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Efficiency and strategy-proofness in object allocation problems with payments: Externalities with income effects*

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Abstract

We consider the problem of allocating an object to $n \geq 2$ agents with payments. We allow agents to have preferences that exhibit (*allocative*) *externalities* and are not necessarily quasi-linear. Thus, agents care not only their own consumption of the object but also other agents' consumption or the owner keeping the object. A preference of an agent is *identity-independent* if he does not care who else (except for the owner) wins the object at the payment of zero. We show that if (i) all the agents have identity-independent preferences, and (ii) at least $n-1$ agents have preferences that exhibit positive externalities, then the *generalized pivotal rule* is the only rule satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness*. We also establish that if we relax one of the assumptions (i) and (ii), then no rule satisfies the four properties. Further, we find the two environments where some agents may have *identity-dependent* preferences, others have quasi-linear preferences exhibiting positive externaliteis, and there is a rule satisfying the four properties. Overall, our results suggest the importance of identity-independence and positive externalities in a non-quasi-linear environment with externalities for the existence of a rule satisfying the four properties.

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1 Introduction

1.1 Purpose

Auctions are widely perceived as an effective method of allocating scarce resources to agents efficiently. Indeed, one of the announced goals of many important auctions such as spectrum auctions is to allocate the resources efficiently. The following two features are pervasive in real-life auctions.

(i) Allocative externalities. Bidders who participate in an auction often care not only whether they win the goods but also who else does, i.e., they often experience (*allocative*) *externalities* (Jehiel and Moldovanu, 2003).[1] In many auctions of importance such as spectrum auctions and auctions for privatization of publicly owned companies, bidders engage in economic activities using the auctioned goods after the auctions. In such an auction, the result of the auction will affect the market structure, and thus the bidders will experience externalities. The sign of externalities depends on a situation. For example, in a spectrum auction, if a group company wins a license, then a company will experience a positive externality, while if a hostile company does, then it will do a negative externality.[2] Agents may also experience externalities because of concerns of fairness. The large literature on experimental economics suggests that some real-life people have preferences that are altruistic, philanthropic, or inequity-averse, which will cause positive externaliteis, while others have spiteful preferences, which will cause negative externalities (Fehr and Schmidt, 2003).

(ii) Non-quasi-linear preferences. The assumption of quasi-linear preferences makes the analysis simple and tractable, but is plausible only when a payment that each bidder makes

[1] Indeed, in the 3G spectrum auctions in the U.K. and Germany, "a major investment bank estimated license value as a function of the various possible market constellations" (Jehield and Mpldovanu, 2003), which reflects the fact that the bidders in the 3G auctions in these countries had preferences that exhibit externalities.

[2] Jehiel and Moldovanu (2000) illustrate via an example that when bidders engage in the Cournot competition in the downstream market, if an inventor sells a patent for a cost-reducing technical innovation, then they will experience negative externalities. They also illustrate that if a firm in a Cournot competition is up for sale through a second-price auction, then the other firms will experience positive externalities.

is so small that both budget constraints and income effects are negligible. In many auctions of importance, however, bidders often make large-scale payments, so that neither budget constraints nor income effects is negligible. Indeed, in the 3G spectrum auction in the U.K., the average winning bid was 7.41 billion euros (Jehiel and Moldovanu, 2003), which is not negligible for a firm.

We consider the problem of allocating a single object to $n \geq 2$ agents with payments. An *object allocation* specifies who receives the object. It is possible that the owner of the object keeps it. An agent not only cares whether he receives the object but also who else does (or the owner keeps it). Thus, a (*consumption*) *bundle* of an agent is a pair consisting of an object allocation and a payment. Each agent has a preference over the set of bundles that may exhibit externalities and is not necessarily quasi-linear. We assume *desirability of own consumption* which means that at a given payment, an agent prefers the own consumption the most. Such an assumption is plausible in auction environments which we regard one of the most important applications of this paper.

An *allocation* is a pair of an object allocation and a profile of payments. An (*allocation*) *rule* is a function from a set of preference profiles (a *domain*) to the set of allocations. It satisfies *efficiency* if no other allocation makes some agent better off without making any agent worse off, or decreasing the revenue of the owner. It satisfies *weak individual rationality* if each agent finds his outcome bundle of the rule at least as desirable as a bundle consisting of some object allocation and the payment of zero. This property is weaker than standard *individual rationality* which requires no agent get worse off than the bundle consisting of the owner keeping the object and the payment of zero, and is a minimal requirement of a participation constraint in an environment with externalities. A rule satisfies *no subsidy for losers* if the payment of an agent who does not win the object is non-negative. It satisfies *strategy-proofness* if no agent ever benefits from misrepresenting his preferences. We regard these four properties as basic desiderata.

It is already known that there is a rule satisfying the four properties if agents have quasi-linear preferences (Vickrey, 1961, Clarke, 1971, Groves, 1973), or if they have preferences exhibiting no externality (Saitoh and Serizawa, 2008; Sakai, 2008). The purpose of this paper is to study the effect of externalieis coupled with non-quasi-linear preferences on the class of rules satisfying the four properties. To be more specific, we attempt to identity

(i) the domains on which there is a rule satisfying the four properties and (ii) the class of rules satisfying the four properties when such a rules exists.

1.2 Main results

A preference of an agent exhibits *positive externalities* if at each payment, he finds the consumption of the object by other agents at least as desirable as the owner keeping it. *Negative externalities* are defined analogously. A preference is (*partly*) *identity-independent* if at the payment of zero, he is indifferent among the consumption of the object by other agents, i.e., he does not care the identity of the other agents (except for the owner) at the payments of zero. Note that identity-independence impose a restriction only the bundles with the payments of zero, and is a weak property.

We propose a new extension of the pivotal rule (Vickrey, 1961; Clarke, 1971; Groves, 1971) for quasi-linear preferences to non-quasi-linear environments. A *generalized pivotal rule* applies the pivotal rule to the valuations at the bundles consisting of the worst object allocation among the agents (i.e., without the owner) and the payments of zero. Note that non-quasi-linearity implies that the valuation depends on a bundle. Thus, the choice of the reference bundles to which an extension of the pivotal rule to a non-quasi-linear environment is evaluated matters for the properties of the rule. The generalized pivotal rule is different from the previous extensions of the pivotal rule such as the generalized Vickrey rule (Saitoh and Serizawa, 2008; Sakai, 2008) and the extended pivotal rule (Hashimoto and Saitoh, 2010) in the choice of the reference bundles.[3]

First, we study an environment where all the agents have identity-independent preferences. Identity-independence makes the situation somewhat close to an environment without externalities, and is a good place to begin the analysis.[4] We establishes that *if all the agents have identity-independent preferences, and at least* $n-1$ *agents have preferences that exhibit positive externalities, then the generalized pivotal rule is the only rule satisfying* efficiency, weak individual rationality, *no subsidy for losers, and* strategy-proofness (Theorem 1).

Once we obtain a positive result for the existence of a rule satisfying the desirable prop-

[3]In Section 3.1, we will discuss this point in detail.

[4]Some authors who take externalities into account assume conditions similar to our identity-independence to make their analysis tractable. See, for example, Jehiel et al. (1996, 1999) for the analysis of revenue optimal rules and Velez (2016) for the analysis of fair allocations.

erties, it is interesting to ask whether we can we relax the assumptions while guaranteeing the existence of such a rule. Recall that in Theorem 1, we make the two assumptions for a positive result, i.e., (i) all the agents have identity-independent preferences, and (ii) at least $n-1$ agents have preferences exhibiting positive externalities. We show that tightness of both the assumptions for the existence of a rule satisfying the four properties. That is, we show that *if all the agents have non-quasi-linear preferences, then once we relax one of the assumptions, no rule satisfies the four properties* (Theorems 2 and 3; see also Corollaries 3 and 4). Thus, both identity-independence and positive externalities are important for the existence of a rule satisfying the four properties in a non-quasi-linear environment with externalities.

Then, we move on to an environment where some agents may have identity-dependent preferences. A lesson from our results in an identity-independent environment is that if all the agents have non-quasi-linear preferences, then the existence of a rule satisfying the four properties is no longer guaranteed once we relax the assumption of identity-independence (Theorem 2). Thus, we focus on an environment where some agents have identity-dependent preferences, while others have quasi-linear preferences. We identity two such environments where there is a rule satisfying the four properties.

First, we establish that *even if we add agents who have identity-dependent and quasi-linear preferences that exhibit positive externalities to the environment in Theorem 1, the generalized pivotal rule is still the only rule satisfying the four properties* (Theorem 4).

Second, we consider an environment where a single agent (agent i) may have identity-dependent and non-quasi-linear preferences, while all the other agents have identity-independent and quasi-linear preferences that exhibit positive externalities. We propose an alternative and novel extension of the pivotal rule to such an environment. To do so, note that when preferences are quasi-linear, *strategy-proofness* of the pivotal rule implies that the pivotal rule is equivalent to the following rule: First, an agent chooses his most preferred bundle among the possible bundles given other agents' preferences under a pivotal rule. Second, the payments of the other agents are equal to that under a pivotal rule, i.e., the maximal impacts on the other agents. We call such a rule the *respect for the choice of an agent rule*. The *generalized pivotal rule respecting agent i* extends the pivotal rule to an environment under consideration on the basis of the equivalence between the pivotal rule and the respect for the choice of agent i rule for quasi-linear preferences. Thus, under the generalized pivotal rule respecting agent i, agent i first chooses the best bundle among the possible bundles given other agents' (quasi-linear) preferences under a pivotal rule, and then the other agents' payments are equal to the maximal impact on the other agents. We establish that *in an environment under consideration, the generalized pivotal rule respecting agent i is the only rule satisfying the four properties* (Theorem 5).

Finally, we show the tightness of the assumptions in the two positive results in environments where some agents have identity-dependent preferences, while others have quasi-linear preferences (Theorems 4 and 5). That is, we establish that *if we drop one of the assumptions in Theorem 4, then no rule satisfies the four properties* (Theorem 6 and Proposition 1; see also Corollaries 7, 8, and 9). We also establish that *if we drop one the assumptions in Theorem 5, then no rule satisfies the four properties* (Corollaries 10, 11, 12, and 13).

1.3 Related literature

This paper is the first one that investigates the class of rules satisfying *efficiency* and *strategy-proofness* together with the other desirable properties in a non-quasi-linear environment with externalities.

There is a growing literature on object allocation problems with payments that takes non-quasi-linear preferences into account (Saitoh and Serizawa, 2008; Sakai, 2008; Morimoto and Serizawa, 2015, Velez, 2016, etc.). One of the most closely related papers in this literature are Saitoh and Serizawa (2008) and Sakai (2008), both of which establish that in a single object model without externalities, the generalized Vickrey rule is the only rule satisfying *efficiency*, (*weak*) *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.[5] This paper extends their results to an environment with externalities, and in particular, because our generalized pivotal rule coincides with their generalized Vickrey rule in an environment without externalities, our first result (Theorem 1) implies their results as a corollary. This paper also contributes to the literature by proposing two new extensions of the pivotal rule to non-quasi-linear environments. In particular, the generalized pivotal rule respecting an agent is a novel extension such that there is no counterpart in the literature to the best of our knowledge. Apart from *strategy-proof* rules, Velez (2016) studies the

[5]Note that in an environment without externalities, *weak individual rationality* is equivalent to *individual rationality*.

structure of the set of envy-free allocations in a model with unit-demand agents, heterogeneous objects, non-quasi-linear preferences, and externalities.[6] His model is more general than ours because he considers agents who care not only the other agents' assignments of objects but also their payments.[7] Our results are logically independent from his because ours are concerned with the class of rules satisfying *efficiency* and *strategy-proofness* together with the other desirable properties, while he investigates envy-free allocations.

The literature on object allocation problems with payments that takes externalities into account has assumed quasi-linear preferences, and mainly focused on revenue optimal rules (Jehiel and Moldovanu, 1996; Jehiel et al., 1996, 1999; Aseff and Chade, 2008), the equilibrium analysis of particular auction rules (Jehiel and Moldovanu, 2000; Das Varma, 2002), or the properties of core allocations (Jehiel and Moldovanu, 1996; Jeong, 2020). This paper is different from this literature in that we are interested in *efficient* and *strategy-proof* rules in a non-quasi-linear environment. Note that the characterization result by Holmström (1979) implies that on quasi-linear and convex domains with externalities, the pivotal rule is the only rule satisfying the four properties. The results in this paper can be regarded as endeavors to extend his result to non-quasi-linear domains with externalities.

The model with externalities can be interpreted as a public goods model with payments if we regard the object allocations as public projects. Hashimoto and Saitoh (2010) consider the two public projects model with non-quasi-linear preferences, and characterize an extension of the pivotal rule that is different from ours by a weak property of *efficiency* that they call *partial efficiency*, *strategy-proofness*, *weak individual rationality*, and *no deficit*.[8] Ma et al. (2018) also study the public goods model with non-quasi-linear preferences, and identify a maximal domain for the existence of a rule that satisfies *non-dictatorship*, *ontoness*, *weak individual rationality*, *no subsidy*, and *strategy-proofness*.[9] The difference between this paper and these papers lies in domains and the properties of rules. Indeed, we consider preferences that satisfy not only the properties of externalities such as positive or negative externalities and identity-independence but also desirability of own consumption.

[6]An allocation is *envy-free* for a given preference profile if no agent prefers other agent's bundle. Velez (2016) calls an envy-free allocation a *noncontestable* allocation.

[7]Note that we study only agents with allocative externalities, i.e., agents who care only the other agents' consumption of the object.

[8]A rule satisfies *partial efficiency* if for each quasi-linear preference profile, it selects an efficient allocation. It satisfies *no deficit* if the sum of payments is always non-negative.

[9]A rule satisfies *non-dictatorship* if it is not a fixed-price dictatorship rule. It satisfies *ontoness* if for each public project, there is a preference profile at which the rule selects it. It satisfies *no subsidy* if the payment of each agent is always non-negative.

Such restrictions on preferences are essential for auctions with externalities, which gives the independent importance from these papers to this paper. Further, they consider the different sets of properties, so that our results are logically independent from theirs.

1.4 Organization

The remainder of this paper is organized as follows. In Section 2, we introduce the model. In Section 3, we introduce the pivotal rule and its extensions to non-quasi-linear environments. In Section 4, we present the results for an identity-independent environment. In Section 5, we give the results for an identity-dependent environment. In Section 6, we conclude the paper. All the proofs are relegated to the Appendix.

2 Model

There are $n \geq 2$ agents and a single object. The set of agents is $N = \{1, \ldots, n\}$. Our generic notation for an agent is i, j, k, l, etc. Let 0 denote the owner of the object. An object allocation is $x \in N \cup \{0\}$. Let $X = N \cup \{0\}$ denote the set of object allocations. Our generic notation for an object allocation is x, x', x'', etc. The amount of a payment made by an agent $i \in N$ is denoted by $t_i \in \mathbb{R}$. We consider agents who care not only their own consumption of the object but also the consumption of the other agents. Thus, the **consumption set** of an agent $i \in N$ is $X \times \mathbb{R}$, and his (**consumption**) **bundle** is a pair $z_i = (x, t_i) \in X \times \mathbb{R}$.

2.1 Preferences

An agent $i \in N$ has a complete and transitive preference R_i over $X \times \mathbb{R}$. Let P_i and I_i denote the strict and indifference relations associated with R_i, respectively. We assume that a preference R_i of an agent $i \in N$ satisfies the following four properties.

Desirability of own consumption. For each $x \in X \backslash \{i\}$ and each $t_i \in \mathbb{R}$, $(i, t_i)\ P_i\ (x, t_i)$.

Money monotonicity. For each $x \in X$ and each pair t_i,$t'_i \in \mathbb{R}$ with $t_i < t'_i$, $(x, t_i)\ P_i\ (x, t'_i)$.

Possibility of compensation. For each $z_i \in X \times \mathbb{R}$ and each $x \in X$, there is a pair

$t_i, t_i' \in \mathbb{R}$ such that $(x, t_i)\ R_i\ z_i$ and $z_i\ R_i\ (x, t_i')$.

Continuity. For each $z_i \in M \times \mathbb{R}$, the upper contour set at z_i, $\{z_i' \in X \times \mathbb{R} : z_i'\ R_i\ z_i\}$, and the lower contour set at z_i, $\{z_i' \in X \times \mathbb{R} : z_i\ R_i\ z_i'\}$, are both closed.

Desirability of own consumption means that given a payment, an agent prefers the own consumption of the object the most. The other three properties are standard in the literature. Given $i \in N$, let $\overline{\mathcal{R}}_i$ denote the class of all preferences of agent i satisfying the above four properties. Our generic notation for a class of preferences of an agent $i \in N$ satisfying the above four properties is $\mathcal{R}_i$. Thus, $\mathcal{R}_i \subseteq \overline{\mathcal{R}}_i$. Because of desirability of own consumption, a class of preferences depends on the identity of an agent.

Given a preference $R_i \in \mathcal{R}_i$ of an agent $i \in N$, a bundle $z_i \in X \times \mathbb{R}$, and an object allocation $x \in X$, we can choose a payment $t_i \in \mathbb{R}$ such that $(x, t_i)\ I_i\ z_i$.[10] By money monotonicity, such a payment is unique. The **valuation of** $x \in X$ **at** $z_i \in X \times \mathbb{R}$ **for** R_i is the unique payment $t_i \in \mathbb{R}$ such that $(x, t_i)\ I_i\ z_i$. Given $x \in X$, $z_i \in X \times \mathbb{R}$, and $R_i \in \mathcal{R}_i$, let $V_i(x, z_i)$ denote the valuation of x at z_i for R_i.

Given $i \in N$ and $R_i \in \mathcal{R}_i$, let $\underline{x}_i(R_i) \in X$ denote the worst object allocation at the payment of zero according to R_i, i.e., $(x, 0)\ R_i\ (\underline{x}_i(R_i), 0)$ for each $x \in X$. Also, given $i \in N$ and $R_i \in \mathcal{R}_i$, let $\overline{x}_i(R_i) \in N$ denote the worst object allocation among N at the payment of zero according to R_i, i.e., $(j, 0)\ R_i\ (\overline{x}_i(R_i), 0)$ for each $j \in N$. Then, $(\overline{x}_i(R_i), 0)\ R_i\ (\underline{x}_i(R_i), 0)$, and in general, $(\overline{x}(R_i), 0)$ is not indifferent to $(\underline{x}_i(R_i), 0)$. By desirability of own consumption, for each $i \in N$ and each $R_i \in \mathcal{R}_i$, $\underline{x}_i(R_i), \overline{x}_i(R_i) \in X \backslash \{i\}$.

Given a preference $R_i \in \mathcal{R}_i$ of an agent $i \in N$ and a set $A \subseteq X \times \mathbb{R}$, let $B(R_i, A) = \{z_i \in A : \forall z_i' \in A,\ z_i\ R_i\ z_i'\}$, i.e., $B(R_i, A)$ is the set of the best bundles according to R_i.

2.1.1 Externalities

In this section, we introduce the properties of (allocative) externalities. First, the following property has been (implicitly) assumed in the literature without externalities (e.g., Saitoh and Serizawa, 2008; Sakai, 2008, etc.).

Definition 1. A preference $R_i \in \mathcal{R}_i$ of an agent $i \in N$ **exhibits no externality** if for each $x \in X \backslash \{i, 0\}$ and each $t_i \in \mathbb{R}$, $(x, t_i)\ I_i\ (0, t_i)$.

[10] The existence of such a payment is guaranteed by the possibility of compensation and continuity.

Given $i \in N$, let $\mathcal{R}_i^0$ denote the class of agent i's preferences that exhibit no externality.

Then, we introduce the two properties of externalities. The first property states that an agent prefers other agents' consumption of the object to the owner keeping the object. By contrast, the second property states that an agent prefers the owner keeping the object to other agents' consumption.

Definition 2. (i) A preference $R_i \in \mathcal{R}_i$ of an agent $i \in N$ **exhibits** (resp. **strictly**) **positive externalities** if for each $x \in X \backslash \{i, 0\}$ and each $t_i \in \mathbb{R}$, $(x, t_i)\ R_i\ (0, t_i)$ (resp. $(x, t_i)\ P_i\ (0, t_i)$).
(ii) A preference $R_i \in \mathcal{R}_i$ of an agent $i \in N$ **exhibits** (resp. **strictly**) **negative externalities** if for each $x \in X \backslash \{i, 0\}$ and each $t_i \in \mathbb{R}$, $(0, t_i)\ R_i\ (x, t_i)$ (resp. $(0, t_i)\ P_i\ (x, t_i)$).

Given $i \in N$, let $\mathcal{R}_i^+$ and $\mathcal{R}_i^{++}$ denote the classes of agent i's preferences that exhibit positive externalities and strictly positive externalities, respectively. Also, given $i \in N$, let $\mathcal{R}_i^-$ and $\mathcal{R}_i^{--}$ denote the classes of agent i's preferences that exhibit negative externalities and strictly negative externalities, respectively. For each $i \in N$, $\mathcal{R}_i^+ \cap \mathcal{R}_i^- = \mathcal{R}_i^0$, and so $\mathcal{R}_i^0 \subsetneq \mathcal{R}_i^+$ and $\mathcal{R}_i^0 \subsetneq \mathcal{R}_i^-$. Also, for each $i \in N$, $\mathcal{R}_i^{++} \subsetneq \mathcal{R}_i^+$, $\mathcal{R}_i^{--} \subsetneq \mathcal{R}_i^-$, $\mathcal{R}_i^{++} \cap \mathcal{R}_i^{--} = \varnothing$, $\mathcal{R}_i^{++} \cap \mathcal{R}_i^0 = \varnothing$, and $\mathcal{R}_i^{--} \cap \mathcal{R}_i^0 = \varnothing$.

The next property describes an agent's preference which does not care about the identity of other agents at the payment of zero.

Definition 3. A preference $R_i \in \mathcal{R}_i$ of an agent $i \in N$ is (**partly**) **identity-independent** if for each pair $x, x' \in X \backslash \{i, 0\}$, $(x, 0)\ I_i\ (x', 0)$.

Given $i \in N$, let $\mathcal{R}_i^I$ denote the class of agent i's preferences that are identity-independent. For each $i \in N$, we have $\mathcal{R}_i^0 \subsetneq \mathcal{R}_i^I$, i.e., any preference that exhibits no externality is identity-independent. For the sake of the simplicity of notation, given $i \in N$, let $\mathcal{R}_i^{+I} = \mathcal{R}_i^+ \cap \mathcal{R}_i^I$ and $\mathcal{R}_i^{-I} = \mathcal{R}_i^- \cap \mathcal{R}_i^I$. We say that a preference R_i of an agent $i \in N$ is *identity-dependent* if it is not identity-independent, i.e., $R_i \notin \mathcal{R}_i^I$.

The above definition does not require identity-independence at each payment: for each pair $x, x' \in X \backslash \{i, 0\}$ and *for each* $t_i \in \mathbb{R}$, $(x, t_i)\ I_i\ (x', t_i)$. Instead, it only requires identity-independence at the payment of zero. Thus, even if a preference R_i of an agent $i \in N$ is identity-independent, he may care about the identify of other agents at a payment other than t_i, i.e., for some pair $j, k \in N \backslash \{i\}$ and for some $t_i \notin \mathbb{R} \backslash \{0\}$, we may have $(j, t_i)\ P_i\ (k, t_i)$. Note that even if a preference R_i of an agent $i \in N$ is identity-independent, there is no restriction on the relationship between the owner keeping the object (i.e., $x = 0$) and the consumption of another agent (i.e. $x = j$ for some $j \in N \backslash \{i\}$). Thus, identity-independence only imposes a few restrictions on a preference.

2.1.2 Income effects

In this section, we introduce the properties of income effects. For a given pair $x, x' \in X$ and a payment $t_i \in \mathbb{R}$, $V_i(x,(x',t_i)) - t_i$ is the *willingness to pay of* x *at* (x', t_i) for R_i. Note that if $(x, t_i)\ P_i\ (x', t_i)$, then $V_i(x,(x',t_i)) - t_i > 0$, i.e., the willingness to pay of x at (x', t_i) for R_i is positive. First, we introduce preferences that exhibit no income effect.

Definition 4. A preference $R_i \in \mathcal{R}_i$ of an agent $i \in N$ is **quasi-linear** if there is a (*quasi-linear*) *valuation function* $v_i : X \to \mathbb{R}$ such that (i) $v_i(\overline{x}_i(R_i)) = 0$, and (ii) for each pair $(x,t_i),(x',t_i') \in X \times \mathbb{R}$, $(x, t_i)\ R_i\ (x', t_i')$ if and only if $v_i(x) - t_i \geq v_i(x') - t_i'$.

Given $i \in N$, let $\mathcal{R}_i^Q$ denote the class of agent i's quasi-linear preferences. For each each $x \in X$ and each $(x', t_i) \in X \times \mathbb{R}$, we have $V_i(x,(x',t_i)) - t_i = v_i(x) - v_i(x')$. Thus, if $R_i \in \mathcal{R}_i^Q$, then the willingness to pay of x at (x', t_i) is independent of a preference t_i. Also, for each $x \in X$, we have $V_i(x,(\overline{x}_i(R_i), 0)) = v_i(x)$.

Next, we introduce the positive and negative income effects.

Definition 5. (i) A preference R_i of an agent $i \in N$ **exhibits positive income effects** if for each pair $x, x' \in X$, each $t_i \in \mathbb{R}$, and each $\delta \in \mathbb{R}_{++}$, $(x, t_i)\ P_i\ (x', t_i)$ implies $V_i(x,(x',t_i - \delta)) - (t_i - \delta) > V_i(x,(x',t_i)) - t_i$.
(ii) A preference R_i of an agent $i \in N$ **exhibits negative income effects** if for each pair $x, x' \in X$, each $t_i \in \mathbb{R}$, and each $\delta \in \mathbb{R}_{++}$, $(x, t_i)\ P_i\ (x', t_i)$ implies $0 < V_i(x,(x',t_i - \delta)) - (t_i - \delta) < V_i(x,(x',t_i)) - t_i$.

Although we do not explicitly take an agent's income into account, the payment of zero can be regarded as his initial income. Thus, a payment corresponds to an agent's (relative) income. The positive income effects mean that when a payment decreases (i.e., when an income increases), the willingness to pay of a preferable object allocation increases, i.e., a preferable object allocation gets more preferable. By contrast, the negative income effects mean that when a payment decreases, the willingness to pay of a preferable object allocation decreases.

Given $i \in N$, let $\mathcal{R}_i^{PIE}$ denote the class of agent i's preferences that exhibit positive income effects. Also, given $i \in N$, let $\mathcal{R}_i^{NIE}$ denote the class of agent i's preferences that exhibit negative income effects. For each $i \in N$, we have $\mathcal{R}_i^{PIE} \cap \mathcal{R}_i^{NIE} = \varnothing$, $\mathcal{R}_i^{PIE} \cap \mathcal{R}_i^{Q} = \varnothing$, and $\mathcal{R}_i^{NIE} \cap \mathcal{R}_i^{Q} = \varnothing$.

2.2 Allocations and rules

A (**feasible**) **allocation** is an $(n+1)$-tuple $z = (x,t) = (x, t_1, \ldots, t_n) \in X \times \mathbb{R}^n$, where $t = (t_1, \ldots, t_n)$ is the profile of payments associated with z.

A **domain** is $\mathcal{R}_N = \times_{i \in N} \mathcal{R}_i$. Given $N' \subseteq N$, let $\mathcal{R}_{N'} = \times_{i \in N'} \mathcal{R}_i$ and $\mathcal{R}_{-N'} = \times_{i \in N \setminus N'} \mathcal{R}_i$. Given a distinct pair $i, j \in N$, let $\mathcal{R}_{i,j} = \mathcal{R}_{\{i,j\}}$, $\mathcal{R}_{-i} = \mathcal{R}_{-\{i\}}$, and $\mathcal{R}_{-i,j} = \mathcal{R}_{-\{i,j\}}$. A **preference profile** is an n-tuple $R = (R_1, \ldots, R_n) \in \mathcal{R}_N$. Given $R \in \mathcal{R}_N$ and $N' \subseteq N$, let $R_{N'} = (R_i)_{i \in N'} \in \mathcal{R}_{N'}$ and $R_{-N'} = (R_i)_{i \in N \setminus N'} \in \mathcal{R}_{-N'}$. Given $R \in \mathcal{R}_N$ and a distinct pair $i, j \in N$, let $R_{i,j} = R_{\{i,j\}}$, $R_{-i} = R_{-\{i\}}$, and $R_{-i,j} = R_{-\{i,j\}}$.

Recall that $\mathcal{R}_i^{+}$ is the class of agent i's preferences that exhibit positive externalities. We employ the notation $\mathcal{R}_N^{+}$ to indicate the domain where the preferences of each agent exhibit positive externalities, i.e., $\mathcal{R}_N^{+} = \times_{i \in N} \mathcal{R}_i^{+}$. We will apply the parallel notations to indicate the domains where the preferences of each agent satisfy the corresponding properties, e.g., $\mathcal{R}_N^{PIE} = \times_{i \in N} \mathcal{R}_i^{PIE}$, $\mathcal{R}_N^{I} = \times_{i \in N} \mathcal{R}_i^{I}$, etc. Given $N' \subseteq N$ and a pair $i, j \in N$, we will also employ the notations $\mathcal{R}_{N'}^{+}$, $\mathcal{R}_{N'}^{PIE}$, $\mathcal{R}_{i,j}^{+}$, $\mathcal{R}_{-N'}^{+}$, $\mathcal{R}_{-i}^{+}$, , and $\mathcal{R}_{-i,j}^{+}$, etc.

In all of our results, we require a domain be rich in the following sense. A domain $\mathcal{R}_N$ is **rich** if $\mathcal{R}^N \supseteq \mathcal{R}_N^0 \cap \mathcal{R}_N^Q$. That is, a domain is said to be rich if it includes all quasi-linear preference profiles at which the preferences of each agent exhibit no externality. Since almost all domains of interest are rich, our requirement of richness is natural.

An (**allocation**) **rule** on $\mathcal{R}_N$ is a function $f : \mathcal{R}_N \to X \times \mathbb{R}^n$. With a slight abuse of notation, we may write $f = (x,t)$, where $x : \mathcal{R}_N \to X$ and $t : \mathcal{R}_N \to \mathbb{R}^n$ are the object allocation and the payment rules associated with f, respectively. Agent i's outcome bundle of a rule f at a preference profile $R \in \mathcal{R}_N$ is $f_i(R) = (x(R), t_i(R))$, where $x(R)$ and $t_i(R)$ are the object allocation and the payment made by agent i for R under f, respectively.

We introduce the properties of rules. Given $R \in \mathcal{R}_N$, an allocation $z = (x,t) \in X \times \mathbb{R}^n$ is (**Pareto-**)**efficient for** $R \in \mathcal{R}_N$ if there is no $z' = (x', t') \in X \times \mathbb{R}^n$ such that (i) for each $i \in N$, $z'_i \; R_i \; z_i$, (ii) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$, and (iii) for some $i \in N$, $z'_i \; P_i \; z_i$, or $\sum_{i \in N} t'_i > \sum_{i \in N} t_i$. The next remark states an allocation $z = (x,t)$ is efficient for a given preference profile if and only if the object allocation x associated with z maximizes the sum of valuations at z.

Remark 1. Let $R \in \mathcal{R}_N$. An allocation $z = (x, t) \in Z$ is efficient for R if and only if

$$x \in \arg\max_{x' \in X} \sum_{i \in N} V_i(x', z_i).$$

The first property requires that a rule should select an efficient allocation for each preference profile.

Efficiency. For each $R \in \mathcal{R}_N$, $f(R)$ is efficient for R.

Next, we introduce the two properties of participation constraints. The second property requires that each agent should find his outcome bundle of a rule at least as desirable as the bundle $(0, 0)$.

Individual rationality. For each $R \in \mathcal{R}_N$ and each $i \in N$, $f_i(R)\ R_i\ (0, 0)$.

Individual rationality corresponds to a participation constraint in an environment without externalities. In such an environment, an agent $i \in N$ is indifferent between $(0, 0)$ and $(x, 0)$ for each $x \in N \backslash \{i\}$, and thus the bundle $(0, 0)$ represents an outside option. In an environment with externalities, however, even if an agent chooses not to participate in a rule, he is concerned with who receives the object (or the owner keeps it), and so *individual rationality* may be demanding as a property of a participation constraint. The third property is a minimal property of a participation constraint in an environment with externalities, which requires that each agent should find his outcome bundle of a rule at least as desirable as the bundle $(x, 0)$ *for some* $x \in X$ instead of the bundle $(0, 0)$.

Weak individual rationality. For each $R \in \mathcal{R}_N$ and each $i \in N$, $f_i(R)\ R_i\ (\underline{x}_i(R_i), 0)$.

Clearly, *individual rationality* implies *weak individual rationality*.

The fourth property requires that if an agent does not receive the object (i.e., if he is a "loser"), then his payment should be nonnegative.

No subsidy for losers. For each $R \in \mathcal{R}_N$ and each $i \in N$, if $x(R) \neq i$, then $t_i(R) \geq 0$.

Note that *no subsidy for losers* is a natural extension of the property with the same name in a model without externalities to a model with externalities.

The last property is a dominant strategy incentive compatibility, which requires that no agent ever benefit from misrepresenting his preferences.

Strategy-proofness. For each $R \in \mathcal{R}_N$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $f_i(R) \; R_i \; f_i(R'_i, R_{-i})$.

3 The pivotal rule and its extensions

In this section, we introduce two new extensions of the pivotal rule (Vickrey, 1961; Clarke, 1971; Groves. 1973) to non-quasi-linear environments.

The pivotal rule is defined for quasi-linear preferences. Under the pivotal rule, the outcome object allocation maximizes the sum of quasi-linear valuations, and the outcome payment of each agent is equivalent to the maximal impact on the other agents.

Definition 6. A rule $f = (x, t)$ on $\mathcal{R}_N \subseteq \mathcal{R}_N^Q$ is a **pivotal rule** if for each $R \in \mathcal{R}_N$, the following conditions hold.
(i) We have

$$x(R) \in \arg\max_{x \in X} \sum_{i \in N} v_i(x).$$

(ii) For each $i \in N$,

$$t_i(R) = \max_{x \in X} \sum_{j \in N \setminus \{i\}} v_j(x) - \sum_{j \in N \setminus \{i\}} v_j(x(R)).$$

It is well-known that on a convex and rich quasi-linear domain, the pivotal rule is the only rule satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness*.[11]

[11] A domain $\mathcal{R}_N \subseteq \mathcal{R}_N^Q$ is *convex* if for each $i \in N$, each pair $R_i, R'_i \in \mathcal{R}_i$, and each $\lambda \in [0,1]$, a quasi-linear preference $R_i^\lambda \in \mathcal{R}_i^Q$ with a (quasi-linear) valuation function $v_i^\lambda(x) = \lambda v_i(x) + (1-\lambda)v'_i(x)$ belongs to $\mathcal{R}_i$.

Fact 1 (Holmström, 1979). *Let $\mathcal{R}_N \subseteq \mathcal{R}_N^Q$ be convex and rich. A rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a pivotal rule.*

Note that almost all quasi-linear domains of interest are convex and rich. Indeed, $\mathcal{R}_N^Q$, $\mathcal{R}_N^0 \cap \mathcal{R}_N^Q$, $\mathcal{R}_N^+ \cap \mathcal{R}_N^Q$, $\mathcal{R}_N^- \cap \mathcal{R}_N^Q$, $\mathcal{R}_N^I \cap \mathcal{R}_N^Q$, $\mathcal{R}_N^{+I} \cap \mathcal{R}_N^Q$, and $\mathcal{R}_N^{-I} \cap \mathcal{R}_N^Q$ are all convex and rich. Thus, we can apply Fact 1 to these domains.[12]

3.1 The generalized pivotal rule

We introduce our first new extension of the pivotal rule to a non-quasi-linear environment. Notice that the pivotal rule is defined by means of the quasi-linear valuation functions. Recall that if a preference R_i of an agent i is quasi-linear, then for each $x \in X$ and each $(x', t_i) \in X \times \mathbb{R}$, $V_i(x, (x', t_i)) - t_i = v_i(x') - v_i(x)$. Thus, when defining the pivotal rule for quasi-linear preferences, we do not have to care about the *reference bundles* at which the rule evaluates the valuations of agents. If a preference is not quasi-linear, however, the valuation may vary depending on a bundle. Thus, when extending the pivotal rule to a non-quasi-linear environment, the choice of reference bundles matters. The next extension of the pivotal rule chooses $(\overline{x}_i(R_i), 0)_{i \in N}$ as the reference bundles.

Definition 7. A rule $f = (x, t)$ on $\mathcal{R}_N$ is a **generalized pivotal rule** if for each $R \in \mathcal{R}_N$, the following conditions hold.
(i) We have

$$x(R) \in \arg\max_{x \in X} \sum_{i \in N} V_i(x, (\overline{x}_i(R_i), 0)).$$

(ii) For each $i \in N$,

$$t_i(R) = \max_{x \in X} \sum_{j \in N \setminus \{i\}} V_j(x, (\overline{x}_j(R_j), 0)) - \sum_{j \in N \setminus \{i\}} V_j(x(R), (\overline{x}_j(R_j), 0)).$$

Note that the generalized pivotal rule coincides with the pivotal rule on $\mathcal{R}_N \subseteq \mathcal{R}_N^Q$.

The previous literature has proposed several extensions of the pivotal rule to non-quasi-linear environments. The difference between the generalized pivotal rule and the previous

[12]Note that both $\mathcal{R}_N^{++} \cap \mathcal{R}_N^Q$ and $\mathcal{R}_N^{--} \cap \mathcal{R}_N^Q$ are convex but not rich, and we cannot apply Fact 1 to these domains. However, it is straightforward to see that the parallel characterization results hold on these domains.

ones lies in the choice of reference bundles. As we will show in the next sections, the choice of reference bundles will have a significant effect on the properties of a rule. Here, we compare the generalized pivotal rule with the generalized Vickrey rule (Saitoh and Serizawa, 2008; Sakai, 2008) and the extended pivotal rule (Hashimoto and Saitoh, 2010).

The generalized Vickrey rule has played a central role in the literature on object allocation problems with payments for non-quasi-linear preferences without externalities (Saitoh and Serizawa, 2008; Sakai, 2008, Malik and Mishra, 2021; Kazumura, 2022, Shinozaki et al., 2022). It is an extension of the pivotal rule with the reference bundles $(0,0)_{i\in N}$. It is different from the generalized pivotal rule unless for each $i \in N$, $(\overline{x}_i(R_i), 0)\ I_i\ (0,0)$ (as in the case of no externality). If preferences exhibit no externality, then it is the only rule satisfying *efficiency*, (*weak*) *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

Fact 2 (Saitoh and Serizawa, 2008; Sakai, 2008)**.** *Let $\mathcal{R}_N$ be rich and satisfy $\mathcal{R}_N \subseteq \mathcal{R}_N^0$. A rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

The extended pivotal rule was introduced in the model with two public projects (Hashimoto and Saitoh, 2010).[13] It is an extension of the pivotal rule with the reference bundles $(\underline{x}_i(R_i), 0)_{i\in N}$. It is different from the generalized pivotal rule unless for each $i \in N$, $(\overline{x}_i(R_i), 0)\ I_i\ (\underline{x}_i(R_i), 0)$ (as in the case of negative externaities).

3.2 The generalized pivotal rule respecting an agent

We introduce our second extension of the pivotal rule to an environment where a single agent may have identity-dependent and quasi-linear preferences, and all the other agents have identity-independent and quasi-linear preferences that exhibit positive externalities.

Note that *strategy-proofness* of the pivotal rule implies that under the rule, the payment of each agent depends only on the outcome object allocation chosen by the rule and other agents' preferences.[14] Given an agent $i \in N$, if agents have quasi-linear preferences, then the pivotal rule is equivalent to the following rule:

[13]Note that the object allocation model with externalities can be interpreted as the public goods model if we regard an object allocation as a public project.

[14]See Lemma 3 and the discussion in the two paragraphs just before Lemma 3 in Appendix A for the detail.

STEP 1. Let $R \in \mathcal{R}_N^Q$ be given. Agent i faces the possible payment for R_{-i} under a pivotal rule for each possible object allocation for R_{-i}. He chooses the best bundle among the pairs of object allocations and the payments that he faces, and receives it.

STEP 2. Note that the outcome object allocation is already determined in Step 1. For each $j \in N \backslash \{i\}$, the payment of agent j is equivalent to that under a pivotal rue.

We call the above rule the *respect for the choice of agent i rule*. The next remark states that if preferences are quasi-linear, then the respect for the choice of each agent rule produces the equivalent outcome to a pivotal rule. Thus, it does not matter to the rule who makes the choice in the first step.

Remark 2. Let $R \in \mathcal{R}_N^Q$ and $i \in N$. Then, the outcome of the respect for the choice of agent i rule coincides with that of agent j rule (except for ties), both of which are equivalent to the outcome of a pivotal rule.

If at least $n-1$ agents have preferences that exhibit positive externalities, then because of desirability of own consumption of an agent who may not have a preference that exhibits positive externalities and positive externalities of other agents' preferences, 0 is never chosen under an efficient allocation, and so under a pivotal rule.[15] Thus, if agents have such preferences, then 0 is excluded when an agent $i \in N$ makes a choice in Step 1 of the respect for the choice of agent i rule.

To illustrate that the outcome of a pivotal rule is equivalent to that of the respect for the choice of an agent $i \in N$ rule, we consider the following example.

Example 1. Let $n = 4$. Let $R_1 \in \mathcal{R}_1^Q$ be such that $v_1(1) = 12$, $v_1(2) = 8$, $v_1(3) = 4$, and $v_1(4) = v_1(0) = 3$. Let $R_2 \in \mathcal{R}_2^Q$ be such that $v_2(2) = 3$, and $v_2(1) = v_2(3) = v_2(4) = v_2(0) = 0$. Let $R_3 \in \mathcal{R}_3^Q$ be such that $v_3(3) = 10$, $v_3(1) = v_3(2) = v_3(4) = 0$, and $v_3(0) = -2$. Let $R_4 \in \mathcal{R}_4^Q$ be such that $v_4(4) = 6$, $v_4(1) = v_4(2) = v_4(3) = 0$, and $v_4(0) = -1$. Note that $R_{-1} \in \mathcal{R}_{-1}^+$, and so a pivotal rule never chooses 0 for R.

We identity the outcome of the respect for the choice of agent 1 rule for R. Given R_{-1}, the possible payments of agent 1 under a pivotal rule are as follows. When the rule chooses $x = 1$, agent 1 pays $\max_{x \in X} \sum_{i \in N \backslash \{1\}} v_i(x) - \sum_{i \in N \backslash \{1\}} v_i(1) = v_3(3) = 10$.

[15] See Lemma 1 for the formal statement of this claim.

When the rule chooses $x = 2$, agent 1 pays $v_3(3) - v_2(2) = 7$. When the rule choose $x = 3$, agent 1 pays $v_3(3) - v_3(3) = 0$. When the rule choose $x = 4$, agent 1 pays $v_3(3) - v_4(4) = 4$. Then, it is straightforward to check that agent 1 prefers $(3, 0)$ the most among $\{(1, 10), (2, 7), (3, 0), (4, 4)\}$, i.e., $B(R_1, \{(1, 10), (2, 7), (3, 0), (4, 4)\}) = \{(3, 0)\}$. Thus, in Step 1 of the rule, agent 1 receives $(3, 0)$. By Step 2, the payments of the other agents are equal to that under a pivotal rule. Thus, agent 2 pays 0, agent 3 pays $\max_{x \in X} \sum_{i \in N \setminus \{3\}} v_i(x) - \sum_{i \in N \setminus \{3\}} v_i(3) = v_1(1) - v_1(3) = 8$, and agent 4 pays 0. Thus, the outcome allocation of the rule is $z = (x, t_1, t_2, t_3, t_4) = (3, 0, 0, 8, 0)$, which is equivalent to the outcome allocation of a pivotal rule for R. □

On the basis of the observation that the pivotal rule is equivalent to the respect for the choice of an agent rule for quasi-linear preferences, we introduce an extension of a generalized pivotal rule to an environment where a single agent may have identity-dependent and quasi-linear preferences, and the other agents have identity-independent and quasi-linear preferences that exhibit positive externalities.

We introduce some notations. Given $i \in N$ and $R_i \in \mathcal{R}_i$, let $\succeq_i$ be the binary relation on $N \setminus \{i\}$ such that for each pair $j, k \in N \setminus \{i\}$, $j \succeq_i k$ if and only if $(j, 0)\ R_i\ (k, 0)$. Since R_i is complete and transitive, $\succeq_i$ is also complete and transitive. Also, since we may have $(j, 0)\ I_i\ (k, 0)$ for some distinct pair $j, k \in N \setminus \{i\}$, $\succeq_i$ is not necessarily antisymmetric. Let $\succ_i$ denote the asymmetric part of $\succeq_i$. Given a distinct pair $i, j \in N$ and $R_{-i} \in \mathcal{R}^Q_{-i}$, let

$$N_{-i,j}(R_{-j}) = \Big\{k \in N \setminus \{i, j\} : v_k(k) = \max_{l \in N \setminus \{i,j\}} v_l(l), \forall l \in N \setminus \{i, j\} \text{ with } v_l(l) = v_k(k),\ k \succeq_i l\Big\}.$$

Thus, $N_{-i,j}(R_{-j})$ is the set of agents other than agents i and j with the highest order according to $\succeq_i$ among the agents who have the highest (quasi-linear) valuation of own consumption among all the agents other than agents i and j.

Now, we are ready to introduce our second extension of the pivotal rule.

Definition 8. Given $i \in N$, let $\mathcal{R}_N$ be such that $\mathcal{R}_{-i} \subseteq \mathcal{R}^{+I}_{-i} \cap \mathcal{R}^Q_{-i}$. A rule f on $\mathcal{R}_N$ is a **generalized pivotal rule respecting agent** i if for each $R \in \mathcal{R}_N$, the following conditions hold.

(i) We have

$$f_i(R) \in B\Big(R_i, \Big\{\Big(j, \max_{k \in N \setminus \{i\}} v_k(k) - v_j(j)\Big) : j \in N \setminus \{i\}\Big\} \bigcup \Big\{\Big(i, \max_{j \in N \setminus \{i\}} v_j(j)\Big)\Big\}\Big).$$

(ii) For each $j \in N\backslash\{i\}$, if $x(R) = j$, then there is $k \in N_{-i,j}(R_{-j})$ such that (ii-i) if $k \succeq_i j$, then

$$t_j(R) = \max\Big\{\tau_{j,i}(R_{-j}), \max_{l \in N\backslash\{i,j\}: l \succeq_i j} \tau_{j,l}(R_{-j})\Big\},$$

and (ii-ii) if $x(R) = j$ and $j \succ_i k$, then

$$t_j(R) = \max\Big\{\tau_{j,i}(R_{-j}), \max_{l \in N\backslash\{i,j\}: l \succeq_i k} \tau_{j,l}(R_{-j})\Big\},$$

where

$$\tau_{j,i}(R_{-j}) = \begin{cases} V_i(i,(j,0)) & \text{if } k \succeq_i j \text{ or } v_k(k) \leq V_i(i,(j,0)) \\ v_k(k) - V_i(j,(i,v_k(k))) & \text{if } j \succ_i k \text{ and } V_i(i,(j,0)) \leq v_k(k) \leq V_i(i,(k,0)), \\ v_k(k) - V_i(j,(k,0)) & \text{if } j \succ_i k \text{ and } v_k(k) \geq V_i(i,(k,0)), \end{cases}$$

for each $l \in N\backslash\{i,j\}$ with $l \succeq_i j$,

$$\tau_{j,l}(R_{-j}) = \begin{cases} v_l(l) + V_i(l,(j,0)) & \text{if } v_k(k) - v_l(l) \leq V_i(l,(j,0)), \\ v_k(k) - V_i(j,(l,v_k(k) - v_l(l))), & \text{if } v_k(k) - v_l(l) \geq V_i(l,(j,0)), \end{cases}$$

and for each $l \in N\backslash\{i,j\}$ with $j \succ_i l$,

$$\tau_{j,l}(R_{-j}) = v_k(k) - V_i(j,(l,v_k(k) - v_l(l))).$$

For each $j \in N\backslash\{i\}$, if $x(R) \neq j$, then $t_j(R) = 0$.

The first condition of the definition corresponds to Step 1 of the respect for the choice of agent i rule, which states that agent i faces the possible payments under a pivotal rule, and chooses his best object allocation under the payments.[16] Note that by $\mathcal{R}_{-i} \subseteq \mathcal{R}^+_{-i}$, a pivotal rule never chooses 0, and so 0 is excluded from the choice of agent i in the above definition. Note that for each $l \in N\backslash\{j\}$, $\tau_{j,l}(R_{-j})$ corresponds to the impact of agent j's consumption of the object from agent l's consumption on the other agents. Because of non-quasi-linear preferences, such impacts depend on bundles. The second condition corresponds to Step 2 of the respect for the choice of agent i rule, which states that the

[16] Note that for each $R_{-i} \in \mathcal{R}^{+I}_{-i} \cap \mathcal{R}^{Q}_{-i}$, if a pivotal rule chooses $x = 1$, then the payment of agent i is $\max_{x \in X} \sum_{j \in N\backslash\{i\}} v_j(x) = \max_{j \in N\backslash\{i\}} v_j(j)$, and if it chooses $x = j$ for some $j \in N\backslash\{i\}$, then the payment of agent i is $\max_{x \in X} \sum_{k \in N\backslash\{i\}} v_k(x) - \sum_{k \in N\backslash\{i\}} v_k(j) = \max_{k \in N\backslash\{i\}} v_k(k) - v_j(j)$.

payment of each agent $j \in N\backslash\{i\}$ is equal to the maximum impact of agent j's consumption on the other agents. If preferences are quasi-linear, then such impacts do not depend on bundles, and so the generalized pivotal rule respecting agent i coincides with the pivotal rule on $\mathcal{R}_N \subseteq \mathcal{R}_N^Q$. Also, if $\mathcal{R}_N \subseteq \mathcal{R}_N^I$, i.e., if preferences are identity-independent, then the generalized pivotal rule respecting agent i coincides with the generalized pivotal rule.

The definition of the rule does not depend on the choice of an agent $k \in N_{-i,j}(R_{-j})$ in the second condition above. Indeed, except for ties, for any choice of an agent $k \in N_{-i,j}(R_{-j})$, the rule produces the same outcome.[17] The definition of the rule, however, depends on the choice of agent $i \in N$ in contrast to Remark 1.

To illustrate the generalized pivotal rule respecting a single agent, we consider the following example which applies the same preference profile as in Example 1 to the rule, and verify that if preferences are quasi-linear, then the outcome of the rule is indeed equivalent to that of the pivotal rule (or the respect for the choice of agent i rule).

Example 2. Let $n = 4$. Let $\mathcal{R}_N$ be such that $\mathcal{R}_1 = \mathcal{R}_1^Q$ and $\mathcal{R}_{-1} = \mathcal{R}_{-1}^{+I} \cap \mathcal{R}_{-1}^Q$. Let $f = (x, t)$ be a generalized pivotal rule respecting agent 1 on $\mathcal{R}_N$. Consider the same preference profile $R = (R_1, R_2, R_3, R_4)$ as in Example 1. Recall that we have argued in Example 1 that $B(R_1, \{(1, 10), (2, 7), (3, 0), (4, 4)\}) = \{(3, 0)\}$. Thus, $f_1(R) = (3, 0)$. By $x(R) = 3$, for each $i \in N\backslash\{1, 3\}$, $t_i(R) = 0$. We identify the payment of agent 3 on the basis of the definition of the rule. Note that $2 \succ_1 3 \succ_1 4$. By $v_4(4) > v_2(2)$, $N_{-1,3}(R_{-3}) = \{4\}$. By $R_1 \in \mathcal{R}_1^Q$, $V_1(1, (3, 0)) = v_1(1) - v_1(3) = 8$. Thus, by $v_4(4) = 6 < 8 = V_1(1, (3, 0))$, $\tau_{3,1}(R_{-3}) = V_1(1, (3, 0)) = 8$. By $R_1 \in \mathcal{R}_1^Q$, $V_1(2, (3, 0)) = v_1(2) - v_1(3) = 4$. Thus, by $v_4(4) - v_2(2) = 3 < 4 = V_1(2, (3, 0))$, $\tau_{3,2}(R_{-3}) = v_2(2) + V_1(2, (3, 0)) = 7$. By $R_1 \in \mathcal{R}_1^Q$, we have $V_1(3, (4, v_4(4) - v_4(4))) = V_1(3, (4, 0)) = v_1(3) - v_1(4) = 4$. Thus, $\tau_{3,4}(R_{-3}) = v_4(4) - V_1(3, (4, v_4(4) - v_4(4))) = 6 - 4 = 2$. Thus, by $3 \succ_1 4$, $t_3(R) = \max\{\tau_{3,1}(R_{-3}), \max_{i \in N\backslash\{1,3\}: i \succeq_1 4} \tau_{3,i}(R_{-3})\} = \max\{8, 7, 2\} = 8$. In summary, $f(R) = (x(R), t_1(R), t_2(R), t_3(R), t_4(R)) = (3, 0, 0, 8, 0)$. By Example 1, this coincides with an outcome of a pivotal rule for R (and that of a respect for the choice of agent 1 rule). □

[17] To see this, let $k, l \in N_{-i,j}(R_{-j})$, and $\tau_{j,i}^k(R_{-j})$ and $\tau_{j,i}^l(R_{-j})$ denote the functions in the definition of the generalized pivotal rule respecting agent i defined on the basis of k and l, respectively. By $k, l \in N_{-i,j}(R_{-j})$, $v_k(k) = v_l(l)$. Also, $k \succeq_i l$ and $l \succeq_i k$. Thus, $\tau_{j,i}^k(R_{-j}) = \tau_{j,i}^l(R_{-j})$, and so the generalized pivotal rule respecting agent i does not depend on the choice of an agent in $N_{i,j}(R_{-i})$.

4 Identity-independent preferences

In this section, we study an environment where all the agents have identity-independent preferences.

4.1 Characterization

The next result states that if at least $n-1$ agents have preferences that exhibit positive externalities, then the generalized pivotal rule is the only rule satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

Theorem 1. *Let $i \in N$. Let $\mathcal{R}_N$ be rich and satisfy $\mathcal{R}_N \subseteq \mathcal{R}_i^I \times \mathcal{R}_{-i}^{+I}$. A rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a generalized pivotal rule.*

One of the two key observations behind Theorem 1 is that if at least $n-1$ agents have preferences that exhibit positive externalities, then the owner never keeps the object under an efficient allocation.[18] The other one is that under identity-independence, each agent $i \in N$ finds the bundle $(j, 0)$ indifferent to his reference bundle $(\overline{x}_i(R_i), 0)$ for each $j \in N\backslash\{i\}$, so that his reference bundle is essentially unique. These two observations together imply that the behavior of the generalized pivotal rule is similar to the generalized Vickrey rule in an environment without externalities. Recall Fact 2 that in an environment without externalities, the generalize Vickrey rule is the only rule satisfying the four properties. Theorem 1 shows that a parallel characterization result extends to an environment under consideration. We note, however, that Theorem 1 is not a trivial extension of Fact 2 because identity-independence only imposes a restriction on the payment of zero, and so how externalities at the payments of other than zero affect not only the properties of the generalized pivotal rule but also the class of rules satisfying the four properties is not obvious a priori.

For a rich domain R_N such that $\mathcal{R}_N \subseteq \mathcal{R}_i^I \times \mathcal{R}_{-i}^{+I}$ and an agent $j \in N$, $(\overline{x}_j(R_j), 0)$ is indifferent neither to $(0, 0)$ nor to $(\underline{x}_j(R_j), 0)$ according to R_j in general. Thus, on such a domain, a generalized pivotal rule is equivalent neither to a generalized Vickrey rule (Saitoh and Serizawa, 2008; Sakai, 2008) nor to an extended pivotal rule (Hashimoto and Saitoh,

[18]See Lemma 1 in Appendix A for the formal statement of and proof. Note that we already discussed this point when introducing the generalized pivotal rule respecting agent i in Section 3.2.

2010) in general.[19] Theorem 1 shows that the generalized pivotal rule stands out as the unique rule satisfying the four properties instead of the other extensions of the pivotal rule to non-quasi-linear environments, highlighting the importance of the choice of reference bundles in a non-quasi-linear environment.

We discuss the implications from Theorem 1. First, it implies that if all the agents have identity-independent preferences that exhibit positive externalities, then the generalized pivotal rule is the only rule satisfying the four properties.

Corollary 1. *Let $\mathcal{R}_N$ be rich and satisfy $\mathcal{R}_N \subseteq \mathcal{R}_N^{+I}$. A rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a generalized pivotal rule.*

It also implies that under identity-independence, if one agent has preferences that exhibit negative externalities, and all the other agents have those that exhibit positive externalities, then the generalized pivotal rule is the only rule satisfying the four properties.

Corollary 2. *Let $i \in N$. Let $\mathcal{R}_N$ be rich and satisfy $\mathcal{R}_N \subseteq \mathcal{R}_i^{-I} \times \mathcal{R}_{-i}^{+I}$. A rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a generalized pivotal rule.*

4.2 Tightness

Theorem 1 implies that for a given $i \in N$, there is a rule satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness* on the domain $\mathcal{R}_i^{I} \times \mathcal{R}_{-i}^{+I}$. Thus, in Theorem 1, we make the two assumptions for a positive result, i.e., (i) all the agents have identity-independent preferences, and (ii) at least $n-1$ agents have preferences that exhibit positive externalities. In this section, we ask whether we can relax the assumptions while guaranteeing the existence of a rule satisfying the four properties. Then, we show the tightness of the assumptions in Theorem 1, i.e., show that if we relax one of the assumptions, then no rule satisfies the four properties.

To show the tightness of the first assumption in Theorem 1 that all the agents have identity-independent preferences, we show the next result. It states that if a class of preferences of an agent includes at least one *identity-dependent* preference that exhibits either positive or negative income effects, and a domain includes all non-quasi-linear preference

[19] Recall the discussion in Section 3.1 after the definition of the generalized pivotal rule.

profiles at which the preferences of each agent exhibit no externality, then no rule satisfies the four properties.

Theorem 2. *Let $i \in N$. Let $R_i \in (\mathcal{R}_i^{PIE} \cup \mathcal{R}_i^{NIE}) \setminus \mathcal{R}_i^I$. Let $\mathcal{R}_N$ be such that $R_i \in \mathcal{R}_i$ and $\mathcal{R}_N \supseteq \mathcal{R}_N^0$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Theorem 2 implies that if we add at least one arbitrary *identity-dependent* preference of an agent that exhibits either positive or negative externalities to the domain $\mathcal{R}_i^I \times \mathcal{R}_{-i}^{+I}$, then no rule satisfies the four properties. Thus, we cannot relax the assumption of identity-independence in Theorem 1.

Corollary 3. *Let $i, j \in N$ be a pair.*[20] *Let $R_j \in (\mathcal{R}_j^{PIE} \cup \mathcal{R}_j^{NIE}) \setminus \mathcal{R}_j^I$. Let $\mathcal{R}_N$ satisfy $R_j \in \mathcal{R}_j$, and $\mathcal{R}_N \supseteq \mathcal{R}_i^I \times \mathcal{R}_{-i}^{+I}$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Next, we show the tightness of the second assumption in Theorem 1 that at least $n-1$ agents have preferences that exhibit positive externalities. To do so, we show the following result which states that if an agent has at least one preference that exhibits *strictly negative externalities* and either positive or negative income effects, and another agent has identity-independent and quasi-linear preferences that exhibit negative externalities, then no rule satisfies the four properties.

Theorem 3. *Let $i, j \in N$ be a distinct pair. Let $R_i \in \mathcal{R}_i^{--} \cap (\mathcal{R}_i^{PIE} \cup \mathcal{R}_i^{NIE})$. Let $\mathcal{R}_N$ be rich and satisfy $R_i \in \mathcal{R}_i$ and $\mathcal{R}_j \supseteq \mathcal{R}_j^{-I} \cap \mathcal{R}_j^Q$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Theorem 3 implies that if we add one arbitrary preference of an agent $j \in N \setminus \{i\}$ that exhibits *strictly negative externalities* and either positive or negative externalities to the domain $\mathcal{R}_i^I \times \mathcal{R}_{-i}^{+I}$, then no rule satisfies the four properties. Put differently, it implies that if *at least two agents* have identity-independent preferences that *do not necessarily exhibit positive externalities*, then no rule satisfies the four properties. Thus, we cannot relax the assumption that at least $n-1$ agents have preferences that exhibit positive externalities in Theorem 1.

[20] Note that we allow the case where $i = j$.

Corollary 4. *Let $i, j \in N$ be a distinct pair. Let $R_j \in \mathcal{R}_j^{--} \cap (\mathcal{R}_j^{PIE} \cup \mathcal{R}_j^{NIE})$. Let $\mathcal{R}_N$ satisfy $R_j \in \mathcal{R}_j$, and $\mathcal{R}_N \supseteq \mathcal{R}_i^I \times \mathcal{R}_{-i}^{+I}$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Before moving on to the next section, we discuss Theorems 2 and 3. We first discuss Theorem 2. Notice that a preference R_i in Theorem 2 may exhibit either positive or negative externalities. Thus, as a corollary of Theorem 2, we obtain the following result.

Corollary 5. *Assume $n \geq 3$. Let $\mathcal{R}_N$ be either $\mathcal{R}_N = \mathcal{R}_N^+$ or $\mathcal{R}_N = \mathcal{R}_N^-$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

In Theorem 2, we require a domain include all non-quasi-linear preference profiles at which the preferences of each agent exhibit no externality, which is stronger than our richness condition. We cannot replace this stronger richness condition by our richness condition in Theorem 2. Indeed, we will show in Section 5.1 that if a single agent may have identity-dependent preferences, and all the other agents have identity-independent and quasi-linear preferences that exhibit positive externalities, then a generalized pivotal rule respecting an agent satisfies the four properties (Theorem 5 in Section 5.1).

Next, we discuss Theorem 3. Recall Corollary 2 shows that if there is only one agent who has preferences that exhibit negative externalities, then there is a rule satisfying the four properties. It is interesting to ask whether there is a rule satisfying the four properties if several agents may have preferences that exhibit negative externalities. Theorem 3 helps us answer this question. It implies that if there are at least two agents who have identity-independent preferences that exhibit negative externalities, and at least one such agent has non-quasi-linear preferences, then no rule satisfies the four properties.

Corollary 6. *Let $i, j \in N$ be a distinct pair. Let $\mathcal{R}_N$ be a rich domain that satisfies $\mathcal{R}_{i,j} \supseteq \mathcal{R}_i^{-I} \times (\mathcal{R}_j^{-I} \cap \mathcal{R}_j^Q)$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

5 Identity-dependent preferences

Next, we study an environment where some agents may have identity-dependent preferences. An implication from Theorem 2 is that if all the agents have non-quasi-linear preferences, then a single identity-dependent preference exhibiting either positive or negative income effects leads to an impossibility result for *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Thus, we focus on environments where some agents have identity-dependent preferences, while others have quasi-linear preferences.

5.1 Characterization

First, we consider an environment where some agents have identity-independent and non-quasi-linear preferences, while others have identity-dependent and quasi-linear preferences that exhibit positive externalities. Let $NQL \subseteq N$ denote the set of agents who have non-quasi-linear preferences, and $QL \subseteq N$ those who have quasi-linear preferences. The next result states that in such an environment, if all but one agents in NQL have preferences that exhibit positive externalities, then the generalized pivotal rule is the only rule satisfying the four properties. Put differently, it states that even if we add agents who have identity-dependent and quasi-linear preferences to the environment in Theorem 1, the generalized pivotal rule is still the only rule satisfying the four properties.

Theorem 4. *Let $NQL, QL \subseteq N$ be a pair such that $NQL \cap QL = \varnothing$ and $N = NQL \cup QL$. Let $i \in NQL$. Let $\mathcal{R}_N = \mathcal{R}_{NQL} \times \mathcal{R}_{QL}$ be rich and satisfy $\mathcal{R}_{NQL} \subseteq \mathcal{R}_i^I \times \mathcal{R}_{NQL\setminus\{i\}}^{+I}$ and $\mathcal{R}_{QL}^{+I} \cap \mathcal{R}_{QL}^{Q} \subseteq \mathcal{R}_{QL} \subseteq \mathcal{R}_{QL}^{+} \cap \mathcal{R}_{QL}^{Q}$. A rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a generalized pivotal rule.*

Notice that in Theorem 4, we allow the case where $N = NQL$ or $N = QL$. If $N = NQL$, then Theorem 4 reduces to Theorem 1. Thus, Theorem 4 subsumes Theorem 1 as a corollary, and so it is sufficient to provide the proof of Theorem 4 instead of Theorem 1.[21] If $N = QL$, then all the agents have quasi-linear preferences, and the result reduces to a characterization of the pivotal rule. However, a characterization in a case where $N = QL$ is still a new result because it allows non-convex domains, and so does not follow from previous characterization results of the pivotal rule for quasi-linear preferences such as Fact 1.

Next, we consider an environment where a single agent may have identity-dependent

[21] We begin with the results in an environment where all the agents have identity-independent preferences in Section 4 instead of introducing Theorem 4 first in order to make the motivations for considering the domains in Theorems 4 and 5 clearer. Indeed, Theorem 2 tells us that we need to restrict our attention to domains where some agents have identity-dependent preferences whereas others have quasi-linear preferences, which makes the reason for considering the domains in Theorems 4 and 5 clearer.

and non-quasi-linear preferences, while all the other agents have identity-independent and quasi-linear preferences that exhibit positive externalities. The next theorem establishes that in such an environment, the generalized pivotal rule respecting the agent who may have identity-dependent preferences is the only rule satisfying the four properties.

Theorem 5. *Let $i \in N$. Let $\mathcal{R}_N$ be rich and satisfy $\mathcal{R}_N \subseteq \overline{\mathcal{R}}_i \times (\mathcal{R}^Q_{-i} \cap \mathcal{R}^{+I}_{-i})$. A rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness if and only if it is a generalized pivotal rule respecting agent i.*

Recall Remark 2 states that the respect for the choice of an agent rule is equivalent to the pivotal rule. Recall also that for a given $i \in N$, the generalized pivotal rule respecting agent i extends the pivotal rule to a non-quasi-linear environment on the basis of this equivalence. Although several extensions of the pivotal rule to non-quasi-linear environments has been proposed such as the generalized Vickrey rule in several environments without externalities (Saitoh and Serizawa, 2008; Sakai, 2008, etc.), the minimum price Warlasian rule in a unit-demand environment without externalities (Demange and Gale, 1985; Morimoto and Serizawa, 2015), etc., there has been no extension similar to the generalized pivotal rule respecting an agent. Theorem 5 is a novel characterization result in that it shows the pivotal rule is extended to a non-quasi-linear environment in a distinct way while keeping its uniqueness of a rule satisfying the four properties.

Theorem 4 helps us prove the "only if" of Theorem 5. The proof of the "only if" part of Theorem 5 is divided into two steps: First, we show that the outcome bundle of agent i under a rule f satisfying the four properties coincides with that under a generalized pivotal rule respecting agent i. Second, we show that the payments of the other agents under the rule f coincide with those under a generalized pivotal rule respecting agent i. Theorem 4 implies that the payment of agent i under the rule f coincides with that under a generalized pivotal rule respecting agent i. Thus, it enables us to focus on the proof that the outcome object allocation under the rule f coincides with that under a generalized pivotal rule respecting agent i in the first step.[22] Moreover, we will repeatedly exploit Theorem 4 in the second step to identify the payments of agent i under the rule f.[23]

A byproduct of Theorem 5 is that it helps us prove the impossibility theorems (Theorems 2, 3, and 6 in Section 5.2). Indeed, the common proof strategy underlying the

[22]For the detail, see Lemma 21 in Appendix C.2.

[23]See Lemmas 22 and 23 in Appendix C.2 for the detail.

impossibility theorems is that we (i) first suppose for a contradiction that there is a rule satisfying the four properties, (ii) attempt to identify the payment of an agent under the rule when another single agent has an identity-dependent preference, and (iii) finally show that the rule violates either *efficiency* or *strategy-proofness* at a certain preference profile. We will exploit the "only if" part of Theorem 5 (in particular, Lemmas 22 and 23 in Appendix C.2) to identify such a payment in (ii) of the proof strategy.

5.2 Tightness

In this section, we show the tightness of the assumptions in Theorems 4 and 5. To do so, together with Theorems 2 and 3, we need to show the following two results. First, the following theorem states that if an agent has an identity-dependent preference that exhibits either positive or negative income effects and another agent has quasi-linear and *identity-dependent* preferences that exhibit positive externalities, then no rule satisfies the four properties.

Theorem 6. *Let $i, j \in N$ be a distinct pair. Let $R_i \in (\mathcal{R}_i^{PIE} \cup \mathcal{R}_i^{NIE}) \backslash \mathcal{R}_i^I$. Let $\mathcal{R}_N$ be rich and satisfy $R_i \in \mathcal{R}_i$ and $\mathcal{R}_j \supseteq \mathcal{R}_j^+ \cap \mathcal{R}_j^Q$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Second, the following proposition states that if an agent has *identity-dependent* preferences that exhibit positive externalities, and another agent has identity-independent preferences that exhibit positive externalities, then no rule satisfies the four properties. Since its proof is essentially same as that of Theorem 2, we omit it.[24]

Proposition 1. *Assume that $n \geq 3$. Let $i, j \in N$ be a distinct pair. Let $\mathcal{R}_N$ be rich and satisfy $\mathcal{R}_{i,j} \supseteq \mathcal{R}_i^+ \times \mathcal{R}_j^{+I}$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Recall that when establishing the tightness of Theorem 1 in Section 4.2, we show that adding at least one preference that violates one of the assumptions in Theorem 1 to the domain leads to an impossibility result (Corollaries 3 and 4). Such an approach is striking because just a single preference that violates one the assumptions leads to an impossibility

[24]Indeed, we choose a preference $R_i \in \mathcal{R}_i^+$ of agent i such that $R_i \in \mathcal{R}_i^{PIE} \cup \mathcal{R}_i^{NIE}$, and for each $k \in N \backslash \{i, j\}$, $(j, 0)\ P_i\ (k, 0)$. Then, we can apply the proof of Theorem 2 to such R_i, and we can derive Proposition 1.

result. In Theorems 4 and 5, however, we make more assumptions than Theorem 1, and it is difficult to apply the same approach as the tightness of Theorem 1 to some of the assumptions in Theorems 4 and 5. We instead establish the tightness of the assumptions in Theorems 4 and 5 by showing that to *drop* one of the assumptions leads to an impossibility result.

5.2.1 Tightness of Theorem 4

First, we show the tightness of the assumptions in Theorem 4. Throughout the section, let $NQL, QL \subseteq N$ be a pair such that $NQL, QL \neq \varnothing$, $NQL \cap QL = \varnothing$, and $NQL \cup QL = N$.[25] Also, let $i \in NQL$. Theorem 4 implies that there is a rule satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness* on the domain $\mathcal{R}_i^I \times \mathcal{R}_{NQL\setminus\{i\}}^{+I} \times (\mathcal{R}_{QL}^{+} \cap \mathcal{R}_{QL}^{Q})$. Thus, in Theorem 4, we make the assumptions that (i) all the agents who belong to NQL have identity-independent preferences, (ii) all but one agents in NQL have preferences that exhibit positive externalities, and (iii) the agents who belong to QL have identity-dependent and quasi-linear preferences that exhibit positive externalities.

First, we show that tightness of the first assumption (i) that all the agents in NQL have identity-independent preferences. Proposition 1 implies that if an agent in NQL has *identity-dependent* preference, then no rule satisfies the four properties. Thus, we cannot drop the first assumption (i) in Theorem 4.

Corollary 7. *Assume that $n \geq 3$. Let $j \in NQL\setminus\{i\}$. Then, let $\mathcal{R}_N$ be either $\mathcal{R}_N = \overline{\mathcal{R}}_i \times \mathcal{R}_{NQL\setminus\{i\}}^{+I} \times (\mathcal{R}_{QL}^{+I} \times \mathcal{R}_{QL}^{Q})$ or $\mathcal{R}_N = \mathcal{R}_i^I \times \mathcal{R}_j^{+} \times \mathcal{R}_{NQL\setminus\{i,j\}}^{+I} \times (\mathcal{R}_{QL}^{+I} \times \mathcal{R}_{QL}^{Q})$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Second, we show that we cannot drop the second assumption (ii) that all but one agents in NQL have preferences that exhibit positive externalities. Theorem 3 implies that if at least two agents have identity-independent preferences that *do not necessarily exhibit positive externalities*, then no rule satisfies the four properties.

[25] If $NQL = \varnothing$, then $N = QL$, i.e., all the agents have quasi-linear preferences. In such a case, regardless of the properties of externalities, a pivotal rule satisfies the four properties. Instead, if $QL = \varnothing$, then $NQL = N$, and Theorem 4 is equivalent to Theorem 1. We have already shown the tightness of the assumptions in Theorem 1 in Section 4.2.

Corollary 8. *Let* $j \in NQL \backslash \{i\}$. *Let* $\mathcal{R}_N = \mathcal{R}^I_{i,j} \times \mathcal{R}^{+I}_{NQL\backslash\{i\}} \times (\mathcal{R}^{+}_{QL} \times \mathcal{R}^{Q}_{QL})$. *No rule on* $\mathcal{R}_N$ *satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Finally, we show the tightness of the third assumption (iii) that the agents who belong to QL have quasi-linear preferences that exhibit positive externalities. Theorem 3 implies that if an agent in QL has quasi-linear preferences that *do not necessarily exhibit positive externalities*, then no rule satisfies the four properties.

Corollary 9. *Let* $j \in QL$. *Let* $\mathcal{R}_N = \mathcal{R}^I_{i} \times \mathcal{R}^{+I}_{NQL\backslash\{i\}} \times \mathcal{R}^Q_j \times (\mathcal{R}^{+}_{QL\backslash\{j\}} \cap \mathcal{R}^{Q}_{QL\backslash\{j\}})$. *No rule on* $\mathcal{R}_N$ *satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

5.2.2 Tightness of Theorem 5

Next, we show the tightness of the assumptions in Theorem 5. Note that Theorem 5 implies for an agent $i \in N$, there is a rule satisfying the four properties on $\overline{\mathcal{R}}_i \times (\mathcal{R}^{+I}_{-i} \cap \mathcal{R}^{Q}_{-i})$. Thus, in Theorem 5, we assume that (i) there is only a single agent (agent i) who may have identity-dependent and non-quasi-linear preferences, and (ii) the other agents have identity-independent and quasi-linear preferences that exhibit positive externalities.

First, we show that we cannot drop the first assumption (i). Theorem 3 implies that if *at least two agents* have *identity-dependent* and non-quasi-linear preferences, then no rule satisfies the four properties. Thus, we cannot drop the first assumption in Theorem 5.

Corollary 10. *Let* $i, j \in N$ *be a distinct pair. Let* $\mathcal{R}_N = \overline{\mathcal{R}}_{i,j} \times (\mathcal{R}^{+I}_{-i,j} \cap \mathcal{R}^{Q}_{-i,j})$. *No rule on* $\mathcal{R}_N$ *satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Next, we show that we cannot drop the second assumption (ii) in Theorem 5 that except for a single agent, all the agents have identity-independent and quasi-linear preferences that exhibit positive externalities. Notice that the second assumption consists of the three subassumptions: (ii-i) quasi-linear preferences, (ii-ii) identity-independent preferences, and (ii-iii) positive externalities.

First, we show that we cannot drop the assumption (ii-i). Proposition 1 implies that if an agent has identity-dependent and non-quasi-linear preferences, and another agent has identity-independent and *non-quasi-linear* preferences that exhibit positive externalities, then no rule satisfies the four properties.

Corollary 11. *Assume that $n \geq 3$. Let $i, j \in N$ be a distinct pair. Then, let $\mathcal{R}_N = \overline{\mathcal{R}}_i \times \mathcal{R}_j^{+I} \times (\mathcal{R}_{-i,j}^{+I} \cap \mathcal{R}_{-i,j}^{Q})$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Second, we show the tightness of the assumption (ii-ii). Theorem 6 implies that if an agent has identity-dependent and non-quasi-linear preferences, and another agent has *identity-dependent* and quasi-linear preferences that exhibit positive externalities, then no rule satisfies the four properties.

Corollary 12. *Assume that $n \geq 3$. Let $i, j \in N$ be a distinct pair. Then, let $\mathcal{R}_N = \overline{\mathcal{R}}_i \times (\mathcal{R}_j^{+} \cap \mathcal{R}^{Q}) \times (\mathcal{R}_{-i,j}^{+I} \cap \mathcal{R}_{-i,j}^{Q})$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

Finally, Theorem 3 implies that if an agent has identity-dependent and non-quasi-linear preferences, and another agent has identity-independent and quasi-linear preferences that *do not necessarily exhibit positive externalities*, then no rule satisfies the four properties. Thus, we cannot drop the assumption (ii-iii) in Theorem 5.

Corollary 13. *Let $i, j \in N$ be a distinct pair. Let $\mathcal{R}_N = \overline{\mathcal{R}}_i \times (\mathcal{R}_j^{I} \cap \mathcal{R}^{Q}) \times (\mathcal{R}_{-i,j}^{+I} \cap \mathcal{R}_{-i,j}^{Q})$. No rule on $\mathcal{R}_N$ satisfies efficiency, weak individual rationality, no subsidy for losers, and strategy-proofness.*

6 Conclusion

We have considered the problem of allocation a single object to $n \geq 2$ agents with payments who have preferences that may exhibit externalities and are not necessarily quasi-linear. We have established that if (i) all the agents have identity-independent preferences and (ii) at least $n-1$ agents have those that exhibit positive externalities, then the generalized pivotal rule is the only rule satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness* (Theorem 1). We have also shown that if we relax one of the two assumptions (i) and (ii) in Theorem 1, no rule satisfies the four properties (Theorems 2 and 3; see also Corollaries 3 and 4). We have further found the two environments where some agents may have identity-dependent preferences, others have quasi-linear preferences exhibiting positive externaliteis, and there is a rule satisfying the four properties (Theorems 4 and 5). Overall, our results suggest the importance of identity-independence and positive externalities in a non-quasi-linear environment with externalities for the existence of a rule satisfying the four properties.

In this paper, we have focused on the problem of allocating a single object to agents with payments. Such a simplification of the model may be helpful to emphasize the effect of externalities coupled with non-quasi-linear preferences on the class of rules satisfying the four properties, but may not be unrealistic in many situations. An interesting open question is how externalies and non-quasi-linear preferences together affect the class of rules satisfying the four properties in multi-object models (particularly with unit-demand agents),[26] and we hope that the results and the technique that we have developed in this paper are helpful.

Appendix

A Preliminaries

In this section, we provide the lemmas that will be used throughout the proofs of the theorems.

First, we provide the two lemmas concerning efficient allocations. The next lemma states that if at least $n-1$ agents have preferences that exhibit positive externalities, then the owner never keeps the object under an efficient allocation.

Lemma 1. *Let $R \in \mathcal{R}_N$. Let $z = (x, t) \in X \times \mathbb{R}^n$ be efficient for R. Let $i \in N$. If $R_{-i} \in \mathcal{R}^+_{-i}$, then $x \neq 0$.*

Proof. Suppose $R_{-i} \in \mathcal{R}^+_{-i}$, but $x = 0$. Then,

$$\sum_{j \in N} V_j(i, z_j) > V_i(0, z_i) + \sum_{j \in N \setminus \{i\}} V_j(i, z_j) \geq \sum_{j \in N} V_j(0, z_j),$$

[26]In a multi-object model without externalities, if agents have unit-demand, then there is a rule satisfying the four properties (Demange and Gale, 1985; Morimoto and Serizawa, 2015, etc.), while if they do multi-demand, then typically no rule satisfies the four properties (Baisa, 2020; Malik and Mishra, 2021, Kazumura, 2022, etc.).

where the first inequality follows from desirability of own consumption of R_i, and the second one from $R_{-i} \in \mathcal{R}^+_{-i}$. By Remark 1, this contradicts that z is efficient for R. $\square$

The following lemma follows from Remark 1 and the fact that for each $i \in N$ and each $z_i = (x, t_i) \in X \times \mathbb{R}$, $t_i = V_i(x, z_i)$. Thus, we omit the proof.

Lemma 2. *Let $i \in N$ and $R \in \mathcal{R}_N$ be such that $R_{-i} \in \mathcal{R}^I_{-i}$. Let $z = (x, t) \in X \times \mathbb{R}^n$ be efficient for R.*
(i) If $x = i$, then for each $j \in N \backslash \{i\}$, then $(t_i - V_i(j, z_i)) + (t_j - V_j(j, z_j)) \geq 0$.
(ii) If $x \in N \backslash \{i\}$, then for each $j \in N \backslash \{i\}$, $(t_i - V_i(j, z_i)) + (t_x - V_x(j, z_x)) \geq V_j(j, z_j) - t_j$.
(iii) If $x \in N \backslash \{i\}$, then $(t_i - V_i(i, z_i)) + (t_x - V_x(i, z_x)) \geq 0$.

Next, we provide the definitions and a lemma that are related to *strategy-proofness*. Given a rule $f = (x, t)$ on $\mathcal{R}_N$, an agent $i \in N$, and preferences $R_{-i} \in \mathcal{R}_{-i}$, let

$$o_i^f(R_{-i}) = \{z_i \in X \times \mathbb{R} : \exists R_i \in \mathcal{R}_i \text{ s.t. } f_i(R_i, R_{-i}) = z_i\}$$

denote the set of available bundles of agent i for some preferences of agent i under a rule f. Also, let

$$X_i^f(R_{-i}) = \{x \in X : \exists R_i \in \mathcal{R}_i \text{ s.t. } x(R_i, R_{-i}) = x\}$$

denote the set of available object allocations for some preference of agent i under a rule f.

If a rule f on $\mathcal{R}_N$ is *strategy-proof*, then for each $i \in N$, each $R_{-i} \in \mathcal{R}_{-i}$, and each $x \in X_i^f(R_{-i})$, there is a unique payment $t_i \in \mathbb{R}$ such that $(x, t_i) \in o_i^f(R_{-i})$. Let $t_i^f(R_{-i}; x) \in \mathbb{R}$ denote the unique payment such that $(x, t_i^f(R_{-i}; x)) \in o_i^f(R_{-i})$. Let $z_i^f(R_{-i}; x) = (x, t_i^f(R_{-i}; x))$.

The following lemma states that if a rule f on $\mathcal{R}_N$ is *strategy-proof*, then for each $R \in \mathcal{R}_N$, each agent $i \in N$ receives the best bundle among $o_i^f(R_{-i})$ under the rule. Since its proof is straightforward from *strategy-proofness*, we omit it.

Lemma 3. *Let f be a rule on $\mathcal{R}_N$ satisfying strategy-proofness. Let $R \in \mathcal{R}_N$ and $i \in N$. For each $x \in X_i^f(R_{-i})$, $f_i(R)\ R_i\ z_i^f(R_{-i}; x)$.*

We provide the lemmas concerning rules satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness*. From now until the end of this section, let $\mathcal{R}_N$ be rich and f be a rule on $\mathcal{R}_N$ satisfying the four properties.

The next lemma states that under a rule satisfying the four properties, given other agents' preferences, there is a quasi-linear preference of an agent that exhibits no externality such that he obtains the object.

Lemma 4. *Let* $i \in N$ *and* $R_{-i} \in \mathcal{R}_{-i}$. *There is* $R_i \in \mathcal{R}_i^0 \cap \mathcal{R}_i^Q$ *such that* $x(R_i, R_{-i}) = i$.

Proof. Let $N' \subseteq N \backslash \{i\}$. For each $j \in N'$, $B_j = \{z_j \in X \times \mathbb{R}_+ : z_j \; R_j \; (\underline{x}_j(R_j), 0)\}$ is bounded, and by continuity of R_j, is closed. Thus, for each $j \in N'$, B_j is compact, and so $B_{N'} = \times_{j \in N'} B_j$ is also compact. Let $h_{N'} : B_{N'} \to \mathbb{R}$ be a function such that for each $z_{N'} = (z_j)_{j \in N'} \in B_{N'}$, $h_{N'}(z_{N'}) = \max_{x \in X} \sum_{j \in N'} \Big(V_j(x, z_j) - V_j(i, z_j) \Big)$. By continuity of preferences $R_{N'}$, $h_{N'}$ is a continuous function. Thus, since $B_{N'}$ is compact, $h_{N'}(B_{N'})$ is also compact. Thus, we can choose $R_i \in \mathcal{R}_i^0 \cap \mathcal{R}_i^Q$ such that for each nonempty $N' \subseteq N \backslash \{i\}$ and each $(z_j)_{j \in N'} \in B_{N'}$, we have

$$v_i(i) > \max_{x \in X} \sum_{j \in N'} \Big(V_j(x, z_j) - V_j(i, z_j) \Big). \quad (1)$$

We show $x(R) = i$. By contradiction, suppose $x(R) \neq i$. There are two cases.

Suppose $x(R) = 0$. Then, by *no subsidy for losers*, for each $j \in N \backslash \{i\}$, $t_j(R) \geq 0$. Thus, by *weak individual rationality*, for each $j \in N \backslash \{i\}$, $f_j(R) \in B_j$, and so $(f_j(R))_{j \in N \backslash \{i\}} \in B_{N \backslash \{i\}}$. Thus, by (1), $v_i(i) > \sum_{j \in N \backslash \{i\}} (V_j(0, f_j(R)) - V_j(i, f_j(R)))$, which implies

$$\sum_{j \in N} \Big(V_j(i, f_j(R)) - V_j(x(R), f_j(R)) \Big) = v_i(i) - \sum_{j \in N \backslash \{i\}} \Big(V_j(0, f_j(R)) - V_j(i, f_j(R)) \Big) > 0,$$

where the equality follows from $R_i \in \mathcal{R}_i^0$. By Remark 1, this contradicts *efficiency*.

Suppose $x(R) = j$ for some $j \in N \backslash \{i\}$. First, suppose $t_j(R) \geq 0$. Then, by *weak individual rationality*, $f_j(R) \in B_j$. Also, by *weak individual rationality* and *no subsidy for losers*, for each $k \in N \backslash \{i, j\}$, $f_k(R) \in B_k$. Thus, $(f_k(R))_{k \in N \backslash \{i\}} \in B_{N \backslash \{i\}}$, and by the same discussion as in the previous case, we can derive a contradiction to *efficiency*. Then, suppose $t_j(R) < 0$. Let $R'_j \in \mathcal{R}_j^0 \cap \mathcal{R}_j^Q$ be such that for each $(z_j)_{k \in N \in \{i,j\}} \in B_{N \backslash \{i,j\}}$,

$$v'_j(j) < v_i(i) - \max_{x \in X} \sum_{j \in N \backslash \{i,j\}} \Big(V_j(x, z_j) - V_j(i, z_j) \Big). \quad (2)$$

Note that by (1), we can choose such a preference R'_j. For each $x \in X_j^f(R_{-j}) \backslash \{j\}$, we have $v'_j(j) - t_j^f(R_{-j}; j) > 0 \geq v_j(x) - t_j^f(R_{-j}; x)$, where the first inequality follows from desirability of own consumption of R'_j and $t^f_j(R_{-j};j) = t_j(R) < 0$, and the second one from *no subsidy for losers*. Thus, by Lemma 3, $x(R'_j, R_{-j}) = j$. By *weak individual rationality* and *no subsidy for losers*, for each $k \in N\backslash\{i,j\}$, $f_k(R'_j, R_{-j}) \in B_k$. By (2),

$$v_i(i) - v'_j(j) + \sum_{k \in N\backslash\{i,j\}} \Big(V_k(i, f_k(R'_j, R_{-j})) - V_k(j, f_k(R'_j, R_{-j})) \Big) > 0.$$

By Remark 1, this contradicts *efficiency*. □

The following lemma states that under a rule satisfying the four properties, for each agent $i \in N$ and each $R_{-i} \in \mathcal{R}_{-i}$, the payment of agent i when he receives the object is greater than that when he does not. Note that by Lemma 4, $i \in X^f_i(R_{-i})$.

Lemma 5. *Let $i \in N$, $R_{-i} \in \mathcal{R}_{-i}$, and $x \in X^f_i(R_{-i})\backslash\{i\}$. Then, $t^f_i(R_{-i};i) > t^f_i(R_{-i};x)$.*

Proof. By contradiction, suppose $t^f_i(R_{-i};i) \leq t^f_i(R_{-i};x)$. By $x \in X^f_i(R_{-i})$, there is $R_i \in \mathcal{R}_i$ such that $x(R_i, R_{-i}) = x$. Then,

$$z^f_i(R_{-i};i) = (i, t^f_i(R_{-i};i)) \ R_i \ (i, t^f_i(R_{-i};x)) \ P_i \ (x, t^f_i(R_{-i};x)) = f_i(R),$$

where the first relation follows from $t^f_i(R_{-i};i) \leq t^f_i(R_{-i};x)$, and the second one from desirability of own consumption of R_i. However, this contradicts Lemma 3. □

The following lemma states that under a rule satisfying the four properties, if an agent with an identity-independent preference does not receive the object, then his payment is equal to zero.

Lemma 6. *Let $i \in N$. Assume that $\mathcal{R}_{-i} \subseteq \mathcal{R}^+_{-i}$. For each $R \in \mathcal{R}_N$ and each $j \in N$, if $R_j \in \mathcal{R}^I_j$ and $x(R) \neq j$, then $t_j(R) = 0$.*

Proof. Let $R \in \mathcal{R}_N$ and $j \in N$ be such that $R_j \in \mathcal{R}^I_j$ and $x(R) \neq j$. By *no subsidy for losers*, $t_j(R) \geq 0$. By contradiction, suppose $t_j(R) > 0$. By Lemma 4, $j \in X^f_j(R_{-j})$. By Lemma 5, $t^f_j(R_{-j};j) > t^f_j(R_{-j};x(R)) = t_j(R)$. Let $R'_j \in \mathcal{R}^0_j \cap \mathcal{R}^Q_j$ be a preference such that $v'_j(j) < t^f_j(R_{-j};j)$. Then, by $R'_j \in \mathcal{R}^0_j$ and *weak individual rationality*, $x(R'_j, R_{-j}) \neq j$. By $R'_j \in \mathcal{R}^0_j$, *weak individual rationality* and *no subsidy for losers* together imply that $t_j(R'_j, R_{-j}) = 0$. By Lemma 1, $x(R), x(R'_j, R_{-j}) \in N$. Thus, by $R_j \in \mathcal{R}^I_j$, $f_j(R'_j, R_{-j}) = (x(R'_j, R_{-j}), 0) \ I_j \ (x(R), 0)$. By $t_j(R) > 0$, $(x(R), 0) \ P_j \ f_j(R)$. Combining these, we get $f_j(R'_j, R_{-j}) \ P_j \ f_j(R)$, which contradicts *strategy-proofness*. □

The following lemma states that under a rule satisfying the four properties, an agent $j \in N$ with an identity-independent preference finds his outcome bundle at least as desirable as $(\overline{x}_j(R_j), 0)$.

Lemma 7. *Let $i \in N$. Assume that $\mathcal{R}_{-i} \subseteq \mathcal{R}^+_{-i}$. For each $R \in \mathcal{R}_N$ and each $j \in N$, if $R_j \in \mathcal{R}^I_j$, then $f_j(R)\ R_j\ (\overline{x}_j(R_j), 0)$.*

Proof. Let $R \in \mathcal{R}_N$ and $j \in N$ be such that $R_j \in \mathcal{R}^I_j$.

Suppose $x(R) \neq j$. Then, by $R_j \in \mathcal{R}^I_j$, Lemma 6 implies that $t_j(R) = 0$. By Lemma 1, $x(R) \neq 0$. Thus, by $R_j \in \mathcal{R}^I_j$, $f_j(R)\ I_j\ (\overline{x}_j(R_j), 0)$.

Suppose $x(R) = j$. By contradiction, suppose that $(\overline{x}_j(R_j), 0)\ P_j\ f_j(R)$. Then, we have $V_j(j, (\overline{x}_j(R_j), 0)) < t_j(R)$. By desirability of own consumption of R_j, $V_j(j, (\overline{x}_j(R_j), 0)) > 0$. Let $R'_j \in \mathcal{R}^0_j \cap \mathcal{R}^Q_j$ be such that $v'_j(j) < V_j(j, (\overline{x}_j(R_j), 0))$. By $V_j(j, (\overline{x}_j(R_j), 0)) < t_j(R)$, $v'_j(j) < t_j(R) = t^f_j(R_{-j}; j)$. By $R'_j \in \mathcal{R}^0_j$, *weak individual rationality* implies $x(R'_j, R_{-j}) \neq j$. Thus, by Lemma 6, $t_j(R'_j, R_{-j}) = 0$. By Lemma 1, $x(R'_j, R_{-j}) \neq 0$. Thus, by $R_j \in \mathcal{R}^I_j$, $f_j(R'_j, R_{-j})\ I_j\ (\overline{x}_j(R_j), 0)\ P_j\ f_j(R)$, which contradicts *strategy-proofness*. □

The following lemma states that under a rule satisfying the four properties, if an agent $j \in N$ has an identity-independent preference whose valuation of own consumption at $(\underline{x}_j(R_j), 0)$ is greater than the payment $t^f_j(R_{-j}; j)$ when he receives the object, then he indeed receives the object. Note that by Lemma 4, $j \in X^f_j(R_{-j})$.

Lemma 8. *Let $i \in N$. Assume that $\mathcal{R}_{-i} \subseteq \mathcal{R}^+_{-i}$. For each $R \in \mathcal{R}_N$ and each $j \in N$, if $R_j \in \mathcal{R}^I_j$ and $V_j(j, (\overline{x}_j(R_j), 0)) > t^f_j(R_{-j}; j)$, then $x(R) = j$.*

Proof. Let $R \in \mathcal{R}_N$ and $j \in N$ be such that $R_j \in \mathcal{R}^I_j$ and $V_j(j, (\overline{x}_j(R_j), 0)) > t^f_j(R_{-j}; j)$. By contradiction, suppose $x(R) \neq j$. By $R_j \in \mathcal{R}^I_j$, Lemma 6 implies $t_j(R) = 0$. By $V_j(j, (\overline{x}_j(R_j), 0)) > t^f_j(R_{-j}; j)$ and $R_j \in \mathcal{R}^I_j$, $z^f_j(R_{-j}; j)\ P_j\ (\overline{x}_j(R_i), 0)\ I_j\ (x(R), 0) = f_j(R)$. This contradicts Lemma 3. □

The following lemma states that under a rule satisfying the four properties, if other agents have quasi-linear preferences, then the payment of an agent is equivalent to that of a pivotal rule. Since it follows from Fact 1, we omit the proof.

Lemma 9. *Assume that $\mathcal{R}_N \cap \mathcal{R}^Q_N$ is convex. Let $i \in N$, $R_{-i} \in \mathcal{R}_{-i} \cap \mathcal{R}^Q_{-i}$, and $x \in X^f_i(R_{-i})$. Then, $t^f_i(R_{-i}; x) = \max_{x' \in X} \sum_{j \in N \setminus \{i\}} v_j(x') - \sum_{j \in N \setminus \{i\}} v_j(x)$.*

Finally, the following lemma states that under a rule satisfying the four properties, given other agents' quasi-linear preferences, there is a preference of an agent such that he does not receive the object and makes no payment.

Lemma 10. *Assume that $\mathcal{R}_N \cap \mathcal{R}_N^Q$ is convex. Let $i, j \in N$ be a distinct pair. Let $R_{-i} \in \mathcal{R}_{-i} \cap \mathcal{R}_{-i}^Q$ be such that for each $k \in N \backslash \{i, j\}$ and each $x \in X \backslash \{i, k\}$, $v_k(x) = 0$, and $v_j(j) > \max_{k \in N \backslash \{i,j\}} v_k(k)$. Then, there is $R_i \in \mathcal{R}_i^0 \cap \mathcal{R}_i^Q$ such that $f_i(R) = (j, 0)$.*

Proof. Let $R_i \in \mathcal{R}_i^0 \cap \mathcal{R}_i^Q$ be such that $v_i(i) < v_j(j)$. By $R \in \mathcal{R}_N^Q$, *efficiency* and Remark 1 together imply $x(R) \in \arg\max_{x \in X} \sum_{k \in N} v_k(x)$. Thus, by construction of R, $x(R) = j$. By Lemma 9, $t_i(R) = t_i^f(R_{-i}; j) = \max_{x \in X} \sum_{k \in N \backslash \{i\}} v_k(x) - v_j(j) = v_j(j) - v_j(j) = 0$. Thus, $f_i(R) = (j, 0)$. □

Part I

Proofs of Characterization Theorems

In the first part, we prove the characterization theorems (Theorems 1, 4, and 5). Since Theorem 1 corresponds to Theorem 4 for the case where $NQL = N$, we only provide the proofs of Theorems 4 and 5.

B Proof of Theorem 4

In this section, we prove Theorem 4. Without loss of generality, let $i = 1$ throughout the proof.

B.1 Proof of the "if" part

We prove the "if" part of Theorem 4. Let $f = (x, t)$ be a generalized pivotal rule.

We begin with the following two lemmas. First, the following lemma states that if an agent $i \in NQL$ receives the object, then his valuations of own consumption at $(\overline{x}_i(R_i), 0)$ is no less than his payment. Since its proof is straightforward, we omit it.

Lemma 11. *Let $R \in \mathcal{R}_N$ and $i \in NQL$. If $x(R) = i$, then $V_i(i, (\overline{x}_i(R_i), 0)) \geq t_i(R)$.*

The following lemma states the if an agent $i \in NQL$ does not receive the object, then his payment is equal to zero. Since its proof is straightforward, we omit it.

Lemma 12. *Let $R \in \mathcal{R}_N$ and $i \in NQL$. If $x(R) \neq i$, then $t_i(R) = 0$.*

We show that f satisfies the four properties. Since the proof of *no subsidy for losers* is straightforward, we omit it. By the definition of the object allocation rule under a generalized pivotal rule and Remark 1, for each $R \in \mathcal{R}_N$, the allocation $(x(R), t)$ is efficient for R, where for each $i \in N$, $t_i = V_i(x(R), (\overline{x}_i(R_i), 0))$. Thus, by Lemma 1, $x(R) \neq 0$.

Efficiency. Let $R \in \mathcal{R}_N$. Let $x \in X \backslash \{x(R)\}$. Let $i = x(R) \in N$. By Lemma 12, for each $j \in NQL$ with $x(R) \neq j$, $t_j(R) = 0$, and so by $R_j \in \mathcal{R}^I_j$, for each $x \in X$,

$$V_j(x, f_j(R)) = V_j(x, (\overline{x}_j(R_j), 0)). \tag{1}$$

If $x = 0$, then by $R_{-1} \in \mathcal{R}^+_{-1}$ and desirability of own consumption of R_1, we have $\sum_{j \in N} V_j(x(R), f_j(R)) > \sum_{j \in N} V_j(x, f_j(R))$. Thus, suppose $x \neq 0$, and let $j = x$.

Suppose that $i \in NQL$. By Lemma 11 and $R_i \in \mathcal{R}^I_i$, $f_i(R)\ R_i\ (j, 0)$, which implies $V_i(j, f_i(R)) \leq 0$. Then,

$$\begin{aligned}
&\sum_{k \in N} \Big(V_k(i, f_k(R)) - V_k(j, f_k(R)) \Big) \\
&= t_i(R) - V_i(j, f_i(R)) + \sum_{k \in N \backslash \{i\}} \Big(V_k(i, (\overline{x}_k(R_k), 0)) - V_k(j, (\overline{x}_k(R_k), 0)) \Big) \\
&\geq t_i(R) + \sum_{k \in N \backslash \{i\}} \Big(V_k(i, (\overline{x}_k(R_k), 0)) - V_k(j, (\overline{x}_k(R_k), 0)) \Big) \\
&= \max_{x' \in X} \sum_{k \in N \backslash \{i\}} V_k(x', (\overline{x}_k(R_k), 0)) - \sum_{k \in N \backslash \{i\}} V_k(j, (\overline{x}_k(R_k), 0)) \geq 0,
\end{aligned}$$

where the first equality follows from (1) and $R_{QL} \in \mathcal{R}^Q_{QL}$, the first inequality from $V_i(j, f_i(R)) \leq 0$, and the second equality from $t_i(R) = \max_{x' \in X} \sum_{k \in N \backslash \{i\}} V_k(x', (\overline{x}_k(R_k), 0)) - \sum_{k \in N \backslash \{i\}} V_k(i, (\overline{x}_k(R_k), 0))$.

Next, suppose that $i \in QL$. Then,

$$\sum_{k \in N} \Big(V_k(i, f_k(R)) - V_k(j, f_k(R)) \Big) = \sum_{k \in N} \Big(V_k(i, (\overline{x}_k(R_k), 0)) - V_k(j, (\overline{x}_k(R_k), 0)) \Big) \geq 0,$$

where the equality follows from (1) and $R_{QL} \in \mathcal{R}^Q_{QL}$, and the inequality from the definition of the object allocation rule under f.

Thus, in any case, by Remark 1, $f(R)$ is efficient for R.

Weak individual rationality. Let $i \in NQL$ and $R \in \mathcal{R}_N$. If $x(R) = i$, then by Lemma 11, $f_i(R)\ R_i\ (\overline{x}_i(R_i), 0)\ R_i\ (\underline{x}_i(R_i), 0)$. If $x(R) \neq i$, then by Lemma 12, $f_i(R) = (x(R), 0)\ R_i\ (\underline{x}_i(R_i), 0)$.

Let $i \in QL$ and $R \in \mathcal{R}_N$. Note that $f(R)$ is equivalent to an outcome of a pivotal rule for (R'_{NQL}, R_{QL}), where for each $j \in NQL$, $R'_j \in \mathcal{R}_j^{+I} \cap \mathcal{R}_j^Q$ is such that $v'_j(j) = V_j(j, (\overline{x}_j(R_j), 0))$. Thus, by *weak individual rationality* of a pivotal rule, $f_i(R)\ R_i\ (\underline{x}_i(R_i), 0)$.

Strategy-proofness. Let $i \in NQL$, $R \in \mathcal{R}_N$, and $R'_i \in \mathcal{R}_i$. If $x(R) = x(R'_i, R_{-i})$, then $t_i(R) = t_i(R'_i, R_{-i})$. Suppose $x(R) = i$ and $x(R'_i, R_{-i}) \neq i$. Then, by Lemma 11, $f_i(R)\ R_i\ (\overline{x}_i(R_i), 0)$. By Lemma 12, $t_i(R'_i, R_{-i}) = 0$. Thus, by $R_i \in \mathcal{R}_i^I$ and $x(R'_i, R_{-i}) \in N\backslash\{i\}$, $f_i(R'_i, R_{-i})\ I_i\ (\overline{x}_i(R_i), 0)$. Thus, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$. Suppose $x(R) \neq i$ and $x(R'_i, R_{-i}) = i$. Then, by Lemma 12, $t_i(R) = 0$. We show $V_i(i, f_i(R)) \leq t_i(R'_i, R_{-i})$. By contradiction, suppose $V_i(i, f_i(R)) > t_i(R'_i, R_{-i})$. By $R_i \in \mathcal{R}_i^I$ and $x(R) \in N\backslash\{i\}$, $V_i(i, f_i(R)) = V_i(i, (\overline{x}_i(R_i), 0))$, and $V_i(x(R), (\overline{x}_i(R_i), 0)) = 0$. Then,

$$\begin{aligned}
&\sum_{j \in N} V_j(i, (\overline{x}_j(R_j), 0)) > \max_{x \in X} \sum_{j \in N\backslash\{i\}} V_j(x, (\overline{x}_j(R_j), 0)) \\
&\geq \sum_{j \in N\backslash\{i\}} V_j(x(R), (\overline{x}_j(R_j), 0)) = \sum_{j \in N} V_j(x(R), (\overline{x}_j(R_j), 0)),
\end{aligned}$$

where the first inequality follows from $V_i(i, (\overline{x}_i(R_i), 0)) = V_i(i, f_i(R)) > t_i(R)$ and the definition of the payment rule under f, and the equality from $V_i(x(R), (\overline{x}_i(R_i), 0)) = 0$. However, this contradicts the definition of the object allocation rule under f. Thus, we have $V_i(i, f_i(R)) \leq t_i(R'_i, R_{-i})$, which implies $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

Let $i \in QL$. Note that for each $R \in \mathcal{R}_N$, $f(R)$ is equivalent to an outcome of a pivotal rule for (R'_{NQL}, R_{QL}), where for each $j \in NQL$, $R'_j \in \mathcal{R}_j^{+I} \cap \mathcal{R}_j^Q$ is such that $v'_j(j) = V_j(j, (\overline{x}_j(R_j), 0))$. Thus, by *strategy-proofness* of a pivotal rule, for each $R \in \mathcal{R}_N$ and each $R'_i \in \mathcal{R}_i$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$. ■

B.2 Proof of the "only if" part

We prove the "only if" part of Theorem 4. Let $f = (x, t)$ be a rule on $\mathcal{R}_N$ satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness*. The proof is in a series of lemmas.

Lemma 13. *Let $R \in \mathcal{R}_N$. For each $i \in NQL\backslash\{x(R)\}$ and each $x \in X$, $V_i(x, f_i(R)) = V_i(x, (\overline{x}_i(R_i), 0))$.*

Proof. Let $i \in NQL\backslash\{x(R)\}$. By $R_i \in \mathcal{R}_i^I$ and $x(R) \neq i$, Lemma 6 implies $t_i(R) = 0$. Thus, by $R_i \in \mathcal{R}_i^I$, $f_i(R)\ I_i\ (\overline{x}_i(R_i), 0)$, which implies that for each $x \in X$, $V_i(x, f_i(R)) = V_i(x, (\overline{x}_i(R_i), 0))$. □

By the following two lemmas, we show if an agent with a non-quasi-linear preference receives the object, then his payment under the rule f coincides with that under a generalized pivotal rule. Note that for each $i \in N$ and each $R_{-i} \in \mathcal{R}_{-i}$, by Lemma 4, $i \in X_i^f(R_{-i})$.

Lemma 14. *Let $i \in NQL$ and $R_{-i} \in \mathcal{R}_{-i}$. Then, $t_i^f(R_{-i}; i) \geq \max_{x \in X} \sum_{j \in N\backslash\{i\}} V_j(x, (\overline{x}_j(R_j), 0)) - \sum_{j \in N\backslash\{i\}} V_j(i, (\overline{x}_j(R_j), 0))$.*

Proof. By contradiction, suppose that $t_i^f(R_{-i}; i) < \max_{x \in X} \sum_{j \in N\backslash\{i\}} V_j(x, (\overline{x}_j(R_j), 0)) - \sum_{j \in N\backslash\{i\}} V_j(i, (\overline{x}_j(R_j), 0))$. Let $R_i \in \mathcal{R}_i^0 \cap \mathcal{R}_i^Q$ be such that $v_i(i) > t_i^f(R_{-i}; i)$, and

$$v_i(i) < \max_{x \in X} \sum_{j \in N\backslash\{i\}} V_j(x, (\overline{x}_j(R_j), 0)) - \sum_{j \in N\backslash\{i\}} V_j(i, (\overline{x}_j(R_j), 0)). \tag{1}$$

By $v_i(i) > t_i^f(R_{-i}; i)$, Lemma 8 implies $x(R) = i$. Let $x \in \arg\max_{x' \in X} \sum_{j \in N\backslash\{i\}} V_j(x', (\overline{x}_j(R_j), 0))$. Then,

$$\sum_{j \in N} \Big(V_j(x, f_j(R)) - V_j(x(R), f_j(R))\Big) = \sum_{j \in N\backslash\{i\}} \Big(V_j(x, (\overline{x}_j(R_j), 0)) - V_j(x, (\overline{x}_j(R_j), 0))\Big) - v_i(i) > 0,$$

where the equality follows from Lemma 13 and $R_{QL} \in \mathcal{R}_{QL}^Q$, and the inequality from (1). By Remark 1, this contradicts *efficiency*. □

Lemma 15. *Let $i \in NQL$ and $R_{-i} \in \mathcal{R}_{-i}$. Then, $t_i^f(R_{-i}; i) \leq \max_{x \in X} \sum_{j \in N\backslash\{i\}} V_j(x, (\overline{x}_j(R_j), 0)) - \sum_{j \in N\backslash\{i\}} V_j(i, (\overline{x}_j(R_j), 0))$.*

Proof. Suppose by contradiction that $t_i^f(R_{-i}; i) > \max_{x \in X} \sum_{j \in N \setminus \{i\}} V_j(x, (\overline{x}_j(R_j), 0)) - \sum_{j \in N \setminus \{i\}} V_j(i, (\overline{x}_j(R_j), 0))$. Let $R_i \in \mathcal{R}_i^0 \cap \mathcal{R}_i^Q$ be such that $v_i(i) < t_i^f(R_{-i}; i)$, and

$$v_i(i) > \max_{x \in X} \sum_{j \in N \setminus \{i\}} V_j(x, (\overline{x}_j(R_j), 0)) - \sum_{j \in N \setminus \{i\}} V_j(i, (\overline{x}_j(R_j), 0)). \tag{2}$$

By $v_i(i) < t_i^f(R_{-i}; i)$ and *weak individual rationality*, $x(R) \neq i$. By Lemma 1, $x(R) \neq 0$. Thus, $x(R) \in N \setminus \{i\}$. Let $j = x(R)$.

First, suppose $j \in NQL$. Then,

$$\begin{aligned} v_i(i) &> \sum_{k \in N \setminus \{i\}} V_k(j, (\overline{x}_k(R_k), 0)) - \sum_{k \in N \setminus \{i\}} V_k(i, (\overline{x}_k(R_k), 0)) \\ &= V_j(j, (\overline{x}_j(R_j), 0)) + \sum_{k \in N \setminus \{j\}} V_k(j, (\overline{x}_k(R_k), 0)) - \sum_{k \in N \setminus \{i.j\}} V_k(i, (\overline{x}_k(R_k), 0)), \end{aligned}$$

where the inequality follows from (2), and the equality from $R_{i,j} \in \mathcal{R}_{i,j}^I$. Thus,

$$\begin{aligned} V_j(j, (\overline{x}_j(R_j), 0)) &< \Big(v_i(i) + \sum_{k \in N \setminus \{i.j\}} V_k(i, (\overline{x}_k(R_k), 0)) \Big) - \sum_{k \in N \setminus \{j\}} V_k(j, (\overline{x}_k(R_k), 0)) \\ = \sum_{k \in N \setminus \{j\}} V_k(i, (\overline{x}_k(R_k), 0)) &- \sum_{k \in N \setminus \{j\}} V_k(j, (\overline{x}_k(R_k), 0)) \leq t_j(R), \end{aligned}$$

where the last inequality follows from Lemma 14. This implies $(\overline{x}_j(R_j), 0) \; P_j \; f_j(R)$, which contradicts Lemma 7.

Next, suppose $j \in QL$. We have

$$\sum_{k \in N} \Big(V_k(i, f_k(R)) - V_k(j, f_k(R)) \Big) = v_i(i) - \sum_{k \in N \setminus \{i\}} \Big(V_k(j, (\overline{x}_k(R_k), 0)) - V_k(i, (\overline{x}(R_k), 0)) \Big) > 0,$$

where the first equality follows from Lemma 13 and $R_{QL} \in \mathcal{R}_{QL}^Q$, and the inequality from (2). By Remark 1, this contradicts *efficiency*. □

The next lemma shows that the payment of an agent with a quasi-linear preference under the rule f coincides with that under a generalized pivotal rule.

Lemma 16. *Let $i \in QL$ and $R_{-i} \in \mathcal{R}_{-i}$. For each $x \in X_i^f(R_{-i})$, we have $t_i^f(R_{-i}; x) = \max_{x' \in X} \sum_{j \in N \setminus \{i\}} V_j(x', (\overline{x}_j(R_j), 0)) - \sum_{j \in N \setminus \{i\}} V_j(x, (\overline{x}_j(R_j), 0))$.*

Proof. Let $x \in X_i^f(R_{-i})$. Note that by Lemma 1, $x \neq 0$. By contradiction, suppose that

$t_i^f(R_{-i};x) \neq \max_{x' \in X} \sum_{j \in N\setminus\{i\}} V_j(x', (\overline{x}_j(R_j), 0)) - \sum_{j \in N\setminus\{i\}} V_j(x, (\overline{x}_j(R_j), 0))$. There are two cases.

Case 1. $t_i^f(R_{-i};x) < \max_{x' \in X} \sum_{j \in N\setminus\{i\}} V_j(x', (\overline{x}_j(R_j), 0)) - \sum_{j \in N\setminus\{i\}} V_j(x, (\overline{x}_j(R_j), 0))$.

Let $R_i \in \mathcal{R}_i^{+I} \cap \mathcal{R}_i^Q$ be such that $v_i(x) > t_i^f(R_{-i};x)$,

$$v_i(x) < \max_{x' \in X} \sum_{j \in N\setminus\{i\}} V_j(x', (\overline{x}_j(R_j), 0)) - \sum_{j \in N\setminus\{i\}} V_j(x, (\overline{x}_j(R_j), 0)), \tag{3}$$

for each $x' \in X\setminus\{i, x\}$, $v_i(x') = 0$, and if $x \neq i$, then $v_i(i) - v_i(x) < t_i^f(R_{-i};i) - t_i^f(R_{-i};x)$. Note that we can choose such R_i by Lemma 5. By $v_i(x) > t_i^f(R_{-i};x)$ and *no subsidy for losers*, for each $x' \in X_i^f(R_{-i})\setminus\{i, x\}$, $v_i(x) - t_i^f(R_{-i};x) > 0 \geq v_i(x') - t_i^f(R_{-i};x')$. If $x \neq i$, then by $v_i(i) - v_i(x) < t_i^f(R_{-i};i) - t_i^f(R_{-i};x)$, $v_i(x) - t_i^f(R_{-i};x) > v_i(i) - t_i^f(R_{-i};i)$. Thus, by Lemma 3, $x(R) = x$. Let $x' \in \underset{x'' \in X}{\arg\max} \sum_{j \in N\setminus\{i\}} V_j(x'', (\overline{x}_j(R_j), 0))$. Then,

$$\sum_{j \in N\setminus\{i\}} V_j(x, (\overline{x}_j(R_j), 0)) \leq v_i(x) + \sum_{j \in N\setminus\{i\}} V_j(x, (\overline{x}_j(R_j), 0)) < \sum_{j \in N\setminus\{i\}} V_j(x', (\overline{x}_j(R_j), 0)),$$

where the second inequality follows from (3). Thus, $x \neq x'$. By $R_{-1} \in \mathcal{R}_{-1}^+$ and desirability of own consumption of R_1, $x' \neq 0$.

Suppose $x \in NQL$. By $R_x \in \mathcal{R}_x^I$ and $x' \neq x, 0$, $V_x(x', (\overline{x}_x(R_x), 0)) = 0$. We have

$$\begin{aligned} V_x(x, (\overline{x}_x(R_x), 0)) &< \sum_{j \in N\setminus\{i\}} V_j(x', (\overline{x}_j(R_j), 0)) - \Big(\sum_{j \in N\setminus\{i,x\}} V_j(x, (\overline{x}_j(R_j), 0)) + v_i(x)\Big) \\ &= \sum_{j \in N\setminus\{i\}} V_j(x', (\overline{x}_j(R_j), 0)) - \sum_{j \in N\setminus\{x\}} V_j(x, (\overline{x}_j(R_j), 0)) \\ &\leq \sum_{j \in N\setminus\{x\}} V_j(x', (\overline{x}_j(R_j), 0)) - \sum_{j \in N\setminus\{x\}} V_j(x, (\overline{x}_j(R_j), 0)) \leq t_x(R), \end{aligned}$$

where the first inequality follows from (3), the second one from $V_x(x', (\overline{x}_x(R_x), 0)) = 0$, and the last one from $x \in NQL$ and Lemma 14. Thus, $(\overline{x}_x(R_x), 0)\ P_x\ f_x(R)$, which contradicts Lemma 7.

Next, suppose $x \in QL$. Then,

$$\sum_{j\in N}\Big(V_j(x,(\overline{x}_j(R_j),0)) - V_j(x',(\overline{x}_j(R_j),0))\Big) = \sum_{j\in N}\Big(V_j(x,f_j(R)) - V_j(x',f_j(R))\Big) \geq 0, \quad (4)$$

where the equality follows from Lemma 13 and $R_{QL} \in \mathcal{R}^Q_{QL}$, and the inequality from *efficiency* and Remark 1. We also have

$$\begin{aligned}\sum_{j\in N} V_j(x,(\overline{x}_j(R_j),0)) &= v_i(x) + \sum_{j\in N\setminus\{i\}} V_j(x,(\overline{x}_j(R_j),0)) \\ &< \sum_{j\in N\setminus\{i\}} V_j(x',(\overline{x}_j(R_j),0)) \leq \sum_{j\in N} V_j(x',(\overline{x}_j(R_j),0)),\end{aligned}$$

where the first inequality follows from (3). This contradicts (4).

Case 2. $t^f_i(R_{-i};x) > \max_{x'\in X}\sum_{j\in N\setminus\{i\}} V_j(x',(\overline{x}_j(R_j),0)) - \sum_{j\in N\setminus\{i\}} V_j(x,(\overline{x}_j(R_j),0))$.

Let $R_i \in \mathcal{R}^{+I}_i \cap \mathcal{R}^Q_i$ be such that $v_i(x) < t^f_i(R_{-i};x)$,

$$v_i(x) > \max_{x'\in X}\sum_{j\in N\setminus\{i\}} V_j(x',(\overline{x}_j(R_j),0)) - \sum_{j\in N\setminus\{i\}} V_j(x,(\overline{x}_j(R_j),0)),$$

for each $x' \in X\setminus\{i,x\}$, $v_i(x') = 0$, and if $x \neq i$, then $v_i(i) - v_i(x) < t^f_i(R_{-i};i) - t^f_i(R_{-i};x)$. By Lemma 5, we can choose such R_i. By $v_i(x) < t^f_i(R_{-i};x)$, $0 > v_i(x) - t^f_i(R_{-i};x)$. If $x \neq i$, then by $v_i(i) - v_i(x) < t^f_i(R_{-i};i) - t^f_i(R_{-i};x)$, $v_i(x) - t^f_i(R_{-i};x) > v_i(i) - t^f_i(R_{-i};i)$. Thus, by *weak individual rationality*, $x(R) \in X\setminus\{x,i\}$. Also, by Lemma 1, $x(R) \neq 0$. Let $j = x(R)$. Then, in the same way as in the proof of Lemma 15, we can derive a contradiction. Thus, we omit the detail. □

The next lemma shows that the object allocation rule under the rule f coincides with that under a generalized pivotal rule.

Lemma 17. *For each $R \in \mathcal{R}_N$, $x(R) \in \arg\max_{x\in X}\sum_{i\in N} V_i(x,(\overline{x}_i(R_i),0))$.*

Proof. Let $R \in \mathcal{R}_N$. By Lemma 1, $x(R) \neq 0$. Let $x \in \arg\max_{x'\in X}\sum_{i\in N} V_i(x',(\overline{x}_i(R_i),0))$. By $R_{-1} \in \mathcal{R}^+_{-1}$ and desirability of own consumption of R_1, $x \neq 0$. Let $i = x(R)$ and $j = x$.

First, suppose $i \in NQL$. Then.

$$\begin{aligned}
&\sum_{k\in N}\Big(V_k(i,(\overline{x}_k(R_k),0)) - V_k(j,(\overline{x}_k(R_k),0))\Big) \\
&= V_i(i,(\overline{x}_i(R_i),0)) + \sum_{k\in N\setminus\{i\}}\Big(V_k(i,(\overline{x}_k(R_k),0)) - V_k(j,(\overline{x}_k(R_k),0))\Big) \\
&\geq t_i(R) + \sum_{k\in N\setminus\{i\}}\Big(V_k(i,(\overline{x}_k(R_k),0)) - V_k(j,(\overline{x}_k(R_k),0))\Big) \\
&\geq \max_{x'\in X}\sum_{k\in N\setminus\{i\}} V_k(x',(\overline{x}_k(R_k),0)) - \sum_{k\in N\setminus\{i\}} V_k(j,(\overline{x}_k(R_k),0)) \geq 0,
\end{aligned}$$

where the equality follows from $R_i \in \mathcal{R}_i^I$, the first inequality from $R_i \in \mathcal{R}_i^I$ and Lemma 7, and the second one from Lemma 14.

Next, suppose $i \in QL$. Then,

$$\sum_{k\in N}\Big(V_k(i,(\overline{x}_k(R_k),0)) - V_k(j,(\overline{x}_k(R_k),0))\Big) = \sum_{k\in N}\Big(V_k(i,f_k(R)) - V_k(j,f_k(R))\Big) \geq 0,$$

where the equality follows from Lemma 13 and $R_{QL} \in \mathcal{R}_{QL}^Q$, and the inequality from Remark 1 and *efficiency*. □

Now, we are ready to complete the proof of the "only if" part of Theorem 4. By Lemma 17, the object allocation rule under f coincides with that under a generalized pivotal rule. By Lemmas 14, 15, and 16, the payment rule under f also coincides with that under a generalized pivotal rule. Thus, f is a generalized pivotal rule. ■

C Proof of Theorem 5

In this section, we prove Theorem 5. Throughout the proof, without loss of generality, let $i = 1$.

C.1 Proof of the "if" part

Let $f = (x, t)$ be a generalized pivotal rule respecting agent 1. By the definition of the rule, for each $R \in \mathcal{R}_N$,

$$f_1(R) \in B\Big(R_1, \Big\{\Big(i, \max_{j \in N\setminus\{1\}} v_j(j) - v_i(i)\Big) : i \in N\setminus\{1\}\Big\} \bigcup \Big\{\Big(1, \max_{i \in N\setminus\{1\}} v_i(i)\Big)\Big\}\Big). \tag{1}$$

We begin with the following lemma.

Lemma 18. *Let $R \in \mathcal{R}_N$. For each $i \in N\setminus\{1\}$, if $x(R) = i$, then $v_i(i) \geq t_i(R)$.*

Proof. Suppose $x(R) = i$. Let $j \in N_{-1,i}(R_{-i})$. Without loss of generality, let $i = 2$ and $j = 3$. By $3 \in N_{-1,2}(R_{-2})$, $v_3(3) = \max_{i \in N\setminus\{1,2\}} v_i(i)$. There are two cases.

Case 1. $3 \succeq_1 2$.

Then, $t_2(R) = \max\{V_1(1,(2,0)), \max_{i \in N\setminus\{1,2\}: i \succeq_1 2} \tau_{2,i}(R_{-2})\}$. We claim $v_2(2) \geq v_3(3)$. By contradiction, suppose $v_3(3) > v_2(2)$. Then, we have $v_3(3) = \max_{i \in N\setminus\{1\}} v_i(i)$, and $(3, v_3(3) - v_3(3)) = (3,0)\ R_1\ (2,0)\ P_1\ (2, v_3(3) - v_2(2)) = f_1(R)$, where the first relation follows from $3 \succeq_1 2$, and the second one from $v_3(3) > v_2(2)$. However, this contradicts (1). Thus, $v_2(2) \geq v_3(3)$, and so $v_2(2) = \max_{i \in N\setminus\{1\}} v_i(i)$. Thus, by (1), we have $(2,0) = (2, v_2(2) - v_2(2)) = f_1(R)\ R_1\ (1, v_2(2))$, which implies

$$v_2(2) \geq V_1(1,(2,0)). \tag{2}$$

Let $i \in N\setminus\{1,2\}$ be such that $i \succeq_1 2$. Then, by (1), $(2,0) = f_1(R)\ R_1\ (i, v_2(2) - v_i(i))$, which implies $v_2(2) - v_i(i) \geq V_1(i,(2,0))$. Thus, if $v_3(3) - v_i(i) \leq V_1(i,(2,0))$, then

$$v_2(2) \geq v_i(i) + V_1(i,(2,0)) = \tau_{2,i}(R_{-2}). \tag{3}$$

If $v_3(3) - v_i(i) \geq V_1(i,(2,0))$, then $V_1(2,(i, v_3(3) - v_i(i))) \geq 0$, and so by $v_2(2) \geq v_3(3)$,

$$v_2(2) \geq v_3(3) - V_1(2,(i, v_3(3) - v_i(i))) = \tau_{2,i}(R_{-2}). \tag{4}$$

By (2), (3), and (4), $v_2(2) \geq \max\Big\{V_1(1,(2,0)), \max_{i \in N\setminus\{1,2\}: i \succeq_1 2} \tau_{2,i}(R_{-2})\Big\} = t_2(R)$.

Case 2. $2 \succ_1 3$.

Then, $t_2(R) = \max\{\tau_{2,1}(R_{-2}), \max_{i\in N\setminus\{1,2\}:i\succeq_1 3} \tau_{2,i}(R_{-2})\}$. The proof consists of three claims.

Claim 1. *We have* $v_2(2) \geq \tau_{2,1}(R_{-2})$.

Proof. Suppose first that $v_3(3) \leq V_1(1,(2,0))$. We claim that $v_2(2) \geq v_3(3)$. By contradiction, suppose that $v_3(3) > v_2(2)$. Then, we have $v_3(3) = \max_{i\in N\setminus\{1\}} v_i(i)$, and $(1, v_3(3))\ R_1\ (2,0)\ P_1\ (2, v_3(3) - v_2(2)) = f_1(R)$, where the first relation follows from $v_3(3) \leq V_1(1,(2,0))$, and the second one from $v_3(3) > v_2(2)$. This contradicts (1). Thus, $v_2(2) \geq v_3(3)$, which implies $v_2(2) = \max_{i\in N\setminus\{1\}} v_i(i)$. Thus, by (1), we have $(2,0) = (2, v_2(2) - v_2(2)) = f_1(R)\ R_1\ (1, v_2(2))$, which implies $v_2(2) \geq V_1(1,(2,0)) = \tau_{2,1}(R_{-2})$.

Next, suppose $V_1(1,(2,0)) \leq v_3(3) \leq V_1(1,(3,0))$. By $v_3(3) \geq V_1(1,(2,0))$, we have $V_1(2,(1,v_3(3))) \geq 0$. Thus, if $v_2(2) \geq v_3(3)$, then $v_2(2) \geq v_3(3) - V_1(2,(1,v_3(3))) = \tau_{2,1}(R_{-2})$. If $v_3(3) \geq v_2(2)$, then $v_3(3) = \max_{i\in N\setminus\{1\}} v_i(i)$. By (1), $(2, v_3(3) - v_2(2)) = f_1(R)\ R_1\ (1, v_3(3))$, which implies $V_1(2,(1,v_3(3))) \geq v_3(3) - v_2(2)$. Thus, $v_2(2) \geq v_3(3) - V_1(2,(1,v_3(3))) = \tau_{2,1}(R_{-2})$.

Finally, suppose $v_3(3) \geq V_1(1,(3,0))$. By $2 \succ_1 3$, $V_1(2,(3,0)) > 0$. Thus, if $v_2(2) \geq v_3(3)$, then $v_2(2) \geq v_3(3) - V_1(2,(3,0)) = \tau_{2,1}(R_{-2})$. If $v_3(3) \geq v_2(2)$, then $v_3(3) = \max_{i\in N\setminus\{1\}} v_i(i)$, and so by (1), $(2, v_3(3) - v_2(2)) = f_1(R)\ R_1\ (3, v_3(3) - v_3(3)) = (3,0)$. This implies $V_1(2,(3,0)) \geq v_3(3) - v_2(2)$. Thus, $v_2(2) \geq v_3(3) - V_1(2,(3,0)) = \tau_{2,1}(R_{-2})$. □

Claim 2. *For each* $i \in N\setminus\{1,2\}$ *with* $i \succeq_1 2$, *we have* $v_2(2) \geq \tau_{2,i}(R_{-2})$.

Proof. Let $i \in N\setminus\{1,2\}$ be such that $i \succeq_1 2$.

First, suppose $v_3(3) - v_i(i) \geq V_1(i,(2,0))$. Then, $(2,0)\ R_1\ (i, v_3(3) - v_i(i))$, which implies $V_1(2,(i, v_3(3) - v_i(i))) \geq 0$. Thus, if $v_2(2) \geq v_3(3)$, then we have $v_2(2) \geq v_3(3) - V_1(2,(i, v_3(3) - v_i(i))) = \tau_{2,i}(R_{-i})$. If $v_3(3) \geq v_2(2)$, then $v_3(3) = \max_{j\in N\setminus\{1\}} v_j(j)$, and so by (1), we have $(2, v_3(3) - v_2(2)) = f_1(R)\ R_1\ (i, v_3(3) - v_i(i))$. This implies that $V_1(2,(i, v_3(3) - v_i(i))) \geq v_3(3) - v_2(2)$. Thus, $v_2(2) \geq v_3(3) - V_1(2,(i, v_3(3) - v_i(i))) = \tau_{2,i}(R_{-2})$.

Next, suppose $v_3(3) - v_i(i) \leq V_1(i,(2,0))$. We claim that $v_2(2) \geq v_3(3)$. By contradiction, suppose that $v_3(3) > v_2(2)$. Then, $v_3(3) = \max_{j\in N\setminus\{1\}} v_j(j)$, and so we have $(i, v_3(3) - v_i(i))\ R_1\ (2,0)\ P_1\ (2, v_3(3) - v_2(2)) = f_1(R)$, where the first relation follows from $v_3(3) - v_i(i) \leq V_1(i,(2,0))$, and the second one from $v_3(3) > v_2(2)$. This contradicts (1). Thus, $v_2(2) \geq v_3(3)$. Then, $v_2(2) = \max_{j\in N\setminus\{1\}} v_j(j)$, and by (1), $(2,0) =$

$(2, v_2(2) - v_2(2)) = f_1(R)\ R_1\ (i, v_2(2) - v_i(i))$. This implies $v_2(2) - v_i(i) \geq V_1(i, (2, 0))$, or equivalently, $v_2(2) \geq v_i(i) + V_1(i, (2, 0)) = \tau_{2,i}(R_{-2})$. $\square$

Claim 3. *For each $i \in N \backslash \{1, 2\}$ with $2 \succ_1 i \succeq_1 3$, we have $v_2(2) \geq \tau_{2,i}(R_{-2})$.*

Proof. First, suppose $v_2(2) \geq v_3(3)$. By $v_3(3) \geq v_i(i)$, $V_1(2, (i, v_3(3) - v_i(i))) \geq 0$. Thus, by $v_2(2) \geq v_3(3)$, $v_2(2) \geq v_3(3) - V_1(2, (i, v_3(3) - v_i(i))) = \tau_{2,i}(R_{-2})$.

Next, suppose $v_3(3) \geq v_2(2)$. Then, $v_3(3) = \max_{j \in N \backslash \{1\}} v_j(j)$, and so by (1), we have $(2, v_3(3) - v_2(2)) = f_1(R)\ R_1\ (i, v_3(3) - v_i(i))$. Thus, $V_1(2, (i, v_3(3) - v_i(i))) \geq v_3(3) - v_2(2)$, or equivalently, $v_2(2) \geq v_3(3) - V_1(2, (i, v_3(3) - v_i(i))) = \tau_{2,i}(R_{-i})$. $\square$

By Claims 1, 2, and 3, $v_2(2) \geq \max\{\tau_{2,1}(R_{-2}), \max_{i \in N \backslash \{1,2\} : i \succeq_1 3} \tau_{2,i}(R_{-2})\} = t_2(R)$. $\square$

Now, we proceed to the proof of the "if" part of Theorem 5. Since *no subsidy for losers* follows from the definition of the rule, we show that f satisfies the other three properties.

Efficiency. Let $R \in \mathcal{R}_N$. By (1), $x(R) \in N$, i.e., $x(R) \neq 0$. There are two cases.

Case 1. $x(R) = 1$.

Then, $t_1(R) = \max_{i \in N \backslash \{1\}} v_i(i)$. Let $x \in X \backslash \{x(R)\}$. If $x = 0$, then

$$\sum_{i \in N} \Big(V_i(x(R), f_i(R)) - V_i(x, f_i(R)) \Big) \geq V_1(1, f_1(R)) - V_1(0, f_1(R)) > 0,$$

where the first inequality follows from $R_{-1} \in \mathcal{R}^+_{-1}$, and the second one from desirability of own consumption of R_1. Next, suppose that $x \in N \backslash \{1\}$. Note that by (1), we have $f_1(R)\ R_1\ (x, \max_{i \in N \backslash \{1\}} v_i(i) - v_x(x))$. Thus, $\max_{i \in N \backslash \{1\}} v_i(i) - v_x(x) \geq V_1(x, f_1(R))$. Thus,

$$t_1(R) - V_1(x, f_1(R)) = \max_{i \in N \backslash \{1\}} v_i(i) - V_1(x, f_1(R)) \geq v_x(x). \tag{5}$$

Then,

$$\sum_{i \in N} \Big(V_i(x(R), f_i(R)) - V_i(x, f_i(R)) \Big) = t_1(R) - V_1(x, f_1(R)) - v_x(x) \geq 0,$$

where the equality follows from $R_{-1} \in \mathcal{R}^I_{-1}$, and the inequality from (5). Thus, by Remark 1, $f(R)$ is efficient for R.

Case 2. $x(R) \in N\backslash\{1\}$.

Let $i = x(R)$. Then, $t_1(R) = \max_{j \in N\backslash\{1\}} v_j(j) - v_i(i)$. By (1), $f_1(R)\ R_1\ (1, \max_{j \in N\backslash\{1\}} v_j(j))$, which implies that $\max_{j \in N\backslash\{1\}} v_j(j) \geq V_1(1, f_1(R))$. Thus,

$$t_1(R) - V_1(1, f_1(R)) = \Big(\max_{j \in N\backslash\{1\}} v_j(j) - v_i(i)\Big) - V_1(1, f_1(R)) \geq \ - v_i(i). \tag{6}$$

Let $x \in X\backslash\{x(R)\}$. If $x = 0$, then

$$\sum_{j \in N} \Big(V_j(x(R), f_j(R)) - V_j(x, f_j(R))\Big) \geq t_1(R) - V_1(0, f_1(R)) + v_i(i) > t_1(R) - V_1(1, f_1(R)) + v_i(i) \geq 0,$$

where the first inequality follows from $R_{-1} \in \mathcal{R}^+_{-1}$, the second one from desirability of own consumption of R_1, and the last one from (6). If $x = 1$, then

$$\sum_{j \in N} \Big(V_j(x(R), f_j(R)) - V_j(x, f_j(R))\Big) = t_1(R) - V_1(1, f_1(R)) + v_i(i) \geq 0,$$

where the equality follows from $R_{-1} \in \mathcal{R}^I_{-1}$, and the inequality from (6). Finally, suppose that $x \in N\backslash\{1\}$. By (1), $f_1(R)\ R_1\ (x, \max_{j \in N\backslash\{1\}} v_j(j) - v_x(x))$, which implies that $\max_{j \in N\backslash\{1\}} v_j(j) - v_x(x) \geq V_1(x, f_1(R))$. Thus,

$$t_1(R) - V_1(x, f_1(R)) = \Big(\max_{j \in N\backslash\{1\}} v_j(j) - v_i(i)\Big) - V_1(x, f_1(R)) \geq v_x(x) - v_i(i). \tag{7}$$

Then,

$$\sum_{j \in N} \Big(V_j(x(R), f_j(R)) - V_j(x, f_j(R))\Big) = t_1(R) - V_1(x, f_1(R)) + v_i(i) - v_x(x) \geq 0,$$

where the equality follows from $R_{-1} \in \mathcal{R}^I_{-1}$, and the inequality from (7). Thus, by Remark 1, $f(R)$ is efficient for R.

Weak individual rationality. Let $R \in \mathcal{R}_N$. First, we consider agent 1. Let $i \in N\backslash\{1\}$ be such that $v_i(i) = \max_{j \in N\backslash\{1\}} v_j(j)$. By (1), $f_1(R)\ R_1\ (i, \max_{j \in N\backslash\{1\}} v_j(j) - v_i(i)) = (i, 0)\ R_1\ (\underline{x}_1(R_1), 0)$. Then, we consider the other agents. Let $i \in N\backslash\{1\}$. If $x(R) \neq i$, then $t_i(R) = 0$, and so $f_i(R)\ R_i\ (\underline{x}_i(R_i), 0)$. If $x(R) = i$, then by Lemma 18, we have that $f_i(R)\ R_i\ (\overline{x}_i(R_i), 0)\ R_i\ (\underline{x}_i(R_i), 0)$.

Strategy-proofness. Let $R \in \mathcal{R}_N$. First, consider agent 1. Let $R'_1 \in \mathcal{R}_1$. By (1),

$$f_1(R'_1, R_{-1}) \in \Big\{\Big(i, \max_{j \in N\setminus\{i\}} v_j(j) - v_i(i)\Big) \colon j \in N\setminus\{1\}\Big\} \bigcup \Big\{\Big(1, \max_{i \in N\setminus\{1\}} v_i(i)\Big)\Big\}.$$

Thus, by (1), $f_1(R)\ R_1\ f_1(R'_1, R_{-1})$.

Next, we consider the other agents. Let $i \in N\setminus\{1\}$. Let $j \in N_{-1,i}(R_{-i})$. Without loss of generality, let $i = 2$ and $j = 3$. Note that $v_3(3) = \max_{i \in N\setminus\{1,2\}} v_i(i)$.

We show the following two lemmas, both of which show that if agent 2's (quasi-linear) valuation of own consumption is greater than the payment when he receives the object, then he indeed receives the object.

Lemma 19. *Suppose* $3 \succeq_1 2$ *and* $v_2(2) > \max\{V_1(1, (2, 0)), \max_{i \in N\setminus\{1,2\}: i \succeq_1 2} \tau_{2,i}(R_{-2})\}$. *Then,* $x(R) = 2$.

Proof. Suppose by contradiction that $x(R) \neq 2$. Note that by (1), $x(R) \in N$, i.e., $x(R) \neq 0$. Thus, there are three cases.

Case 1. $x(R) = 1$.

Suppose $v_2(2) \geq v_3(3)$. Then, $v_2(2) = \max_{i \in N\setminus\{1\}} v_i(i)$. Thus, $t_1(R) = v_2(2)$. By $v_2(2) > V_1(1, (2, 0))$, $(2, v_2(2) - v_2(2)) = (2, 0)\ P_1\ (1, v_2(2)) = f_1(R)$, which contradicts (1).

Next, suppose $v_3(3) \geq v_2(2)$. Then, $v_3(3) = \max_{i \in N\setminus\{1\}} v_i(i)$. Thus, $t_1(R) = v_3(3)$. Then, $(3, v_3(3) - v_3(3)) = (3, 0)\ R_1\ (2, 0)\ P_1\ (1, v_2(2))\ R_1\ (1, v_3(3)) = f_1(R)$, where the first relation follows from $3 \succeq_1 2$, the second one from $v_2(2) > V_1(1, (2, 0))$, and the third one from $v_3(3) \geq v_2(2)$. This contradicts (1).

Case 2. $x(R) \in N\setminus\{1, 2\}$ and $x(R) \succeq_1 2$.

Let $i = x(R)$. Suppose that $v_3(3) - v_i(i) \leq V_1(i, (2, 0))$. Then, $\tau_{2,i}(R_{-2}) = v_i(i) + V_1(i, (2, 0))$. Thus, $v_2(2) > \tau_{2,i}(R_{-2}) = v_i(i) + V_1(i, (2, 0))$. By $v_3(3) - v_i(i) \leq V_1(i, (2, 0))$, $v_i(i) + V_1(i, (2, 0)) \geq v_3(3)$. Thus, $v_2(2) > v_3(3)$, and so $v_2(2) = \max_{j \in N\setminus\{1\}} v_j(j)$. By

$v_2(2) > v_i(i) + V_1(i,(2,0))$, $v_2(2) - v_i(i) > V_1(i,(2,0))$. Thus, we have $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ (i, v_2(2) - v_i(i)) = f_1(R)$. This contradicts (1).

Next, suppose that $v_3(3) - v_i(i) \geq V_1(i,(2,0))$. Then, we have $\tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i)))$. First, suppose $v_3(3) \geq v_2(2)$. Then, $v_3(3) = \max_{j \in N \setminus \{1\}} v_j(j)$. By $v_2(2) > \tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i)))$, $V_1(2,(i, v_3(3) - v_i(i))) > v_3(3) - v_2(2)$. Thus, $(2, v_3(3) - v_2(2))\ P_1\ (i, v_3(3) - v_i(i)) = f_1(R)$, which contradicts (1). Suppose instead $v_2(2) > v_3(3)$. Then, $v_2(2) = \max_{j \in N \setminus \{1\}} v_j(j)$. Recall that we have assumed $v_3(3) - v_i(i) \geq V_1(i,(2,0))$. Then, $v_2(2) - v_i(i) > v_3(3) - v_i(i) \geq V_1(i,(2,0))$, which implies that $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ (i, v_2(2) - v_i(i)) = f_1(R)$. This contradicts (1).

Case 3. $x(R) \in N \setminus \{1,2\}$ and $2 \succ_1 x(R)$.

Let $i = x(R)$. If $v_2(2) \geq v_3(3)$, then $v_2(2) = \max_{j \in N \setminus \{1\}} v_j(j)$, and $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ (i,0)\ R_1\ (i, v_2(2) - v_i(i)) = f_1(R)$, where the first relation follows from $2 \succ_1 i$, and the second one from $v_2(2) \geq v_i(i)$. This contradicts (1). Thus, $v_3(3) > v_2(2)$, and so $v_3(3) = \max_{j \in N \setminus \{1\}} v_j(j)$. Then, by $v_2(2) > \tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i)))$, $V_1(2,(i, v_3(3) - v_i(i))) > v_3(3) - v_2(2)$. Thus, $(2, v_3(3) - v_2(2))\ P_1\ (i, v_3(3) - v_i(i)) = f_1(R)$, which contradicts contradicts (1). □

Lemma 20. *Suppose* $2 \succ_1 3$ *and* $v_2(2) > \max\{\tau_{2,1}(R_{-2}), \max_{i \in N \setminus \{1,2\}: i \succeq_1 3} \tau_{2,i}(R_{-2})\}$. *Then,* $x(R) = 2$.

Proof. By contradiction, suppose $x(R) \neq 2$. By (1), $x(R) \in N$. There are four cases.

Case 1. $x(R) = 1$.

First, suppose $v_3(3) \leq V_1(1,(2,0))$. Then, $\tau_{2,1}(R_{-2}) = V_1(1,(2,0))$. By $v_2(2) > \tau_{2,1}(R_{-2}) = V_1(1,(2,0))$ and $V_1(1,(2,0)) \geq v_3(3)$, $v_2(2) > v_3(3)$. Thus, $v_2(2) = \max_{i \in N \setminus \{1\}} v_i(i)$, and $t_1(R) = v_2(2)$. By $v_2(2) > V_1(1,(2,0))$, $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ (1, v_2(2)) = f_1(R)$, which contradicts (1).

Next, suppose $V_1(1,(2,0)) \leq v_3(3) \leq V_1(1,(3,0))$. Then, $\tau_{2,1}(R_{-2}) = v_3(3) - V_1(2,(1, v_3(3)))$. By $v_2(2) > \tau_{2,1}(R_{-2}) = v_3(3) - V_1(2,(1, v_3(3)))$, $V_1(2,(1, v_3(3))) > v_3(3) - v_2(2)$. Thus, if $v_3(3) \geq v_2(2)$, then $v_3(3) = \max_{i \in N \setminus \{1\}} v_i(i)$, and $(2, v_3(3) - v_2(2))\ P_1\ (1, v_3(3)) = f_1(R)$, which contradicts (1). Suppose $v_2(2) > v_3(3)$. Then, $v_2(2) = \max_{i \in N \setminus \{1\}} v_i(i)$, and $t_1(R) =$

$v_2(2)$. By $v_3(3) \geq V_1(1,(2,0))$, $(2,0)\ R_1\ (1,v_3(3))$. By $v_2(2) > v_3(3)$, $(1,v_3(3))\ P_1\ (1,v_2(2)) = f_1(R)$. Thus, we get $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ f_1(R)$, which contradicts (1).

Finally, suppose $v_3(3) \geq V_1(1,(3,0))$. Then, $\tau_{2,1}(R_{-2}) = v_3(3) - V_1(2,(3,0))$. By $v_2(2) > \tau_{2,1}(R_{-2}) = v_3(3) - V_1(2,(3,0))$, $V_1(2,(3,0)) > v_3(3) - v_2(2)$. Suppose $v_3(3) \geq v_2(2)$. Then, $v_3(3) = \max_{i \in N\backslash\{1\}} v_i(i)$, and $t_1(R) = v_3(3)$. By $V_1(2,(3,0)) > v_3(3) - v_2(2)$, $(2, v_3(3) - v_2(2))\ P_1\ (3,0)$. By $v_3(3) \geq V_1(1,(3,0))$, $(3,0)\ R_1\ (1, v_3(3)) = f_1(R)$. Thus, $(2, v_3(3) - v_2(2))\ P_1\ f_1(R))$, which contradicts (1). Suppose instead $v_2(2) \geq v_3(3)$. Then, $v_2(2) = \max_{i \in N\backslash\{1\}} v_i(i)$, and $t_2(R) = v_2(2)$. By $v_2(2) \geq v_3(3)$ and $v_3(3) \geq V_1(1,(3,0))$, $v_2(2) \geq V_1(3,0)$, which implies $(3,0)\ R_1\ (1, v_2(2)) = f_1(R)$. By $2 \succ_1 3$, $(2,0)\ P_1\ (3,0)$. Thus, $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ f_1(R)$, which contradicts (1).

Case 2. $x(R) \in N\backslash\{1,2\}$ and $x(R) \succeq_1 2$.

By inspection, one can find that the discussion in Case 2 of Lemma 19 does not depend on the relationship between 2 and 3 according to $\succeq_1$. Thus, by the same discussion as in Case 2 of Lemma 19, we can derive a contradiction.

Case 3. $x(R) \in N\backslash\{1,2\}$ and $2 \succ_1 x(R) \succeq_1 3$.

Let $i = x(R)$. Suppose $v_2(2) \geq v_3(3)$. Then, $v_2(2) = \max_{j \in N\backslash\{1\}} v_j(j)$, and $t_1(R) = v_2(2) - v_i(i) \geq 0$. Then, $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ (i,0)\ R_1\ f_1(R)$, where the first relation follows from $2 \succ_1 i$, and the second one from $t_1(R) \geq 0$. However, this contradicts (1). Next, suppose $v_3(3) \geq v_2(2)$. Then, $v_3(3) = \max_{j \in N\backslash\{1\}} v_j(j)$, and so $t_1(R) = v_3(3) - v_i(i)$. By $v_2(2) > \tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i)))$, $V_1(2,(i, v_3(3) - v_i(i))) > v_3(3) - v_2(2)$. Thus, $(2, v_3(3) - v_2(2))\ P_1\ (i, v_3(3) - v_i(i)) = f_1(R)$, which contradicts (1).

Case 4. $x(R) \in N\backslash\{1,2\}$ and $3 \succ_1 x(R)$.

Let $i = x(R)$. Suppose $v_2(2) \geq v_3(3)$. Then, $v_2(2) = \max_{j \in N\backslash\{1\}} v_j(j)$, and $t_1(R) = v_2(2) - v_i(i) \geq 0$. We have $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ (3,0)\ P_1\ (i,0)\ R_1\ f_1(R)$, where the first relation follows from $2 \succ_1 3$, the second one from $3 \succ_1 i$, and the third one from $t_1(R) \geq 0$. However, this contradicts (1). Next, suppose $v_3(3) \geq v_2(2)$. Then,

$v_3(3) = \max_{j\in N\setminus\{1\}} v_j(j)$, and $t_1(R) = v_3(3) - v_i(i) \geq 0$. We have $(3, v_3(3) - v_3(3)) = (3, 0)\ P_1\ (i, 0)\ R_1\ f_1(R)$, where the first relation follows from $3 \succ_1 i$, and the second one from $t_1(R) \geq 0$. This contradicts (1). □

Now, we complete the proof that agent 2 never benefits from misrepresenting his preferences. Let $R'_2 \in \mathcal{R}_2$. If $x(R) = x(R'_2, R_{-2})$, then $t_2(R) = t_2(R'_2, R_{-2})$. If $x(R) = 2$ and $x(R'_2, R_{-2}) \neq 2$, then $t_2(R'_2, R_{-2}) = 0$, and so by Lemma 18 and $R_2 \in \mathcal{R}^I_2$, we have that $f_2(R)\ R_2\ f_2(R'_2, R_{-2})$. Suppose $x(R) \neq 2$ and $x(R'_2, R_{-2}) = 2$. Then, $t_2(R) = 0$. By Lemmas 19 and 20, $v_2(2) \leq t_2(R'_2, R_{-2})$. Thus, by $R_2 \in \mathcal{R}^I_2$, $f_2(R)\ R_2\ f_2(R'_2, R_{-2})$. ■

C.2 Proof of the "only if" part

We prove the "only if" part of Theorem 5. Let $\mathcal{R}_N$ be rich.[27] Let $f = (x, t)$ be a rule on $\mathcal{R}_N$ satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

The following lemma states that an outcome bundle of agent 1 under the rule f coincides with that under a generalized pivotal rule respecting agent 1.

Lemma 21. *For each $R \in \mathcal{R}_N$ with $R_{-1} \in \mathcal{R}^{+I}_{-1} \cap \mathcal{R}^Q_{-1}$,*

$$f_1(R) \in B\Big\{\Big(i, \max_{j\in N\setminus\{1\}} v_j(j) - v_i(i)\Big) : i \in N\setminus\{1\}\Big\} \bigcup \Big\{\Big(1, \max_{i\in N\setminus\{1\}} v_i(i)\Big)\Big\}\Big).$$

Proof. Let $R \in \mathcal{R}_N$. By Lemma 1, $x(R) \neq 0$, i.e., $x(R) \in N$. Let $i = x(R)$. For notational convenience, let $v_1(1) = 0$. By $R_{-1} \in \mathcal{R}^{+I}_{-1}$, Theorem 1 implies that $t_1(R) = t^f_1(R_{-1}; i) = \max_{x\in X} \sum_{j\in N\setminus\{1\}} v_j(x) - v_i(i) = \max_{j\in N\setminus\{1\}} v_j(j) - v_i(i)$.[28] Thus, we have $f_1(R) \in \{(j, \max_{k\in N\{1\}} v_k(k) - v_j(j)) : j \in N\}$. By contradiction, suppose that there is $j \in N\setminus\{i\}$ such that $(j, \max_{k\in N\{1\}} v_k(k) - v_j(j))\ P_1\ f_1(R)$. Thus,

$$V_1(j, f_1(R)) > \max_{k\in N\setminus\{1\}} v_k(k) - v_j(j). \tag{1}$$

[27] Because the lemmas in this section will be used to prove the impossibility theorems (Theorems 2, 3, and 6), we do not assume that $\mathcal{R}_{-1} \subseteq \mathcal{R}^{+I}_{-1} \cap \mathcal{R}^Q_{-1}$ in the following lemmas.

[28] Note that in Theorem 2, we do not assume that $\mathcal{R}_N \cap \mathcal{R}^Q_N$ is convex, and so we cannot apply Lemma 9 here to identity the payment of agent 1. Instead, we apply Theorem 1 to do so.

If $i = 1$, then

$$t_1(R) - V_1(j, f_1(R)) = \max_{k \in N\setminus\{1\}} v_k(k) - V_1(j, f_1(R)) < v_j(j) = V_j(j, f_j(R)) - t_j(R),$$

where the inequality follows from (1). This contradicts Lemma 2 (i). If $i \neq 1$ and $j \neq 1$, then

$$\Big(t_1(R) - V_1(j, f_1(R))\Big) + \Big(t_i(R) - V_i(j, f_i(R))\Big) = \Big(\max_{k \in N\setminus\{1\}} v_k(k) - v_i(i)\Big) - V_1(j, f_1(R)) + v_i(i)$$

$$= \max_{k \in N\setminus\{1\}} v_k(k) - V_1(j, f_1(R)) < v_j(j) = V_j(j, f_j(R)) - t_j(R),$$

where the inequality follows from (1). This contradicts Lemma 2 (ii). Finally, if $i \neq 1$ and $j = 1$, then

$$t_1(R) - V_1(j, f_1(R)) = \Big(\max_{k \in N\setminus\{1\}} v_k(k) - v_i(i)\Big) - V_1(j, f_1(R)) < -v_i(i) = V_i(j, f_i(R)) - t_i(R),$$

where the inequality follows from (1). This contradicts Lemma 2 (iii). □

In the following two lemmas, we will show that the payments of agents other than agent 1 under a rule f coincide with those under a generalized pivotal rule respecting agent 1. Note that by Lemma 4, for each $i \in N\setminus\{1\}$ and each $R_{-i} \in \mathcal{R}_{-i}$, $i \in X_i^f(R_{-i})$.

Lemma 22. *Let $i \in N\setminus\{1\}$ and $R_{-i} \in \mathcal{R}_{-i}$ be such that $R_{-1,i} \in \mathcal{R}_{-1,i}^{+I} \cap \mathcal{R}_{-1,i}^{Q}$. Let $j \in N_{-1,i}(R_{-i})$. If $j \succeq_1 i$, then $t_i^f(R_{-i}; i) = \max\{V_1(1,(2,0)), \max_{k \in N\setminus\{1,i\}: k \succeq_1 i} \tau_{i,k}(R_{-i})\}$.*

Proof. Without loss of generality, let $i = 2$ and $j = 3$. By contradiction, suppose $3 \succeq_1 2$, but $t_2^f(R_{-2}; 2) \neq \max\{V_1(1,(2,0)), \max_{i \in N\setminus\{1,2\}: i \succeq_1 2} \tau_{2,i}(R_{-2})\}$. There are three cases.

Case 1. $t_2^f(R_{-2}; 2) < V_1(1,(2,0))$.

Let $R_2 \in \mathcal{R}_2^0 \cap \mathcal{R}_2^Q$ be such that $t_2^f(R_{-2}; 2) < v_2(2) < V_1(1,(2,0))$. By $v_2(2) > t_2^f(R_{-2}; 2)$, Lemma 8 implies $x(R) = 2$.

First, suppose $v_3(3) < V_1(1,(2,0))$. Then, without loss of generality, we can assume $v_2(2) > v_3(3)$. Then, $v_2(2) = \max_{i \in N\setminus\{1\}} v_i(i)$. Theorem 1 implies that $t_1(R) = t_1^f(R_{-1}; 2) = \max_{x \in X} \sum_{i \in N\setminus\{1\}} v_i(x) - v_2(2) = v_2(2) - v_2(2) = 0$. By $v_2(2) < V_1(1,(2,0)) = V_1(1, f_1(R))$, $(1, v_2(2))\ P_1\ f_1(R)$. This contradicts Lemma 21.

Next, suppose $v_3(3) \geq V_1(1,(2,0))$. By $V_1(1,(2,0)) > v_2(2)$, we have $v_3(3) > v_2(2)$. Thus, $v_3(3) = \max_{i \in N\setminus\{1\}} v_i(i)$. By $R_{-1} \in \mathcal{R}_{-1}^{+I}$, Theorem 1 implies $t_1(R) = t_1^f(R_{-1};2) = v_3(3) - v_2(2)$. By $v_3(3) > v_2(2)$, $t_1(R) > 0$. Thus, $(2,0)\ P_1\ (2, t_1(R)) = f_1(R)$. By $3 \succeq_1 2$, $(3,0)\ R_1\ (2,0)$. Thus, $(3, v_3(3) - v_3(3)) = (3,0)\ P_1\ f_1(R)$, which contradicts Lemma 21.

Case 2. $t_2^f(R_{-2};;2) < \max_{i \in N\setminus\{1,2\}: i \succeq_1 2} \tau_{2,i}(R_{-2})$.

Then, there is $i \in N\setminus\{1,2\}$ such that $i \succeq_1 2$ and $t_2^f(R_{-2};2) < \tau_{2,i}(R_{-2})$. Let $R_2 \in \mathcal{R}_2^0 \cap \mathcal{R}_2^Q$ be such that $t_2^f(R_{-2};2) < v_2(2) < \tau_{2,i}(R_{-2})$. By $v_2(2) > t_2^f(R_{-2};2)$, Lemma 8 implies $x(R) = 2$.

First, suppose that $v_3(3) - v_i(i) < V_1(i,(2,0))$. Then, $\tau_{2,i}(R_{-2}) = v_i(i) + V_1(i,(2,0))$. By $v_3(3) - v_i(i) < V_1(i,(2,0))$, $\tau_{2,i}(R_{-2}) > v_3(3)$. Thus, without loss of generality, we can assume that $v_2(2) > v_3(3)$. Thus, $v_2(2) = \max_{j \in N\setminus\{1\}} v_j(j)$, and by $R_{-1} \in \mathcal{R}_{-1}^{+I}$, Theorem 1 implies $t_1(R) = t_1^f(R_{-1};2) = v_2(2) - v_2(2) = 0$. Thus, by $v_2(2) < v_i(i) + V_1(i,(2,0))$, $v_2(2) - v_i(i) < V_1(i,(2,0)) = V_1(i, f_1(R))$. Thus, we have $(i, v_2(2) - v_i(i))\ P_1\ f_1(R)$, which contradicts Lemma 21.

Next, suppose that $v_3(3) - v_i(i) \geq V_1(i,(2,0))$. Then, we have $\tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i)))$. By $v_3(3) - v_i(i) \geq V_1(i,(2,0))$, we have $(2,0)\ R_1\ (i, v_3(3) - v_i(i))$, which implies $V_1(2,(i, v_3(3) - v_i(i))) \geq 0$. Thus, by $v_2(2) < \tau_{2,i}(R_{-2})$, $v_2(2) < v_3(3)$. Thus, $v_3(3) = \max_{j \in N\setminus\{1\}} v_j(j)$. By Theorem 1, $t_1(R) = t_1^f(R_{-1};2) = v_3(3) - v_2(2)$. By $v_2(2) < v_3(3) - V_1(2,(i, v_3(3) - v_i(i)))$, $V_1(2,(i, v_3(3) - v_i(i))) < v_3(3) - v_2(2)$, which implies $(i, v_3(3) - v_i(i))\ P_1\ (2, v_3(3) - v_2(2)) = f_1(R)$. This contradicts Lemma 21.

Case 3. $t_2^f(R_{-2};2) > \max\{V_1(1,(2,0)), \max_{i \in N\setminus\{1,2\}: i \succeq_1 2} \tau_{2,i}(R_{-2})\}$.

Let $R_2 \in \mathcal{R}_2^0 \cap \mathcal{R}_2^Q$ be such that

$$\max\Big\{V_1(1,(2,0)), \max_{i \in N\setminus\{1,2\}: i \succeq_1 2} \tau_{2,i}(R_{-2})\Big\} < v_2(2) < t_2^f(R_{-2};2).$$

By $3 \succeq_1 2$, $V_1(3,(2,0)) \geq 0$, and so we have $v_2(2) > \tau_{2,3}(R_{-2}) = v_3(3) + V_1(3,(2,0)) \geq v_3(3)$.

Thus, $v_2(2) = \max_{i \in N\setminus\{1\}} v_i(i)$, and by $R_{-1} \in \mathcal{R}^{+I}_{-1}$, Theorem 1 implies

$$t_1(R) = t^f_1(R_{-1}; x(R)) = v_2(2) - \sum_{i \in N\setminus\{1\}} v_i(x(R)) = \begin{cases} v_2(2) & \text{if } x(R) = 1, \\ v_2(2) - v_{x(R)}(x(R)) & \text{if } x(R) \neq 1. \end{cases} \tag{2}$$

. By $v_2(2) < t^f_2(R_{-2}; 2)$, *weak individual rationality* implies $x(R) \neq 2$. By Lemma 1, $x(R) \neq 0$. Thus, there are the following three cases.

First, suppose $x(R) = 1$. By (2), $t_1(R) = v_2(2)$. By $v_2(2) > V_1(1,(2,0))$, we have $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ (1, v_2(2)) = f_1(R)$, which contradicts Lemma 21.

Next, suppose that $x(R) \in N\setminus\{1,2\}$, and $x(R) \succeq_1 2$. Let $i = x(R)$. First, we show $v_2(2) - v_i(i) > V_1(i,(2,0))$. By contradiction, suppose $v_2(2) - v_i(i) \leq V_1(i,(2,0))$. By $v_3(3) < v_2(2)$, $v_3(3) - v_i(i) < v_2(2) - v_i(i) \leq V_1(i,(2,0))$. Thus, $\tau_{2,i}(R_{-2}) = v_i(i) + V_1(i,(2,0))$. By $v_2(2) > \tau_{2,i}(R_{-2}) = v_i(i) + V_1(i,(2,0))$, $v_2(2) - v_i(i) > V_1(i,(2,0))$, which contradicts the assumption that $v_2(2) - v_i(i) \leq V_1(i,(2,0))$. Thus, $v_2(2) - v_i(i) > V_1(i,(2,0))$. By (2), $t_1(R) = v_2(2) - v_i(i)$. Thus, $t_1(R) > V_1(i,(2,0))$, which implies $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ (i, t_1(R)) = f_1(R)$. This contradicts Lemma 21.

Finally, suppose that $x(R) \in N\setminus\{1,2\}$, and $2 \succ_1 x(R)$. Let $i = x(R)$. By (2), $t_1(R) = v_2(2) - v_i(i) \geq 0$. By $2 \succ_1 i$, $(2,0)\ P_1\ (i,0)$. By $t_1(R) \geq 0$, $(i,0)\ R_1\ f_1(R)$. Thus, we have $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ f_1(R)$, which contradicts Lemma 21. □

Lemma 23. *Let $i \in N\setminus\{1\}$ and $R_{-i} \in \mathcal{R}_{-i}$ be such that $R_{-1,i} \in \mathcal{R}^{+I}_{-1,i} \cap \mathcal{R}^{Q}_{-1,i}$. Let $j \in N_{-1,i}(R_{-i})$. If $i \succ_1 j$, then $t^f_i(R_{-i}; i) = \max\{\tau_{i,1}(R_{-i}), \max_{k \in N\setminus\{1,i\}: k \succeq_1 j} \tau_{i,k}(R_{-i})\}$.*

Proof. Without loss of generality, let $i = 2$ and $j = 3$. Suppose by contradiction that $2 \succ_1 3$, but $t^f_2(R_{-2}; 2) \neq \max\{\tau_{2,1}(R_{-2}), \max_{i \in N\setminus\{1,2\}: i \succeq_1 3} \tau_{2,i}(R_{-2})\}$. There are three cases.

Case 1. $t^f_2(R_{-2}; 2) < \tau_{2,1}(R_{-2})$.

Let $R_2 \in \mathcal{R}^0_2 \cap \mathcal{R}^Q_2$ be such that $t^f_2(R_{-2}; 2) < v_2(2) < \tau_{2,1}(R_{-2})$. By $v_2(2) > t^f_2(R_{-2}; 2)$, Lemma 8 implies $x(R) = 2$.

First, suppose $v_3(3) < V_1(1,(2,0))$. Then, $\tau_{2,1}(R_{-2}) = V_1(1,(2,0))$. Thus, we have $v_2(2) < \tau_{2,1}(R_{-2}) = V_1(1,(2,0))$, and so we can derive a contradiction to Lemma 21 by the same discussion as in Case 1 of Lemma 22.

Next, suppose $v_3(3) \geq V_1(1,(2,0))$. Then, $\tau_{2,1}(R_{-2}) = v_3(3) - V_1(2,(1, \min\{v_3(3), V_1(1,(3,0))\}))$.

Suppose $v_2(2) > v_3(3)$. Then, $v_2(2) = \max_{i\in N\setminus\{1\}} v_i(i)$, and so by Theorem 1, we have $t_1(R) = t_1^f(R_{-1};2) = \max_{x\in X}\sum_{i\in N\setminus\{1\}} v_i(i) - v_2(2) = v_2(2) - v_2(2) = 0$. By $v_2(2) > v_3(3)$ and $v_3(3) \geq V_1(1,(2,0))$, $v_2(2) > V_1(1,(2,0)) = V_1(1, f_1(R))$. Thus, $(1, v_2(2))\ P_1\ f_1(R)$, which contradicts Lemma 21. Suppose $v_3(3) \geq v_2(2)$. Then, $v_3(3) = \max_{i\in N\setminus\{1\}} v_i(i)$, and by Theorem 1, $t_1(R) = t_1^f(R_{-1};2) = v_3(3) - v_2(2)$. By $v_2(2) > \tau_{2,1}(R_{-2}) = v_3(3) - V_1(2,(1,\min\{v_3(3), V_1(1,(3,0))\}))$, $V_1(2,(1,\min\{v_3(3), V_1(1,(3,0))\})) > v_3(3)-v_2(2) = t_1(R)$. By $v_3(3) \geq \min\{v_3(3), V_1(1,(3,0))\}$, $V_1(2,(1,v_3(3))) \geq V_1(2,(1,\min\{v_3(3), V_1(1,(3,0))\}))$. Thus, $V_1(2,(1,v_3(3))) > t_1(R)$, which implies $(1, v_3(3))\ P_1\ f_1(R)$. This contradicts Lemma 21.

Case 2. $t_2^f(R_{-2};2) < \max_{i\in N\setminus\{1,2\};i\succeq_1 3} \tau_{2,i}(R_{-2})$.

Then, there is $i \in N\setminus\{1,2\}$ such that $i \succeq_1 3$ and $t_2^f(R_{-2};2) < \tau_{2,i}(R_{-2})$. If $i \succeq_1 2$, then we can derive a contradiction as in the proof of Case 2 of Lemma 22. Thus, assume $2 \succ_1 i \succeq_1 3$. Then, $\tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i)))$. Let $R_2 \in \mathcal{R}_2^0 \cap \mathcal{R}_2^Q$ be such that $t_2^f(R_{-2};2) < v_2(2) < \tau_{1,2}(R_{-2})$. Then, by $v_2(2) > t_2^f(R_{-2};2)$, Lemma 8 implies $x(R) = 2$. By $2 \succ_1 i$, $(2,0)\ P_1\ (i,0)$. By $v_3(3) \geq v_i(i)$, $(i,0)\ R_1\ (i, v_3(3) - v_i(i))$. Thus, $(2,0)\ P_1\ (i, v_3(3)-v_i(i))$, which implies $V_1(2,(i,v_3(3)-v_i(i))) > 0$. Thus, by $v_2(2) < \tau_{2,i}(R_{-2})$, $v_2(2) < v_3(3)$. Thus, $v_3(3) = \max_{j\in N\setminus\{1\}} v_j(j)$, and so by Theorem 1, we have $t_1(R) = t_1^f(R_{-1};2) = v_2(2) - v_i(i)$. By $v_2(2) < \tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i)))$, we have $V_1(2,(i, v_3(3) - v_i(i))) < v_3(3) - v_2(2) = t_1(R)$, which implies $(i, v_3(3) - v_i(i))\ P_1\ f_1(R)$. This contradicts Lemma 21.

Case 3. $t_2^f(R_{-2};2) > \max\{\tau_{2,1}(R_{-2}), \max_{i\in N\setminus\{1,2\}:i\succeq_1 3} \tau_{2,i}(R_{-2})\}$.

Let $R_2 \in \mathcal{R}_2^0 \cap \mathcal{R}_2^Q$ be such that

$$\max\Big\{\tau_{2,1}(R_{-2}), \max_{i\in N\setminus\{1,2\}:i\succeq_1 3} \tau_{2,i}(R_{-2})\Big\} < v_2(2) < t_2^f(R_{-2};2).$$

By $2 \succ_1 3$, $\tau_{2,3}(R_{-2}) = v_3(3) - V_1(2,(3,0))$. By $v_2(2) < t_2^f(R_{-2};2)$, *weak individual rationality* implies that $x(R) \neq 2$. Also, by Lemma 1, $x(R) \neq 0$.

First, suppose $x(R) = 1$. Suppose $v_2(2) \geq v_3(3)$. Then, $v_2(2) = \max_{i\in N\setminus\{1\}} v_i(i)$, and by Theorem 1, $t_1(R) = t_1^f(R_{-1};1) = \max_{i\in N\setminus\{1\}} v_i(i) = v_2(2)$. If $v_3(3) \leq V_1(1,(2,0))$, then $\tau_{2,1}(R_{-2}) = V_1(1,(2,0))$. Thus, by $v_2(2) > \tau_{2,1}(R_{-2})$, we have $v_2(2) > V_1(1,(2,0))$. Instead, if $v_3(3) > V_1(1,(2,0))$, then by $v_2(2) \geq v_3(3)$, $v_2(2) > V_1(1,(2,0))$. Thus, in either case, we have $v_2(2) > V_1(1,(2,0))$, which implies $(2, v_2(2) - v_2(2)) = (2,0)\ P_1\ (1, v_2(2)) = f_1(R)$. This contradicts Lemma 21. Then, suppose $v_3(3) \geq v_2(2)$. Then, $v_3(3) = \max_{i \in N\setminus\{1\}} v_i(i)$. By Theorem 1, $t_1(R) = t_1^f(R_{-1};1) = v_3(3)$. We claim $v_3(3) > V_1(1,(2,0))$. Indeed, if $v_3(3) \leq V_1(1,(2,0))$, then $\tau_{2,1}(R_{-2}) = V_1(1,(2,0))$. Thus, by $v_2(2) > \tau_{2,1}(R_{-2})$ and $V_1(1,(2,0)) \geq v_3(3)$, $v_2(2) > v_3(3)$, which contradicts the assumption that $v_3(3) \geq v_2(2)$. Thus, $v_3(3) > V_1(1,(2,0))$, and so $\tau_{2,1}(R_{-2}) = v_3(3) - V_1(2,(1, \min\{v_3(3), V_1(1,(3,0))\}))$. By $v_2(2) > \tau_{2,1}(R_{-2})$, we have $V_1(2,(1,\min\{v_3(3), V_1(1,(3,0))\})) > v_3(3) - v_2(2)$. Also, by $v_3(3) \geq \min\{v_3(3), V_1(1,(3,0))\}$, $V_1(2,(1,v_3(3))) \geq V_1(2,(1,\min\{v_3(3), V_1(1,(3,0))\}))$. Combining these, we get $V_1(2, f_1(R)) = V_1(2,(1,v_3(3))) > v_3(3) - v_2(2)$, which implies $(2, v_3(3) - v_2(2))\ P_1\ f_1(R)$. This contradicts Lemma 21.

Next, suppose that $x(R) \in N\setminus\{1,2\}$ and $x(R) \succeq_1 2$. Let $i = x(R)$. If $v_2(2) > v_3(3)$, then by the same discussion as in the case where $x(R) \in N\setminus\{1,2\}$ and $x(R) \succeq_1 2$ in Case 3 of Lemma 22, we can derive a contradiction to Lemma 21. Then, suppose $v_3(3) \geq v_2(2)$. Then, $v_3(3) = \max_{j \in N\setminus\{1\}} v_j(j)$, and so by Theorem 1, $t_1(R) = t_1^f(R_{-1};i) = v_3(3) - v_i(i)$. We claim $v_3(3) - v_i(i) > V_1(i,(2,0))$. By contradiction, suppose $v_3(3) - v_i(i) \leq V_1(i,(2,0))$. Then, $\tau_{2,i}(R_{-2}) = v_i(i) + V_1(i,(2,0))$. By $v_2(2) > \tau_{2,i}(R_{-2})$, $v_2(2) - v_i(i) > V_1(i,(2,0))$. Thus, by $V_1(i,(2,0)) \geq v_3(3) - v_i(i)$, $v_2(2) - v_i(i) > v_3(3) - v_i(i)$, which implies $v_2(2) > v_3(3)$. This contradicts $v_3(3) \geq v_2(2)$. Thus, $v_3(3) - v_i(i) > V_1(i,(2,0))$. Then, $\tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i))) = v_3(3) - V_1(2, f_1(R))$. By $v_2(2) > \tau_{2,i}(R_{-2})$, $V_1(2, f_1(R)) > v_3(3) - v_2(2)$, which implies $(2, v_3(3) - v_2(2))\ P_1\ f_1(R)$. This contradicts Lemma 21.

Finally, suppose that $x(R) \in N\setminus\{1,2\}$ and $2 \succ_1 x(R)$. Let $i = x(R)$. By $2 \succ_1 i$, $\tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i)))$. If $v_2(2) > v_3(3)$, then by the same discussion as in the case where $x(R) \in N\setminus\{1,2\}$ and $2 \succ_1 x(R)$ in Case 3 of Lemma 22, we can derive a contradiction. Thus, suppose $v_3(3) \geq v_2(2)$. Then, $v_3(3) = \max_{j \in N\setminus\{1\}} v_j(j)$. Thus, by Theorem 1, $t_1(R) = t_1^f(R_{-1};i) = v_3(3) - v_i(i)$. By $v_2(2) > \tau_{2,i}(R_{-2})$, we have $V_1(2,(i, v_3(3) - v_i(i))) > v_3(3) - v_2(2)$. Thus, $(2, v_3(3) - v_2(2))\ P_1\ (i, v_3(3) - v_i(i)) = f_1(R)$, which contradicts Lemma 21. □

Now, we complete the proof of the "only if" part of Theorem 5. Suppose $\mathcal{R}_{-1} \subseteq \mathcal{R}_{-1}^{+I} \cap \mathcal{R}_{-1}^{Q}$. Let $R \in \mathcal{R}_N$. By Lemma 21, an outcome bundle $f_1(R)$ of agent 1 for R under f is equivalent to that under a generalized pivotal rule respecting agent 1. Let $i \in N\setminus\{1\}$. If $x(R) = i$, then by Lemmas 22 and 23, $t_i(R) = t_i^f(R_{-i};i)$ is equivalent to the payment of agent i

for R under a generalized pivotal rule respecting agent 1. If $x(R) \neq i$, then by $R_i \in \mathcal{R}_i^I$, Lemma 6 implies $t_i(R) = 0$. Thus, f is a generalized pivotal rule respecting agent 1. ■

Part II

Proofs of Impossibility Theorems

In the second part, we prove the impossibility theorems (Theorems 2, 3, and 6).

D Proof of Theorem 2

In this section, we prove Theorem 2. Without loss of generality, let $i = 1$. Suppose by contradiction that there is rule $f = (x, t)$ on $\mathcal{R}_N$ satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

By $R_1 \notin \mathcal{R}^I$, there is a pair $i, j \in N \backslash \{1\}$ such that $(i, 0)\ P_1\ (j, 0)$. Without loss of generality, let $i = 2$ and $j = 3$. Let $R_3 \in \mathcal{R}_3^0 \cap \mathcal{R}_3^Q$ be such that $v_3(3) = V_1(1, (3, 0))$. By desirability of own consumption of R_1, for each $i \in N \backslash \{1\}$, $v_3(3) = V_1(1, (3, 0)) > V_1(i, (3, 0))$. Also, for each $i \in N \backslash \{1\}$, by $2 \succ_1 3$, $V_1(i, (3, 0)) > V_1(i, (2, 0))$, and so $v_3(3) > V_1(i, (2, 0))$. If $n \geq 4$, then for each $i \in N \backslash \{1, 2, 3\}$, let $R_i \in \mathcal{R}_i^0 \cap \mathcal{R}_i^Q$ be a preference such that $v_i(i) < \min\{v_3(3), v_3(3) - V_1(i, (2, 0)), v_3(3) - V_1(i, (3, 0))\}$. By Lemma 4, $2 \in X_2^f(R_{-2})$.

By $2 \succ_1 3$ and $v_3(3) = V_1(1, (3, 0))$, $\tau_{2,1}(R_{-2}) = v_3(3) - V_1(2, (3, 0))$. For each $i \in N \backslash \{1, 2, 3\}$ with $i \succeq_1 2$, by $v_i(i) < v_3(3) - V_1(i, (2, 0))$, $\tau_{2,i}(R_{-2}) = v_3(3) - V_1(2, (i, v_3(3) - v_i(i)))$. Thus, for each $i \in N \backslash \{1, 2, 3\}$, $\tau_{2,i}(R_{-2}) = v_3(3) - V_1(2, (i, v_3(3) - v_i(i)))$. For each $i \in N \backslash \{1, 2, 3\}$, by $v_i(i) < v_3(3) - V_1(i, (3, 0))$, $(3, 0)\ P_1\ (i, v_3(3) - v_i(i))$, which implies $V_1(2, (i, v_3(3) - v_i(i))) > V_1(2, (3, 0))$. Thus, for each $i \in N \backslash \{1, 2, 3\}$, $\tau_{2,1}(R_{-2}) = v_3(3) - V_1(2, (3, 0)) > v_3(3) - V_1(2, (i, v_3(3) - v_i(i))) = \tau_{2,i}(R_{-2})$. Thus, by $2 \succ_1 3$, Lemma 23 implies that $t_2^f(R_{-2}; 2) = \tau_{2,1}(R_{-2}) = v_3(3) - V_1(2, (3, 0)) = v_3(3) - V_1(2, (1, v_3(3)))$.

By $R_1 \in \mathcal{R}_1^{PIE} \cup \mathcal{R}_1^{NIE}$, either $R_1 \in \mathcal{R}_1^{PIE}$ or $R_1 \in \mathcal{R}_1^{NIE}$. Thus, there are two cases.

Case 1. $R_1 \in \mathcal{R}_1^{PIE}$.

By $(2, 0)\ P_1\ (3, 0)$, $V_1(1, (2, 0)) < V_1(1, (3, 0)) = v_3(3)$. Thus, by $R_1 \in \mathcal{R}_1^{PIE}$, we have

$V_1(1,(2,0)) > v_3(3) - V_1(2,(1,v_3(3))) = t_2^f(R_{-2};2)$.[29] Thus, we can choose $\varepsilon \in \mathbb{R}$ such that

$$0 < \varepsilon < V_1(1,(2,0)) - t_2^f(R_{-2};2). \tag{1}$$

Let $R_2 \in \mathcal{R}_2^0$ be a preference such that $V_2(2,(\overline{x}_2(R_2),2)) > \max\{v_3(3), t_2^f(R_{-2};0)\}$, and $V_2(1, z_2^f(R_{-2};2)) = -\varepsilon$. By $V_2(2,(\overline{x}_2(R_2),0)) > t_2^f(R_{-2};2)$, Lemma 8 implies $x(R) = 2$. Thus, we have $t_2(R) = t_2^f(R_{-2};2) = v_3(3) - V_1(2,(1,v_3(3)))$. By $V_2(2,(\overline{x}_2(R_2),0)) > v_3(3)$ and $v_3(3) > \max_{i\in N\setminus\{1,2,3\}} v_i(i)$, we have $V_2(2,(\overline{x}_2(R_2),0)) > \max_{i\in N\setminus\{1,2\}} v_i(i)$. Then, by $R_{-1} \in \mathcal{R}_{-1}^0$, Theorem 1 implies $t_1(R) = t_1^f(R_{-1};2) = \max_{x\in X}\sum_{i\in N\setminus\{1\}} V_i(x,(\overline{x}_i(R_i),0)) - \sum_{i\in N\{1\}} V_i(2,(\overline{x}_i(R_i),0)) = V_2(2,(\overline{x}_2(R_2),0)) - V_2(2,(\overline{x}_2(R_2),0)) = 0$. Then,

$$\Big(t_1(R) - V_1(1,f_1(R))\Big) + \Big(t_2(R) - V_2(1,f_2(R))\Big) = -V_1(1,(2,0)) + t_2^f(R_{-2};2) + \varepsilon < 0,$$

where the equality follows from $f_1(R) = (2,0)$ and $V_2(1,f_2(R)) = -\varepsilon$, and the inequality from (1). This contradicts Lemma 2 (iii).

Case 2. $R_1 \in \mathcal{R}_1^{NIE}$.

By $(2,0)\ P_1\ (3,0)$, $V_1(2,(3,0)) > 0$. Thus, by $R_1 \in \mathcal{R}_1^{NIE}$, we have $-V_1(3,(2,0)) = 0 - V_1(3,(2,0)) < V_1(2,(3,0))$.[30] Thus, we can choose $\delta \in \mathbb{R}$ such that

$$0 < \delta < V_1(2,(3,0)) - \Big(-V_1(3,(2,0))\Big) = V_1(2,(3,0)) + V_1(3,(2,0)). \tag{2}$$

Let $R_2 \in \mathcal{R}_2^0$ be a preference such that $V_2(2,(\overline{x}_2(R_2),2)) > \max\{v_3(3), t_2^f(R_{-2};2)\}$, and $V_2(3, z_2^f(R_{-2};2)) = -\delta$. By $V_2(2,(\overline{x}_2(R_2),0)) > t_2^f(R_{-2};2)$, Lemma 8 implies $x(R) = 2$. Thus, $f_2(R) = z_2^f(R_{-2};2)$. By $V_2(2,(\overline{x}_2(R_2),0)) > v_3(3)$ and $v_3(3) > \max_{i\in N\setminus\{1,2,3\}} v_i(i)$, we have that $V_2(2,(\overline{x}_2(R_2),0)) > \max_{i\in N\setminus\{1,2\}} v_i(i)$. By $R_{-1} \in \mathcal{R}_{-1}^0$, Theorem 1 implies that $t_1(R) = t_1^f(R_{-1};2) = V_2(2,(\overline{x}_2(R_2),0)) - V_2(2,(\overline{x}_2(R_2),0)) = 0$. Then,

$$\Big(t_1(R) - V_1(3,f_1(R))\Big) + \Big(t_2(R) - V_2(3,f_2(R))\Big) = -V_1(3,(2,0)) + \Big(v_3(3) - V_1(2,(3,0))\Big) + \delta$$
$$< v_3(3) = V_3(3,f_3(R)) - t_3(R),$$

[29] Note that $V_1(2,(1,V_1(1,(2,0)))) = 0$.
[30] Note that $V_1(3,(2,V_1(2,(3,0)))) = 0$.

where the first equality follows from $f_1(R) = (2,0)$, $t_2(R) = v_3(3) - V_1(2,(3,0))$, and $V_2(3, f_2(R)) = -\delta$, and the inequality from (2). This contradicts Lemma 2 (ii). ■

E Proof of Theorem 3

In this section, we prove Theorem 3. Without loss of generality, let $i = 1$ and $j = 2$. By contradiction, suppose that there is a rule g on $\mathcal{R}_N$ satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Let $f = (x,t)$ denote the restriction of g to $((\mathcal{R}_1^0 \cap \mathcal{R}_1^Q) \cup \{R_1\}) \times (\mathcal{R}_2^{-I} \cap \mathcal{R}_2^Q) \times (\mathcal{R}_{-1,2}^0 \cap \mathcal{R}_{-1,2}^Q)$. Then, f satisfies the four properties. Note that the intersection of the domain of f and the quasi-linear domain $\mathcal{R}_N$ is convex.

By desirability of own consumption of R_1, for each $i \in N\backslash\{1\}$, $V_1(1,(2,0)) > V_1(i,(2,0))$. Let $w \in \mathbb{R}_{++}$ be such that for each $i \in N\backslash\{1\}$, $w < \min\{V_1(1,(2,0)) - V_1(i,(2,0)), V_1(1,(2,0))\}$. If $n \geq 3$, then for each $i \in N\backslash\{1,2\}$, let $R_i \in \mathcal{R}_i^0 \cap \mathcal{R}_i^Q$ be such that $v_i(i) = w$. By Lemma 4, $2 \in X_2^f(R_{-2})$. Let $i \in N_{-1,2}(R_{-2})$. By $v_i(i) < V_1(1,(2,0))$, $\tau_{2,1}(R_{-2}) = V_1(1,(2,0))$. For each $j \in N\backslash\{1,2\}$ with $j \succeq_1 2$, by $v_j(j) < V_1(1,(2,0)) - V_1(j,(2,0))$, $\tau_{2,j}(R_{-2}) = v_j(j) + V_1(j,(2,0)) < V_1(1,(2,0)) = \tau_{2,1}(R_{-2})$. For each $j \in N\backslash\{1,2\}$ with $2 \succ_1 j$, $\tau_{2,j}(R_{-j}) = v_j(j) - V_1(2,(j,0)) < v_j(j) < V_1(1,(2,0)) = \tau_{2,1}(R_{-2})$, where the first inequality follows from $2 \succ_1 j$. Thus, by Lemmas 22 and 23, $t_2^f(R_{-2}; 2) = \tau_{2,1}(R_{-2}) = V_1(1,(2,0))$.

The following lemma states that given R_{-2} as above, the rule f selects the object allocation 0 for some preference of agent 2.

Lemma 24. *We have* $0 \in X_2^f(R_{-2})$.

Proof. By $R_1 \in \mathcal{R}^{--}$, $V_1(0,(2,0)) > 0$. Thus, we can choose $R_2 \in \mathcal{R}_2^{-I} \cap \mathcal{R}_2^Q$ such that $v_2(2) > V_1(1,(2,0))$, $v_2(2) - v_2(0) < V_1(0,(2,0))$, and for each $x \in X\backslash\{0,2\}$, $v_2(x) = 0$. By $v_2(2) > V_1(1,(2,0)) = t_2^f(R_{-2};2)$ and *no subsidy for losers*, for each $x \in X_2^f(R_{-2})\backslash\{0,2\}$, $v_2(2) - t_2^f(R_{-2};2) > 0 \geq v_2(x) - t_2^f(R_{-2};x)$. Thus, by Lemma 3, $x(R) \in \{0,2\}$. We show $x(R) = 0$. By contradiction, suppose $x(R) = 2$.

By $v_2(2) > V_1(1,(2,0))$ and $V_1(1,(2,0)) > \max_{i \in N\backslash\{1,2\}} v_i(i)$, $v_2(2) > \max_{i \in N\backslash\{1,2\}} v_i(i)$. Thus, by Lemma 9, $t_1(R) = t_1^f(R_{-1};2) = \max_{x \in X} \sum_{i \in N\backslash\{1\}} v_i(x) - v_2(2) = v_2(2) - v_2(2) = 0$.

Let $x = 0$. Then,

$$\sum_{i \in N} \Big(V_i(x, f_i(R)) - V_i(x(R), f_i(R)) \Big) = \Big(V_1(0, f_1(R)) - t_1(R) \Big) - (v_2(2) - v_2(0))$$
$$= V_1(0, (2,0)) - (v_2(2) - v_2(0)) > 0,$$

where the first equality follows from $R_{-1,2} \in \mathcal{R}^0_{-1,2}$, the second one from $f_1(R) = (2,0)$, and the inequality from $v_2(2) - v_2(0) < V_1(0,(2,0))$. However, by Remark 1, this contradicts *efficiency*. □

By Lemma 24, $t_2^f(R_{-2}; 0)$ is well-defined. Given R_{-2}, the following lemma identifies the payment of agent 2 when f selects the object allocation 0.

Lemma 25. *We have* $t_2^f(R_{-2}; 0) = V_1(1,(2,0)) - V_1(0,(2,0))$.

Proof. By contradiction, suppose $t_2^f(R_{-2}; 0) \neq V_1(1,(2,0)) - V_1(0,(2,0))$. There are two cases.

Case 1. $t_2^f(R_{-2}; 0) < V_1(1,(2,0)) - V_1(0,(2,0))$.

By $R_1 \in \mathcal{R}_1^{--}$, $V_1(0,(2,0)) > 0$. Thus, we can choose $R_2 \in \mathcal{R}_2^{-I} \cap \mathcal{R}_2^Q$ such that $v_2(2) = V_1(1,(2,0))$,

$$t_2^f(R_{-2}; 0) < v_2(0) < V_1(1,(2,0)) - V_1(0,(2,0)),$$

and for each $x \in X \backslash \{0,2\}$, $v_2(x) = 0$. By $v_2(0) > t_2^f(R_{-2}; 0)$ and $t_2^f(R_{-2}; 2) = V_1(1,(2,0)) = v_2(2)$, $v_2(0) - t_2^f(R_{-2}; 0) > 0 = v_2(2) - t_2^f(R_{-2}; 2)$. For each $x \in X_2^f(R_{-2}) \backslash \{0,2\}$, by *no subsidy for losers*, $v_2(2) - t_2^f(R_{-2}; 2) = 0 \geq v_2(x) - t_2^f(R_{-2}; x)$. Thus, by Lemma 3, $x(R) = 0$.

Note that $v_2(2) = V_1(1,(2,0)) > \max_{i \in N \backslash \{1,2\}} v_i(i)$. Thus, by Lemma 9, $t_1(R) = t_1^f(R_{-1}; 0) = \max_{x \in X} \sum_{i \in N} v_i(x) - v_2(0) = v_2(2) - v_2(0) = V_1(1,(2,0)) - v_2(0)$. By $v_2(0) < V_1(1,(2,0)) - V_1(0,(2,0))$, $t_1(R) = V_1(1,(2,0)) - v_2(0) > V_1(0,(2,0))$, which implies $(2,0)\ P_1\ f_1(R)$. By Lemma 10, there is $R_1' \in \mathcal{R}_1$ such that $f_1(R_1', R_{-1}) = (2,0)$. Thus, $f_1(R_1', R_{-1})\ P_1\ f_1(R)$, which contradicts *strategy-proofness*.

Case 2. $t_2^f(R_{-2}; 0) > V_1(1,(2,0)) - V_1(0,(2,0))$.

By desirability of own consumption of R_1, $V_1(1,(2,0)) > 0$. Thus, by $t_2^f(R_{-2};0) > V_1(1,(2,0)) - V_1(0,(2,0))$, we can choose $\delta \in \mathbb{R}_{++}$ such that $\delta < V_1(1,(2,0))$, and

$$2\delta < t_2^f(R_{-2};0) - \Big(V_1(1,(2,0)) - V_1(0,(2,0))\Big). \tag{1}$$

Let $R_2 \in \mathcal{R}_2^{-I} \cap \mathcal{R}_2^Q$ be such that $v_2(2) = V_1(1,(2,0)) + \delta$, $v_2(0) = V_1(1,(2,0)) - V_1(0,(2,0)) + 2\delta$, and for each $x \in X\backslash\{0,2\}$, $v_2(x) = 0$. Note that by $\delta < V_1(1,(2,0))$, $R_2 \in \mathcal{R}_2^-$. By (1), $v_2(2) - t_2^f(R_{-2};2) = \delta > 0 > v_2(0) - t_2^f(R_{-2};0)$, For each $x \in X_2^f(R_{-2})\backslash\{0,2\}$, by *no subsidy for losers*, $v_2(2) - t_2^f(R_{-2};2) = \delta > 0 \geq v_2(x) - t_2^f(R_{-2};x)$. By Lemma 3, $x(R) = 2$.

By $V_1(1,(2,0)) > \max_{i\in N\backslash\{1,2\}} v_i(i)$, $v_2(2) = V_1(1,(2,0)) + \delta > \max_{i\in N\backslash\{1,2\}} v_i(i)$. Thus, by Lemma 9, $t_1(R) = t_1^f(R_{-1};2) = v_2(2) - v_2(2) = 0$. Let $x = 0$. Then,

$$\sum_{i\in N}\Big(V_i(x, f_i(R)) - V_i(x(R), f_i(R))\Big) = \Big(V_1(0, f_1(R)) - t_1(R)\Big) - (v_2(2) - v_2(0))$$
$$= V_1(0,(2,0)) - \Big(V_1(0,(2,0)) - \delta\Big) = \delta > 0,$$

where the first equality follows from $R_{-1,2} \in \mathcal{R}^0_{-1,2}$, and the second one from $f_1(R) = (2,0)$. By Remark 1, this contradicts *efficiency*. $\square$

By $R_1 \in \mathcal{R}^{--}$, $(0,0)\ P_1\ (2,0)$, which implies $V_1(1,(0,0)) < V_1(1,(2,0))$. Recall that for each $i \in N\backslash\{1,2\}$, $v_i(i) = w < \min\{V_1(1,(2,0)) - V_1(i,(2,0)), V_1(1,(2,0))\}$. Thus, we can choose $\overline{v} \in \mathbb{R}_{++}$ such that

$$\max\Big\{V_1(1,(0,0)), \max_{i\in N\backslash\{1,2\}} v_i(i), \max_{i\in N\backslash\{1,2\}}\Big(v_i(i) + V_1(i,(2,0))\Big)\Big\} < \overline{v} < V_1(1,(2,0)). \tag{2}$$

By $R_1 \in \mathcal{R}_1^{PIE} \cup \mathcal{R}_1^{NIE}$, either $R_1 \in \mathcal{R}_1^{PIE}$ or $R_1 \in \mathcal{R}_1^{NIE}$. Thus, there are two cases.

Case 1. $R_1 \in \mathcal{R}_1^{PIE}$.

By (2), $\overline{v} < V_1(1,(2,0))$. Thus, by $R_1 \in \mathcal{R}_1^{PIE}$, $\overline{v} - V_1(0,(1,\overline{v})) > V_1(1,(2,0)) - V_1(0,(2,0))$.[31] Thus, we can choose $R_2 \in \mathcal{R}_2^{-I} \cap \mathcal{R}_2^Q$ such that $v_2(2) = \overline{v}$,

$$V_1(1,(2,0)) - V_1(0,(2,0)) < v_2(0) < \overline{v} - V_1(0,(1,\overline{v})), \tag{3}$$

[31] Note that $V_1(0,(1,V_1(1,(2,0)))) = V_1(0,(2,0))$.

and for each $x \in X \backslash \{0, 2\}$, $v_2(x) = 0$.[32] For each $x \in X_2^f(R_{-2}) \backslash \{0, 2\}$, $v_2(0) - t_2^f(R_{-2}; 0) = v_2(0) - (V_1(1, (2, 0)) - V_1(0, (2, 0))) > 0 \geq v_2(x) - t_2^f(R_{-2}; x)$, where the equality follows from Lemma 25, the first inequality from (3), and the last one from *no subsidy for losers*. Also, $v_2(0) - t_2^f(R_{-2}; 0) > 0 > \overline{v} - V_1(1, (2, 0)) = v_2(2) - t_2^f(R_{-2}; 2)$, where the second inequality follows from (2). Thus, by Lemma 3, $x(R) = 0$.

By (2), $v_2(2) = \overline{v} > \max_{i \in N \backslash \{1,2\}} v_i(i)$. Thus, by Lemma 9, $t_1(R) = t_1^f(R_{-1}; 0) = v_2(2) - v_2(0)$. By (3), $t_1(R) = v_2(2) - v_2(0) = \overline{v} - v_2(0) > V_1(0, (1, \overline{v}))$, which implies $(1, \overline{v})\ P_1\ f_1(R)$. By Lemma 4, there is $R_1' \in \mathcal{R}_1^0 \cap \mathcal{R}_1^Q$ such that $x(R_1', R_{-1}) = 1$. By Lemma 9, $t_1(R_1', R_{-1}) = t_1^f(R_{-1}; 1) = v_2(2)$. Thus, $f_1(R_1', R_{-1}) = (1, v_2(2)) = (1, \overline{v})$. By $(1, \overline{v})\ P_1\ f_1(R)$, $f_1(R_1', R_{-1})\ P_1\ f_1(R)$, which contradicts *strategy-proofness*.

Case 2. $R_1 \in \mathcal{R}_1^{NIE}$.

By (2), $\overline{v} < V_1(1, (2, 0))$. Thus, by $R_1 \in \mathcal{R}_1^{NIE}$, $\overline{v} - V_1(0, (1, \overline{v})) < V_1(1, (2, 0)) - V_1(0, (2, 0))$.[33] Let $R_2 \in \mathcal{R}_2^{-I} \cap \mathcal{R}_2^Q$ be such that $v_2(2) = \overline{v}$,

$$\overline{v} - V_1(0, (1, \overline{v})) < v_2(0) < \min\Big\{V_1(1, (2, 0)) - V_1(0, (2, 0)), \overline{v}\Big\}, \tag{4}$$

and for each $x \in X \backslash \{0, 2\}$, $v_2(x) = 0$. Then, we have $v_2(0) - t_2^f(R_{-2}; 0) = v_2(0) - (V_1(1, (2, 0)) - V_1(0, (2, 0))) < 0$, where the equality follows from Lemma 25, and the inequality from (4). We also have $v_2(2) - t_2^f(R_{-2}; 2) = \overline{v} - V_1(1, (2, 0)) < 0$, where the inequality follows from (2). Thus, by *weak individual rationality*, $x(R) \in X \backslash \{0, 2\}$.

By $v_2(2) = \overline{v} > \max_{i \in N \backslash \{1,2\}} v_i(i)$, Lemma 9 implies that

$$t_1(R) = t_1^f(R_{-1}; x(R)) = \max_{x \in X} \sum_{i \in N \backslash \{1\}} v_i(x) - \sum_{i \in N \backslash \{1\}} v_i(x(R)) = v_2(2) - \sum_{i \in N \backslash \{1\}} v_i(x(R)). \tag{5}$$

First, suppose $x(R) = 1$. By (5), $t_1(R) = v_2(2) = \overline{v}$. Let $x = 0$. Then,

$$\begin{aligned}\sum_{i \in N} \Big(V_i(x, f_i(R)) - V_i(x(R), f_i(R))\Big) &= v_2(0) - \Big(t_1(R) - V_1(0, f_1(R))\Big) \\ &= v_2(0) - \Big(\overline{v} - V_1(0, (1, \overline{v}))\Big) > 0,\end{aligned}$$

[32] By $\overline{v} > V_1(1, (0, 0))$, $(0, 0)\ P_1\ (1, \overline{v})$. This implies $V_1(0, (1, \overline{v})) > 0$. Thus, by (3), $v_2(0) < \overline{v}$, and so R_2 satisfies desirability of own consumption.

[33] Note that $V_1(0, (1, V_1(1, (2, 0)))) = V_1(0, (2, 0))$.

where the first equality follows from $R_{-1,2} \in \mathcal{R}^0_{-1,2}$, the second one from $f_1(R) = (1, \overline{v})$, and the inequality from (4). By Remark 1, this contradicts *efficiency*.

Next, suppose $x(R) \in N \backslash \{1,2\}$. Let $i = x(R)$. By (5), $t_1(R) = v_2(2) - v_i(i) = \overline{v} - v_i(i)$. By (2), $t_1(R) > V_1(i,(2,0))$, which implies $(2,0)\ P_1\ f_1(R)$. By Lemma 10, there is $R_1' \in \mathcal{R}_1$ such that $f_1(R_1', R_{-1}) = (2,0)$. Thus, we have $f_1(R_1', R_{-1})\ P_1\ f_1(R)$, which contradicts *strategy-proofness*. ■

F Proof of Theorem 6

In this section, we prove Theorem 6. By $R_i \notin \mathcal{R}^I_i$, there is $k \in N \backslash \{i,j\}$ such that either $(j,0)\ P_i\ (k,0)$ or $(k,0)\ P_i\ (j,0)$, i.e., either $j \succ_i k$ or $k \succ_i j$. Without loss of generality, let $i = 1$, $j = 2$, and $k = 3$. By contradiction, suppose that there is a rule g on $\mathcal{R}_N$ satisfying *efficiency*, *weak individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Let $f = (x,t)$ denote the restriction of g to $((\mathcal{R}^0_1 \cap \mathcal{R}^Q_1) \cup \{R_1\}) \times (\mathcal{R}^+_2 \cap \mathcal{R}^Q_2) \times (\mathcal{R}^0_{-1,2} \cap \mathcal{R}^Q_{-1,2})$. Then, f satisfies the four properties. Note that the intersection of the domain of f and the quasi-linear domain $\mathcal{R}_N$ is convex.

We begin with the following lemma which states that given R_{-2}, the rule f selects the object allocation 1 for some preference of agent 2.

Lemma 26. *For each $R_{-1,2} \in \mathcal{R}^0_{-1,2} \cap \mathcal{R}^Q_{-1,2}$, we have $1 \in X^f_2(R_{-2})$.*

Proof. Let $R_{-1,2} \in \mathcal{R}^0_{-1,2} \cap \mathcal{R}^Q_{-1,2}$. Let $R_2 \in \mathcal{R}^+_2 \cap \mathcal{R}^Q_2$ be such that for each $i \in N \backslash \{1,2\}$, $v_2(2) > v_i(i) + V_1(i, (\underline{x}_1(R_1), 0))$ and $v_2(2) - v_2(1) < V_1(1,(2,0))$, and for each $x \in X \backslash \{1,2\}$, $v_2(x) = 0$. Note that by desirability of own consumption of R_1, $V_1(1,(2,0)) > 0$, and so we can choose such R_2. Note also that for each $i \in N \backslash \{1,2\}$, by $(i,0)\ R_1\ (\underline{x}_1(R_1), 0)$, $V_1(i, \underline{x}_1(R_1), 0) \geq 0$, and so $v_2(2) > v_i(i)$. To show $1 \in X^f_2(R_{-2})$, it suffices to show that $x(R) = 1$. By contradiction, suppose not. Note that by Lemma 1, $x(R) \neq 0$.

First, suppose $x(R) = 2$. By Lemma 9 and $v_2(2) > \max_{i \in N \backslash \{1,2\}} v_i(i)$, we have $t_1(R) = \max_{x \in X} \sum_{i \in N \backslash \{1\}} v_i(x) - v_2(2) = v_2(2) - v_2(2) = 0$. Then,

$$\Big(t_1(R) - V_1(1, f_1(R))\Big) + \Big(t_2(R) - V_2(1, f_2(R))\Big) = -V_1(1,(2,0)) + (v_2(2) - v_2(1)) < 0,$$

where the equality follows from $f_1(R) = (2,0)$, and the inequality from $v_2(2) - v_2(1) < V_1(1,(2,0))$. This contradicts Lemma 2 (iii).

Next, suppose $x(R) = i$ for some $i \in N\backslash\{1,2\}$. By Lemma 9 and $v_2(2) > \max_{i\in N\backslash\{1,2\}} v_i(i)$, $t_1(R) = v_2(2) - v_i(i)$. By $v_2(2) > v_i(i) + V_1(i, (\underline{x}_1(R_1), 0))$, $t_1(R) = v_2(2) - v_i(i) > V_1(i, (\underline{x}_1(R_1), 0))$, which implies $(\underline{x}_1(R_1), 0)\ P_1\ f_1(R)$. This contradicts *weak individual rationality*. □

Recall that either $2 \succ_1 3$ or $3 \succ_1 2$. The proof consists of two parts. In the first part, we consider the case where $2 \succ_1 3$, and in the second one, we do the other case.

PART 1. Suppose $2 \succ_1 3$. Let $R_3 \in \mathcal{R}_3^0 \cap \mathcal{R}_3^Q$ be a preference such that for each $i \in N\backslash\{3\}$, $v_3(3) > \max\{V_1(i, (2,0)), V_1(i, (3,0))\}$. If $n \geq 4$, then for each $i \in N\backslash\{1,2,3\}$, let $R_i \in \mathcal{R}_i^0 \cap \mathcal{R}_i^Q$ be a preference such that $v_i(i) < \min\{v_3(3), v_3(3) - V_1(i,(2,0)), v_3(3) - V_1(i,(3,0))\}$.

By $v_3(3) > \max_{i\in N\backslash\{1,2,3\}} v_i(3)$, $3 \in N_{-1,2}(R_{-2})$. By $2 \succ_1 3$ and $v_3(3) > V_1(1,(3,0))$, $\tau_{2,1}(R_{-2}) = v_3(3) - V_1(2,(3,0))$. For each $i \in N\backslash\{1,2,3\}$ with $i \succeq_1 2$, $\tau_{2,i}(R_{-2}) = v_3(3) - V_1(2, (i, v_3(3) - v_i(i))) < v_3(3) - V_1(2,(3,0))$, where both the equality and the inequality follow from $v_i(i) < v_3(3) - V_1(i,(3,0))$. For each $i \in N\backslash\{1,2,3\}$ with $2 \succ_1 i \succeq_1 3$, $\tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i))) < v_3(3) - V_1(2,(3,0))$, where the inequality follows from $v_i(i) < v_3(3) - V_1(i,(3,0))$. Thus, by $2 \succ_1 3$, Lemma 23 implies $t_2^f(R_{-2}; 2) = v_3(3) - V_1(2,(3,0))$.

By Lemma 26, $1 \in X_2^f(R_{-2})$. We show the following lemma which identifies the payment of agent 2 given R_{-2} under the rule f when it selects the object allocation 1.

Lemma 27. *We have* $t_2^f(R_{-2}; 1) = v_3(3) - V_1(1,(3,0))$.

Proof. Suppose by contradiction that $t_2^f(R_{-2}; 1) \neq v_3(3) - V_1(1,(3,0))$. There are two cases.

CASE 1. $t_2^f(R_{-2}; 1) < v_3(3) - V_1(1,(3,0))$.

Let $R_2 \in \mathcal{R}_2^+ \cap \mathcal{R}_2^Q$ be a preference such that $v_2(2) < v_3(3) - V_1(2,(3,0)) = t_2^f(R_{-2}; 2)$, $t_2^f(R_{-2}; 1) < v_2(1) < v_3(3) - V_1(1,(3,0))$, and for each $x \in X\backslash\{1,2\}$, $v_2(x) = 0$. By $v_2(1) > t_2^f(R_{-2}; 1)$ and $v_2(2) < t_2^f(R_{-2}; 2)$, $v_2(1) - t_2^f(R_{-2}; 1) > 0 > v_2(2) - t_2^f(R_{-2}; 2)$. For each $x \in X_2^f(R_{-2})\backslash\{1,2\}$, by *no subsidy for losers*, $0 \geq v_2(x) - t_2^f(R_{-2}; x)$. Thus, by Lemma 3, $x(R) = 1$. By $v_2(2) < v_3(3) - V_1(2,(3,0))$ and $2 \succ_1 3$, $v_2(2) < v_3(3)$. Thus, by $v_3(3) > \max_{i\in N\backslash\{1,2,3\}} v_i(i)$, Lemma 9 implies $t_1(R) = t_1^f(R_{-1}; 1) = \max_{x\in X} \sum_{i\in N\backslash\{1\}} v_i(x) - v_2(1) = v_3(3) - v_2(1)$. By $v_2(1) < v_3(3) - V_1(1,(3,0))$, $V_1(1,(3,0)) < t_1(R)$, which implies $(3,0)\ P_1\ f_1(R)$. By Lemma 10, there is $R_1' \in \mathcal{R}_1^0 \cap \mathcal{R}_1^Q$ such that $f_1(R_1', R_{-1}) = (3,0)$.

Thus, $f_1(R_1', R_{-1})\ P_1\ f_1(R)$, which contradicts *strategy-proofness*.

CASE 2. $t_2^f(R_{-2}; 1) > v_3(3) - V_1(1, (3, 0))$.

Let $R_2 \in \mathcal{R}_2^+ \cap \mathcal{R}_2^Q$ be a preference such that $v_2(2) < v_3(3) - V_1(2, (3, 0)) = t_2^f(R_{-2}; 2)$, $v_3(3) - V_1(1, (3, 0)) < v_2(1) < t_2^f(R_{-2}; 1)$, and for each $x \in X \backslash \{1, 2\}$, $v_2(x) = 0$. By $v_2(1) < t_2^f(R_{-2}; 1)$, $0 > v_2(1) - t_2^f(R_{-2}; 1)$. By $v_2(2) < t_2^f(R_{-2}; 2)$, $0 > v_2(2) - t_2^f(R_{-2}; 2)$. Thus, by *weak indiviudal rationality*, $x(R) \in X \backslash \{1, 2\}$. By Lemma 1, $x(R) \neq 0$. Thus, $x(R) \in N \backslash \{1, 2\}$. Let $i = x(R)$. By $v_2(2) < v_3(3) - V_1(2, (3, 0))$ and $2 \succ_1 3$, $v_2(2) < v_3(3)$. Thus, by $v_3(3) > \max_{i \in N \backslash \{1,2,3\}} v_i(i)$, Lemma 9 implies $t_1(R) = t_1^f(R_{-1}; i) = v_3(3) - v_i(i)$. By $v_i(i) < v_3(3) - V_1(i, (3, 0))$, $V_1(i, (3, 0)) < t_1(R)$, which implies $(3, 0)\ P_1\ f_1(R)$. By Lemma 4, there is $R_1' \in \mathcal{R}_1^0 \cap \mathcal{R}_1^Q$ such that $x(R_1', R_{-1}) = 1$. By Lemma 9, we have $t_1(R_1', R_{-1}) = t_1^f(R_{-1}; 1) = v_3(3) - v_2(1)$. By $v_2(1) > v_3(3) - V_1(1, (3, 0))$, we have $V_1(1, (3, 0)) > t_1(R_1', R_{-1})$, which implies that $f_1(R_1', R_{-1})\ P_1\ (3, 0)$. This, together with $(3, 0)\ P_1\ f_1(R)$, implies $f_1(R_1', R_{-1})\ P_1\ f_1(R)$. This contradicts *strategy-proofness*. □

By $R_1 \in \mathcal{R}_1^{PIE} \cup \mathcal{R}_1^{NIE}$, either $R_1 \in \mathcal{R}_1^{PIE}$ or $R_1 \in \mathcal{R}_1^{NIE}$. Thus, there are two cases.

CASE 1. $R_1 \in \mathcal{R}_1^{PIE}$.

By $2 \succ_1 3$, $V_1(2, (3, 0)) > 0$. Thus, $v_3(3) - V_1(2, (3, 0)) < v_3(3)$. Let $\overline{v} \in \mathbb{R}_{++}$ be such that $v_3(3) - V_1(2, (3, 0)) < \overline{v} < v_3(3)$. By $v_3(3) - V_1(2, (3, 0)) < \overline{v}$, $v_3(3) - \overline{v} < V_1(2, (3, 0))$. By $R_1 \in \mathcal{R}_1^{PIE}$, $V_1(1, (2, v_3(3) - \overline{v})) - (v_3(3) - \overline{v}) > V_1(1, (3, 0)) - V_1(2, (3, 0))$.[34] Thus, $\overline{v} - (V_1(1, (3, 0)) - V_1(2, (3, 0))) > v_3(3) - V_1(1, (2, v_3(3) - \overline{v}))$. Let $\underline{v} \in \mathbb{R}_{++}$ be such that

$$v_3(3) - V_1(1, (2, v_3(3) - \overline{v})) < \underline{v} < \overline{v} - \Big(V_1(1, (3, 0)) - V_1(2, (3, 0))\Big). \tag{1}$$

Let $R_2 \in \mathcal{R}_2^+ \cap \mathcal{R}_2^Q$ be such that $v_2(2) = \overline{v}$, $v_2(1) = \underline{v}$, and for each $x \in X \backslash \{1, 2\}$, $v_2(x) = 0$. We have

$$v_2(2) - v_2(1) = \overline{v} - \underline{v} > V_1(1, (3, 0)) - V_1(2, (3, 0)) = t_2^f(R_{-2}; 2) - t_2^f(R_{-2}; 1),$$

where the inequality follows from (1), and the last equality from Lemma 27 and $t_2^f(R_{-2}; 2) =$

[34] Note that $V_1(1, (2, V_1(2, (3, 0)))) = V_1(1, (3, 0))$.

$v_3(3) - V_1(2,(3,0))$. Thus, $v_2(2) - t_2^f(R_{-2};2) > v_2(1) - t_2^f(R_{-2};1)$. For each $x \in X_2^f(R_{-2}) \backslash \{1,2\}$,

$$v_2(2) - t_2^f(R_{-2};2) = \overline{v} - \Big(v_3(3) - V_1(2,(3,0))\Big) > 0 \geq v_2(x) - t_2^f(R_{-2};x),$$

where the first inequality follows from $\overline{v} > v_3(3) - V_1(2,(3,0))$, and the second one from *no subsidy for losers*. Thus, by Lemma 3, $x(R) = 2$. Then, by $v_3(3) > \overline{v} = v_2(2)$ and $v_3(3) > \max_{i \in N \backslash \{1,2,3\}} v_i(i)$, Lemma 9 implies $t_1(R) = t_1^f(R_{-1};2) = \max_{x \in X} \sum_{i \in N \backslash \{1\}} v_i(x) - v_2(2) = v_3(3) - v_2(2) = v_3(3) - \overline{v}$. Then,

$$\Big(t_1(R) - V_1(1, f_1(R))\Big) + \Big(t_2(R) - V_2(1, f_2(R))\Big) = \Big((v_3(3) - \overline{v}) - V_1(1,(2, v_3(3) - \overline{v}))\Big) + (\overline{v} - \underline{v})$$
$$= v_3(3) - V_1(1,(2, v_3(3) - \overline{v})) - \underline{v} < 0,$$

where the first equality follows from $f_1(R) = (2, v_3(3) - \overline{v})$, and the inequality from (1). This contradicts Lemma 2 (iii).

Case 2. $R_1 \in \mathcal{R}_1^{NIE}$.

By desirability of own consumption of R_1, $V_1(1,(3,0)) > 0$. Thus, $v_3(3) - V_1(1,(3,0)) < v_3(3)$. Let $\underline{v} \in \mathbb{R}_{++}$ be such that $v_3(3) - V_1(1,(3,0)) < \underline{v} < v_3(3)$. By $v_3(3) - V_1(1,(3,0)) < \underline{v}$, $v_3(3) - \underline{v} < V_1(1,(3,0))$. Thus, by $R_1 \in \mathcal{R}_1^{NIE}$, $(v_3(3) - \underline{v}) - V_1(2,(1, v_3(3) - \underline{v})) < V_1(1,(3,0)) - V_1(2,(3,0))$.[35] Thus, $v_3(3) - V_1(2,(1, v_3(3) - \underline{v})) < \underline{v} + V_1(1,(3,0)) - V_1(2,(3,0))$. Let $\overline{v} \in \mathbb{R}_{++}$ be such that $\underline{v} < \overline{v} < v_3(3)$, and

$$v_3(3) - V_1(2,(1, v_3(3) - \underline{v})) < \overline{v} < \underline{v} + V_1(1,(3,0)) - V_1(2,(3,0)). \qquad (2)$$

Let $R_2 \in \mathcal{R}_2^+ \cap \mathcal{R}_2^Q$ be such that $v_2(2) = \overline{v}$, $v_2(1) = \underline{v}$, and for each $x \in X \backslash \{1,2\}$, $v_2(x) = 0$. Then,

$$v_2(1) - v_2(2) = \underline{v} - \overline{v} > V_1(2,(3,0)) - V_1(1,(3,0)) = t_2^f(R_{-2};1) - t_2^f(R_{-2};2),$$

where the inequality follows from (2), and the last equality from Lemma 27 and $t_2^f(R_{-2};2) =$

[35] Note that $V_1(2,(1, V_1(1,(3,0)))) = V_1(2,(3,0))$.

$v_3(3)-V_1(2,(3,0))$. Thus, $v_2(1)-t_2^f(R_{-2};1) > v_2(2)-t_2^f(R_{-2};2)$. For each $x \in X_2^f(R_{-2})\backslash\{1,2\}$,

$$v_1(1) - t_2^f(R_{-2};1) = \underline{v} - \Big(v_3(3) - V_1(1,(3,0))\Big) > 0 \geq v_2(x) - t_2^f(R_{-2};x),$$

where the equality follows from Lemma 27, the first inequality from $\underline{v} > v_3(3)-V_1(1,(3,0))$, and the second one from *no subsidy for losers*. Thus, by Lemma 3, $x(R) = 1$. By $v_3(3) > \overline{v} = v_2(2)$ and $v_3(3) > \max_{i\in N\backslash\{1,2,3\}} v_i(i)$, Lemma 9 implies that $t_1(R) = v_3(3) - v_2(1) = v_3(3) - \underline{v}$. Then,

$$\Big(t_1(R) - V_1(2, f_1(R))\Big)+\Big(t_2(R) - V_2(2, f_2(R))\Big)= \Big((v_3(3) - \underline{v}) - V_1(2,(1, v_3(3) - \underline{v}))\Big)+(\underline{v} - \overline{v})$$
$$= v_3(3) - V_1(2,(1, v_3(3) - \underline{v})) - \overline{v} < 0,$$

where the first equality follows from $f_1(R) = (1, v_3(3) - \underline{v})$, and the inequality from (2). This contradicts Lemma 2 (i).

PART 2. Suppose $3 \succ_1 2$. Without loss of generality, assume that for each $i \in N\backslash\{1,2\}$, $3 \succeq_1 i$. Let $i \in N\backslash\{1,2\}$ be such that $i \succeq_1 2$. By $3 \succeq_1 i$, $V_1(3,(i,0)) \geq 0$. Thus, by $R_1 \in \mathcal{R}_1^{PIE} \cup \mathcal{R}_1^{NIE}$, $V_1(3,(2,0)) - V_1(i,(2,0)) \geq 0$. Thus, for each $i \in N\backslash\{1,2\}$ with $i \succeq_1 2$, $V_1(3,(2,0)) \geq V_1(i,(2,0))$.

If $n \geq 4$, then for each $i \in N\backslash\{1,2,3\}$, let $R_i \in \mathcal{R}_i^0 \cap \mathcal{R}_i^Q$ be a preference such that $v_i(i) < \min\{V_1(1,(2,0)) - V_1(3,(2,0)), V_1(1,(3,0))\}$. By desirability of own consumption of R_1, we can choose such R_1. Let $R_3 \in \mathcal{R}_3^0 \cap \mathcal{R}_3^Q$ be such that $v_3(3) > \max_{i\in N\backslash\{1,2,3\}} v_i(i)$. Then, $3 \in N_{-1,2}(R_{-2})$. Let $i \in N\backslash\{1,2\}$ be such that $i \succeq_1 2$. If $i = 3$, then by $3 \succ_1 2$, $V_1(3,(2,0)) > 0 = v_3(3) - v_3(3)$. Thus, we have $\tau_{2,3}(R_{-2}) = v_3(3) + V_1(3,(2,0)) > v_3(3)$. Suppose that $i \geq 4$. If $v_3(3) - v_i(i) \geq V_1(i,(2,0))$, then we have $\tau_{2,i}(R_{-2}) = v_3(3) - V_1(2,(i, v_3(3) - v_i(i))) \leq v_3(3) < \tau_{2,3}(R_{-2})$. If $v_3(3) - v_i(i) < V_1(i,(2,0))$, then $\tau_{2,i}(R_{-2}) = v_i(i) + V_1(i,(2,0)) < v_3(3) + V_1(3,(2,0)) = \tau_{2,3}(R_{-2})$, where the inequality follows from $v_3(3) > v_i(i)$ and $V_1(3,(2,0)) \geq V_1(i,(2,0))$. Thus, $\max_{i\in N\backslash\{1,2\} i\succeq_1 2} \tau_{2,i}(R_{-2}) = \tau_{2,3}(R_{-2}) = v_3(3) + V_1(3,(2,0))$. By Lemma 4, $2 \in X_2^f(R_{-2})$. By $3 \succ_1 2$, Lemma 22 implies

$$t_2^f(R_{-2};2) = \max\Big\{V_1(1,(2,0)), v_3(3) + V_1(3,(2,0))\Big\}. \tag{3}$$

By $R_1 \in \mathcal{R}_1^{PIE} \cup \mathcal{R}_1^{NIE}$, either $R_1 \in \mathcal{R}_1^{PIE}$ or $R_1 \in \mathcal{R}_1^{NIE}$. Thus, there are two cases.

Case 1. $R_1 \in \mathcal{R}_1^{PIE}$.

By $3 \succ_1 2$, $V_1(3,(2,0)) > 0$. Thus, by $R_1 \in \mathcal{R}_1^{PIE}$, $V_1(1,(2,0)) - V_1(3,(2,0)) < V_1(1,(3,0))$.[36] Thus, we can choose $R_3 \in \mathcal{R}_3^0 \cap \mathcal{R}_3^Q$ such that $V_1(1,(2,0)) - V_1(3,(2,0)) < v_3(3) < V_1(1,(3,0))$. By $v_3(3) > V_1(1,(2,0)) - V_1(3,(2,0))$, $v_3(3) + V_1(3,(2,0)) > V_1(1,(2,0))$. Thus, by (3), $t_2^f(R_{-2};2) = v_3(3) + V_1(3,(2,0))$.

By Lemma 26, $1 \in X_2^f(R_{-2})$. We show the following lemma which states that given R_{-2}, if the rule f selects the object allocation 1, then the payment of agent 2 is equal to 0.

Lemma 28. *We have* $t_2^f(R_{-2};1) = 0$.

Proof. By *no subsidy for losers*, $t_2^f(R_{-2};1) \geq 0$. By contradiction, suppose that $t_2^f(R_{-2};1) > 0$. Let $R_2 \in \mathcal{R}_2^+ \cap \mathcal{R}_2^Q$ be such that $v_2(2) < v_3(3)$, $0 < v_2(1) < t_2^f(R_{-2};1)$, and for each $x \in X \setminus \{1,2\}$, $v_2(x) = 0$. By $3 \succ_1 2$ and $t_2^f(R_{-2};2) = v_3(3) + V_1(3,(2,0))$, $t_2^f(R_{-2};2) > v_3(3)$. Thus, by $v_3(3) > v_2(2)$, we have $t_2^f(R_{-2};2) > v_2(2)$, or equivalently, $0 > v_2(2) - t_2^f(R_{-2};2)$. By $v_2(1) < t_2^f(R_{-2};1)$, $0 > v_2(1) - t_2^f(R_{-2};1)$. Thus, by *weak individual rationality*, $x(R) \in X \setminus \{1,2\}$. By Lemma 1, $x(R) \neq 0$. Thus, $x(R) \in N \setminus \{1,2\}$.

Suppose $x(R) = 3$. By $v_3(3) > v_2(2)$ and $v_3(3) > \max_{i \in N \setminus \{1,2,3\}} v_i(i)$, Lemma 9 implies $t_1(R) = \max_{x \in X} \sum_{i \in N \setminus \{1\}} v_i(x) - v_3(3) = v_3(3) - v_3(3) = 0$. Then,

$$\begin{aligned}\Big(t_1(R) - V_1(1, f_1(R))\Big) + \Big(t_3(R) - V_3(1, f_3(R))\Big) &= -V_1(1,(3,0)) + v_3(3) \\ < 0 < v_2(1) = V_2(1, f_2(R)) - t_2(R),\end{aligned}$$

where the first equality follows from $f_1(R) = (3,0)$, the first inequality from $v_3(3) < V_1(1,(3,0))$, and the second inequality from $v_2(1) > 0$. This contradicts Lemma 2 (ii).

Suppose $x(R) = i$ for some $i \in N \setminus \{1,2,3\}$. By $v_3(3) > v_2(2)$ and $v_3(3) > \max_{i \in N \setminus \{1,2,3\}} v_i(i)$, Lemma 9 implies $t_1(R) = v_3(3) - v_i(i) > 0$. By $3 \succeq_1 i$, $(3,0)\ R_1\ (i,0)$. Thus, by $t_1(R) > 0$, $(3,0)\ P_1\ f_1(R)$. By Lemma 10, there is $R_1' \in \mathcal{R}_1^0 \cap \mathcal{R}_1^Q$ such that $f_1(R_1', R_{-1}) = (3,0)$. Thus, $f_1(R_1', R_{-1})\ P_1\ f_1(R)$, which contradicts *strategy-proofness*. $\square$

Recall that $V_1(1,(2,0)) < v_3(3) + V_1(3,(2,0))$. Thus, we can choose $R_2 \in \mathcal{R}_2^+ \cap \mathcal{R}_2^Q$ such that $v_2(2) > v_3(3)$, $V_1(1,(2,0)) < v_2(2) - v_2(1) < v_3(3) + V_1(3,(2,0))$, $v_2(1) > 0$, and

[36] Note that $V_1(1,(3,V_1(3,(2,0)))) = V_1(1,(2,0))$.

for each $x \in X\backslash\{1,2\}$, $v_2(x) = 0$. By $t_2^f(R_{-2};2) = v_3(3) + V_1(3,(2,0))$ and Lemma 28, $t_2^f(R_{-2};2) - t_2^f(R_{-2};1) = v_3(3) + V_1(3,(2,0))$. Thus, by $v_2(2) - v_2(1) < v_3(3) + V_1(3,(2,0))$, $v_2(1) - t_2^f(R_{-2};1) > v_2(2) - t_2^f(R_{-2};2)$. For each $x \in X_2^f(R_{-2})\backslash\{1,2\}$, by $v_2(1) > 0 = t_2^f(R_{-2};1)$ and *no subsidy for losers*, $v_2(1) - t_2^f(R_{-2};1) > 0 \geq v_2(x) - t_2^f(R_{-2};x)$. Thus, by Lemma 3, $x(R) = 1$. By $v_2(2) > v_3(3)$ and $v_3(3) > \max_{i \in N\backslash\{1,2,3\}} v_i(i)$, Lemma 9 implies $t_2(R) = t_2^f(R_{-2};1) = v_2(2) - v_2(1)$. By $v_2(2) - v_2(1) > V_1(1,(2,0))$, $t_1(R) > V_1(1,(2,0))$, which implies $(2,0)\ P_1\ f_1(R)$. By Lemma 10, there is $R_1' \in \mathcal{R}_1^0 \cap \mathcal{R}_1^Q$ such that $f_1(R_1', R_{-1}) = (2,0)$. Thus, $f_1(R_1', R_{-1})\ P_1\ f_1(R)$, which contradicts *strategy-proofness.*

Case 2. $R_1 \in \mathcal{R}_1^{NIE}$.

By $3 \succ_1 2$, $V_1(3,(2,0)) > 0$. Thus, by $R_1 \in \mathcal{R}_1^{NIE}$, $V_1(1,(2,0)) - V_1(3,(2,0)) > V_1(1,(3,0))$.[37] Thus, we can choose $R_3 \in \mathcal{R}_3^0 \cap \mathcal{R}_3^Q$ such that $V_1(1,(3,0)) < v_3(3) < V_1(1,(2,0)) - V_1(3,(2,0))$. By $v_3(3) < V_1(1,(2,0)) - V_1(3,(2,0))$, $v_3(3) + V_1(3,(2,0)) < V_1(1,(2,0))$. Thus, by (3), $t_2^f(R_{-2};2) = V_1(1,(2,0))$.

The following lemma states that given R_{-2}, the rule f selects the object allocation 3 for some preference of agent 2.

Lemma 29. *We have* $3 \in X_2^f(R_{-2})$.

Proof. Let $R_2 \in \mathcal{R}_2^0 \cap \mathcal{R}_2^Q$ be such that $v_2(2) < v_3(3)$. To show that $3 \in X_2^f(R_{-2})$, it suffices to show $x(R) = 3$. By contradiction, suppose not. By Lemma 1, $x(R) \neq 0$.

We claim $(3,0)\ P_1\ f_1(R)$. Suppose $x(R) = 1$. By $v_3(3) > \max_{i \in N\backslash\{1,3\}} v_i(i)$, Lemma 9 implies that $t_1(R) = t_1^f(R_{-1};1) = \max_{i \in N\backslash\{1\}} v_i(x) = v_3(3)$. By $v_3(3) > V_1(1,(3,0))$, $(3,0)\ P_1\ f_1(R)$. Next, suppose $x(R) = i$ for some $i \in N\backslash\{1,3\}$. By $v_3(3) > \max_{j \in N\backslash\{1,3\}} v_j(j)$, Lemma 9 implies that $t_1(R) = v_3(3) - v_i(i) > 0$. By $3 \succeq_1 i$, $(3,0)\ R_1\ (i,0)$. Thus, by $t_1(R) > 0$, $(3,0)\ P_1\ f_1(R)$. Thus, in either case, $(3,0)\ P_1\ f_1(R)$. By Lemma 10, there is $R_1' \in \mathcal{R}_1^0 \cap \mathcal{R}_1^Q$ such that $f_1(R_1', R_{-1}) = (3,0)$. Thus, $f_1(R_1', R_{-1})\ P_1\ f_1(R)$, which contradicts *strategy-proofness.* □

By Lemma 29, $t_2^f(R_{-2};3)$ is well-defined. The following lemma states that given R_{-2}, if the rule f selects the object allocation 3, then the payment of agent 2 is equal to zero.

Lemma 30. *We have* $t_2^f(R_{-2};3) = 0$.

[37]Note that $V_1(1,(3,V_1(3,(2,0)))) = V_1(1,(2,0))$.

Proof. Suppose by contradiction that $t_2^f(R_{-2};3) > 0$. Let $R_2 \in \mathcal{R}_2^+ \cap \mathcal{R}_2^Q$ be a preference such that $v_2(2) < v_3(3)$, $0 < v_2(3) < t_2^f(R_{-2};3)$, and for each $x \in X\backslash\{1,2\}$, $v_2(x) = 0$. By $v_3(3) + V_1(3,(2,0)) < V_1(1,(2,0))$ and $3 \succ_1 2$, we have $v_3(3) < V_1(1,(2,0))$. Thus, by $v_2(2) < v_3(3)$, $v_2(2) < V_1(1,(2,0)) = t_2^f(R_{-2};2)$, or equivalently, $0 > v_2(2) - t_2^f(R_{-2};2)$. By $v_2(3) < t_2^f(R_{-2};3)$, $0 > v_2(3) - t_2^f(R_{-2};3)$. Thus, by *weak individual rationality*, we have $x(R) \in X\backslash\{2,3\}$. By Lemma 1, $x(R) \neq 0$. Thus, $x(R) \in N\backslash\{2,3\}$.

Suppose $x(R) = 1$. By $v_3(3) > v_2(2)$ and $v_3(3) > \max_{i \in N\backslash\{1,2,3\}} v_i(i)$, Lemma 9 implies $t_1(R) = t_1^f(R_{-1};1) = v_3(3)$. By $v_3(3) > V_1(1,(3,0))$, $(3,0)\ P_1\ (1, v_3(3)) = f_1(R)$. By Lemma 10, there is $R_1' \in \mathcal{R}_1^0 \cap \mathcal{R}_1^Q$ such that $f_1(R_1', R_{-1}) = (3,0)$. Thus, $f_1(R_1', R_{-1})\ P_1\ f_1(R)$. This contradicts *strategy-proofness*.

If $x(R) = i$ for some $i \in N\backslash\{1,2,3\}$, then we can derive a contradiction by the same discussion as in the case where $x(R) = i$ for some $i \in N\backslash\{1,2,3\}$ in the proof of Lemma 28. Thus, we omit the detail. □

By $v_3(3) < V_1(1,(2,0)) - V_1(3,(2,0))$, $v_3(3) + V_1(3,(2,0)) < V_1(1,(2,0))$. Thus, we can choose $R_2 \in \mathcal{R}_2^+ \cap \mathcal{R}_2^Q$ such that $v_2(2) > V_1(1,(2,0))$, $v_3(3) + V_1(3,(2,0)) < v_2(2) - v_2(3) < V_1(1,(2,0))$, and for each $x \in X\backslash\{2,3\}$, $v_2(x) = 0$. By $v_2(2) - v_2(3) < V_1(1,(2,0)) = t_2^f(R_{-2};2)$ and Lemma 30, we have $v_2(3) - t_2^f(R_{-2};3) > v_2(2) - t_2^f(R_{-2};2)$. For each $x \in X_2^f(R_{-2})\backslash\{2,3\}$, by $v_2(2) > V_1(1,(2,0)) = t_2^f(R_{-2};2)$ and *no subsidy for losers*, we have $v_2(2) - t_2^f(R_{-2};2) > 0 \geq v_2(x) - t_2^f(R_{-2};x)$. Thus, by Lemma 3, $x(R) = 3$. Then, by $v_2(2) - v_2(3) > v_3(3) + V_1(3,(2,0))$ and $3 \succ_1 2$, $v_2(2) - (v_2(3) + v_3(3)) > V_1(3,(2,0)) > 0$. Thus, by $v_3(3) > \max_{i \in N\backslash\{1,2,3\}} v_i(i)$, Lemma 9 implies that $t_1(R) = t_1^f(R_{-1};3) = v_2(2) - (v_2(3) + v_3(3))$. Thus, by $v_2(2) - v_2(3) > v_3(3) + V_1(3,(2,0))$, we have $t_1(R) = v_2(2) - (v_2(3) + v_3(3)) > V_1(3,(2,0))$, which implies that $(2,0)\ P_1\ f_1(R)$. By Lemma 10, there is $R_1' \in \mathcal{R}_1^0 \cap \mathcal{R}_1^Q$ such that $f_1(R_1', R_{-1}) = (2,0)$. Thus, $f_1(R_1', R_{-1})\ P_1\ f_1(R)$, which contradicts *strategy-proofness*. ■

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