A NOTE ON TECHNICAL PROGRESS*

By Kenjiro Ara**

One of the production functions which are in "disembodied" technical progress is shown by

(1) \[ Y = F(\lambda(t)A(t), \lambda(t)B(t)L), \]

where \( Y \) = output, \( K \) = the existing stock of capital, \( L \) = the number of labour employed, \( t \) = time, \( A(t) \) = capital-augmenter and \( B(t) \) = labour-augmenter. \( A(t) \) and \( B(t) \) are respectively some non-decreasing function of \( t \). Choosing suitable units, we may put \( A(0) = B(0) = 1 \) without loss of generality. The only condition which we impose on the production function (1) is that the first derivatives of (1) are all positive, namely

(2) \[ \frac{\partial Y}{\partial K} > 0 \quad \text{and} \quad \frac{\partial Y}{\partial L} > 0. \]

Then let us call that a technical progress is "purely capital-augmenting" if \( B(t) \) is independent of \( t \), namely

(3) \[ Y = F(\lambda(t)A(t), L) \]

and "purely labour-augmenting" if \( A(t) \) is independent of \( t \), namely

(4) \[ Y = F(K, \lambda(t)L). \]

Now we want to prove the following

Theorem 1: In order for a technical progress which is purely capital-augmenting to be also purely labour-augmenting, it is necessary and sufficient that the production function is described by

\[ Y = \Psi(C(t)\lambda(t)L), \]

where \( \Psi \) = any differentiable function, \( C(t) \) = an increasing function of \( t \), and \( \alpha \) and \( \beta \) = some constants.

Sufficiency is self-evident. To prove the necessity of the theorem, it would be useful to define \( A(t) \equiv X_1 \) and \( B(t) \equiv X_2 \). Thus

(5) \[ Y = F(X_1, X_2). \]

Let us further put \( \log Y = y, \log X_1 = x_1 \) and \( \log X_2 = x_2 \). Thus it follows

(6) \[ y = f(x_1, x_2). \]

The first derivatives of (6) with respect to \( x_1 \) and \( x_2 \) are denoted by \( f_1 \) and \( f_2 \) respectively.

Once again we put \( \log A(t) = \phi_1(t), \log B(t) = \phi_2(t), \log K = k \) and \( \log L = l \).

Proof of Necessity: Using the above notations, we have

(7) \[ f(x_1, x_2) = f(k, x_2). \]

The first differentiation of this equation with respect to \( t \) gives us

(8) \[ \phi_1'(t) \cdot f_1(x_1, t) + \phi_2'(t) \cdot f_2(k, x_2) = 0. \]

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where $\phi'_2(t)$ is the first derivative of $\phi_2(t)$ with respect to $t$. Let us also differentiate (7) with respect to $x_2$. Then it follows

\[ \frac{\phi'_2(t)}{\phi_2(t)} = \frac{f(x_1, l)}{f(x_1, l)} \]

because $l = x_2 - \phi_2(t)$. Inserting (9) into (8), we get

\[ f(x_1, l) = f(x_1, l) \frac{\phi'_2(t)}{\phi_2(t)} = \frac{f(x_1, l)}{f(x_1, l)} \]

Thus it must follow

\[ \frac{f(x_1, l)}{f(x_1, l)} = \text{constant} \]

or

\[ \frac{1}{\alpha} \frac{\partial f}{\partial x_1} = \frac{1}{\beta} \frac{\partial f}{\partial l}, \]

where $\alpha$ and $\beta$ are some constants and $f = f(x_1, l)$. The solution of (12) is given by

\[ f(x_1, l) = f(\alpha x_1 + \beta l). \]

Because of $\alpha x_1 + \beta l = \alpha \phi_1(t) + \alpha k + \beta l$, we get

\[ y = f(\alpha \phi_1(t) + \alpha k + \beta l), \]

or, taking anti-logarithm,

\[ y = f(\alpha \phi_1(t) + \alpha k + \beta l), \]

where $\alpha$ and $\beta$ are some constants and $f_i = f(x_1, l)$. The solution of (12) is given by

\[ \frac{\partial f}{\partial x_1} = \frac{1}{\alpha} \frac{\partial f}{\partial x_1}, \]

Theorem 2: If the production function in Theorem 1 is homogeneous of $m$-th degree, it must be

\[ Y = D(t)K^{\alpha'}L^{\beta'}, \]

where $\alpha' + \beta' = m$ and $D(t)$ is an increasing function of $t$.

**Proof:** Being (15) homogeneous of $m$-th degree, we get

\[ \lambda^m Y = \psi(C(t)K^{\alpha'}L^{\beta'}) = \psi(C(t)K^{\alpha'}L^{\beta'}), \]

where $\lambda$ is any real number. Let us put

\[ \lambda^{\alpha + \beta} = (C(t)K^{\alpha'}L^{\beta'})^{-1}. \]

Then we have

\[ \lambda^m = (C(t)K^{\alpha'}L^{\beta'})^{-\frac{m}{\alpha + \beta}}. \]

Putting (17) and (18) into (16), it must follow

\[ (C(t)K^{\alpha'}L^{\beta'})^{-\frac{m}{\alpha + \beta}} \cdot Y = \psi(1) \]

or

\[ Y = \psi(1) \cdot C(t)^{\frac{m}{\alpha + \beta}} \cdot K^{\frac{\alpha}{\alpha + \beta} m} L^{\frac{\beta}{\alpha + \beta} m} \]

or

\[ Y = D(t)K^{\alpha'}L^{\beta'} \]

where $D(t) = \psi(1)C(t)^{\frac{m}{\alpha + \beta}}$, $\alpha' = \frac{\alpha}{\alpha + \beta} m$ and $\beta' = \frac{\beta}{\alpha + \beta} m$. It should be apparent that $\alpha' + \beta' = m$. 

(Q.E.D.)