at inefficient points in the utility space. Given these considerations, it is important to investigate solutions without imposing *Weak Pareto Optimality* a priori. Some researchers have characterized the Nash solution without *Weak Pareto Optimality*. Roth (1977) showed that *Weak Pareto Optimality* can be replaced by *Strong Individual Rationality*. Anbarci and Sun (2011, 2013), Rachmilevitch (2015), and Mori (2018) also provided efficiency-free characterizations of the Nash solution. Lensberg and Thomson (1988) examined the role of *Weak Pareto Optimality* in a model with a variable number of players. Vartiainen (2007) discussed the limited significance of *Weak Pareto Optimality* in bargaining problems without disagreement points. For a survey of the literature about efficiency-free characterizations, see Thomson (2022).

In this paper, we propose a very mild condition of individual rationality called *Possibility of Utility Gain*. This requires that there exists **at least one** bargaining problems in which **at least one player** improves his or her utility at the agreement. Were it violated, in any bargaining problem, no player could achieve a utility gain over the disagreement point. This condition is weaker than the axioms of individual or collective rationality that are used to characterize the Nash solution by Roth (1977), Anbarci and Sun (2011, 2013), Rachmilevitch (2015), and Mori (2018).

We also consider continuity with respect to feasible sets or disagreement points. These axioms of continuity require that small changes in the bargaining situation do not lead to large changes in the solution outcome. Continuity may be also justified from the perspective of uncertainty about bargaining situations. In real life, we can only approximate what agreements are feasible and how much utility levels can be achieved through agreements. Continuity demands that even if the actual bargaining problem varies slightly from our predictions, the outcome of the solution does not alter considerably.

Many researchers have studied axioms related to uncertainty, including Perles and Maschler (1981), Chun and Thomson (1990a,b), Peters and van Damme (1991), Bossert and Peters (2002, 2022), and others. Thomson (1994) reviewed the related literature up to the mid-1990s.

This paper makes three contributions. First, we provide novel characterizations of the Nash solution with almost no individual or collective rationality. We use the aforementioned axioms of individual rationality and continuity with respect to feasible sets or disagreement points in conjunction with Nash's axioms except *Weak Pareto Optimality*. In addition, we provide an alternative characterization using an axiom introduced by Rachmilevitch (2015).

Second, we show that a solution satisfies continuity with respect to feasible sets and Nash's three axioms, *Scale Invariance*, *Symmetry*, and *Contraction Independence* if and only if it satisfies *Weak Individual Rationality* and these three axioms. This result means that two quite different conditions—continuity with respect to feasible sets and *Weak Individual Rationality*—play the same role if we impose either of them together with the three standard axioms.

Finally, we provide unified proofs of other efficiency-free characterizations by using several lemmas that we show to prove our main results. We clarify the role of each axiom introduced in the related literature. Another new characterization is provided in line with this discussion.

This paper is organized as follows. In Section 2, we introduce the bargaining problem. Section 3 defines several axioms, including a novel axiom called *Possibility of Utility Gain*. In Section 4, we characterize the Nash solution and classes of solutions that include it. Section 5 provides unified proofs of related results, and also presents another new axiomatization of the Nash solution. Finally, Section 6 has some concluding comments.

2 The Bargaining Problems

This paper considers n-person bargaining problems. Let $N = \{1, 2, \cdots, n\}$ be the set of players. The n players can attain some utility levels if they reach an agreement, but stay at the *disagreement point*, represented by $d \in \mathbb{R}^n$, if they do not agree. Let $S \subset \mathbb{R}^n$ be the set of all utility vectors that can be achieved by an agreement.[1] A *bargaining problem* is a pair (S, d). We assume that S is a compact and convex set, and that there exists $s \in S$ such that

[1] Let $\mathbb{R}$ (resp. $\mathbb{R}_+$, $\mathbb{R}_{++}$, $\mathbb{R}_-$, $\mathbb{R}_{--}$) denote the set of real numbers (resp. nonnegative numbers, positive numbers, nonpositive numbers, negative numbers). Let $\mathbb{R}^n$ (resp. $\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$, $\mathbb{R}^n_-$, $\mathbb{R}^n_{--}$) denote the n-fold Cartesian product of $\mathbb{R}$ (resp. $\mathbb{R}_+$, $\mathbb{R}_{++}$, $\mathbb{R}_-$, $\mathbb{R}_{--}$).

$s \gg d$.[2] The set of all bargaining problems is denoted by $\mathcal{B}^n$.

A *(bargaining) solution* is a function $f : \mathcal{B}^n \to \mathbb{R}^n$ such that $f(S, d) \in S$ for all $(S, d) \in \mathcal{B}^n$. For each $(S, d) \in \mathcal{B}^n$ and for each $i \in N$, $f_i(S, d)$ represents player i's utility level thorough the agreements. Nash (1950) characterized the following solution f^{Nash}. For all $(S, d) \in \mathcal{B}^n$,

$$f^{\text{Nash}}(S, d) = \arg\max_{d \leq s \in S} \Pi_{i \in N}(s_i - d_i).$$

This solution is called the *Nash solution*. The *disagreement solution* f^{dis} is defined as $f^{\text{dis}}(S, d) = d$ for all $(S, d) \in \mathcal{B}^n$. We define the *anti-Nash solution* f^{AN} on $\mathcal{B}^2$ as follows:

- If there exists a point $s \in S$ such that $s \ll d$, then

$$f^{AN}(S, d) = \arg\max_{d \geq s \in S}(s_1 - d_1)(s_2 - d_2).$$

- If d is on the boundary of S and there is no point $s \in S \backslash \{d\}$ such that $s \leq d$, then $f^{AN}(S, d) = d$.

- If d is on the boundary of S and there is a point $s \in S \backslash \{d\}$ such that $s_1 < d_1$ and $s_2 = d_2$, then $f^{AN}(S, d) = (\min_{s_1 < d_1, s_2 = d_2} s_1,\ d_2)$.

- If d is on the boundary of S and there is a point $s \in S \backslash \{d\}$ such that $s_1 = d_1$ and $s_2 < d_2$, then $f^{AN}(S, d) = (d_1,\ \min_{s_1 = d_1, s_2 < d_2} s_2)$.

This solution extends the maximizer of the product $(s_1 - d_1)(s_2 - d_2)$ in $s \in \{s' \in S \mid s' \ll d\}$ to the case where d is on the boundary of S.

For simplicity, we introduce the following notations. For all $x \in \mathbb{R}$, let $\mathbf{x} = (x, x, \cdots, x) \in \mathbb{R}^n$. For example, $\mathbf{0}$ represents $(0, 0, \cdots, 0) \in \mathbb{R}^n$. For all $A \subset \mathbb{R}^n$, $\text{ch}(A)$ is the convex hull of the set A. For all $s, \alpha, \beta \in \mathbb{R}^n$, let $\alpha s = (\alpha_1 s_1, \alpha_2 s_2, \cdots, \alpha_n s_n)$ and $\alpha s + \beta = (\alpha_1 s_1 + \beta_1, \alpha_2 s_2 + \beta_2, \cdots, \alpha_n s_n + \beta_n)$. Also, let $\alpha S = \{\alpha s \in \mathbb{R}^n \mid s \in S\}$ and $\alpha S + \beta = \{\alpha s + \beta \in \mathbb{R}^n \mid s \in S\}$ for all $\alpha \in \mathbb{R}^n_{++}$, $\beta \in \mathbb{R}^n$.

[2] We write $a \gg b$ if $a_i > b_i$ for all $i \in N$, and $a \geq b$ if $a_i \geq b_i$ for all $i \in N$. We define $\ll$ and $\leq$ in the same way.

3 Axioms for Solutions

Natural or reasonable properties of solutions, which we call *axioms*, have been considered in the literature. Nash (1950) introduced the following four axioms to characterize the Nash solution.

Scale Invariance. For all $(S,d) \in \mathcal{B}^n$ and for all $\alpha \in \mathbb{R}^n_{++}$, $\beta \in \mathbb{R}^n$, $f(\alpha S + \beta, \alpha d + \beta) = \alpha f(S,d) + \beta$.

Weak Pareto Optimality. For all $(S,d) \in \mathcal{B}^n$, if there exists y such that $y \gg x$, then $f(S,d) \neq x$.

We say that a bargaining problem (S,d) is *symmetric* if for all one-to-one functions $\pi : N \to N$, $S = \{(s_{\pi(1)}, s_{\pi(2)}, \cdots, s_{\pi(n)}) \mid s \in S\}$ and for all $i,j \in N$, $d_i = d_j$.

Symmetry. If $(S,d) \in \mathcal{B}^n$ is symmetric, then $f_i(S,d) = f_j(S,d)$ for all $i,j \in N$.

Contraction Independence. For all $(S,d),(T,d) \in \mathcal{B}^n$, if $S \subset T$ and $f(T,d) \in S$, then $f(S,d) = f(T,d)$.

In the real-life bargainings, there are cases where people do not reach Pareto optimal agreements. Therefore, it is important to explore bargaining solutions without imposing *Weak Pareto Optimality* a priori. Roth (1977) showed that *Weak Pareto Optimality* can be replaced by the following axiom.

Strong Individual Rationality. For all $(S,d) \in \mathcal{B}^n$, $f(S,d) \gg d$.

Strong Individual Rationality requires that all players receive utility levels strictly higher than the disagreement point.

Roth (1979) used a weaker version of individual rationality.

Weak Individual Rationality. For all $(S,d) \in \mathcal{B}^n$, $f(S,d) \geq d$.

Roth (1979) showed that if we impose *Weak Individual Rationality* on a solution along with Nash's axioms other than *Weak Pareto Optimality*, then the solution is either the Nash solution f^{Nash} or the disagreement solution f^{dis}. We call this class of solutions the *Roth class*.

We now introduce a new axiom. It requires that there exists **at least one** bargaining problem such that **at least one** player can gain more utility than the disagreement point. If a solution f violates this condition, then $f(S,d) \leq d$ holds for each $(S,d) \in \mathcal{B}^n$, which means that any player cannot improve their utility levels in any situation. This axiom is logically weaker than *Strong Individual Rationality* and the axioms introduced by Anbarci and Sun (2011), Rachmilevitch (2015) or Mori (2018).

Possibility of Utility Gain. There exists $(S,d) \in \mathcal{B}^n$ such that $f_i(S,d) > d_i$ for some $i \in N$.

We also introduce axioms of continuity. They require that small changes in the bargaining situation do not lead to large changes in the outcome of the solution. These axioms are motivated by uncertainty in the bargaining situation, as discussed in Section 1.

Feasible Set Continuity. For all sequences $\{(S^k,d)\}_{k=1}^{\infty} \subset \mathcal{B}^n$ and for all $(S,d) \in \mathcal{B}^n$, if $\{S^k\}_{k=1}^{\infty}$ converges to S in the Hausdorff topology, then $\lim_{k\to\infty} f(S^k,d) = f(S,d)$.

Disagreement Point Continuity. For all sequences $\{(S,d^k)\}_{k=1}^{\infty} \subset \mathcal{B}^n$ and for all $(S,d) \in \mathcal{B}^n$, if $\{d^k\}_{k=1}^{\infty}$ converges to d, then $\lim_{k\to\infty} f(S,d^k) = f(S,d)$.

4 Main Results

Our two main results are the following: the Nash solution is the unique solution satisfying *Possibility of Utility Gain* and either of two continuity axioms together with the Nash's axioms except *Weak Pareto Optimality*.

Theorem 1. A bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence*, *Feasible Set Continuity*, and *Possibility of Utility Gain* if and only if f is the Nash solution f^{Nash}.

Theorem 2. A bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence*, *Disagreement Point Continuity*, and *Possibility of Utility Gain* if and only if f is the Nash solution f^{Nash}.

In this section, we also show several results related to these two theorems.

4.1 Preliminary Lemmas

To establish our main theorems, we prove three preliminary lemmas, which elucidate the implications of *Scale Invariance*, *Symmetry* and *Contraction Independence* together. We use these lemmas again in Section 5 to prove other characterizations in a unified way.

Lemma 1. Suppose that a bargaining solution f satisfies *Scale Invariance* and *Contraction Independence*. If there exists a bargaining problem $(S,d) \in \mathcal{B}^n$ such that d is an interior point of S and that $f(S,d) = d$, then $f(S',d') = d'$ for all $(S',d') \in \mathcal{B}^n$.

Proof. Suppose that there exists a bargaining problem $(S,d) \in \mathcal{B}^n$ such that d is an interior point of S and $f(S,d) = d$. Let (S',d') be an arbitrary element of $\mathcal{B}^n$. By *Scale Invariance*, we can assume that $d = \mathbf{0}$ and $d' = \mathbf{0}$. We show that $f(S',\mathbf{0}) = \mathbf{0}$.

Fix $\alpha \in \mathbb{R}^n_{++}$ such that $S' \subset \alpha S$. (Note that there exists such an α since d is an interior point of S.) By *Scale Invariance*, we have $f(\alpha S, \mathbf{0}) = \mathbf{0}$. Since $S' \subset \alpha S$ and $f(\alpha S,\mathbf{0}) = \mathbf{0} \in S'$, *Contraction Independence* implies $f(S',\mathbf{0}) = \mathbf{0}$. □

For all $c \leq 0$, let $\triangle(c) = \{s \in \mathbb{R}^n \mid s \geq \mathbf{c} \text{ and } s_1 + s_2 + \cdots + s_n \leq n\}$. (Figure 1 illustrates $\triangle(c)$ in the two-person case.) For all $a \in \mathbb{R}^n_-$, let $\mathrm{tr}(a) = \{s \in \mathbb{R}^n \mid s \geq a \text{ and } s_1 + s_2 + \cdots + s_n \leq n\}$. Note that for all $c \leq 0$, $\triangle(c) = \mathrm{tr}(\mathbf{c})$.

For all symmetric bargaining problems (S,d), let $\bar{l}(S) = \mathbf{x}$ where $x = \max_{(y,y,\cdots,y)\in S} y$ and $\underline{l}(S) = \mathbf{x}'$ where $x' = \min_{(y,y,\cdots,y)\in S} y$.

Lemma 2. Suppose that a bargaining solution f satisfies *Scale Invariance*, *Symmetry* and *Contraction Independence*. Then, for all symmetric bargaining problems (S,d), $f(S,d)$ is either $\underline{l}(S)$, d or $\bar{l}(S)$. In particular, $f(\triangle(c),\mathbf{0})$ is either $\mathbf{c}$, $\mathbf{0}$ or $\mathbf{1}$ for each $c \leq 0$.

Proof. Let (S,d) be a symmetric bargaining problem. Since f satisfies *Scale Invariance*, we can assume that $d = \mathbf{0}$. By *Symmetry*, there exists $x \in [\underline{l}_1(S), \bar{l}_1(S)]$ with $f(S,\mathbf{0}) = \mathbf{x}$. Suppose to the contrary that $x \in (\underline{l}_1(S), 0) \cup (0, \bar{l}_1(S))$. Fix $\varepsilon > 0$ small enough that $\mathbf{x} \in \alpha S$ where $\alpha = (1-\varepsilon, \cdots, 1-\varepsilon) \in \mathbb{R}^n$. *Scale Invariance* implies $f(\alpha S, \mathbf{0}) = \alpha\mathbf{x}$. Since $\alpha S \subset S$ and $f(S,\mathbf{0}) = \mathbf{x} \in \alpha S$, *Contraction Independence* implies $f(\alpha S,\mathbf{0}) = \mathbf{x}$, a contradiction. Therefore, for all symmetric bargaining problems (S,d), $f(S,d)$ is either $\underline{l}(S)$, d or $\bar{l}(S)$. □

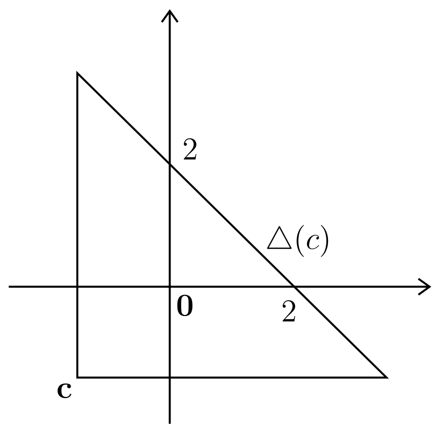


Figure 1: $\triangle(c)$ in the case where $n = 2$

Lemma 3. Suppose that a solution f satisfies *Scale Invariance* and *Contraction Independence*. Then the following statements hold:

(L3-1) If $f(\triangle(c), \mathbf{0}) = \mathbf{0}$ for all $c \leq 0$, then $f = f^{\text{dis}}$.

(L3-2) If $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ for all $c \leq 0$, then $f = f^{\text{Nash}}$.

Proof. If $f(\triangle(c), \mathbf{0}) = \mathbf{0}$ for all $c \leq 0$, then Lemma 2 implies $f(S, d) = d$ for all $(S, d) \in \mathcal{B}^n$. Therefore, we have $f = f^{\text{dis}}$.

When $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ for all $c \leq 0$, we prove $f = f^{\text{Nash}}$. Let (S, d) be an arbitrary bargaining problem. By *Scale Invariance*, without loss of generality, we can assume that $d = \mathbf{0}$ and $f^{\text{Nash}}(S, d) = \mathbf{1}$. There exists $c < 0$ such that $S \subset \triangle(c)$. Since $f(\triangle(c), \mathbf{0}) = \mathbf{1} \in S$, *Contraction Independence* implies $f(S, d) = \mathbf{1} = f^{\text{Nash}}(S, d)$. □

4.2 Characterization with Feasible Set Continuity

Next, we show that the Roth class can be characterized by replacing *Weak Individual Rationality* with *Feasible Set Continuity*. Notice that the two quite different axioms have the same implication when imposed together with Nash's three axioms except *Weak Pareto Optimality*.

Proposition 1. A bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence*, and *Feasible Set Continuity* if and only if f is the Nash solution f^{Nash} or the disagreement solution f^{dis}.

Since f^{dis} does not satisfy *Possibility of Utility Gain*, we obtain Theorem 1 as a corollary of this proposition.

Then we show Proposition 1. To use Lemma 3, we prove that either (i) $f(\triangle(c), \mathbf{0}) = \mathbf{0}$ for all $c < 0$ or (ii) $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ for all $c < 0$ holds.

Lemma 4. Suppose that a bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence* and *Feasible Set Continuity*. There is no f such that $f(\triangle(c), \mathbf{0}) = \mathbf{c}$ for all $c \leq 0$.

Proof. Suppose to the contrary that $f(\triangle(c), \mathbf{0}) = \mathbf{c}$ for all $c \leq 0$.

Step 1. First, consider an arbitrary bargaining problem (S, d) such that d is on the boundary of S and there is no point $s \in S \backslash \{d\}$ with $s \leq d$. We show $f(S, d) = d$. By *Scale Invariance*, we can, without loss of generality, assume that $d = \mathbf{0}$ and $S \subset \{s \in \mathbb{R}^n \mid s_1 + s_2 + \cdots + s_n \geq 0\}$.

(We show the existence of such a positive affine transformation τ. Without loss of generality, we can assume $d = \mathbf{0}$. By the hyperplane separation theorem, there exists $k = (k_1, k_2, \cdots, k_n) \in \mathbb{R}^n_{++}$ such that $k_1 s_1 + k_2 s_2 + \cdots + k_n s_n \geq 0$ for all $s \in S$. Let $\tau(x) = kx$ for all $x \in \mathbb{R}^n$. We have $\tau(\mathbf{0}) = \mathbf{0}$ and for all $s \in S$, $\tau_1(s) + \tau_2(s) + \tau_n(s) = k_1 s_1 + k_2 s_2 + \cdots + k_n s_n \geq 0$, as required.)

Take a symmetric bargaining problem $(T, \mathbf{0})$ satisfying the following conditions:

- $S \subset T \subset \{s \in \mathbb{R}^n \mid s_1 + s_2 + \cdots + s_n \geq 0\}$.
- $T \cap \mathbb{R}^n_+ = \mathbf{x}\triangle(0)$ for some $x \in \mathbb{R}_{++}$.

(See Figure 2 for the two-person case.) By Lemma 2, $f(T, \mathbf{0})$ is either $\mathbf{0}$ or $\mathbf{x}$. If $f(T, \mathbf{0}) = \mathbf{x}$, *Scale Invariance* and *Contraction Independence* imply that $f(\triangle(0), \mathbf{0}) = \mathbf{1}$, a contradiction to $f(\triangle(0), \mathbf{0}) = \mathbf{0}$. Thus, we have $f(T, \mathbf{0}) = \mathbf{0}$. By *Contraction Independence*, it follows that

$$f(S, \mathbf{0}) = \mathbf{0}. \tag{1}$$

Step 2. We show that for all $a \in \mathbb{R}^n_-$, $f(\text{tr}(a), \mathbf{0}) = a$.

For all $a \in \mathbb{R}^n_{--}$, *Scale Invariance* and *Contraction Independence* imply $f(\text{tr}(a), \mathbf{0}) = a$.[3] For each $b \in \mathbb{R}^n_-$, consider a sequence $\{b^k\}_{k=1}^\infty \subset \mathbb{R}^n_{--}$ such

[3] Consider the positive affine transformation $\tau : \mathbb{R}^n \to \mathbb{R}^n$ such that $\tau(a) = \mathbf{c}$ for some $c < 0$ and $\tau(\mathbf{0}) = \mathbf{0}$. Also, consider a bargaining problem $(\mathbf{e}\triangle(\gamma), \mathbf{0})$ with $e\gamma = c$ and $\tau(\text{tr}(a))(= \{\tau(s) \mid s \in \text{tr}(a)\}) \subset \mathbf{e}\triangle(\gamma)$. By *Scale Invariance*, we have $f(\mathbf{e}\triangle(e), \mathbf{0}) = (e\gamma, e\gamma, \cdots, e\gamma) = \mathbf{c}$. *Contraction Independence* implies $f(\tau(\text{tr}(a)), \tau(\mathbf{0})) = \mathbf{c}$. By *Scale Invariance*, it follows that $f(\text{tr}(a), \mathbf{0}) = a$.

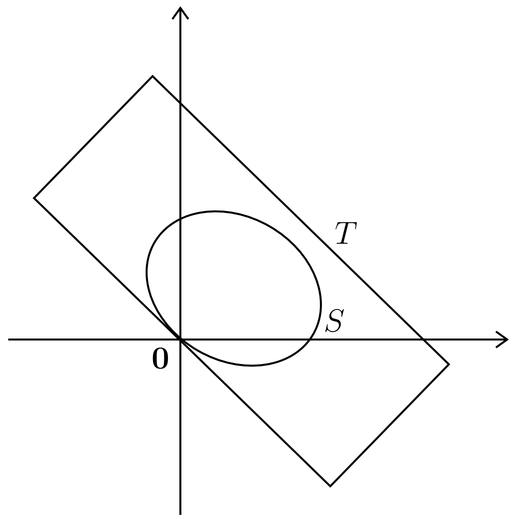


Figure 2

that $\{b^k\}_{k=1}^{\infty}$ converges to b. Since f satisfies *Feasible Set Continuity*, it follws that $f(\mathrm{tr}(b), \mathbf{0}) = \lim_{k\to\infty} f(\mathrm{tr}(b^k), \mathbf{0}) = \lim_{k\to\infty} b^k = b$.

Step 3. Consider the sequence $\{a^k\}_{k=1}^{\infty} \subset \mathbb{R}^2$ with $a^k = (1/k, -1, -1, \cdots, -1)$. Let $a^* = (0, -1, -1, \cdots, -1)(= \lim_{k\to\infty} a^k)$. Let $S^k = \mathrm{ch}(\triangle(0) \cup \{a^k\})$ for each $k \in \mathbb{N}$. This sequence converges to $S^* = \mathrm{ch}(\triangle(0) \cup \{a^*\})$.

Since $S^* \subset \mathrm{tr}(a^*)$, the result of Step 2 and *Contraction Independence* imply $f(S^*, \mathbf{0}) = a^*$. Since $(S^k, \mathbf{0})$ is categorized to the case of Step 1, $f(S^k, \mathbf{0}) = \mathbf{0}$ holds for each $k \in \mathbb{N}$. *Feasible Set Continuity* implies $f(S^*, \mathbf{0}) = \mathbf{0}$, a contradiction. Thus, there is no f such that $f(\triangle(c), \mathbf{0}) = \mathbf{c}$ for all $c \leq 0$. □

Lemma 5. Suppose that a bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence* and *Feasible Set Continuity*. Then, either of following statements holds:

(L5-1) $f(\triangle(c), \mathbf{0}) = \mathbf{0}$ for all $c \leq 0$.

(L5-2) $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ for all $c \leq 0$.

Proof. By Lemma 1, if there exists $c < 0$ such that $f(\triangle(c), \mathbf{0}) = \mathbf{0}$, then $f(\triangle(c'), \mathbf{0}) = \mathbf{0}$ for all $c' \leq 0$, i.e., (L5-1) holds.

Consider the case where there is no $c < 0$ such that $f(\triangle(c), \mathbf{0}) = \mathbf{0}$.

First, we show that, if $f(\triangle(e), \mathbf{0}) = \mathbf{e}$ for some $e < 0$, then $f(\triangle(e'), \mathbf{0}) = \mathbf{e}'$ for all $e' < e$. Suppose to the contrary that $f(\triangle(e'), \mathbf{0}) \neq \mathbf{e}'$. By Lemma 1 and Lemma 2, $f(\triangle(e'), \mathbf{0}) = \mathbf{1}$ holds. Since $\triangle(e) \subset \triangle(e')$ and $f(\triangle(e'), \mathbf{0}) = \mathbf{1} \in \triangle(e)$, *Contraction Independence* implies $f(\triangle(e), \mathbf{0}) = \mathbf{1}$, a contradiction.

Suppose that there exists $c, c' < 0$ such that $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ and $f(\triangle(c'), \mathbf{0}) = \mathbf{c}'$. Let $c^* = \sup\{e < 0 \mid f(\triangle(e), \mathbf{0}) = \mathbf{e}\}(< 0)$. By the result of the last paragraph, the set $\{e < 0 \mid f(\triangle(e), \mathbf{0}) = \mathbf{e}\}$ is either (i) $(-\infty, c^*)$ or (ii) $(-\infty, c^*]$.

(i) When $\{e < 0 \mid f(\triangle(e), \mathbf{0}) = \mathbf{e}\} = (-\infty, c^*)$, consider a sequence $\{c^k\}_{k=1}^{\infty} \subset \mathbb{R}$ with $c^k \to c^*$ as $k \to \infty$ and $c^k < c^*$ for all $k \in \mathbb{N}$. Since $c^k \in (-\infty, c^*) = \{c < 0 \mid f(\triangle(c), \mathbf{0}) = \mathbf{c}\}$, we have $f(\triangle(c^k), \mathbf{0}) = (c^k, c^k, \cdots, c^k)$ for all $k \in \mathbb{N}$. *Feasible Set Continuity* implies $f(\triangle(c^*), \mathbf{0}) = \mathbf{c}^*$. However, since $c^* \notin (-\infty, c^*)$, $f(\triangle(c^*), \mathbf{0}) = \mathbf{1}$ holds, a contradiction.

(ii) When $\{e < 0 \mid f(\triangle(e), \mathbf{0}) = \mathbf{e}\} = (-\infty, c^*]$, consider a sequence $\{c^k\}_{k=1}^{\infty} \subset \mathbb{R}$ with $c^k \to c^*$ as $k \to \infty$ and $c^k > c^*$ for all $k \in \mathbb{N}$. Since $c^k \notin (-\infty, c^*] = \{c < 0 \mid f(\triangle(c), \mathbf{0}) = \mathbf{c}\}$, we have $f(\triangle(c^k), \mathbf{0}) = \mathbf{1}$ for all $k \in \mathbb{N}$. *Feasible Set Continuity* implies $f(\triangle(c^*), \mathbf{0}) = \mathbf{1}$. However, since $c^* \in (-\infty, c^*]$, $f(\triangle(c^*), \mathbf{0}) = \mathbf{c}^*$ holds, a contradiction.

Therefore, if there is no $c < 0$ such that $f(\triangle(c), \mathbf{0}) = \mathbf{0}$, then either of the followings holds: $f(\triangle(c'), \mathbf{0}) = \mathbf{1}$ for all $c' < 0$ or $f(\triangle(c'), \mathbf{0}) = \mathbf{c}'$ for all $c' < 0$. By *Feasible Set Continuity*, we have either (a) $f(\triangle(c'), \mathbf{0}) = \mathbf{1}$ for all $c' \leq 0$ or (b) $f(\triangle(c'), \mathbf{0}) = \mathbf{c}'$ for all $c' \leq 0$. By Lemma 4, (b) does not hold. Hence, if there is no $c < 0$ such that $f(\triangle(c), \mathbf{0}) = \mathbf{0}$, then (L5-2) holds. □

Proof of Proposition 1. It is clear that f^{Nash} and f^{dis} satisfy four axioms. Suppose that f satisfies four axioms: *Scale Invariance*, *Symmetry*, *Contraction Independence*, and *Feasible Set Continuity*. By Lemma 3 and Lemma 5, f is the Nash solution f^{Nash} or the disagreement solution f^{dis}. □

4.3 Characterization with Disagreement Point Continuity

In two-person bargaining problems, if we impose *Disagreement Point Continuity* instead of *Feasible Set Continuity*, then a different class of solutions is characterized.

Proposition 2. In two-person bargaining problems, a bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence*, and *Dis-*

agreement Point Continuity if and only if f is the Nash solution f^{Nash}, the disagreement solution f^{dis} or the anti-Nash solution f^{AN}.

See Appendix for a proof. (Note that in the proof, we use the result of Lemma 6 and the argument of Lemma 7.) This proposition implies that at least in the two-person case, *Disagreement Point Continuity* does not play the same role as *Feasible Set Continuity* (or *Weak Individual Rationality*) together with Nash's axioms except *Weak Pareto Optimality*. On the other hand, in Theorem 1, *Feasible Set Continuity* can be replaced by *Disagreement Point Continuity*. This result is described in Theorem 2. The rest of this section provides a proof of this theorem.

First, we consider bargaining problems represented by $(\triangle(c), \mathbf{x})$ with $c \leq x \leq 0$. Set $\alpha_i = 1/(1-x)$ and $\beta_i = -x/(1-x)$ for each $i \in N$. Note that

$$(\alpha\triangle(c) + \beta, \alpha\mathbf{x} + \beta) = \Big(\triangle\Big(\frac{c-x}{1-x}\Big), \mathbf{0}\Big).$$

If f satisfies *Scale Invariance*, then we have

$$f_i(\triangle(c), \mathbf{x}) = (1-x) f_i\Big(\triangle\Big(\frac{c-x}{1-x}\Big), \mathbf{0}\Big) + x \tag{2}$$

for each $i \in N$.

Lemma 6. Suppose that a bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence* and *Disagreement Point Continuity*. Then, either of following statements holds:

(L6-1) $f(\triangle(c), \mathbf{0}) = \mathbf{0}$ for all $c \leq 0$.

(L6-2) $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ for all $c \leq 0$.

(L6-3) $f(\triangle(c), \mathbf{0}) = \mathbf{c}$ for all $c \leq 0$.

Proof. By Lemma 1, if there exists $c < 0$ such that $f(\triangle(c), \mathbf{0}) = \mathbf{0}$, then $f(\triangle(c'), \mathbf{0}) = \mathbf{0}$ for all $c' \leq 0$, i.e., (L6-1) holds.

Suppose that there exists $c, c' < 0$ such that $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ and $f(\triangle(c'), \mathbf{0}) = \mathbf{c}'$. Let $c^* = \sup\{e < 0 \mid f(\triangle(e), \mathbf{0}) = \mathbf{e}\} (< 0)$. In the same way as the proof of Lemma 5, the set $\{e < 0 \mid f(\triangle(e), \mathbf{0}) = \mathbf{e}\}$ is either (i) $(-\infty, c^*)$ or (ii) $(-\infty, c^*]$.

(i) When $\{e < 0 \mid f(\triangle(e), \mathbf{0}) = \mathbf{e}\} = (-\infty, c^*)$, consider a sequence $\{x^k\}_{k=1}^{\infty} \subset \mathbb{R}$ with $x^k \to 0$ as $k \to \infty$ and $0 < x^k < 1$ for all $k \in \mathbb{N}$. Since $0 < x^k < 1$, we have $\frac{c^* - x^k}{1 - x^k} < c^*$ for each $k \in \mathbb{N}$. By (2), we have

$$f_i(\triangle(c^*), (x^k, x^k, \cdots, x^k)) = (1 - x^k)\frac{c^* - x^k}{1 - x^k} + x^k = c^*$$

for each $i \in N$ and for each $k \in \mathbb{N}$. By *Disagreement Point Continuity*, $f(\triangle(c^*), \mathbf{0}) = (c^*, c^*)$. However, since $c^* \notin (-\infty, c^*)$, $f(\triangle(c^*), \mathbf{0}) = \mathbf{1}$ holds, a contradiction.

(ii) When $\{e < 0 \mid f(\triangle(e), \mathbf{0}) = \mathbf{e}\} = (-\infty, c^*]$, consider a sequence $\{x^k\}_{k=1}^{\infty} \subset \mathbb{R}$ with $x^k \to 0$ as $k \to \infty$ and $x^k < 0$ for all $k \in \mathbb{N}$. Since $x^k < 0$, we have $\frac{c^* - x^k}{1 - x^k} > c^*$ for each $k \in \mathbb{N}$. By (2), we have $f_i(\triangle(c^*), (x^k, x^k, \cdots, x^k)) = (1 - x^k)1 + x^k = 1$ for all $i \in N$ and for each $k \in \mathbb{N}$. By *Disagreement Point Continuity*, $f(\triangle(c^*), \mathbf{0}) = \mathbf{1}$. However, since $c^* \in (-\infty, c^*]$, $f(\triangle(c^*), \mathbf{0}) = \mathbf{c}^*$ holds, a contradiction.

Therefore, if there is no $c < 0$ such that $f(\triangle(c), \mathbf{0}) = \mathbf{0}$, then either of the followings holds: (a) $f(\triangle(c'), \mathbf{0}) = \mathbf{1}$ for all $c' < 0$ or (b) $f(\triangle(c'), \mathbf{0}) = \mathbf{c}'$ for all $c' < 0$.

In the case (a), *Contraction Independence* implies that $f(\triangle(0), \mathbf{0}) = \mathbf{1}$. In the case (b), consider a sequence $\{x^k\}_{k=1}^{\infty} \subset \mathbb{R}$ with $x^k \to -1$ as $k \to \infty$ and $x^k \in (-1, 0)$. By *Scale Invariance*, it follws that $f(\triangle(-1), (x^k, x^k, \cdots, x^k)) = -\mathbf{1}$ for all $k \in \mathbb{N}$. By *Disagreement Point Continuity*, we have $f(\triangle(-1), -\mathbf{1}) = -\mathbf{1}$. *Scale Invariance* implies $f(\triangle(0), \mathbf{0}) = \mathbf{0}$.

Hence, either (L6-1), (L6-2) or (L6-3) holds. □

Lemma 7. Suppose that a bargaining solution f satisfies *Scale Invariance*, *Symmetry*, and *Contraction Independence*. If $f(\triangle(c), \mathbf{0}) = \mathbf{c}$ for all $c \leq 0$, then $f(S, d) \leq d$ for all $(S, d) \in \mathcal{B}^n$.

Proof. Step 1. Consider bargaining problems (S, d) such that d is in the interior of S. By *Scale Invariance*, we can assume that $f^{AN}(S, d) = -\mathbf{1}$ and $d = \mathbf{0}$. There exists symmetric bargaining problems $(T^1, \mathbf{0})$ satisfying following conditions:

- $S \subset T^1 \subset \{s \in \mathbb{R}^n \mid s_1 + s_2 + \cdots + s_n \geq -n\}$.
- $T^1 \cap \mathbb{R}^n_+ = \mathbf{x}\triangle(0)$ for some $x \in \mathbb{R}_{++}$.

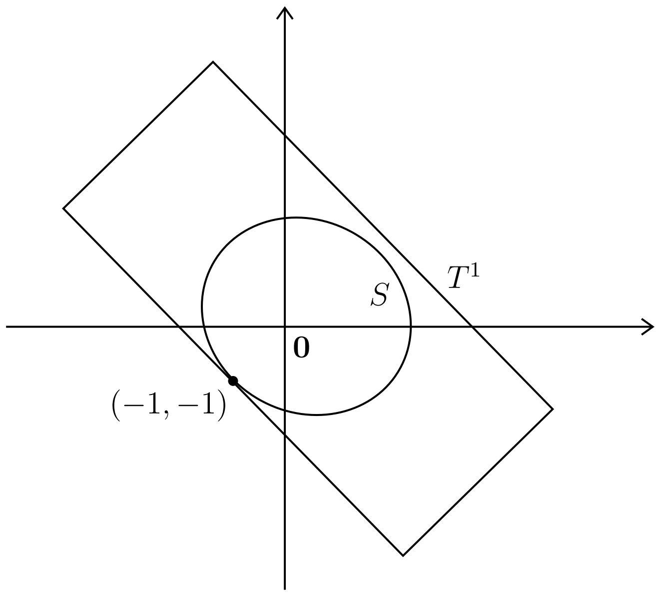


Figure 3: $(T^1, 0)$ in the case where $n = 2$

(See Figure 3 for $(T^1, \mathbf{0})$ in the two-person case.) By Lemma 1 and Lemma 2, $f(T^1, \mathbf{0})$ is either $-\mathbf{1}$ or $\mathbf{x}$. If $f(T^1, \mathbf{0}) = \mathbf{x}$, *Scale Invariance* and *Contraction Independence* imply that $f(\triangle(0), \mathbf{0}) = \mathbf{1}$, a contradiction. Thus, $f(T^1, \mathbf{0}) = -\mathbf{1}$ holds. By *Contraction Independence*, we have $f(S, \mathbf{0}) = -\mathbf{1} = f^{AN}(S, d) \leq \mathbf{0}(= d)$.

Step 2. Consider bargaining problems (S, d) such that d is on the boundary of S and that there is no $s \in S \backslash \{d\}$ satisfying $s \leq d$. In the same way as the proof of Lemma 4, we obtain $f(S, d) = d = f^{AN}(S, d)$.

Step 3. Consider bargaining problems (S, d) such that d is on the boundary of S and there is $s \in S \backslash \{d\}$ satisfying $s \leq d$. By *Scale Invariance*, we can set $d = \mathbf{0}$ without loss of generality. Suppose to the contrary that $f(S, \mathbf{0}) \leq \mathbf{0}$ does not hold. Let $T^2 = \text{ch}((S \cap \mathbb{R}^n_+) \cup \{f(S, \mathbf{0})\})$ and consider the bargaining problem $(T^2, \mathbf{0})$. *Contraction Independence* implies that $f(T^2, \mathbf{0}) = f(S, \mathbf{0})$. Since this bargaining problem is categorized to the case of Step 2, we have $f(T^2, \mathbf{0}) = \mathbf{0} \neq f(S, d)$, a contradiction. Hence, we obtain $f(S, d) \leq d$. □

Proof of Theorem 2. It is clear that f^{Nash} satisfies five axioms. Suppose that f satisfies five axioms: *Scale Invariance*, *Symmetry*, *Contraction Independence*, *Disagreement Point Continuity* and *Possibility of Utility Gain*. By Lemma 3, Lemma 6 and Lemma 7, f is the Nash solution f^{Nash}. □

4.4 The Independence of axioms

In the following, we show the independence of the axioms in Theorem 1 and Theorem 2.

Example 1. The egalitarian solution f^E is defined as the maximizer of $\min_{i\in N}(s_i - d_i)$ in $s \in S$ for each $(S,d) \in \mathcal{B}^n$. (See Kalai, 1977a.) This solution satisfies all of the axioms except *Scale Invariance*.

Example 2. Let $\theta \in \mathbb{R}^n_{++}\backslash\{(1/n, 1/n, \cdots, 1/n)\}$ with $\sum_{i\in N}\theta_i = 1$. The asymmetric Nash solution $f^{\theta,N}$ associated with θ is the maximizer of the product $\Pi_{i\in N}(s_1 - d_1)^{\theta_i}$ in $s \in \{s' \in S \mid s' \geq d\}$ for each $(S,d) \in \mathcal{B}^n$. (See Kalai, 1977b.) This solution does not satisfy *Symmetry*, but satisfies the other axioms.

Example 3. The Kalai-Smorodinsky solution satisfies all axioms other than *Contraction Independence*. (See Kalai and Smorodinsky, 1975.)

Example 4. Let $NP^+(S,d) = \max_{d\leq s\in S}\Pi_{i\in N}(s_i - d_i)$ and $NP^-(S,d) = \max_{d\geq s\in S}\Pi_{i\in N}|s_i - d_i|$. Consider a solution f defined by, for all $(S,d) \in \mathcal{B}^n$,

$$f(S,d) = \begin{cases} f^{\text{Nash}}(S,d) & (NP^+(S,d) \geq NP^-(S,d)) \\ \underset{d\geq s\in S}{\arg\max}\,\Pi_{i\in N}|s_i - d_i| & (NP^+(S,d) < NP^-(S,d)). \end{cases}$$

This solution does not satisfy *Feasible Set Continuity* and *Disagreement Point Continuity*, but satisfies the other axioms.

Example 5. The disagreement solution f^{dis} satisfies all of the axioms except *Possibility of Utility Gain*.

5 Unified Proofs of Other Characterizations

In this section, we use the lemmas in Section 4.1 to provide unified proofs of other efficiency-free characterizations. This clarifies the technical role of each axiom introduced in the literature.

Using these lemmas, we provide a new characterization of the Nash solution. In Rachmilevitch (2015), the class of asymmetric Nash solutions is characterized by several axioms. We demonstrate that if we impose *Symmetry*, one of these axioms becomes redundant. This result is presented in Theorem 7.

5.1 Two characterizations by Roth (1977, 1979)

First, we show two theorems shown by Roth (1977, 1979).

Theorem 3 (Roth, 1977)**.** A bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence*, and *Strong Individual Rationality* if and only if f is the Nash solution f^{Nash}.

Proof of Theorem 3. By Lemma 2 and *Strong Individual Rationality*, we have $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ for each $c \leq 0$. By Lemma 3, $f = f^{\text{Nash}}$ holds. □

Theorem 4 (Roth, 1979)**.** A bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence*, and *Weak Individual Rationality* if and only if f is the Nash solution f^{Nash} or the disagreement solution f^{dis}.

Proof of Theorem 4. By Lemma 2 and *Weak Individual Rationality*, $f(\triangle(c), \mathbf{0})$ is either $\mathbf{0}$ or $\mathbf{1}$ for each $c \leq 0$. By Lemma 1, either $f(\triangle(c), \mathbf{0}) = \mathbf{0}$ for all $c \leq 0$ or $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ for all $c < 0$ holds. *Contraction Independence* implies that either $f(\triangle(c), \mathbf{0}) = \mathbf{0}$ for all $c \leq 0$ or $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ for all $c \leq 0$ holds. By Lemma 3, we have $f = f^{\text{Nash}}$ or f^{dis}. □

5.2 Rachmilevitch (2015)

Let $\text{id}_i(S) = \max_{s \in S} s_i$ and $\text{id}(S) = (\text{id}_1(S), \text{id}_2(S), \cdots, \text{id}_n(S))$. Rachmilevitch (2015) characterized the class of asymmetric Nash solutions with the following axiom.

Conflict-Freeness. For all $(S, d) \in \mathcal{B}^n$, if $\text{id}(S) \in S$, then $f(S, d) = \text{id}(S)$.

This axiom is stronger than *Possibility of Utility Gain*. In Rachmilevitch (2015), the class of asymmetric Nash solutions is characterized by *Scale Invariance*, *Contraction Independence*, *Feasible Set Continuity*, and *Conflict-Freeness*. As a corollary, if we impose *Symmetry* in addition to these axioms, then the Nash solution is characterized.

By using the lemmas in Section 4.1, we can easily show the characterization of the Nash solution. Moreover, *Feasible Set Continuity* is not necessary for this result.

Theorem 5. A bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence*, and *Conflict-Freeness* if and only if f is the Nash solution f^{Nash}.

Proof. For all $c \leq 0$, let $T_c = \{s \in \mathbb{R}^n \mid \mathbf{c} \leq s \leq \mathbf{1}\}$. If $f(\triangle(c), \mathbf{0}) = \mathbf{0}$ or $\mathbf{c}$, then *Contraction Independence* implies that $f(T_c, \mathbf{0})$ is either $\mathbf{0}$ or $\mathbf{c}$. This is a contradiction to *Conflict-Freeness*. By Lemma 2, we have $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ for all $c \leq 0$. Lemma 3 implies $f = f^{\text{Nash}}$. □

5.3 Anbarci and Sun (2011)

Anbarci and Sun (2011) used the following axiom to characterize the Nash solution.

Weakest Collective Rationality. For all $(S, d) \in \mathcal{B}^n$ and for all $s \in S$, if there is no $t \in S \backslash \{s\}$ such that $s \geq t$, then $f(S, d) \neq s$.

Theorem 6 (Anbarci and Sun, 2011)**.** A bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence*, and *Weakest Collective Rationality* if and only if f is the Nash solution f^{Nash}.

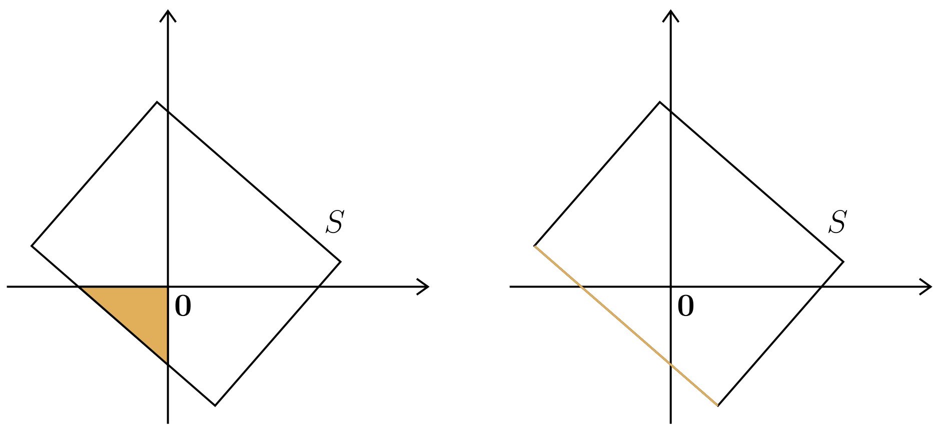
Proof. We show that, for all $c \leq 0$, $f(\triangle(c), \mathbf{0}) = \mathbf{1}$. By Lemma 2, $f(\triangle(c), \mathbf{0})$ is $\mathbf{c}$, $\mathbf{0}$ or $\mathbf{1}$ for each $c \leq 0$. If there exists $c < 0$ such that $f(\triangle(c'), \mathbf{0}) = \mathbf{0}$, then Lemma 1 implies $f(\triangle(0), \mathbf{0}) = \mathbf{0}$. This is a contradiction to *Weakest Collective Rationality*. Also, by *Weakest Collective Rationality*, there is no $c'' \leq 0$ satisfying $f(\triangle(c''), \mathbf{0}) = \mathbf{c}''$. Therefore, $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ holds for all $c \leq 0$. By Lemma 3, we have $f = f^{\text{Nash}}$. □

5.4 Mori (2018)

Mori (2018) also characterized the Nash solution. He used an axiom that requires that the solution outcome should not be weakly less than the disagreement point (in the sense of vector inequality). He argues that this axiom is more natural and intuitive than *Weakest Collective Rationality*.

Strong Undominatedness. For all $(S, d) \in \mathcal{B}^n$, $f(S, d) \leq d$ does not hold.

Note that this axiom and *Weakest Collective Rationality* are independent of each other. Figure 4 illustrates the distinction between these axioms in the case where $n = 2$. The colored area, including its boundary, is excluded by each axiom. Since there is no relationship of inclusion, the independence of these axioms is guaranteed.



(a) *Strong Undominatedness* **(b)** *Weakest Collective Rationality*

Figure 4: The difference between two axioms

Theorem 7 (Mori, 2018)**.** A bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence*, and *Strong Undominatedness* if and only if f is the Nash solution f^{Nash}.

Proof. By Lemma 2, $f(\triangle(c), \mathbf{0})$ is $\mathbf{c}$, $\mathbf{0}$ or $\mathbf{1}$ for each $c < 0$. By *Strong Undominatedness*, we have $f(\triangle(c), \mathbf{0}) = \mathbf{1}$ for all $c \leq 0$. Lemma 3 implies that $f = f^{\text{Nash}}$. □

6 Conclusion

In this paper, we have provided novel characterizations of the Nash solution without *Weak Pareto Optimality*. Our new axiom, *Possibility of Utility Gain*, requires weaker rationality than the axioms imposed to characterize the Nash solution in the literature. We have also shown that *Weak Individual Rationality* can be replaced by *Feasible Set Continuity* in the theorem of Roth (1979). The two quite different axioms have the same implication when each is imposed in conjunction with Nash's three axioms. Furthermore, we have provided unified proofs of other efficiency-free characterizations of the Nash solution. These proofs clarify the role of each axiom in the literature.

Our key axioms are *Possibility of Utility Gain*, *Feasible Set Continuity*, and *Disagreement Point Continuity*. Although we have investigated the properties of solutions satisfying either of two continuity axioms and Nash's axioms except *Weak Pareto Optimality*, we have not examined solutions satisfying *Possibility of Utility Gain* and these three axioms. Identifying the class of solutions that satisfy these axioms will clarify the role of *Feasible Set Continuity* and *Disagreement Point Continuity* in our results. We leave the investigation of this question for future work.

Appendix: Proof of Proposition 2

Lemma 8. In two-person bargaining problems, suppose that a bargaining solution f satisfies *Scale Invariance*, *Symmetry*, *Contraction Independence* and *Disagreement Point Continuity.* If $f(\triangle(c), \mathbf{0}) = \mathbf{c}$ for all $c \leq 0$, then $f = f^{AN}$ holds.

Proof. Step 1. Consider bargaining problems (S, d) such that d is in the interior of S. In the same way as Lemma 7, we have $f(S, d) = f^{AN}(S, d)$

Step 2. Let (S, d) be a bargaining problem such that d is on the boundary of S and that there is no $s \in S \backslash \{d\}$ satisfying $s \leq d$. In the same way as the proof of Lemma 4, we obtain $f(S, d) = d = f^{AN}(S, d)$.

Step 3. Consider bargaining problems (S, d) satisfying the following conditions:

- The disagreement point d is on the boundary of S.
- There exists $s \in S \backslash \{d\}$ such as $s_1 \leq d_1$ and $s_2 = d_2$.

By *Scale Invariance*, we can assume that $d = \mathbf{0}$. We show that $f(S, d) = (\min_{d \geq s} s_1,\ d_2)(= f^{AN}(S, d))$.

Let $S^* = \{s \in S \mid s \geq (\min_{d \geq s} s_1,\ d_2)\ (= f^{AN}(S, d))\ \}$. If $S = S^*$, there exist $a \in \mathbb{R}^2_-$ and $\beta \in \mathbb{R}^2_{++}$ such that $\beta a (= f^{AN}(S, d)) \in S$ and $S \subset \beta \mathrm{tr}(a)$. In the same way as Lemma 4, $f(\mathrm{tr}(a), \mathbf{0}) = a$ holds. (We use *Disagreement Point Continuity* instead of *Feasible Set Continuity.*) *Scale Invariance* and *Contraction Independence* imply that $f(S, \mathbf{0}) = \beta a = f^{AN}(S, d)$.

When $S \neq S^*$, suppose to the contrary that $f(S, d) \neq f^{AN}(S, d)$. Note that by Lemma 7, we have $f(S, d) \leq d$, which implies that $f(S, d) \in S^*$ *Contraction Independence* implies that $f(S^*, \mathbf{0}) = f(S, d) \neq f^{AN}(S, d) = f^{AN}(S^*, d)$. This is a contradiction to the result of the last paragraph. Therefore, $f(S, d) = (\min_{d \geq s} s_1,\ d_2) = f^{AN}(S, d)$ holds.

Step 4. Consider bargaining problems that satisfy the following conditions:

- The disagreement point d is on the boundary of S.

- There exists $s \in S \backslash \{d\}$ such as $s_1 = d_1$ and $s_2 \leq d_2$.

In the same way as Step 3, we have $f(S, d) = f^{AN}(S, d)$ for all (S, d) satisfying these conditions.

Hence, we have shown $f(S, d) = f^{AN}(S, d)$ for all $(S, d) \in \mathcal{B}^2$. □

Proof of Proposition 2. It is clear that f^{Nash}, f^{dis} and f^{AN} satisfy four axioms. Suppose that f satisfies four axioms: *Scale Invariance*, *Symmetry*, *Contraction Independence*, and *Disagreement Point Continuity*. By Lemma 3, Lemma 6 and Lemma 8, f is either of three solutions. □

References

Anbarci, Nejat and Ching-jen Sun (2011) "Weakest collective rationality and the Nash bargaining solution," *Social Choice and Welfare*, 37 (3), 425–429.

——— (2013) "Robustness of intermediate agreements and bargaining solutions," *Games and Economic Behavior*, 77 (1), 367–376.

Bossert, Walter and Hans Peters (2002) "Efficient solutions to bargaining problems with uncertain disagreement points," *Social Choice and Welfare*, 19 (3), 489–502.

——— (2022) "Individual disagreement point concavity and the bargaining problem," *International Journal of Economic Theory*, 18 (1), 6–15.

Chun, Youngsub and William Thomson (1990a) "Bargaining with Uncertain Disagreement Points," *Econometrica*, 58 (4), 951–59.

——— (1990b) "Nash solution and uncertain disagreement points," *Games and Economic Behavior*, 2 (3), 213–223.

Kalai, Ehud (1977a) "Proportional Solutions to Bargaining Situations: Interpersonal Utility Comparisons," *Econometrica*, 45 (7), 1623–1630.

——— (1977b) "Nonsymmetric Nash Solutions and Replication of 2-Person Bargaining," *International Journal of Games Theory*, 6, 129–133.

Kalai, Ehud and Meir Smorodinsky (1975) "Other Solutions to Nash's Bargaining Problem," *Econometrica*, 43 (3), 513–18.

Lensberg, Terje and William Thomson (1988) "Characterizing the Nash Bargaining Solution Without Pareto-Optimality," *Social Choice and Welfare*, 5 (2/3), 247–259.

Mori, Osamu (2018) "Two simple characterizations of the Nash bargaining solution," *Theory and Decision*, 85 (2), 225–232.

Nash, John F. (1950) "The Bargaining Problem," *Econometrica*, 18 (2), 155–162.

Perles, M. A. and Michael Maschler (1981) "The super-additive solution for the Nash bargaining game," *International Journal of Game Theory*, 10 (3-4), 163–193.

Peters, Hans and Eric van Damme (1991) "Characterizing the Nash and Raiffa Bargaining Solutions by Disagreement Point Axioms," *Mathematics of Operations Research*, 16 (3), 447–461.

Rachmilevitch, Shiran (2015) "Nash bargaining with (almost) no rationality," *Mathematical Social Sciences*, 76 (C), 107–109.

Roth, Alvin E. (1977) "Individual Rationality and Nash's Solution to the Bargaining Problem," *Mathematics of Operations Research*, 2 (1), 64–65.

——— (1979) *Axiomatic models of bargaining*, Lecture notes in economics and mathematical systems 170, Berlin, Heidelberg: Springer.

Thomson, William (1994) "Cooperative models of bargaining," in Aumann, R.J. and S. Hart eds. *Handbook of Game Theory with Economic Applications*, 2, 1st edition, Chap. 35, 1237–1284: Elsevier.

——— (2022) "On the axiomatic theory of bargaining: a survey of recent results," *Review of Economic Design*, 26 (4), 491–542.

Vartiainen, Hannu (2007) "Collective choice with endogenous reference outcome," *Games and Economic Behavior*, 58 (1), 172–180.