Graduate School of Economics, Hitotsubashi University

Discussion Paper Series No. 2023-04

Change-point estimators with the weighted objective function when estimating breaks one at a time

Toshikazu Tayanagi and Eiji Kurozumi

November 2023

Change-point estimators with the weighted objective function when estimating breaks one at a time*

Toshikazu Tayanagi and Eiji Kurozumi
Department of Economics, Hitotsubashi University

November 14, 2023

Abstract

This study investigates the change-point estimators based on the recently proposed weighted objective function, wherein the model has two structural breaks and these breaks are estimated one at a time. The asymptotic distributions of the first-step and second-step estimators are investigated under the long-span asymptotic scheme, which is valid when the break sizes are large. We show that they are unimodal and asymmetric in general. However, for the small sizes of the breaks, the limiting distribution based on the long-span scheme cannot approximate the finite sample distribution; the finite sample distribution of the first-step estimator tends to have two peaks, while the corresponding asymptotic distribution under the long-span asmptotic scheme is unimodel. This finite sample property is approximated under the in-fill asymptotic scheme.

JEL classification: C13; C22;

Keywords: Structural break; structural change; break point; break fraction

*We thank the participants at economics and statistics seminar held at Hitotsubashi University and at Kansai Keiryo Keizaigaku Kenkyukai held at Osaka University for helpful discussions and comments. All errors are our responsibility. Kurozumi's research was supported by JSPS KAKENHI Grant Number 22K01422 and 23H00804. Address correspondence to Toshikazu Tayanagi, Department of Economics, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo 186-8601, Japan; E-mail: ed215004@g.hit-u.ac.jp

1 Introduction

This study investigates the behavior of the change-point estimators obtained by maximizing the weighted objective function, proposed by Baek (2023), when estimating the breaks one at a time for the level-shift model with two breaks.

The estimation of the change point has been investigated for more than 50 years and early contributions include Hinkley (1970) and Yao (1987) who studied the maximum likelihood estimator (MLE) of the break point, and Bai (1994, 1997b), who developed the asymptotic property of the least squares estimator (LSE). Further, multiple breaks were investigated by Bai and Perron (1998) and Qu and Perron (2007), among others. These studies considered estimating the break dates simultaneously, whereas Chong (1995) and Bai (1997a) proposed to estimate the change points step by step; first, the largest break is estimated, and then, the second break is estimated from the subsample. They considered a level-shift model and proposed to fit the model with only one break. Although the estimation model is misspecified in that it includes only a one-time break, they proved that the estimated break fraction, which is defined as the proportion of the break date that is relative to the whole sample size, is consistent for the largest break fraction. This step-by-step method was further extended to the regression model with multiple breaks by Bai et al. (2008). For a review of testing and estimating structural changes, see, for example, Perron (2006), Aue and Horváth (2013), and Casini and Perron (2019).

These studies demonstrate that the asymptotic distribution of the break point estimator(s) based on the least squares method is typically unimodal under the long-span scheme, wherein both the subsamples before and after the break go to infinity at the same rate, and the finite sample distribution of the estimator is approximated effectively by the asymptotic counterpart when the magnitude of the break is large. However, this is not the case when the break size is small. For the one-time break model, the finite sample distribution of the break point estimator tends to have three peaks at around the true break point and the beginning and end of the sample points; see Jiang et al. (2018, 2020) and Casini and Perron (2021a,b, 2022). These authors explained this finite sample property by introducing the in-fill asymptotic scheme, whereby the sampling frequency goes to infinity in the fixed interval or, equivalently, the sampling interval goes to zero. Casini and Perron (2021b, 2022) proposed a Laplace-based procedure to estimate the break point estimator, which is defined by an integration.

Recently, Baek (2023) tackled this problem from a different perspective by noting that the break point estimator at the beginning or end of the sample point has no information regarding a structural break, and this lowers the probability of estimating the true change point. Baek (2023) proposed to estimate the break date by maximizing the objective function weighted by the positive weight function, which places more weights at around the middle, but less weights at the beginning and end, of the sample points. Baek (2023) shows that the new estimator is easy to calculate and consistent for the true break fraction, that both the finite and asymptotic distributions of the estimators have only one peak around the true break point, and that the mean squared error tends to be smaller than that of the LSE.

In this study, we consider the level-shift model with two breaks and estimate each break point step by step, as in Chong (1995), Bai (1997a), and Tayanagi and Kurozumi (2023), who estimate each break date based on the least squares method; however, we estimate it by maximizing the weighted objective function proposed by Baek (2023). We investigate the asymptotic property of the step-by-step estimators under the long-span scheme and demonstrate that the break fraction estimated at first is consistent for the largest one in the sense of the weighted objective function and that the second estimator from the subsample split by the first estimator is also consistent for the other break fraction. Further, we also derive the asymptotic distributions of the two estimators, which are unimodal and approximate the finite sample distribution effectively when the magnitude of the breaks is large. When the magnitude of the breaks is small, the finite sample distribution of the first estimator based on the least squares method tends to have four peaks at around two break dates and the beginning and end of the sample points, as demonstrated by Tayanagi and Kurozumi (2023), whereas the corresponding estimator based on the weighted objective function is demonstrated as having only two peaks at around the true break points. We demonstrate that this finite sample property can be approximated under the in-fill asymptotic scheme. By Monte Carlo simulations, the estimators based on the weighted objective function tend to have the smaller root mean squared error and the standard deviation compared to those based on the least squares method, while the former estimator is slightly more biased than the latter.

The remainder of the paper is organized as follows: The model and assumptions are introduced in Section 2. The estimators estimated one at a time based on the weighted objective function are investigated under the long-span scheme in Section 3. We consider the small shift case and develop asymptotic theory under the in-fill asymptotic scheme in Section 4. The finite sample property is investigated using Monte Carlo simulations in Section 5. Concluding remarks are provided in Section 6. All the proofs are relegated to the Appendix.

2 Model and Assumptions

Let us consider the following level-shift model comprising two breaks:

$$y_t = \mu_t + X_t \quad \text{for} \quad 1, 2, \ldots, T, \tag{1}$$

where

$$\mu_t = \begin{cases} \mu_1 = \mu_0 + \delta_1 \lambda_T & 1 \le t \le k_1^0, \\ \mu_2 = \mu_0 + \delta_2 \lambda_T & k_1^0 < t \le k_2^0, \\ \mu_3 = \mu_0 + \delta_3 \lambda_T & k_2^0 < t \le T \end{cases} \tag{2}$$

with $\delta_1 \neq \delta_2$ and $\delta_2 \neq \delta_3$, and λ_T controls the magnitude of the breaks. In model (1), the level of y_t shifts from μ_1 to μ_2 at $t = k_1^0 + 1$ and from μ_2 to μ_3 at $t = k_2^0 + 1$. We define the break fractions corresponding to k_j^0 as $\tau_j^0 = k_j^0/T$ for $j = 1, 2$, and 3, respectively. Without loss of generality, we assume that $[T\tau_j]$ is an integer value and, thus, $k_j^0 = [T\tau_j^0]$ holds for $j = 1, 2$, and 3, respectively. The stochastic part $\{X_t\}$ is a sequence of unobservable disturbances generated as the linear process expressed as follows:

$$X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, \quad \text{where} \quad \sum_{j=0}^{\infty} j|a_j| < \infty \quad \text{and} \quad a(1) = \sum_{j=0}^{\infty} a_j \neq 0.$$

Assumption 1 *$\{\varepsilon_t\}$ is a martingale difference sequence satisfying $E[\varepsilon_t|\mathcal{F}_{t-1}] = 0$, $E[\varepsilon_t^2] = \sigma^2$, and $\sup_t[E|\varepsilon_t|^{2+\nu}] < \infty$ for some $\nu > 0$, where $\mathcal{F}_t$ is the σ-field generated by ε_s for $s \le t$.*

For the break fractions, we assume throughout this study that they are distinct, which is a standard assumption in the literature, while the magnitude of the breaks is assumed to either be fixed or shrink to zero.

Assumption 2 $0 < \tau_1^0 < \tau_2^0 < 1$.

Assumption 3 *(i) λ_T is fixed. (ii) $\lambda_T \to 0$ as $T \to \infty$ and $T^{\frac{1}{2}-\gamma}\lambda_T \to \infty$ for some $\gamma \in \left(0, \frac{1}{2}\right)$.*

When Assumption 3(i) holds, we set $\lambda_T = 1$, $\mu_0 = 0$, and $\mu_i = \delta_i$ for $i = 1, 2$, and 3, respectively, without loss of generality. This is called the "fixed shift case," whereas the "shrinking shift case" corresponds to Assumption 3(ii). As elucidated in the following section, the restriction on the shrinking speed of λ_T guarantees the consistency of the break date estimators.

3 Asymptotics under the Long-Span Scheme

3.1 Estimation method of the break points

In this section, we investigate the asymptotic properties of the break point estimators based on the weighted objective function proposed by Baek (2023) when estimating the break dates one at a time, as in Bai (1997a). The motivation for the new estimation method is that it is expected to reduce the mean squared error (MSE) of the break date estimators, in particular when the magnitude of the breaks is not excessively large, as demonstrated by Baek (2023) for a model with a one-time break.

First, we briefly review the estimation method of the break point by Baek (2023) and consider the model with only a one-time break. As investigated by Bai (1994), the natural estimator for the break date is the minimizer of the sum of the squared residuals (SSR) given by $S_T^2(k)$:

$$\hat{k}_{LS,1} = \arg\min_{1 \le k < T} S_T^2(k), \tag{3}$$

where

$$S_T^2(k) = \sum_{t=1}^{k} (y_t - \bar{Y}_k)^2 + \sum_{t=k+1}^{T} (y_t - \bar{Y}_k^*)^2, \tag{4}$$

with $\bar{Y}_k$ being the average of the first k observations and $\bar{Y}_k^*$ being that of the last $T-k$ observations:

$$\bar{Y}_k = \frac{1}{k}\sum_{t=1}^{k} y_t, \quad \bar{Y}_k^* = \frac{1}{T-k}\sum_{t=k+1}^{T} y_t.$$

We note that, as indicated by Bai (1994), because $\sum_{t=1}^{T}(y_t - \bar{Y}_T)^2 = S_T^2(k) + TV_T^2(k)$ holds where $V_T(k)$ is defined below, $\hat{k}_{LS,1}$ in (3) is equivalent to

$$\hat{k}_{LS,1} = \arg\max_{1 \le k < T} V_T^2(k), \quad \text{where} \quad V_T(k) = \sqrt{\frac{k(T-k)}{T^2}} \left| \bar{Y}_k^* - \bar{Y}_k \right|. \tag{5}$$

Denote $\hat{\tau}_{LS,1} = \hat{k}_{LS,1}/T$. Bai (1994) demonstrated that $\hat{\tau}_{LS,1}$ is consistent for the true break fraction and derived its limiting distribution under the assumption of the shrinking shift.

Instead of minimizing the SSR or maximizing $V_T^2(k)$, Baek (2023) proposed modifying the objective function $V_T^2(k)$ to improve the efficiency of the break point estimator.[1] More precisely, Baek (2023) proposed maximizing the weighted objective function given by $Q_T^2(k/T)$, which is defined as follows:

$$Q_T^2(k/T) = w_T(k)V_T^2(k) \quad \text{where} \quad w_T(k) = \frac{k(T-k)}{T^2}. \tag{6}$$

Then, the new estimator, which we call the weighted estimator in this study, expressed as follows:

$$\hat{k}_{W,1} = \underset{1 \le k \le T-1}{\arg\max}\, Q_T^2(k/T). \tag{7}$$

We define $\hat{\tau}_{W,1} = \hat{k}_{W,1}/T$. Figure 1 illustrates the weight function $w_T(\tau)$ for $0 \le \tau \le 1$, which clearly imposes greater weight on the middle, and less weight on both ends, of the sample. Motivation for this weight is that, because the finite sample distribution of $\hat{k}_{LS,1}$ tends to have peaks at both the ends of the sample, in particular, when the break size is small, even if the true break point is located at an inner point of the sample, placing less weight on the sample ends results in unimodality of $\hat{k}_{W,1}$ in finite samples. As demonstrated by Baek (2023), the MSE of $\hat{k}_{W,1}$ tends to be smaller than that of $\hat{k}_{LS,1}$.

Let us now consider the case wherein the model has two breaks, as in (1)–(2). Following Chong (1995) and Bai (1997a), we propose estimating each break point step by step, which, compared to the simultaneous estimation of breaks, is computationally less expensive and makes it easier to obtain the estimators. In this study, we consider maximizing the objective function given by (6), instead of $V_T^2(k)$, following Baek (2023). More precisely, first, we fit the model with a one-time break and obtain $\hat{k}_{W,1}$ as defined in (7). Although the number of the breaks is misspecified in this case, it is presented in the following subsection that $\hat{k}_{W,1}$, is consistent (in the sense of the fraction relative to the sample size T) for the largest break among the two breaks in model (1)–(2). We call $\hat{k}_{W,1}$ the first-step estimator. Given the first-step estimator, the second break date is estimated similarly from either $t = 1, \ldots, \hat{k}_{W,1}$ or $t = \hat{k}_{W,1} + 1, \ldots, T$, by maximizing the corresponding objective function (6).[2] We denote the second break point estimator as $\hat{k}_{W,2}$ and call it the second-step estimator. Notably, the difference between Bai (1997a) and this study is that the former estimates the break dates based on the objective function $V_T^2(k)$, whereas we use $Q_T^2(k/T)$. We denote the step-by-step estimators

[1] Baek (2023) proposed a general weight function but recommended $w_T(k) = k(T-k)/T^2$ for a level-shift model.

[2] In practice, we may test for structural change in the subsamples, and if we find statistical evidence of the additional break, we estimate the second break in the corresponding subsample.

proposed by Bai (1997a) as $\hat{k}_{LS,1}$ and $\hat{k}_{LS,2}$, respectively. As in the case of the model with a one-time structural change, the weighted estimators are expected to have the smaller MSE, which is confirmed in finite samples in a later section.

3.2 Asymptotic property of the first-step estimator

Suppose that (1)–(2) is a true model and estimate the break dates one at a time using (7). Let $\operatorname{plim} \lambda_T^{-2} Q_T^2(k/T) = Q^2(\tau)$, where

$$Q^2(\tau) = \begin{cases} \tau^2 \left[(1-\tau_2^0)(\delta_3-\delta_2)+(1-\tau_1^0)(\delta_2-\delta_1)\right]^2 & : \quad 0 \le \tau \le \tau_1^0 \\ \left[\tau(1-\tau_2^0)(\delta_3-\delta_2)+\tau_1^0(1-\tau)(\delta_2-\delta_1)\right]^2 & : \quad \tau_1^0 < \tau \le \tau_2^0 \\ (1-\tau)^2 \left[\tau_2^0(\delta_3-\delta_2)+\tau_1^0(\delta_2-\delta_1)\right]^2 & : \quad \tau_2^0 < \tau \le 1. \end{cases} \tag{8}$$

The derivation is provided in Appendix. Evidently, $Q^2(\tau)$ is a piece-wise convex function of τ. To identify "the largest break", we make the following assumption:

Assumption 4

$$Q^2(\tau_1^0) - Q^2(\tau_2^0) > 0. \tag{9}$$

This assumption implies that the first break at k_1^0 dominates the second break at k_2^0 considering the weighted objective function. Additionally, as presented in Appendix, (9) implies the following:

$$(1-\tau_2^0)(\delta_3-\delta_2)+(1-\tau_1^0)(\delta_2-\delta_1) \neq 0, \tag{10}$$

$$(1-\tau_2^0)(\delta_3-\delta_2)-\tau_1^0(\delta_2-\delta_1) \neq 0. \tag{11}$$

Two relations (10) and (11) play an important role with respect to the proof of the consistency of the estimator. They imply that $Q^2(\tau)$ is strictly increasing and convex for $\tau \in [0, \tau_1^0]$, strictly convex for $\tau \in [\tau_1^0, \tau_2^0]$, and non-increasing for $\tau \in [\tau_2^0, 1]$ from (8). Notably, if

$$\tau_2^0(\delta_3-\delta_2)+\tau_1^0(\delta_2-\delta_1) \neq 0, \tag{12}$$

$Q^2(\tau)$ for $\tau \in [\tau_2^0, 1]$ is strictly decreasing and convex and, thus, (8) becomes bimodal with two peaks at τ_1^0 and τ_2^0; by contrast, if the inequality in (12) is replaced by the equality, $Q^2(\tau) = 0$ for $\tau \in [\tau_2^0, 1]$ and, thus, $Q^2(\tau)$ is unimodal with the peak at τ_1^0. Therefore, the largest break point τ_1^0 can be identified by the weighted objective function under Assumption 4. Later, we discuss the case wherein Assumption 4 does not hold.

Proposition 1 *Under Assumptions 1–4, we have*

$$\hat{\tau}_{W,1} - \tau_1^0 = O_p\left(\lambda_T^{-1} T^{-1/2}\right).$$

Proposition 1 implies that the break fraction estimator $\hat{\tau}_{W,1}$ converges in probability to the true break fraction τ_1^0, irrespective of whether λ_T is fixed or shrinks to zero. Notably, if inequality (9) is reversed, $\hat{\tau}_{W,1}$ becomes consistent for τ_2^0.

The convergence rate in Proposition 1 is not sharp, and we can refine it by focusing on maximizing $Q_T^2(k/T)$ around the neighborhood of k_1^0. Let η be a small value such that $\tau_1^0 \in \left(\eta, \tau_2^0(1-\eta)\right)$, $D_T = \left\{k : T\eta \leq k \leq T\tau_2^0(1-\eta)\right\}$, and $D_M = \left\{k : |k - k_1^0| \leq M\lambda_T^{-2}\right\}$ for some given large value of $M < \infty$. Define $D_{T,M^c} = \left\{k : T\eta < k < T\tau_2^0(1-\eta), |k - k_1^0| > M\lambda_T^{-2}\right\}$. In Appendix, we demonstrate that $Q_T^2(k/T)$ cannot be maximized on D_{T,M^c} with probability approaching one, which implies the following result:

Proposition 2 *Under Assumptions 1–4, for every $\epsilon > 0$, there exists an $M < \infty$ independent of T, such that, for all large T,*

$$P\left(T|\hat{\tau}_{W,1} - \tau_1^0| > M\lambda_T^{-2}\right) < \epsilon. \tag{13}$$

Proposition 2 implies $T\lambda_T^2(\hat{\tau}_{W,1} - \tau_1^0) = O_p(1)$. Notably, if $\lambda_T = T^{-1/2}$, $\hat{\tau}_{W,1}$ is not consistent. Hence, we need a restriction of γ in Assumption 3(ii). This convergence rate is the same as $\hat{\tau}_{LS,1}$, as proved by Bai (1997a).

Once the refined rate of convergence is obtained, we can derive the limiting distribution of $\hat{k}_{W,1}$ under the assumption of the shrinking shift.

Theorem 1 *Under Assumptions 1, 2, 3(ii), and 4,*

$$\lambda_T^2(\hat{k}_{W,1} - k_1^0) \Rightarrow \sigma^2 a^2(1) \operatorname*{arg\,max}_{u \in (-\infty,\infty)} \{\Gamma(u)\}, \tag{14}$$

where $\Rightarrow$ signifies weak convergence of the associated probability measures,

$$\Gamma(u) = \begin{cases} B_1(u) - |g||u| & \text{if } u \leq 0 \\ B_2(u) - |h||u| & \text{if } u > 0 \end{cases}, \tag{15}$$

$$g = (1-\tau_2^0)(\delta_3 - \delta_2) + (1-\tau_1^0)(\delta_2 - \delta_1), \tag{16}$$

$$h = (1-\tau_2^0)(\delta_3 - \delta_2) - \tau_1^0(\delta_2 - \delta_1), \tag{17}$$

and $B_1(\cdot)$ and $B_2(\cdot)$ are two independent standard Brownian motions on $[0, \infty)$.

Figure 2 depicts the histogram of the finite sample distribution of $\lambda_T^2(\hat{k}_{W,1} - k_1^0)$ in the shrinking shift case with $\tau_1^0 = 0.3, \tau_2^0 = 0.7, \mu_0 = 0, \delta_1 = 0, \delta_2 = 4, \delta_3 = 1, \sigma = 1, \lambda_T = T^{-1/4}, T = 100$ with 5,000 replications, and Figure 3 illustrates its asymptotic distribution given by Theorem 1 with the same parameters. Evidently, the distribution in Figure 3 is unimodal and asymmetric, with a shape similar to that of the left peak in Figure 2. Figure 4 presents the histograms of the finite sample distribution of $\hat{\tau}_{W,1}$ for the same parameters with different sample sizes given by $T = \{100, 200, 600, 1000\}$. We observe that the peak of the histogram becomes higher as T increases while the small right spike tends to disappear.

3.3 Asymptotic property of the second-step estimator

Having shown that the first step break fraction estimator $\hat{\tau}_{W,1}$ is consistent, we subsequently investigate the behavior of the second-step estimator $\hat{k}_{W,2}$. As a preparation, we define the weighted objective function $Q_{T'}(k/T)$ in the subsample of $\hat{k}_{W,1} + 1 \leq t \leq T$ as follows:

$$Q_{T'}^2(k/T) = w_{T'}(k) V_{T'}^2(k), \tag{18}$$

$$\text{where} \quad T' = T - \hat{k}_{W,1}, \quad w_{T'}(k) = \frac{(k - \hat{k}_{W,1})(T - k)}{(T - \hat{k}_{W,1})^2},$$

$$V_{T'}(k) = \sqrt{\frac{(k - \hat{k}_{W,1})(T - k)}{(T - \hat{k}_{W,1})^2}} \left| \bar{Y}_k^* - \bar{Y}_k' \right|, \quad \bar{Y}_k' = \frac{1}{k - \hat{k}_{W,1}} \sum_{t=\hat{k}_{W,1}+1}^{k} y_t,$$

and $\bar{Y}_k^*$ is defined identically as previously stated. The second-step estimator $\hat{k}_{W,2}$ is defined as the location that maximizes (18):

$$\hat{k}_{W,2} = \underset{\hat{k}_{W,1}+1 \leq k \leq T-1}{\arg\max} \; Q_{T'}^2(k/T), \tag{19}$$

and the second step break fraction estimator is defined as $\hat{\tau}_{W,2} = \hat{k}_{W,2}/T$. In exactly the same manner as that applied in the previous section, it can be shown using $\hat{k}_{W,1}/T \xrightarrow{p} \tau_1^0$ that

$$Q_{T'}^2(k/T)/\lambda_T^2 \xrightarrow{p} \begin{cases} \frac{(\tau-\tau_1^0)^2(1-\tau_2^0)^2}{(1-\tau_1^0)^4}(\delta_3 - \delta_2)^2 & : \ \tau_1^0 \leq \tau \leq \tau_2^0 \\ \frac{(1-\tau)^2(\tau_2^0-\tau_1^0)^2}{(1-\tau_1^0)^4}(\delta_3 - \delta_2)^2 & : \ \tau_2^0 \leq \tau \leq 1 \end{cases}$$

uniformly and, thus, the second break date can be identified by the function $Q_{T'}^2(k/T)$.

As in the case of the first-step estimator, first, we show the consistency of the second-step estimator.

Proposition 3 *Under Assumptions 1–4, $\hat{\tau}_{W,2}$ is consistent.*

Subsequently, we improve the order of convergence rate to derive the asymptotic distribution. Using the same manner as that used in Proposition 2, we can obtain the exact order of $\hat{\tau}_{W,2} - \tau_2^0$, but we need a slight modification because the second-step estimator is defined on the post $\hat{k}_{W,1}$ samples. Let $D_{T'} = \{k; T\underline{\eta} \leq k \leq T\overline{\eta}\}, D_{M'} = \{k; |k - k_2^0| \leq M'\lambda_T^{-2}\}$ and $D_{T',M'^c} = \{k : T\underline{\eta} \leq k \leq T\overline{\eta}, |k - k_2^0| > M'\lambda_T^{-2}\}$, where $\tau_1^0 < \underline{\eta} < \tau_2^0 < \overline{\eta} < 1$. These sets are modifications of D_T, D_M, and D_{T,M^c} in Proposition 2 to accommodate the theory to the post $\hat{k}_{W,1}$ sample.

Proposition 4 *Under Assumptions 1–4, for every $\epsilon > 0$, there exists an $M' < \infty$ independent of T, such that, for all large T,*

$$P\left(T(\hat{\tau}_{W,2} - \tau_2^0) > M'\lambda_T^{-2}\right) < \epsilon,$$

and this is equivalent to

$$\hat{\tau}_{W,2} - \tau_2^0 = O_p\left(T^{-1}\lambda_T^{-2}\right).$$

Using Proposition 4, we can derive the asymptotic distribution.

Theorem 2 *Under Assumptions 1, 2, 3(ii), and 4, we have*

$$\lambda_T^2\left(\hat{k}_{W,2} - k_2^0\right) \Rightarrow \frac{(1-\tau_1^0)^2}{(\delta_3 - \delta_2)^2}\sigma^2 a^2(1) \underset{u\in(-\infty,\infty)}{\arg\max}\, W(u),$$

where

$$W(u) = \begin{cases} B_1(|u|) - (1-\tau_2^0)|u| & \text{if } u \leq 0 \\ B_2(|u|) - (\tau_2^0 - \tau_1^0)|u| & \text{if } u > 0 \end{cases}, \tag{20}$$

and $B_1(\cdot)$ and $B_2(\cdot)$ are two independent standard Brownian motions on $[0, \infty)$.

Notably, (20) is assymetric, and this result differs from the asymptotic distribution of the second-step estimator with the LSE, which is symmetric. Figure 5 depicts histograms of the finite sample ($T = 100$) and asymptotic distributions in Theorem 2 with $\delta_1 = 0, \delta_2 = 4, \delta_3 = 1, \tau_1^0 = 0.33, \tau_2^0 = 0.67, \lambda_T = T^{-1/4}$, and $X_t \sim i.i.d.N(0,1)$, with 5,000 replications, while Figures 6 and 7 correspond to the cases wherein (τ_1^0, τ_2^0) are $(0.33, 0.80)$ and $(0.33, 0.54)$, respectively. Considering these results, we can observe that both the finite sample and the asymptotic distributions are skewed depending on the location of τ_2^0 relative to τ_1^0. If $\tau_2^0 - \tau_1^0$ is large (small), there are less (more) observations after $[T\tau_2^0]$ and, thus, the distribution tends to be skewed to the left (right), as in Figure 6 (Figure 7). In the case wherein τ_2^0 is located in the middle of the $[\tau_1^0,\ 1]$ interval, the distribution is symmetric as in Figure 5.

3.4 Violation of the identification condition

We have investigated the asymptotic property of the weighted estimators under Assumption 4. This assumption is required to identify the "largest break." However, considering the case wherein the objective function assumes the same value at τ_1^0 and τ_2^0 is possible. In this case, the two breaks have the same size and thus, the first-step estimator is expected to converge to either τ_1^0 or τ_2^0. In fact, Proposition 3 of Bai (1997a) indicates that this is the case with equal probability for the first-step estimator $\hat{k}_{W,1}$.

Assumption 5

$$Q^2(\tau_1^0) = Q^2(\tau_2^0), \tag{21}$$

and (11) *holds.*

Following the proof of (10) and (11), it can be demonstrated that (21) implies (10) and (12), but not necessarily (11). From (8), it is observed that under Assumption 5, the objective function is asymptotically a piece-wise, strictly convex function with two peaks at τ_1^0 and τ_2^0 with the same height implying that the first-step estimator is consistent for either of the break fractions.

Theorem 3 *Under Assumptions 1, 2, 3(i), and 5, we have*

$$\hat{\tau}_{W,1} \Rightarrow \begin{cases} \tau_1^0 & \text{with probability } \frac{1}{2} \\ \tau_2^0 & \text{with probability } \frac{1}{2} \end{cases}.$$

Theorem 3 is essentially the same as Proposition 3 in Bai (1997a) for $\hat{k}_{LS,1}$. That is, the weighted estimator $\hat{k}_{W,1}$ possesses the desirable property that the estimated break fraction is consistent for either of the true break fractions with equal probability when the objective function becomes indifferent between the two break dates.

Remark 1 *Suppose that* $Q^2(\tau_1^0) = Q^2(\tau_2^0)$ *but that* (11) *does not hold. In this case,* $(1-\tau_2^0)(\delta_3 - \delta_2) - \tau_1^0(\delta_2 - \delta_1) = 0$ *and, thus,* $Q^2(\tau) = (\tau_1^0)^2(\delta_2 - \delta_1)^2$ *for* $\tau \in [\tau_1^0, \tau_2^0]$. *That is, the objective function is asymptotically flat between the two breaks, as demonstrated in Figure 8, wherein* $\tau_1^0 = 0.2$ *and* $\tau_2^0 = 0.8$. *As the objective function attains its maximum at any point between* 0.2 *and* 0.8, *our first-step estimator cannot be consistent in this case. In fact, this is confirmed by Figure 9 with* $\tau_1^0 = 0.2, \tau_2^0 = 0.8, \mu_0 = 0, \delta_1 = 0.3, \delta_2 = 0.6, \delta_3 = 0.3, T = 100$.

4 Asymptotics under the In-fill Asymptotic Scheme

We have investigated the break point estimators under the so-called long-span asymptotic scheme, wherein the break fractions are fixed and the samples before and after the breaks increase proportionally to the whole sample size. As demonstrated in the previous section, the asymptotic distribution of each break point estimator can approximate the finite sample distribution relatively well. However, it is known in the the literature that this approximation under the long-span asymptotic scheme deteriorates when the break size is small. In the case of a one-time break model, the finite sample distribution of the break point estimator based on the minimization of the SSR tends to be trimodal (one peak at the true break point and two peaks at both of the ends of the sample) as demonstrated by Jiang et al. (2018) for a break in the mean, Jiang et al. (2020) for a break in the autoregressive coefficient, and Casini and Perron (2021a,b, 2022) for a regression model. To explain this finite sample property of the estimator theoretically, these authors introduced the in-fill asymptotic scheme, under which the sampling interval is fixed but the sampling frequency goes to infinity. They demonstrated that the limiting distribution derived under the in-fill asymptotic scheme can approximate the finite sample distribution highly effectively. Recently, Tayanagi and Kurozumi (2023) applied this technique to investigate $\hat{k}_{LS,1}$, the first-step estimator based on the minimization of the SSR in the level-shift model with two breaks, and demonstrated that the limiting distribution derived under the in-fill asymptotic scheme can replicate the important finite sample distributional property; the finite sample distribution has four modes, as follows: two peaks at the true break points and the others at both ends of the sample.

The finite sample distribution of $\hat{k}_{W,1}$ based on the weighted objective function differs from that of $\hat{k}_{LS,1}$ when the breaks are small; the former has two peaks at the true break points, as demonstrated in Figure 10, whereas the latter has four peaks, as observed in Tayanagi and Kurozumi (2023). In this section, we derive the limiting distribution of $\hat{k}_{W,1}$ under the in-fill asymptotic scheme and demonstrate that the in-fill asymptotic scheme can replicate the finite sample property observed in Figure 10.

To establish the in-fill asymptotic scheme, suppose that the sampling interval $h \to 0$ with $Th = 1$, which suggests that the sampling span is fixed, but the sample size T increases as $h \to 0$. Following Jiang et al. (2018), we begin with the discretization of the continuous time model with level shifts given by

$$Y_{th} - Y_{th-1} = \mu_i h + \sqrt{h}\upsilon_t, \tag{22}$$

for $t = 1, \ldots, T$, where Y_{th} is a continuous time observation and $\upsilon_t \sim \text{i.i.d.}\,\mathrm{N}(0, \sigma^2)$. Let $y_t = (Y_{th} - Y_{th-1})/\sqrt{h}$, and we divide both sides of (22) by $\sqrt{h}$. Then, (22) becomes

$$y_t = \mu_i\sqrt{h} + \upsilon_t. \tag{23}$$

By replacing $\{\upsilon_t\}$ with $\{X_t\}$ considered in the previous section, the model under the in-fill asymptotic scheme is expressed as follows:

$$y_t = \mu_i\sqrt{h} + X_t, \quad \text{where } \mu_i = \begin{cases} \delta_1 : & 1 \le k \le k_1^0, \\ \delta_2 : & k_1^0 < k \le k_2^0, \\ \delta_3 : & k_2^0 < k \le T. \end{cases} \tag{24}$$

Notably, model (24) differs slightly from (1)–(2) and the magnitude of the breaks shrinks to zero at a rate of $\sqrt{h}$, which corresponds to the rate of $1/\sqrt{T}$ because $Th = 1$, implying that we cannot consistently estimate the break fraction.

Like Assumption 4, we suppose that the first break dominates the second one under the in-fill asymptotic scheme.

Assumption 6

$$\operatorname{plim} \left(\sqrt{h}\right)^{-2} \left[Q_T^2(k_1^0/T) - Q_T^2(k_2^0/T)\right] > 0. \tag{25}$$

Theorem 4 *Under Assumptions 1, 2, and 6*

$$\hat{\tau}_{W,1} - \tau_1^0 \Rightarrow \arg\max_{\tau\in(0,1)} J^2(\tau), \tag{26}$$

where

$$J(\tau) = \begin{cases} J_1(\tau) & \text{if} \quad 0 < \tau \le \tau_1^0 \\ J_2(\tau) & \text{if} \quad \tau_1^0 < \tau \le \tau_2^0 \\ J_3(\tau) & \text{if} \quad \tau_2^0 < \tau < 1 \end{cases},$$

with

$$\begin{aligned} J_1(\tau) =& \tilde{B}(\tau) + \tau(1-\tau_2^0)(\delta_3 - \delta_2) + \tau(1-\tau_1^0)(\delta_2 - \delta_1), \\ J_2(\tau) =& \tilde{B}(\tau) + \tau(1-\tau_2^0)(\delta_3 - \delta_2) + \tau_1^0(1-\tau)(\delta_2 - \delta_1), \\ J_3(\tau) =& \tilde{B}(\tau) + \tau_2^0(1-\tau)(\delta_3 - \delta_2) + \tau_1^0(1-\tau)(\delta_2 - \delta_1), \end{aligned}$$

and $\tilde{B}_\tau = a(1)\sigma\left(-B(\tau) + \tau B(1)\right)$.

Remark 2 *Theorem 4 evidently indicates that* $\hat{\tau}_{W,1}$ *is inconsistent and, thus, there is no room for estimating the second break point. However, as investigated in Tayanagi and Kurozumi (2023) for the step-by-step estimator based on the SSR, we can consider the case wherein* $\mu_i = \mu_0 + \delta_i/\varepsilon$ *and* $\varepsilon \to 0$ *as* $h \to 0$ *with* $\sqrt{h}/\varepsilon \to 0$*. This case indicates that* $\hat{\tau}_{W,1}$ *is consistent for* τ_1^0 *and has the same limiting distribution as provided in Theorem 1. Consequently, we can consider the second step estimator* $\hat{k}_{W,2}$*, which would have the same asymptotic distribution as provided in Theorem 2. That is, even under the in-fill asymptotic scheme, the limiting distributions of the break point estimators are unimodal if the break size is sufficiently large.*

Figure 11 demonstrates the histogram of the in-fill asymptotic distribution of $\hat{\tau}_{W,1}$ with the same parameter as that used in Figure 10. We can observe that the in-fill asymptotic distribution is bimodal as the finite sample distribution presented in Figure 10.

5 Simulation

In this section, we investigate the finite sample property of the break point estimators investigated in the previous sections using Monte Carlo simulations. Throughout the simulations, we consider the model specified by (1)–(2) with $X_t \sim i.i.d.N(0,1)$.

First, we compare the shape of the finite sample distributions of the two first-step estimators. Figure 12 demonstrates the histograms of the finite sample distributions with the parameter values $(\tau_1^0, \tau_2^0) = \{(0.40, 0.60), (0.30, 0.70), (0.10, 0.90), (0.025, 0.975)\}$, $\mu_0 = 0$, $\delta_1 = 0$, $\delta_2 = 0.4$, $\delta_3 = 0.1$, $\lambda_T = 1$, and $T = 100$ with $5,000$ replications. In the case wherein $(\tau_1^0, \tau_2^0) = \{(0.4, 0.6), (0.3, 0.7)\}$, the $\hat{\tau}_{LS,1}$ has four peaks, whereas $\hat{\tau}_{W,1}$ has only two peaks, and we can observe that $\hat{\tau}_{W,1}$ tends to estimate the true break fraction more frequently than $\hat{\tau}_{LS,1}$. On the contrary, when the true break points are located near the boundaries, such as $(\tau_1^0, \tau_2^0) = \{(0.10, 0.90), (0.025, 0.975)\}$, the finite sample distribution of the LSE becomes bimodal, whereas the peaks of the weighted estimator disappear by the influence of the weight function. In this case, $\hat{\tau}_{LS,1}$ estimates the true break fraction more frequently than $\hat{\tau}_{W,1}$.

To observe the finite sample performance of $\hat{\tau}_{W,1}$ when the magnitude of the breaks is the same as supposed in Assumption 5, we set $(\tau_1^0, \tau_2^0) = (0.2, 0.8)$, $\mu_0 = 0.0$, $\delta_1 = 0.1$, $\delta_2 = 0.4$, $\delta_3 = 0.1$, $\lambda_T = 1$, and $T = \{100, 300, 500, 2000\}$. Figure 13 demonstrates that the peaks are concentrated around the true break fractions with the same probability. We observe that the probability at one of the peaks goes to 1/2 as T increases, which supports Theorem 3.

Subsequently, we compare the root mean squared error (RMSE), bias, and standard error (SE) of the two first-step estimators. First, we set $\mu_0 = 0.0$ and select the shrinking speed at $\lambda_T = 1$ (the fixed shift case). The results when $T = 100$ and 300 are summarized in Tables 1 and 2, wherein first-step estimators based on the weighted function and SSR are denoted as WE and LE, respectively. Table 1 indicates that the RMSE and SE of $\hat{k}_{W,1}$ are smaller than those of $\hat{k}_{LS,1}$, whereas the bias of the former estimator is larger than the latter one, except for the case wherein $(\tau_1^0, \tau_2^0) = \{(0.5, 0.7), (0.5, 0.9)\}$. The relative performance is preserved when $T = 300$, as is observed in Table 2. This tendency is like the case of the one-time break model investigated by Baek (2023).

Tables 3 and 4 correspond to the shrinking shift case with $\lambda_T = T^{-1/4}$. The overall tendency is like the fixed shift case, but the difference between the biases becomes marginal.

Once the first step estimator is obtained, we subsequently investigate the finite sample performance of the second step estimator. In the experiment of the second-step estimator based on the least-squares method, we skipped the replication when $\hat{k}_{LS,1} = T - 1$ because estimating $\hat{k}_{LS,2}$ when $\hat{k}_{LS,1} = T - 1$ is impossible. The number of the skipped replications among 5000 is reported in the "# of exclusion" column of the tables. Tables 5 and 6 present the RMSE, bias, and SE of the second-step estimators in the fixed shift case ($\lambda_T = 1$). In Table 5, for $T = 100$, the RMSE and SE of the weighted estimator are smaller than those of the LSE, like the first-step estimators. Moreover, the bias of $\hat{k}_{W,2}$ is smaller than or nearly equal to that of $\hat{k}_{LS,2}$ in most cases. In Table 6, for $T = 300$, we observe a similar tendency to the case of $T = 100$.

Tables 7 and 8 correspond to the shrinking shift case with $\lambda_T = T^{-1/4}$. Again, the relative performance in this case is preserved compared to that in the fixed shift case.

6 Conclusion

In this study, we investigated the behavior of the break point estimator proposed by Baek (2023) in the case wherein the level-shift model having two breaks is estimated one at a time. Under the long-span scheme, we demonstrated that both the first-step and second-step weighted estimators are consistent for the true break fractions and derived the limiting distributions, which are unimodal.

This long-span asymptotic approximation works effectively when the magnitude of the breaks is not excessively small. However, when it is excessively small, the finite sample distribution of the first-step weighted estimator tends to have two peaks. We demonstrate that this bimodality can be captured under the in-fill asymptotic scheme. Using Monte Carlo simulations, we compare the finite sample performance of the first-step and second-step estimators based on the weighted objective function and least squares method and found that the first-step weighted estimator has the smaller RMSE and SE than the corresponding LSE, whereas the bias of the LSE tends to be smaller than that of the weighted estimator. However, the advantage of the LSE disappears in the shrinking shift case; specifically, the performance of the second-step weighted estimator is better than that of the LSE in most cases.

The model considered in this study is a simple level-shift model, and extending our result to the regression model though, the proof would become significantly more complicated in such a case. This alludes to our future research.

7 Appendix

In this appendix, we denote C as a generic constant that differs by place.

Probability limit of the objective function: It is not difficult to observe that the objective function $Q_T^2(k/T)$ is expressed as follows:

$$
\begin{aligned}
Q_T^2(k/T) &= \frac{k^2(T-k)^2}{T^4}\left(\bar{Y}_k^* - \bar{Y}_k\right)^2 \\
&= \begin{cases} w_T^2(k)a_T^2(k) + 2w_T(k)a_T(k)R_T(k) + R_T^2(k) & : \; 1 \le k \le k_1^0 \\ w_T^2(k)b_T^2(k) + 2w_T(k)b_T(k)R_T(k) + R_T^2(k) & : \; k_1^0 < k \le k_2^0 \\ w_T^2(k)c_T^2(k) + 2w_T(k)c_T(k)R_T(k) + R_T^2(k) & : \; k_2^0 < k \le T, \end{cases}
\end{aligned} \tag{27}
$$

where

$$a_T(k) = \frac{T-k_2^0}{T-k}(\mu_3-\mu_2) + \frac{T-k_1^0}{T-k}(\mu_2-\mu_1), \tag{28}$$

$$b_T(k) = \frac{T-k_2^0}{T-k}(\mu_3-\mu_2) + \frac{k_1^0}{k}(\mu_2-\mu_1), \tag{29}$$

$$c_T(k) = \frac{k_2^0}{k}(\mu_3-\mu_2) + \frac{k_1^0}{k}(\mu_2-\mu_1), \tag{30}$$

$$R_T(k) = \frac{k(T-k)}{T^2}(\bar{X}_k^* - \bar{X}_k), \tag{31}$$

with

$$\bar{X}_k = \frac{1}{k}\sum_{t=1}^{k} X_t \quad \text{and} \quad \bar{X}_k^* = \frac{1}{T-k}\sum_{t=k+1}^{T} X_t. \tag{32}$$

We derive the probability limit of $Q_T^2(k/T)/\lambda_T^2$. Let $k = [\tau T]$ for $\tau \in [0,1]$. From Assumption 1, the functional central limit theorem (FCLT) holds

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{k} X_t \Rightarrow a(1)\sigma B(\tau),$$

where $B(\cdot)$ is a standard Brownian motion on $[0,1]$, and thus, it can be demonstrated that $R_T(k)$ converges to zero in probability uniformly. Hence, we have

$$Q_T^2(k/T)/\lambda_T^2 \xrightarrow{p} Q^2(\tau) = \begin{cases} \tau^2\left[(1-\tau_2^0)(\delta_3-\delta_2) + (1-\tau_1^0)(\delta_2-\delta_1)\right]^2 & : \; 0 \le \tau \le \tau_1^0 \\ \left[\tau(1-\tau_2^0)(\delta_3-\delta_2) + \tau_1^0(1-\tau)(\delta_2-\delta_1)\right]^2 & : \; \tau_1^0 < \tau \le \tau_2^0 \\ (1-\tau)^2\left[\tau_2^0(\delta_3-\delta_2) + \tau_1^0(\delta_2-\delta_1)\right]^2 & : \; \tau_2^0 < \tau \le 1 \end{cases}$$

uniformly. Notably, when (10) holds, $Q^2(\tau)$ is strictly convex and increasing in $\tau \in [0, \tau_1^0]$. Similarly, when (11) holds, $Q^2(\tau)$ is strictly convex in $\tau \in [\tau_1^0, \tau_2^0]$. On the contrary, when $\tau_2^0(\delta_3-\delta_2) + \tau_1^0(\delta_2-\delta_1) \neq 0$, which is not necessarily guaranteed by Assumption 4, $Q^2(\tau)$ is strictly convex and decreasing in $\tau \in [\tau_2^0, 1]$; otherwise, it equals 0. We can observe that τ_1^0 is uniquely identified by $Q^2(\tau)$ under Assumption 4.

Proof of (10) **and** (11): The left-hand side of (9) becomes

$$
\begin{aligned}
&Q^2(\tau_1^0) - Q^2(\tau_2^0) \\
=&(\tau_1^0)^2 \left[(1-\tau_2^0)(\delta_3-\delta_2) + (1-\tau_1^0)(\delta_2-\delta_1)\right]^2 - \left[\tau_2^0(1-\tau_2^0)(\delta_3-\delta_2) + \tau_1^0(1-\tau_2^0)(\delta_2-\delta_1)\right]^2 \\
=&(1-\tau_2^0)^2[(\tau_1^0)^2 - (\tau_2^0)^2](\delta_3-\delta_2)^2 + (\tau_1^0)^2\left[(1-\tau_1^0)^2 - (1-\tau_2^0)^2\right](\delta_2-\delta_1)^2 \\
&+ 2\left(\tau_1^0(1-\tau_1^0) - \tau_2^0(1-\tau_2^0)\right)\tau_1^0(1-\tau_2^0)(\delta_3-\delta_2)(\delta_2-\delta_1) \\
=&(\tau_2^0 - \tau_1^0)\left[(\tau_1^0)^2(2-\tau_1^0-\tau_2^0)(\delta_2-\delta_1)^2 - (1-\tau_2^0)^2(\tau_1^0+\tau_2^0)(\delta_3-\delta_2)^2\right. \\
&\left.+2\tau_1^0(1-\tau_2^0)(\tau_1^0+\tau_2^0-1)(\delta_3-\delta_2)(\delta_2-\delta_1)\right] \\
=&(\tau_2^0-\tau_1^0)LR,
\end{aligned}
$$

where

$$
\begin{aligned}
L =&\tau_1^0(\delta_2-\delta_1) - (1-\tau_2^0)(\delta_3-\delta_2), \\
R =&\tau_1^0(2-\tau_1^0-\tau_2^0)(\delta_2-\delta_1) + (1-\tau_2^0)(\tau_1^0+\tau_2^0)(\delta_3-\delta_2) \\
=&\tau_1^0\left[(1-\tau_2^0)(\delta_3-\delta_2) + (1-\tau_1^0)(\delta_2-\delta_1)\right) + (1-\tau_1^0)\left((\tau_1^0(\delta_2-\delta_1) + \tau_2^0(\delta_3-\delta_2)\right] \\
=&\tau_1^0 R_1 + (1-\tau_2^0)R_2,
\end{aligned}
$$

with

$$
\begin{aligned}
R_1 =&(1-\tau_1^0)(\delta_2-\delta_1) + (1-\tau_2^0)(\delta_3-\delta_2), \\
R_2 =&\tau_1^0(\delta_2-\delta_1) + \tau_2^0(\delta_3-\delta_2).
\end{aligned}
$$

Thus, condition (9) is equivalent to (i) $L > 0$ and $R > 0$ or (ii) $L < 0$ and $R < 0$.

First, we consider the case of (i) $L > 0$ *and* $R > 0$. If $R_1 = 0$, we have

$$
(1-\tau_2^0)(\delta_3-\delta_2) = -(1-\tau_1^0)(\delta_2-\delta_1) \tag{33}
$$

and thus, $L = \delta_2 - \delta_1$. As L is supposed to be positive, we have $\delta_2 - \delta_1 > 0$. From this result and $0 < 1-\tau_2^0 < 1-\tau_1^0 < 1$, (33) implies $\delta_2 - \delta_3 > \delta_2 - \delta_1$, leading to $R_2 = \tau_1^0(\delta_2-\delta_1) + \tau_2^0(\delta_3-\delta_2) < 0$. Then, from the definition, we have $R < 0$ in this case, which contradicts $R > 0$. Hence, when $R_1 = 0$, (i) $L > 0$ and $R > 0$ do not hold.

Next, we consider the case of (ii) $L < 0$ and $R < 0$. If $R_1 = 0$, (33) holds again, and thus, $L = \delta_2 - \delta_1$. As L is supposed to be negative, we have $\delta_2 - \delta_1 < 0$, which implies $R > 0$; however, this contradicts $R < 0$.

In sum, we find that $L \neq 0$ and $R_1 \neq 0$ are necessary conditions for Assumption 4, and thus, we obtain (10) and (11). ■

To prove Proposition 1, we need several lemmas.

Lemma 1 *Under Assumption 1, there exists an $M < \infty$ such that, for all i, j, k, l with $i < j < k < l$,*

$$\left| \mathbb{E}\left[\left(\sum_{t=i}^{j} X_t \right) \left(\sum_{s=k}^{l} X_s \right) \right] \right| < M, i < j < k < l,$$

$$\left| \frac{1}{j-i} \mathbb{E}\left[\left(\sum_{t=i}^{j} X_t \right)^2 \right] \right| < M.$$

Proof of Lemma 1: See Bai (1997a). ■

Lemma 2 *Under Assumptions 1 and 2, there exists an $M < \infty$ such that*

$$|\mathbb{E}[R_T^2(k_1^0)] - \mathbb{E}[R_T^2(k)]| \leq \frac{|k - k_1^0|}{T} \frac{1}{T} M.$$

Proof of Lemma 2: From (31), we have

$$\begin{aligned} &\mathbb{E}[R_T^2(k_1^0)] \\ =&\mathbb{E}\left[\frac{{k_1^0}^2 (T-k_1^0)^2}{T^4} \left\{ \left(\frac{1}{T-k_1^0} \sum_{t=k_1^0+1}^{T} X_t \right)^2 + \left(\frac{1}{k_1^0} \sum_{t=1}^{k_1^0} X_t \right)^2 - 2 \left(\frac{1}{k_1^0} \sum_{t=1}^{k_1^0} X_t \right) \left(\frac{1}{T-k_1^0} \sum_{t=k_1^0+1}^{T} X_t \right) \right\} \right], \end{aligned} \tag{34}$$

$$\begin{aligned} &\mathbb{E}[R_T^2(k)] \\ =&\mathbb{E}\left[\frac{k^2 (T-k)^2}{T^4} \left\{ \left(\frac{1}{T-k} \sum_{t=k+1}^{T} X_t \right)^2 + \left(\frac{1}{k} \sum_{t=1}^{k} X_t \right)^2 - 2 \left(\frac{1}{k} \sum_{t=1}^{k} X_t \right) \left(\frac{1}{T-k} \sum_{t=k+1}^{T} X_t \right) \right\} \right]. \end{aligned} \tag{35}$$

We consider the case of $1 \leq k \leq k_1^0$. The difference between the two first terms on the right-hand side of (34) and (35) is

$$
\begin{aligned}
&\left|\frac{{k_1^0}^2(T-k_1^0)^2}{T^4}\frac{1}{(T-k_1^0)^2}\mathbb{E}\left[\left(\sum_{t=k_1^0+1}^{T}X_t\right)^2\right]-\frac{k^2(T-k)^2}{T^4}\frac{1}{(T-k)^2}\mathbb{E}\left[\left(\sum_{t=k+1}^{T}X_t\right)^2\right]\right|\\
=&\left|\frac{{k_1^0}^2}{T^4}\mathbb{E}\left[\left(\sum_{t=k_1^0+1}^{T}X_t\right)^2\right]-\frac{k^2}{T^4}\mathbb{E}\left[\left(\sum_{t=k+1}^{k_1^0}X_t+\sum_{t=k_1^0+1}^{T}X_t\right)^2\right]\right|\\
=&\left|\frac{{k_1^0}^2}{T^4}\mathbb{E}\left[\left(\sum_{t=k_1^0+1}^{T}X_t\right)^2\right]-\frac{k^2}{T^4}\mathbb{E}\left[\left(\sum_{t=k_1^0+1}^{T}X_t\right)^2\right]-\frac{k^2}{T^4}\mathbb{E}\left[\left(\sum_{t=k+1}^{k_1^0}X_t\right)^2\right]\right.\\
&\left.-2\frac{k^2}{T^4}\mathbb{E}\left[\left(\sum_{t=k_1^0+1}^{T}X_t\right)\left(\sum_{t=k+1}^{k_1^0}X_t\right)\right]\right|\\
\le&\frac{({k_1^0}^2-k^2)(T-k_1^0)}{T^4}\frac{1}{T-k_1^0}\left|\mathbb{E}\left[\left(\sum_{t=k_1^0+1}^{T}X_t\right)^2\right]\right|+\frac{k^2(k_1^0-k)}{T^4}\frac{1}{k_1^0-k}\left|\mathbb{E}\left[\left(\sum_{t=k+1}^{k_1^0}X_t\right)^2\right]\right|\\
&+2\frac{k^2}{T^4}\left|\mathbb{E}\left[\left(\sum_{t=k_1^0+1}^{T}X_t\right)\left(\sum_{t=k+1}^{k_1^0}X_t\right)\right]\right|\\
\le&\frac{({k_1^0}^2-k^2)(T-k_1^0)}{T^4}M+\frac{k^2(k_1^0-k)}{T^4}M+2\frac{k^2}{T^4}M\\
=&\frac{|k-k_1^0|}{T}O(T^{-1}),
\end{aligned}
\tag{36}
$$

where the second inequality holds by Lemma 1.

Similarly, the difference between the two second terms on the right-hand side of (34) and (35) is

$$\begin{aligned}
&\left|\frac{{k_1^0}^2(T-k_1^0)^2}{T^4}\mathbb{E}\left[\frac{1}{{k_1^0}^2}\left(\sum_{t=1}^{k_1^0}X_t\right)^2\right]-\frac{k^2(T-k)^2}{T^4}\mathbb{E}\left[\frac{1}{k^2}\left(\sum_{t=1}^{k}X_t\right)^2\right]\right|\\
=&\left|\frac{(T-k_1^0)^2}{T^4}\mathbb{E}\left[\left(\sum_{t=1}^{k}X_t+\sum_{t=k+1}^{k_1^0}X_t\right)^2\right]-\frac{(T-k)^2}{T^4}\mathbb{E}\left[\left(\sum_{t=1}^{k}X_t\right)^2\right]\right|\\
\le&\frac{(T-k_1^0)^2}{T^4}\mathbb{E}\left[\left(\sum_{t=k+1}^{k_1^0}X_t\right)^2\right]+\frac{\left|(T-k_1^0)^2-(T-k)^2\right|}{T^4}\mathbb{E}\left[\left(\sum_{t=1}^{k}X_t\right)^2\right]\\
&+2\frac{(T-k_1^0)^2}{T^4}\mathbb{E}\left[\left|\left(\sum_{t=k+1}^{k_1^0}X_t\right)\left(\sum_{t=1}^{k}X_t\right)\right|\right]\\
\le&\frac{|k-k_1^0|}{T}\frac{1}{T}M+\frac{|k-k_1^0|}{T}\frac{k}{T^2}M+\frac{1}{T^2}M\\
=&\frac{|k-k_1^0|}{T}O(T^{-1}),
\end{aligned} \tag{37}$$

and the difference between the two third terms on the right-hand side of (34) and (35) is

$$\begin{aligned}
&\left|2\frac{{k_1^0}^2(T-k_1^0)^2}{T^4}\frac{1}{k_1^0(T-k_1^0)}\mathbb{E}\left[\left(\sum_{t=1}^{k_1^0}X_t\right)\left(\sum_{t=k_1^0+1}^{T}X_t\right)\right]\right|-2\frac{k^2(T-k)^2}{T^4}\frac{1}{k(T-k)}\mathbb{E}\left[\left(\sum_{t=1}^{k}X_t\right)\left(\sum_{t=k+1}^{T}X_t\right)\right]\\
\le&\frac{1}{T^2}M+\frac{k}{T^3}M=O(T^{-2}).
\end{aligned} \tag{38}$$

From (36), (37), and (38), there exists an $M<\infty$ such that

$$\left|\mathbb{E}[R_T^2(k_1^0)]-\mathbb{E}[R_T^2(k)]\right|\le\frac{|k-k_1^0|}{T}\frac{1}{T}M.$$

The proof for $k\ge k_1^0+1$ is analogous and omitted. ■

Lemma 3 *Under Assumptions 1–4, there exists a $C>0$ for all large T such that*

$$\mathbb{E}[Q_T^2(k_1^0/T)]-\mathbb{E}[Q_T^2(k/T)]\ge\frac{|k-k_1^0|}{T}\lambda_T^2C.$$

Proof of Lemma 3: For $1 \leq k \leq k_1^0$, we have

$$\begin{aligned}\mathbb{E}\left[Q_T^2(k_1^0/T)\right] - \mathbb{E}\left[Q_T^2(k/T)\right] &= \left(Q^2(\tau_1^0) - Q^2(\tau)\right)\lambda_T^2 + \mathbb{E}[R_T^2(k_1^0)] - \mathbb{E}[R_T^2(k)] \\ &= (\tau_1^0 - \tau)(\tau_1^0 + \tau)\left[(1-\tau_2^0)(\delta_3 - \delta_2) + (1-\tau_1^0)(\delta_2 - \delta_1)\right]^2 \lambda_T^2 \\ &\quad + \mathbb{E}[R_T^2(k_1^0)] - \mathbb{E}[R_T^2(k)] \\ &\geq \frac{|k-k_1^0|}{T} C\lambda_T^2 + \frac{|k-k_1^0|}{T} O(T^{-1}) \\ &\geq \frac{|k-k_1^0|}{T} C\lambda_T^2\end{aligned}$$

for all sufficiently large T, where the first inequality holds by Lemma 2.

For $k_1^0 + 1 \leq k \leq k_2^0$, we have

$$\mathbb{E}\left[Q_T^2(k_1^0/T)\right] - \mathbb{E}\left[Q_T^2(k/T)\right] = \left(Q^2(\tau_1^0) - Q^2(\tau)\right)\lambda_T^2 + \mathbb{E}[R^2(k_1^0)] - \mathbb{E}[R^2(k)].$$

As $Q^2(\tau)$ is a quadratic and convex function of τ and maximized at $\tau = \tau_1^0$ in $\tau \in [\tau_1^0, \tau_2^0]$, we can observe that $[Q^2(\tau) - Q^2(\tau_1^0)]/(\tau - \tau_1^0)$ takes negative values and increases in τ. This implies that, by letting $-C = [Q^2(\tau_2^0) - Q^2(\tau_1^0)]/(\tau_2^0 - \tau_1^0)$,

$$\frac{Q^2(\tau) - Q^2(\tau_1^0)}{\tau - \tau_1^0} < -C \quad \text{or equivalently,} \quad Q^2(\tau_1^0) - Q^2(\tau) > C(\tau - \tau_1^0)$$

on $\tau \in [\tau_1^0, \tau_2^0]$. Using this relation and Lemma 2, we have

$$\mathbb{E}\left[Q_T^2(k_1^0/T)\right] - \mathbb{E}\left[Q_T^2(k/T)\right] \geq \frac{|k-k_1^0|}{T} C\lambda_T^2 + \frac{|k-k_1^0|}{T} O(T^{-1}) \geq \frac{|k-k_1^0|}{T} C\lambda_T^2.$$

For $k_2^0 + 1 \leq k \leq T$, we have

$$\mathbb{E}[Q_T^2(k_1^0/T)] - \mathbb{E}[Q_T^2(k/T)] = \left(\mathbb{E}[Q_T^2(k_1^0/T)] - \mathbb{E}[Q_T^2(k_2^0/T)]\right) + \left(\mathbb{E}[Q_T^2(k_2^0/T)] - \mathbb{E}[Q_T^2(k/T)]\right). \tag{39}$$

The second term on the right-hand side of (39) is

$$\begin{aligned}\mathbb{E}[Q_T^2(k_2^0/T)] - \mathbb{E}[Q_T^2(k/T)] &= \left(Q^2(\tau_2^0) - Q^2(\tau)\right)\lambda_T^2 + \mathbb{E}[R_T^2(k_2^0)] - \mathbb{E}[R_T^2(k)] \\ &\geq \mathbb{E}[R_T^2(k_2^0)] - \mathbb{E}[R_T^2(k)] \\ &= O(T^{-1}),\end{aligned}$$

whereas the first term on the right-hand side of (39) becomes, again by Lemma 2,

$$\begin{aligned}\mathbb{E}[Q_T^2(k_1^0/T)] - \mathbb{E}[Q_T^2(k_2^0/T)] &= \left(Q^2(\tau_1^0) - Q^2(\tau_2^0)\right)\lambda_T^2 + \mathbb{E}[R_T^2(k_1^0)] - \mathbb{E}[R_T^2(k_2^0)] \\ &\geq C\lambda_T^2 + \frac{|k-k_1^0|}{T} O(T^{-1}) \geq \frac{|k-k_1^0|}{T} C\lambda_T^2,\end{aligned}$$

because $|k-k_1^0|/T \leq 1$. The above two results yield

$$\mathbb{E}[Q_T^2(k_1^0/T)] - \mathbb{E}[Q_T^2(k/T)] \geq \frac{|k-k_1^0|}{T} C\lambda_T^2$$

and we obtain the lemma. ■

Lemma 4 *Under Assumptions 1–3, we have*

$$\sup_{1\leq k\leq T-1} \left|Q_T^2(k/T) - \mathbb{E}[Q_T^2(k_1^0/T)]\right| = O_p\left(\lambda_T T^{-1/2}\right).$$

Proof of Lemma 4: The first term on the right-hand side of (27) is non-stochastic and hence, for $k \leq k_1^0$, we have

$$\begin{aligned}
&Q_T^2(k/T) - \mathbb{E}[Q_T^2(k/T)] \\
=&R_T^2(k) - \mathbb{E}[R_T^2(k)] + 2w_T(k)a_T(k)R_T(k) \\
=&\frac{k^2(T-k)^2}{T^4}\left(\bar{X}_t^* - \bar{X}_t\right)^2 - \frac{k^2(T-k)^2}{T^4}\mathbb{E}\left[\left(\bar{X}_t^* - \bar{X}_t\right)^2\right] \\
&+ 2\frac{k^2(T-k)^2}{T^4}\left(\frac{T-k_2^0}{T-k}(\delta_3-\delta_2)\lambda_T + \frac{T-k_1^0}{T-k}(\delta_2-\delta_1)\lambda_T\right)\left(\bar{X}_t^* - \bar{X}_t\right). \qquad (40)
\end{aligned}$$

It can be shown that the first and second terms on the right-hand side of (40) are $O_p(T^{-1})$ and $O(T^{-1})$, respectively, whereas the third term is $O_p(\lambda_T T^{-1/2})$. The proof for $k \geq k_1^0 + 1$ is analogous and omitted. ■

Proof of Proposition 1: The proof proceeds similarly to that of Corollary 1 in Bai (1997a). We note the following:

$$\begin{aligned}
Q_T^2(k/T) - Q_T^2(k_1^0/T) =&Q_T^2(k/T) - \mathbb{E}[Q_T^2(k/T)] \\
&+ \mathbb{E}[Q_T^2(k/T)] - \mathbb{E}[Q_T^2(k_1^0/T)] + \mathbb{E}[Q_T^2(k_1^0/T)] - Q_T^2(k_1^0/T) \\
\leq&2\sup_{1\leq k\leq T-1}|Q_T^2(k/T) - \mathbb{E}[Q_T^2(k/T)]| - (\mathbb{E}[Q_T^2(k_1^0/T)] - \mathbb{E}[Q_T^2(k/T)]). \qquad (41)
\end{aligned}$$

As $Q_T^2(\hat{k}_{W,1}/T) - Q_T^2(k_1^0/T) \geq 0$, (41) implies that

$$\mathbb{E}[Q_T^2(k_1^0/T)] - \mathbb{E}[Q_T^2(\hat{k}_{W,1}/T)] \leq 2\sup_{1\leq k\leq T-1}\left|Q_T^2(k/T) - \mathbb{E}[Q_T^2(k/T)]\right|.$$

Using Lemmas 3 and 4, we have

$$|\hat{\tau}_{W,1} - \tau_1^0| \leq O_p(\lambda_T^{-1} T^{-1/2}) = o_p(1).$$

■

To refine the convergence order, we focus on the neighborhood of k_1^0. As the consistency of $\hat{\tau}$ is established by Proposition 1, we consider the interval given by $D_T = \{k : T\eta \leq k \leq T\tau_2^0(1-\eta)\}$ such that $P(\hat{k}_{W,1} \in D_T) \geq 1 - \epsilon$ for all large T and a given $\epsilon > 0$, where η is a small positive value satisfying $\tau_1^0 \in (\eta, \tau_2^0(1-\eta))$. We demonstrate that $Q_T^2(k/T)$ is hardly maximized on $D_{T,M^c} = \{k : T\eta \leq k \leq T\tau_2^0(1-\eta), |k - k_1^0| > M\lambda_T^{-2}\}$. Noting that the maximization of $Q_T^2(k/T)$ is equivalent to the minimization of $Q_T^2(k_1^0/T) - Q_T^2(k/T)$, we present the following lemma:

Lemma 5 *Under Assumptions 1–4, for every $\epsilon > 0$, there exists an $M > 0$ such that*

$$P\left(\min_{k \in D_{T,M^c}} \left\{Q_T^2(k_1^0/T) - Q_T^2(k/T)\right\} \leq 0\right) < \epsilon.$$

Proof of Lemma 5: By Lemma 3, we have

$$\begin{aligned} Q_T^2(k_1^0/T) - Q_T^2(k/T) =& Q_T^2(k_1^0/T) - \mathbb{E}[Q_T^2(k_1^0/T)] - (Q_T^2(k/T) - \mathbb{E}[Q_T^2(k/T)]) \\ &+ \mathbb{E}[Q_T^2(k_1^0/T)] - \mathbb{E}[Q_T^2(k/T)] \\ \geq& Q_T^2(k_1^0/T) - \mathbb{E}[Q_T^2(k_1^0/T)] - (Q_T^2(k/T) - \mathbb{E}[Q_T^2(k/T)]) + \frac{|k - k_1^0|}{T}\lambda_T^2 C. \end{aligned}$$

Then, $Q_T^2(k_1^0/T) - Q_T^2(k/T) \leq 0$ implies

$$C\lambda_T^2 \leq \left|Q_T^2(k_1^0/T) - \mathbb{E}[Q_T^2(k_1^0/T)] - (Q_T^2(k/T) - \mathbb{E}[Q_T^2(k/T)])\right| \frac{T}{|k - k_1^0|}.$$

Therefore, it is sufficient to prove that for any given value of κ,

$$P\left(\sup_{k \in D_{T,M^c}} \left\{\left|Q_T^2(k_1^0/T) - \mathbb{E}[Q_T^2(k_1^0/T)] - (Q_T^2(k/T) - \mathbb{E}[Q_T^2(k/T)])\right| \frac{T}{|k - k_1^0|}\right\} > \kappa\lambda_T^2\right) < \varepsilon.$$

Notably,

$$\begin{aligned} &\left|Q_T^2(k_1^0/T) - \mathbb{E}[Q_T^2(k_1^0/T)] - (Q_T^2(k/T) - \mathbb{E}[Q_T^2(k/T)])\right| \\ &= \left|R_{1,T}^2(k) - R_{i,T}^2(k) - \left(\mathbb{E}[R_T^2(k_1^0)] - \mathbb{E}[R_T^2(k)]\right)\right| \\ &\leq \left|R_{1,T}^2(k) - R_{i,T}^2(k)\right| + \left|\mathbb{E}[R_T^2(k_1^0)] - \mathbb{E}[R_T^2(k)]\right|, \end{aligned} \tag{42}$$

where, for $k \leq k_1^0$,

$$R_{i,T}^2(k) = R_{1,T}^2(k) = 2w_T^2(k)a_T(k)(\bar{X}_k^* - \bar{X}_{Tk}) + R_T^2(k), \tag{43}$$

and, for $k_1^0 + 1 \leq k < k_2^0$,

$$R_{i,T}^2(k) = R_{2,T}^2(k) = 2w_T^2(k)b_T(k)(\bar{X}_k^* - \bar{X}_k) + R_T^2(k). \tag{44}$$

By Lemma 2, the second term on the right-hand side of (42) multiplied by $T/|k - k_1^0|$ is evaluated as

$$\frac{T}{|k - k_1^0|} \left| \mathbb{E}[R_T^2(k_1^0)] - \mathbb{E}[R_T^2(k)] \right| \leq \frac{T}{|k - k_1^0|} \frac{|k - k_1^0|}{T} O(T^{-1}) = O(T^{-1}),$$

which suggests that the left-hand side converges to 0 faster than λ_T^2. Therefore, it suffices to show that, for any $\kappa > 0$ and $\epsilon > 0$,

$$P\left(\sup_{k \in D_{T,M}{}^c} \left\{ \left| R_{1,T}^2(k_1^0) - R_{i,T}^2(k) \right| \frac{T}{|k - k_1^0|} \right\} > \kappa \lambda_T^2 \right) < \epsilon \tag{45}$$

for $i = 1$ and 2. We demonstrate that the left-hand side of the inequality is $o_p(\lambda_T^2)$.

Notably, for $k \leq k_1^0$,

$$\begin{aligned}
&\left| R_{1,T}^2(k_1^0) - R_{i,T}^2(k) \right| \frac{T}{|k - k_1^0|} \\
&\left| 2w_T^2(k_1^0)a_T(k_1^0)(\bar{X}_{k_1^0}^* - \bar{X}_{k_1^0}) + R_T^2(k_1^0) - \left[2w_T^2(k)a_T(k)(\bar{X}_k^* - \bar{X}_k) + R_T^2(k) \right] \right| \frac{T}{|k - k_1^0|} \\
=& \left| 2\frac{{k_1^0}^2(T - k_1^0)^2}{T^4} a_T(k_1^0)(\bar{X}_{k_1^0}^* - \bar{X}_{k_1^0}) + R_T^2(k_1^0) - \left(2\frac{k^2(T-k)^2}{T^4} a_T(k)(\bar{X}_k^* - \bar{X}_k) + R_T^2(k) \right) \right| \frac{T}{|k - k_1^0|} \\
\leq& \left| 2\left(\frac{{k_1^0}^2(T - k_1^0)}{T^4}(\bar{X}_{k_1^0}^* - \bar{X}_{k_1^0}) - \frac{k^2(T-k)}{T^4}(\bar{X}_k^* - \bar{X}_k) \right) \left((\delta_3 - \delta_2)(T - k_2^0) + (\delta_2 - \delta_1)(T - k_1^0) \right) \lambda_T \right| \frac{T}{|k - k_1^0|} \\
&+ \left| R_T^2(k_1^0) - R_T^2(k) \right| \frac{T}{|k - k_1^0|} \\
\leq& \left| \left(\frac{{k_1^0}^2(T - k_1^0)}{T^4}(\bar{X}_{k_1^0}^* - \bar{X}_{k_1^0}) - \frac{k^2(T-k)}{T^4}(\bar{X}_k^* - \bar{X}_k) \right) \right| CT\lambda_T \frac{T}{|k - k_1^0|} + \left| R_T^2(k_1^0) - R_T^2(k) \right| \frac{T}{|k - k_1^0|}.
\end{aligned} \tag{46}$$

The first term on the right-hand side of (46) is

$$
\begin{aligned}
&\left(\frac{{k_1^0}^2(T-k_1^0)}{T^4}(\bar{X}^*_{k_1^0}-\bar{X}_{k_1^0})-\frac{k^2(T-k)}{T^4}(\bar{X}^*_k-\bar{X}_k)\right)CT\lambda_T\frac{T}{|k-k_1^0|}\\
=&\left(\frac{{k_1^0}^2}{T^4}\sum_{t=k_1^0+1}^{T}X_t-\frac{k_1^0(T-k_1^0)}{T^4}\sum_{t=1}^{k_1^0}X_t-\frac{k^2}{T^4}\sum_{t=k+1}^{T}X_t+\frac{k(T-k)}{T^4}\sum_{t=1}^{k}X_t\right)CT\lambda_T\frac{T}{|k-k_1^0|}\\
=&\left(\frac{{k_1^0}^2-k^2}{T^4}\sum_{t=k_1^0+1}^{T}X_t-\frac{k_1^0(T-k_1^0)+k^2}{T^4}\sum_{t=k+1}^{k_1^0}X_t-\frac{k_1^0(T-k_1^0)-k(T-k)}{T^4}\sum_{t=1}^{k}X_t\right)CT\lambda_T\frac{T}{|k-k_1^0|}.
\end{aligned}
\tag{47}
$$

We can observe that the first term on the right-hand side of (47) is

$$
\begin{aligned}
CT\lambda_T\frac{T}{|k-k_1^0|}\frac{{k_1^0}^2-k^2}{T^4}\sum_{t=k_1^0+1}^{T}X_t =&C\lambda_T\frac{k_1^0+k}{T^{3/2}}\frac{1}{T^{1/2}}\sum_{t=k_1^0+1}^{T}X_t\\
=&O_p(\lambda_T T^{-1/2})=o_p(\lambda_T^2),
\end{aligned}
$$

while the third term becomes

$$
\begin{aligned}
CT\lambda_T\frac{T}{|k-k_1^0|}\frac{k_1^0(T-k_1^0)-k(T-k)}{T^4}\sum_{t=1}^{k}X_t =&C\lambda_T\frac{(T-k_1^0-k)}{T^{3/2}}\frac{1}{T^{1/2}}\sum_{t=1}^{k}X_t\\
=&O_p(\lambda_T T^{-1/2})=o_p(\lambda_T^2).
\end{aligned}
$$

On the contrary, the second term is

$$
CT\lambda_T\frac{T}{|k-k_1^0|}\frac{k_1^0(T-k_1^0)+k^2}{T^4}\sum_{t=k+1}^{k_1^0}X_t=O(1)\frac{\lambda_T}{|k-k_1^0|}\sum_{t=k+1}^{k_1^0}X_t.
$$

Using Hájek-Rényi inequality (HRI) in Bai (1994), we have

$$
P\left(\sup_{k\le k_1^0-M\lambda_T^{-2}}\left\{\frac{1}{|k-k_1^0|}\left|\sum_{t=k+1}^{k_1^0}X_t\right|\right\}>\kappa\lambda_T\right)<\frac{C}{\kappa^2 M}. \tag{48}
$$

Taking a large value of M, (48) becomes sufficieintly small, which implies that the second term is $o_p(\lambda_T^2)$.

For the second term on the right-hand side of (46), we observe that $R_T^2(k_1^0)-R_T^2(k)$ is $O_p(T^{-1})$ because (31) is $O_p(T^{-1/2})$ uniformly in D_{T,M^c}. Then, we have

$$
\left|R_T^2(k_1^0)-R_T^2(k)\right|\frac{T}{|k-k_1^0|}\le O_p(T^{-1})\frac{T}{M\lambda_T^{-2}}=O_p(1)\frac{\lambda_T^2}{M}.
$$

By taking a sufficiently large value of M, the left-hand side is shown to be $o_p(\lambda_T^2)$.

In exactly the same manner, we can show that (45) holds for $i = 2$ $(k_1^0 + 1 \leq k < k_2^0)$, and thus, the lemma is established. ■

Proof of Proposition 2: Under Assumptions 1–4, for all ϵ, there exists an $M < \infty$ such that

$$\begin{aligned} P\left(\left|\hat{k}_{W,1} - k_1^0\right| > M\lambda_T^{-2}\right) \leq & P\left(\hat{k}_{W,1} \notin D_T\right) + P\left(\hat{k}_{W,1} \in D_T, \left|\hat{k}_{W,1} - k_1^0\right| > M\lambda_T^{-2}\right) \\ \leq & \epsilon + P\left(\min_{k \in D_{T,M^c}} \left\{Q_T^2(k_1^0/T) - Q_T^2(k/T)\right\} \leq 0\right) \\ \leq & 2\epsilon, \end{aligned}$$

where the last inequality holds by Lemma 5. ■

Proof of Theorem 1: By Proposition 2, we can focus on the $O(\lambda_T^{-2})$ neighborhood of k_1^0. Thus, let us define $k = k_1^0 + \ell$ where $\ell = s\lambda_T^{-2}$ with $s \in [-M, M]$ for some large value of M. Then, we have $\hat{k} = k_1^0 + \hat{\ell}$. To investigate the behavior of $\hat{k}$, we consider the behavior of $\hat{\ell}$ defined as follows:

$$\begin{aligned} \hat{\ell} &= \underset{\ell \in (-\infty, \infty)}{\arg\max} \left\{Q_T^2((k_1^0 + \ell)/T) - Q_T^2(k_1^0/T)\right\} \\ &= \underset{\ell \in (-\infty, \infty)}{\arg\max} \left\{T\left(Q_T^2((k_1^0 + \ell)/T) - Q_T^2(k_1^0/T)\right)\right\}. \end{aligned}$$

For $\ell \leq 0$, we have

$$Q_T^2((k_1^0 + \ell)/T) = w_T^2(k_1^0 + \ell)a_T^2(k_1^0 + \ell) + 2w_T(k_1^0 + \ell)a_T(k_1^0 + \ell)R_T(k_1^0 + \ell) + R_T^2(k_1^0 + \ell), \quad (49)$$

$$Q_T^2(k_1^0/T) = w_T^2(k_1^0)a_T^2(k_1^0) + 2w_T(k_1^0)a_T(k_1^0)R_T(k_1^0) + R_T^2(k_1^0). \quad (50)$$

The difference between the two first terms on the right-hand side of (49) and (50) is

$$\begin{aligned} & w_T^2(k_1^0 + \ell)a_T^2(k_1^0 + \ell) - w_T^2(k_1^0)a_T^2(k_1^0) \\ =& \frac{(k_1^0 + \ell)^2 - {k_1^0}^2}{T^4}[(T - k_2^0)(\mu_3 - \mu_2) + (T - k_1^0)(\mu_2 - \mu_1)]^2 \\ =& \frac{2k_1^0 s + s^2\lambda_T^{-2}}{T^2}[(1 - \tau_2^0)(\delta_3 - \delta_2) + (1 - \tau_1^0)(\delta_2 - \delta_1)]^2 \\ =& \frac{2\tau_1^0 s}{T}[(1 - \tau_2^0)(\delta_3 - \delta_2) + (1 - \tau_1^0)(\delta_2 - \delta_1)]^2 + O_p(T^{-2}\lambda_T^{-2}). \end{aligned} \quad (51)$$

Notably, the first term on the right-hand side of (51) dominates the second term.

The difference between the two third terms on the right-hand side of (49) and (50) becomes

$$\begin{aligned}
&R_T^2(k_1^0+\ell) - R_T^2(k_1^0)\\
=&\frac{(k_1^0+\ell)^2(T-k_1^0-\ell)^2}{T^4}\left(\frac{1}{T-k_1^0-\ell}\sum_{t=k_1^0+\ell+1}^{T} X_t - \frac{1}{k_1^0+\ell}\sum_{t=1}^{k_1^0+\ell} X_t\right)^2\\
&-\frac{{k_1^0}^2(T-k_1^0)^2}{T^4}\left(\frac{1}{T-k_1^0}\sum_{t=k_1^0+1}^{T} X_t - \frac{1}{k_1^0}\sum_{t=1}^{k_1^0} X_t\right)^2\\
=&\left(\frac{k_1^0+\ell}{T^2}\sum_{t=k_1^0+1}^{T} X_t + \frac{k_1^0+\ell}{T^2}\sum_{t=k_1^0+\ell+1}^{k_1^0} X_t - \frac{T-k_1^0-\ell}{T^2}\sum_{t=1}^{k_1^0+\ell} X_t\right)^2\\
&-\left(\frac{k_1^0}{T^2}\sum_{t=k_1^0+1}^{T} X_t - \frac{T-k_1^0}{T^2}\sum_{t=k_1^0+\ell+1}^{k_1^0} X_t - \frac{T-k_1^0}{T^2}\sum_{t=1}^{k_1^0+\ell} X_t\right)^2\\
=&\frac{2k_1^0\ell+\ell^2}{T^4}\left(\sum_{t=k_1^0+1}^{T} X_t\right)^2 + \frac{(k_1^0+\ell)^2-(T-k_1^0)^2}{T^4}\left(\sum_{t=k_1^0+\ell+1}^{k_1^0} X_t\right)^2 + \frac{2(T-k_1^0)\ell+\ell^2}{T^4}\left(\sum_{t=1}^{k_1^0+\ell} X_t\right)^2\\
&+\frac{2\left((k_1^0+\ell)^2+k_1^0(T-k_1^0)\right)}{T^4}\left(\sum_{t=k_1^0+1}^{T} X_t\right)\left(\sum_{t=k_1^0+\ell+1}^{k_1^0} X_t\right)\\
&+\frac{2\left(-(k_1^0+\ell)(T-k_1^0-\ell)-(T-k_1^0)^2\right)}{T^4}\left(\sum_{t=k_1^0+\ell+1}^{k_1^0} X_t\right)\left(\sum_{t=1}^{k_1^0+\ell} X_t\right)\\
&+\frac{2\left(-(k_1^0+\ell)(T-k_1^0-\ell)+k_1^0(T-k_1^0)\right)}{T^4}\left(\sum_{t=k_1^0+1}^{T} X_t\right)\left(\sum_{t=1}^{k_1^0+\ell} X_t\right)\\
=&O_p(\lambda_T^{-1}T^{-3/2}),
\end{aligned} \tag{52}$$

and thus, $T\left(R_T^2(k_1^0+\ell) - R_T^2(k_1^0)\right) = o_p(1)$.

The difference between the two second terms on the right-hand side of (49) and (50) becomes

$$
\begin{aligned}
&2w_T(k_1^0+\ell)a_T(k_1^0+\ell)R_T(k_1^0+\ell)-2w_T(k_1^0)a_T(k_1^0)R_T(k_1^0)\\
=&2\frac{(k_1^0+\ell)^2(T-k_1^0-\ell)}{T^4}\left((T-k_2^0)(\mu_3-\mu_2)+(T-k_1^0)(\mu_2-\mu_1)\right)\left(\frac{1}{T-k_1^0-\ell}\sum_{t=k_1^0+\ell+1}^{T}X_t-\frac{1}{k_1^0+\ell}\sum_{t=1}^{k_1^0+\ell}X_t\right)\\
&-2\frac{{k_1^0}^2(T-k_1^0)}{T^4}\left((T-k_2^0)(\mu_3-\mu_2)+(T-k_1^0)(\mu_2-\mu_1)\right)\left(\frac{1}{T-k_1^0}\sum_{t=k_1^0+1}^{T}X_t-\frac{1}{k_1^0}\sum_{t=1}^{k_1^0}X_t\right)\\
=&2\frac{\lambda_T}{T^3}\left((1-\tau_2^0)(\delta_3-\delta_2)+(1-\tau_1^0)(\delta_2-\delta_1)\right)\\
&\times\left\{(2k_1^0\ell+\ell^2)\sum_{t=k_1^0+1}^{T}X_t+(k_1^0T+\ell)\sum_{t=k_1^0+\ell+1}^{k_1^0}X_t-\ell(T-2k_1^0-\ell)\sum_{t=1}^{k_1^0}X_t\right\}. \qquad (53)
\end{aligned}
$$

By the FCLT, we observe that

$$
\begin{aligned}
(2k_1^0\ell+\ell^2)\sum_{t=k_1^0}^{T}X_t &=O_p(T^{3/2}\lambda_T^{-2}), &(54)\\
(k_1^0T+\ell)\sum_{t=k_1^0+\ell+1}^{k_1^0}X_t &=O_P(T^2\lambda_T^{-1}), &(55)\\
\ell(T-2k_1^0-\ell)\sum_{t=1}^{k_1^0+\ell}X_t &=O_p(T^{3/2}\lambda_T^{-2}), &(56)
\end{aligned}
$$

and hence, (55) dominates (54) and (56). Therefore, (53) becomes

$$
2\frac{\lambda_T}{T^3}\left((1-\tau_2^0)(\delta_3-\delta_2)+(1-\tau_1^0)(\delta_2-\delta_1)\right)\left\{(k_1^0T+\ell)\sum_{t=k_1^0+\ell+1}^{k_1^0}X_t+O_p(T^{3/2}\lambda_T^{-2})\right\}. \qquad (57)
$$

By (51), (52), and (57), we can derive the asymptotic distribution for $\ell\leq 0$ as follows:

$$
\begin{aligned}
T(Q_T^2((k_1^0+\ell)/T)-Q_T^2(k_1^0/T)) \Rightarrow&2\tau_1^0 s\left[(1-\tau_2^0)(\delta_3-\delta_2)+(1-\tau_1^0)(\delta_2-\delta_1)\right]^2\\
&+2\tau_1^0\left[(1-\tau_2^0)(\delta_3-\delta_2)+(1-\tau_1^0)(\delta_2-\delta_1)\right]\sigma a(1)B_1(|s|)\\
=&2\tau_1^0(g^2s+\sigma a(1)gB_1(|s|))\\
\stackrel{d}{=}&2\tau_1^0(-g^2|s|+\sigma a(1)|g|B_1(|s|)), \qquad (58)
\end{aligned}
$$

where $B_1(\cdot)$ is a standard Brownian motion on $[0,\infty)$, $\stackrel{d}{=}$ denotes equality in distribution, and the last expression holds because $s\leq 0$ and $gB_1(\cdot)\stackrel{d}{=}|g|B_1(\cdot)$ as $B_1(\cdot)$ is symmetrically distributed.

For $\ell > 0$, we consider the difference between the following equations:

$$Q_T^2((k_1^0+\ell)/T) = w_T^2(k_1^0+\ell)b_T^2(k_1^0+\ell) + 2w_T(k_1^0+\ell)b_T(k_1^0+\ell)R_T(k_1^0+\ell) + R_T^2(k_1^0+\ell),$$
$$Q_T^2(k_1^0/T) = w_T^2(k_1^0)b_T^2(k_1^0) + 2w_T(k_1^0)b_T(k_1^0)R_T(k_1^0) + R_T^2(k_1^0).$$

Then, we can derive the asymptotic distribution analogously to the case of $\ell \le 0$ as follows:

$$\begin{aligned}
&T(Q_T^2((k_1^0+\ell)/T) - Q_T^2(k_1^0/T)) \\
\Rightarrow &\Big\{2\tau_1^0(1-\tau_2^0)^2 s(\delta_3-\delta_2)^2 - 2{\tau_1^0}^2(1-\tau_1^0)s(\delta_2-\delta_1)^2 + 2\tau_1^0(1-2\tau_1^0)(1-\tau_2^0)s(\delta_3-\delta_2)(\delta_2-\delta_1)\Big\} \\
&-2\left[\tau_1^0(1-\tau_2^0)(\delta_3-\delta_2) + \tau_1^0(1-\tau_1^0)(\delta_2-\delta_1)\right)\sigma a(1)B_2(s) \\
=&2\tau_1^0(ghs - \sigma a(1)gB_2(s)), \\
\stackrel{d}{=}&2\tau_1^0(-|gh|s + \sigma a(1)|g|B_2(s)), \qquad (59)
\end{aligned}$$

where $B_2(\cdot)$ is a standard Brownian motion on $[0,\infty)$ independent of $B_1(\cdot)$ and the last expression holds because $-gB_2(\cdot) \stackrel{d}{=} |g|B_2(\cdot)$ as $B_2(\cdot)$ is symmetrically distributed and gh takes negative values as $2\tau_1^0 ghs$ corresponds to the limit of $w_T^2(k_1^0+\ell)b_T^2(k_1^0+\ell) - w_T^2(k_1^0)b_T^2(k_1^0)$, which takes negative values because $w_T^2(k_1^0+\ell)b_T^2(k_1^0+\ell)$ attains its maximum at $\ell = 0$. Applying the continuous mapping theorem (CMT) to (58) and (59) and using the change in variable with $s = \sigma^2 a^2(1)u$, we obtain the theorem. ■

To prove Theorem 2, we begin with the proof of the following lemma. Notably, we maximize $Q_{T'}^2(k/T)$ or $|Q_{T'}(k/T)|$ in (18) to estimate the second break point.

Lemma 6 *Under Assumptions 1–3, there exists a $C > 0$ such that*

$$\left|\mathbb{E}[Q_{T_0'}(k_2^0/T)]\right| - \left|\mathbb{E}[Q_{T_0'}(k/T)]\right| \ge C\frac{|k-k_2^0|}{T}\lambda_T,$$

for $k_1^0+1 \le k \le T-M_T$, where M_T is a monotonically increasing sequence as defined in the proof of Proposition3 below and

$$Q_{T_0'}(k/T) = \frac{(T-k)(k-k_1^0)}{(T-k_1^0)^2}\left(\bar{Y}_k^* - \bar{Y}_{0,k}'\right) \quad with \quad \bar{Y}_{0,k}' = \frac{1}{k-k_1^0}\sum_{t=k_1^0+1}^{k} y_t. \qquad (60)$$

Proof of Lemma 6: From (60), we have

$$\mathbb{E}[Q_{T_0'}(k/T)] = \begin{cases} \frac{(k-k_1^0)(T-k_2^0)}{(T-k_1^0)^2}(\mu_3-\mu_2) & \text{for } k_1^0 \le k \le k_2^0 \\ \frac{(k_2^0-k_1^0)(T-k)}{(T-k_1^0)^2}(\mu_3-\mu_2) & \text{for } k_2^0+1 \le k < T-M_T \end{cases}, \tag{61}$$

$$\mathbb{E}[Q_{T_0'}(k_2^0/T)] = \frac{(k_2^0-k_1^0)(T-k_2^0)}{(T-k_1^0)^2}(\mu_3-\mu_2). \tag{62}$$

The difference between (61) and (62) is, for $k_1^0+1 \le k \le k_2^0$,

$$\begin{aligned} \left|\mathbb{E}[Q_{T_0'}(k_2^0/T)]\right| - \left|\mathbb{E}[Q_{T_0'}(k/T)]\right| &= \frac{(T-k_2^0)(k_2^0-k)}{(T-k_1^0)^2}|\delta_3-\delta_2|\lambda_T \\ &\ge \frac{k_2^0-k}{T}C\lambda_T \end{aligned}$$

and for $k_2^0+1 \le k < T-M_T$,

$$\begin{aligned} \left|\mathbb{E}[Q_{T_0'}(k_2^0/T)]\right| - \left|\mathbb{E}[Q_{T_0'}(k/T)]\right| &= \frac{(k_2^0-k_1^0)(k-k_2^0)}{(T-k_1^0)^2}|\delta_3-\delta_2|\lambda_T \\ &\ge \frac{k-k_2^0}{T}C\lambda_T. \end{aligned}$$

The above two results yield

$$\left|\mathbb{E}[Q_{T_0'}(k_2^0/T)]\right| - \left|\mathbb{E}[Q_{T_0'}(k/T)]\right| \ge C\frac{|k-k_2^0|}{T}\lambda_T$$

and we obtain the lemma. ■

Proof of Proposition 3: Let M_T be an increasing sequence such that $M_T \to \infty$, $\lambda_T^2 M_T \to \infty$, and $M_T/T \to 0$. First, we show that $|Q_{T'}(k/T)|$ cannot be maximized asymptotically for k in the neighborhood of both of the end points: $\hat{k}_{W,1}$, k_1^0, and T.

Let us consider the neighborhood of the left end point given by $\hat{k}_{W,1} < k \le \max(k_1^0, \hat{k}_{W,1}) + M_T$. We note that $\max(k_1^0, \hat{k}_{W,1}) + M_T < k_2^0$, at least asymptotically, because of the consistency of $\hat{\tau}_{W,1}$ and the definition of M_T. This implies that k takes values less than k_2^0 in this neighborhood. When $\hat{k}_{W,1} < k_1^0$, we have, for $\hat{k}_{W,1} < k \le k_1^0$,

$$\begin{aligned} \bar{Y}_k^* &= \frac{1}{T-k}\left(\sum_{t=k+1}^{k_1^0}\mu_1 + \sum_{t=k_1^0+1}^{k_2^0}\mu_2 + \sum_{t=k_2^0+1}^{T}\mu_3 + \sum_{t=k+1}^{T}X_T\right) \\ &= \mu_0 + \frac{k_1^0-k}{T-k}\delta_1\lambda_T + \frac{k_2^0-k_1^0}{T-k}\delta_2\lambda_T + \frac{T-k_2^0}{T-k}\delta_3\lambda_T + \frac{1}{T-k}\sum_{t=k+1}^{T}X_t, \end{aligned}$$

$$\bar{Y}'_k = \frac{1}{k - \hat{k}_{W,1}} \sum_{t=\hat{k}_{W,1}+1}^{k} (\mu_1 + X_t) = \mu_0 + \delta_1 \lambda_T + \frac{1}{k - \hat{k}_{W,1}} \sum_{t=\hat{k}_{W,1}+1}^{k} X_t,$$

and thus,

$$\bar{Y}^*_k - \bar{Y}'_k = \frac{1}{T-k} \left[(T - k_2^0)(\delta_3 - \delta_2) + (T - k_1^0)(\delta_2 - \delta_1) \right) \lambda_T + \frac{1}{T-k} \sum_{t=k+1}^{T} X_t - \frac{1}{k - \hat{k}_{W,1}} \sum_{t=\hat{k}_{W,1}+1}^{k} X_t.$$

Notably, the coefficient of λ_T does not equal zero by (10). Using the above result, we obtain

$$\begin{aligned} Q_{T'}(k/T) =& \frac{(k - \hat{k}_{W,1})(T-k)}{(T - \hat{k}_{W,1})^2} \left(\bar{Y}^*_k - \bar{Y}'_k \right) \\ =& O_p \left(\lambda_T M_T T^{-1} \right) + O_p \left(M_T T^{-3/2} \right) + O_p \left(M_T^{1/2} T^{-1} \right) \\ =& o_p(\lambda_T). \end{aligned} \tag{63}$$

We have the same order for the case where $k_1^0 < k \leq k_1^0 + M_T$.

On the contrary, we have

$$\bar{Y}^*_{k_2^0} = \frac{1}{T - k_2^0} \sum_{t=k_2^0+1}^{T} (\mu_3 + X_t) = \mu_0 + \delta_3 \lambda_T + \frac{1}{T - k_2^0} \sum_{t=k_2^0+1}^{T} X_t,$$

$$\begin{aligned} \bar{Y}'_{k_2^0} =& \frac{1}{k_2^0 - \hat{k}_{W,1}} \left(\sum_{t=\hat{k}_{W,1}+1}^{k_1^0} \mu_1 + \sum_{t=k_1^0+1}^{k_2^0} \mu_2 + \sum_{t=\hat{k}_{W,1}+1}^{k_2^0} X_t \right) \\ =& \mu_0 + \frac{k_1^0 - \hat{k}_{W,1}}{k_2^0 - \hat{k}_{W,1}} \delta_1 \lambda_T + \frac{k_2^0 - k_1^0}{k_2^0 - \hat{k}_{W,1}} \delta_2 \lambda_T + \frac{1}{k_2^0 - \hat{k}_{W,1}} \sum_{t=\hat{k}_{W,1}+1}^{k_2^0} X_t, \end{aligned}$$

and thus,

$$\bar{Y}^*_{k_2^0} - \bar{Y}'_{k_2^0} = (\delta_3 - \delta_2)\lambda_T + \frac{k_1^0 - \hat{k}_{W,1}}{k_2^0 - \hat{k}_{W,1}} (\delta_2 - \delta_1) \lambda_T + \frac{1}{T - k_2^0} \sum_{t=k_2^0+1}^{T} X_t - \frac{1}{k_2^0 - \hat{k}_{W,1}} \sum_{\hat{k}_{W,1}+1}^{k} X_t.$$

Notably, the first term on the right hand side does not equal zero. Using the above result, we obtain

$$\begin{aligned} Q_{T'}(k_2^0/T) =& \frac{(k_2^0 - \hat{k}_{W,1})(T - k_2^0)}{(T - \hat{k}_{W,1})^2} \left(\bar{Y}^*_{k_2^0} - \bar{Y}'_{k_2^0} \right) \\ =& O_p(\lambda_T) + O_p(\lambda_T^{-1} T^{-1}) + O_p(T^{-1/2}) - O_p(T^{-1/2}) = O_p(\lambda_T). \end{aligned} \tag{64}$$

From (63) and (64), it is observed that $Q_{T'}(k_2^0/T)$ dominates $Q_{T'}(k/T)$ uniformly over $\hat{k}_{W,1} < k \leq k_1^0 + M_T$. We can show that the same relation is obtained when $k_1^0 \leq \hat{k}_{W,1}$.

In exactly the same manner, it can be shown that $Q_{T'}(k_2^0/T) = O_p(\lambda_T)$ dominates $Q_{T'}(k/T) = o_p(\lambda)$ uniformly over $T - M_T \leq k \leq T-1$. Therefore, $Q_{T'}(k/T)$ cannot be maximized asymptotically in the neighborhood of both of the end points.

Now we focus on k satifying $\max(k_1^0, \hat{k}_{W,1}) + M_T < k < T - M_T$. By the triangle inequality, we have

$$\begin{aligned}
&|Q_{T'}(k/T)| - |Q_{T'}(k_2^0/T)| \\
\leq &|Q_{T'}(k/T) - Q_{T_0'}(k/T)| + |Q_{T'}(k_2^0/T) - Q_{T_0'}(k_2^0/T)| + \left(|Q_{T_0'}(k/T)| - |Q_{T_0'}(k_2^0/T)|\right). \qquad (65)
\end{aligned}$$

According to Bai (1994), the third term of the right-hand side of (65) is bounded by

$$\begin{aligned}
&|Q_{T_0'}(k/T)| - |Q_{T_0'}(k_2^0/T)| \\
\leq &|Q_{T_0'}(k/T) - Q_{T_0'}(k_2^0/T)| \\
\leq &|Q_{T_0'}(k/T) - \mathbb{E}[Q_{T_0'}(k/T)]| + |\mathbb{E}[Q_{T_0'}(k/T)]| - (|\mathbb{E}[Q_{T_0'}(k_2^0/T)]| - |Q_{T_0'}(k_2^0/T) - \mathbb{E}[Q_{T_0'}(k_2^0/T)]|) \\
\leq &2\sup_k |Q_{T_0'}(k/T) - \mathbb{E}[Q_{T_0'}(k/T)]| + |\mathbb{E}[Q_{T_0'}(k/T)]| - |\mathbb{E}[Q_{T_0'}(k_2^0/T)]|. \qquad (66)
\end{aligned}$$

where k in the supremum ranges from $\max(k_1^0, \hat{k}_{W,1}) + M_T$ to $T - M_T$. By (65), (66), Lemma 6, and $|Q_{T'}(\hat{k}_2/T)| - |Q_{T'}(k_2^0/T)| \geq 0$, we have

$$|\hat{\tau}_{W,2} - \tau_2^0| \leq C^{-1}\lambda_T^{-1}\left\{2\sup_k |Q_{T'}(k/T) - Q_{T_0'}(k/T)|\} + 2\sup_k |Q_{T_0'}(k/T) - \mathbb{E}[Q_{T_0'}(k/T)]|\right\}. \qquad (67)$$

As $\hat{k}_{W,1} = k_1^0 + O_p(\lambda_T^{-2})$, we have

$$\begin{aligned}
Q_{T'}(k/T) =&\frac{(k - \hat{k}_{W,1})(T-k)}{(T - \hat{k}_{W,1})^2}(\bar{Y}_k^* - \bar{Y}_k') \\
=&\frac{(k - k_1^0)(T-k)}{(T - k_1^0)^2}(1 + o_p(1))(\bar{Y}_k^* - \bar{Y}_k') \\
=&\frac{(k - k_1^0)(T-k)}{(T - k_1^0)^2}(\bar{Y}_k^* - \bar{Y}_k') + o_p(1)(\bar{Y}_k^* - \bar{Y}_k'). \qquad (68)
\end{aligned}$$

As $\mu_i = \mu_0 + \delta_i\lambda_T$, μ_0 is canceled out in $\bar{Y}_k^* - \bar{Y}_k'$, and thus, we can observe that

$$\begin{aligned}
\bar{Y}_k^* - \bar{Y}_k' =&O_p(\lambda_T) + \frac{1}{T-k}\sum_{t=k+1}^{T} X_t - \frac{1}{k - \hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k} X_t \\
=&O_p(\lambda_T) + O_p\left(\frac{1}{\sqrt{M_T}}\right) = O_p(\lambda_T), \qquad (69)
\end{aligned}$$

by the HRI. Then, the first term in the curly braces of (67) is

$$Q_{T'}(k/T) - Q_{T_0'}(k/T) = \frac{(k-\hat{k}_{W,1})(T-k)}{(T-\hat{k}_{W,1})^2}(\bar{Y}_k^* - \bar{Y}_k') - \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2}(\bar{Y}_k^* - \bar{Y}_{0,k}')$$
$$= \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2}(\bar{Y}_{0,k}' - \bar{Y}_k') + o_p(\lambda_T). \tag{70}$$

When $k_1^0 \leq \hat{k}_{W,1}$, we have, by Proposition 2 and because μ_0 in y_t is canceled out,

$$\begin{aligned}
\bar{Y}_{0,k}' - \bar{Y}_k' =& \frac{1}{k-k_1^0}\sum_{t=k_1^0+1}^{k} y_t - \frac{1}{k-\hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k} y_t \\
=& \left(\frac{1}{k-k_1^0} - \frac{1}{k-\hat{k}_{W,1}}\right)\sum_{t=\hat{k}_{W,1}+1}^{k} y_t + \frac{1}{k-k_1^0}\sum_{t=k_1^0+1}^{\hat{k}_{W,1}} y_t \\
=& \frac{k_1^0-\hat{k}_{W,1}}{(k-k_1^0)(k-\hat{k}_{W,1})}\sum_{t=\hat{k}_{W,1}+1}^{k}(\mu_t + X_t) + \frac{1}{k-k_1^0}\sum_{k_1^0+1}^{k}(\mu_t + X_t) \\
=& O_p\left(\frac{\lambda_T^{-2}}{M_T}\lambda_T\right) + O_p\left(\frac{\lambda_T^{-2}}{M_T^{3/2}}\right) + O_p\left(\frac{\lambda_T^{-2}}{M_T}\lambda_T\right) + O_p\left(\frac{\lambda_T^{-1}}{M_T}\right) \\
=& o_p(\lambda_T)
\end{aligned}$$

uniformly over k. The same result is obtained when $\hat{k}_{W,1} < k_1^0$. That is, we have

$$\bar{Y}_{0,k}' - \bar{Y}_k' = o_p(\lambda_T)$$

uniformly over k and, thus,

$$\sup_k |Q_{T'}(k/T) - Q_{T_0'}(k/T)| = o_p(\lambda_T).$$

Subsequently, we consider the second term in the curly braces of (67). By its definition, we have

$$Q_{T_0'}(k/T) - \mathbb{E}[Q_{T_0'}(k/T)] = \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2}\left(\frac{1}{T-k}\sum_{t=k+1}^{T} X_t - \frac{1}{k-k_1^0}\sum_{k_1^0+1}^{k} X_t\right) = o_p(\lambda_T), \tag{71}$$

uniformly over k.

Using the above two results, we conclude that

$$|\hat{\tau}_{W,2} - \tau_2^0| \leq C^{-1}\lambda_T^{-1} o_p(\lambda_T) = o_p(1).$$

■

Proof of Proposition 4: Under Assumptions 1–4, for all ϵ, there exists an $M' < \infty$ such that

$$P(T|\hat{\tau}_{W,2} - \tau_2^0| > M'\lambda_T^{-2}) \leq P(\hat{k}_{W,2} \notin D_{T'}) + P(\hat{k}_{W,2} \in D_{T',M'^c}) \leq \epsilon + P(\hat{k}_{W,2} \in D_{T',M'^c}),$$

where the last inequality holds by Proposition 3. Then, it is sufficient to show that

$$P(\hat{k}_{W,2} \in D_{T',M'^c}) \leq P\left(\sup_{D_{T',M'^c}} |Q_{T'}(k/T)| \geq |Q_{T'}(k_2^0/T)|\right) \leq \epsilon. \tag{72}$$

Notably, by Proposition 2,

$$\begin{aligned}
Q_{T'}(k/T) - Q_{T_0'}(k/T) =& \frac{(k-\hat{k}_{W,1})(T-k)}{(T-\hat{k}_{W,1})^2}(\bar{Y}_k^* - \bar{Y}_k') - \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2}(\bar{Y}_k^* - \bar{Y}_{0,k}') \\
=& \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2}\left(1 + O_p\left(\lambda_T^{-2}T^{-1}\right)\right)(\bar{Y}_k^* - \bar{Y}_k') - \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2}(\bar{Y}_k^* - \bar{Y}_{0,k}') \\
=& \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2}(\bar{Y}_{0,k}' - \bar{Y}_k') + \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2}O_p\left(\lambda_T^{-2}T^{-1}\right)(\bar{Y}_k^* - \bar{Y}_k') \\
=& \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2}(\bar{Y}_{0,k}' - \bar{Y}_k') + O_p(\lambda_T^{-1}T^{-1})
\end{aligned}$$

uniformly over $k \in D_{T',M^c}$, where the last equality holds because the difference between the partial means is

$$\begin{aligned}
\bar{Y}_k^* - \bar{Y}_k' =& \frac{1}{T-k}\sum_{t=k+1}^{T}\mu_t + \frac{1}{T-k}\sum_{t=k+1}^{T}X_t - \frac{1}{k-\hat{k}}\sum_{t=\hat{k}+1}^{k}\mu_t - \frac{1}{k-\hat{k}}\sum_{t=\hat{k}+1}^{k}X_t \\
=& O_p(\lambda_T) + O_p(T^{-1/2}) = O_p(\lambda_T).
\end{aligned}$$

Next, we evaluate $\bar{Y}_{0,k}' - \bar{Y}_k'$. When $k_1^0 \leq \hat{k}$, we have, by Proposition 2,

$$\begin{aligned}
\bar{Y}_{0,k}' - \bar{Y}_k' =& \frac{1}{k-k_1^0}\sum_{t=k_1^0+1}^{k} y_t - \frac{1}{k-\hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k} y_t \\
=& \left(\frac{1}{k-k_1^0} - \frac{1}{k-\hat{k}_{W,1}}\right)\sum_{t=\hat{k}_{W,1}+1}^{k} y_t + \frac{1}{k-k_1^0}\sum_{t=k_1^0+1}^{\hat{k}_{W,1}} y_t \\
=& \frac{k_1^0 - \hat{k}_{W,1}}{(k-k_1^0)(k-\hat{k}_{W,1})}\sum_{t=\hat{k}_{W,1}+1}^{k} y_t + \frac{1}{k-k_1^0}\sum_{t=k_1^0+1}^{\hat{k}_{W,1}} y_t \\
=& \frac{k_1^0 - \hat{k}}{(k-k_1^0)(k-\hat{k}_{W,1})}\sum_{t=\hat{k}_{W,1}+1}^{k} \mu_t + \frac{1}{k-k_1^0}\sum_{t=k_1^0+1}^{\hat{k}_{W,1}} \mu_t \\
&+ \frac{k_1^0 - \hat{k}_{W,1}}{(k-k_1^0)(k-\hat{k}_{W,1})}\sum_{t=\hat{k}_{W,1}+1}^{k} X_t + \frac{1}{k-k_1^0}\sum_{t=k_1^0+1}^{\hat{k}_{W,1}} X_t \\
=& O_p(\lambda_T^{-1}T^{-1}),
\end{aligned}$$

uniformly. Similarly, when $\hat{k} \leq k_1^0$, it holds that $\bar{Y}'_{0,k} - \bar{Y}'_k = O_p(\lambda_T^{-1} T^{-1})$. Hence, we obtain

$$Q_{T'}(k/T) - Q_{T'_0}(k/T) = O_p(\lambda_T^{-1} T^{-1}) \tag{73}$$

uniformly over D_{T',M'^c}. In exactly the same manner, we can show that $Q_{T'}(k_2^0/T) - Q_{T'_0}(k_2^0/T) = O_p(\lambda_T^{-1} T^{-1})$. Therefore, from (72), we will show that

$$P\left(\sup_{D_{T',M'^c}} |Q_{T'_0}(k/T)| \geq |Q_{T'_0}(k_2^0/T)| + O_p(\lambda_T^{-1} T^{-1}) \right) \leq \epsilon. \tag{74}$$

To prove (74), note that

$$\begin{aligned} Q_{T'_0}(k/T) =& Q_{T'_0}(k/T) - \mathbb{E}[Q_{T'_0}(k/T)] + \mathbb{E}[Q_{T'_0}(k/T)] \\ =& \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2} \left(\frac{1}{T-k} \sum_{t=k+1}^{T} X_t - \frac{1}{k-k_1^0} \sum_{t=k_1^0+1}^{k} X_t \right) + \mathbb{E}[Q_{T'_0}(k/T)] \\ =& O_p(T^{-1/2}) + \mathbb{E}[Q_{T'_0}(k/T)] \end{aligned} \tag{75}$$

uniformly by the FCLT. The expectation is evaluated as, for $k \in D_{T',M'^c}$ and $k < k_2^0$,

$$\begin{aligned} \mathbb{E}[Q_{T'_0}(k/T)] =& \frac{(k-k_1^0)(T-k)}{(T-k_1^0)^2} \left(\frac{1}{T-k} \sum_{k+1}^{k_2^0} \mu_2 + \frac{1}{T-k} \sum_{k_2^0+1}^{T} \mu_3 - \frac{1}{k-k_1^0} \sum_{k_1^0+1}^{k} \mu_2 \right) \\ =& \frac{(k-k_1^0)(T-k_2^0)}{(T-k_1^0)^2} (\delta_3 - \delta_2)\lambda_T, \end{aligned}$$

which dominates the $O_p(T^{-1/2})$ term in (75). Then, we have

$$\begin{aligned} |Q_{T'_0}(k/T)| =& \frac{(k-k_1^0)(T-k_2^0)}{(T-k_1^0)^2} |\delta_3 - \delta_2| \lambda_T (1 + o_p(1)), \\ |Q_{T'_0}(k_2^0/T)| =& \frac{(k_2^0-k_1^0)(T-k_2^0)}{(T-k_1^0)^2} |\delta_3 - \delta_2| \lambda_T (1 + o_p(1)), \end{aligned}$$

and thus,

$$\begin{aligned} |Q_{T'_0}(k/T)| - |Q_{T'_0}(k_2^0/T)| =& \frac{(k-k_2^0)(T-k_2^0)}{(T-k_1^0)^2} |\delta_3 - \delta_2| \lambda_T (1 + o_p(1)) \\ \leq& -C \frac{M' \lambda_T^{-2} (T-k_2^0)}{(T-k_1^0)^2} |\delta_3 - \delta_2| \lambda_T \\ \leq& -CM' \frac{1}{\lambda_T T} < 0 \end{aligned}$$

over $k \in D_{T',M'^c}$ and $k < k_2^0$. This implies that $\sup |Q_{T'_0}(k/T)| - |Q_{T'_0}(k_2^0/T)|$ is negative and dominates the $O_p(\lambda^{-1} T^{-1})$ term in (74) when M' is sufficiently large. Therefore, (74) holds for $k < k_2^0$.

Similarly, for $k > k_2^0$, it can be shown that

$$|Q_{T_0'}(k/T)| - |Q_{T_0'}(k_2^0/T)| = \frac{(k_2^0 - k)(T - k_2^0)}{(T - k_1^0)^2} |\delta_3 - \delta_2| \lambda_T (1 + o_p(1)) \tag{76}$$

$$\leq -CM' \frac{1}{\lambda_T T} \tag{77}$$

over $k \in D_{T',M'^c}$ and $k > k_2^0$. Therefore, (74) holds for $k > k_2^0$, and the proof is completed. ■

Proof of Theorem 2: By Proposition 4, we focus on the $O(\lambda_T^{-2})$ neighborhood of k_2^0. Thus, let us define $k = k_2^0 + s\lambda_T^{-2}$ with $s \in [-M, M]$ for some large value of M. Then, we have $\hat{k}_{W,2} = k_2^0 + \hat{s}\lambda_T^{-2}$. To derive the asymptotic distribution of $\hat{k}_{W,2}$, we investigate the behavior of $\hat{s}$ defined as follows:

$$\begin{aligned}
\hat{s} &= \arg\max_{s \in [-M,M]} \{Q_{T'}^2((k_2^0 + s\lambda_T^{-2})/T)\} \\
&= \arg\max_{s \in [-M,M]} \{T(Q_{T'}^2((k_2^0 + s\lambda_T^{-2})/T) - Q_{T'}^2(k_2^0/T))\}.
\end{aligned}$$

First, we consider the case where $k_1^0 < \hat{k}_{W,1}$. For $s \leq 0$, we decompose the objective function into

$$\begin{aligned}
Q_{T'}^2(k/T) =& \frac{(k - \hat{k}_{W,1})^2 (T - k)^2}{(T - \hat{k}_{W,1})^4} \left(\frac{T - k_2^0}{T - k} (\mu_3 - \mu_2) \right)^2 \\
&+ 2 \frac{(k - \hat{k}_{W,1})^2 (T - k)^2}{(T - \hat{k}_{W,1})^4} \left(\frac{T - k_2^0}{T - k} (\mu_3 - \mu_2) \right) \left(\frac{1}{T - k} \sum_{t=k+1}^{T} X_t - \frac{1}{k - \hat{k}_{W,1}} \sum_{t=\hat{k}_{W,1}+1}^{k} X_t \right) \\
&+ \frac{(k - \hat{k}_{W,1})^2 (T - k)^2}{(T - \hat{k}_{W,1})^4} \left(\frac{1}{T - k} \sum_{t=k+1}^{T} X_t - \frac{1}{k - \hat{k}_{W,1}} \sum_{t=\hat{k}_{W,1}+1}^{k} X_t \right)^2,
\end{aligned} \tag{78}$$

$$\begin{aligned}
Q_{T'}^2(k_2^0/T) =& \frac{(k_2^0 - \hat{k}_{W,1})^2 (T - k_2^0)^2}{(T - \hat{k}_{W,1})^4} (\mu_3 - \mu_2)^2 \\
&+ 2 \frac{(k_2^0 - \hat{k}_{W,1})^2 (T - k_2^0)^2}{(T - \hat{k}_{W,1})^4} (\mu_3 - \mu_2) \left(\frac{1}{T - k_2^0} \sum_{t=k_2^0+1}^{T} X_t - \frac{1}{k_2^0 - \hat{k}_{W,1}} \sum_{t=\hat{k}_{W,1}+1}^{k} X_t \right) \\
&+ \frac{(k_2^0 - \hat{k}_{W,1})^2 (T - k_2^0)^2}{(T - \hat{k}_{W,1})^4} \left(\frac{1}{T - k_2^0} \sum_{t=k_2^0+1}^{T} X_t - \frac{1}{k_2^0 - \hat{k}_{W,1}} \sum_{t=\hat{k}_{W,1}+1}^{k_2^0} X_t \right)^2.
\end{aligned} \tag{79}$$

The difference between the two third terms on the right-hand side of (78) and (79) is

$$
\begin{aligned}
&\frac{(k-\hat{k}_{W,1})^2(T-k)^2}{(T-\hat{k}_{W,1})^4}\left(\frac{1}{T-k_2^0}\sum_{t=k_2^0+1}^{T}X_t-\frac{1}{k_2^0-\hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k_2^0}X_t\right)^2\\
&-\frac{(k_2^0-\hat{k}_{W,1})^2(T-k_2^0)^2}{(T-\hat{k}_{W,1})^4}\left(\frac{1}{T-k_2^0}\sum_{t=k_2^0+1}^{T}X_t-\frac{1}{k_2^0-\hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k_2^0}X_t\right)^2\\
=&\left(\frac{(k-\hat{k}_{W,1})^2}{(T-\hat{k}_{W,1})^4}-\frac{(k_2^0-\hat{k}_{W,1})^2}{(T-\hat{k}_{W,1})^4}\right)\left(\sum_{t=k_2^0+1}^{T}X_t\right)^2\\
&+\left(\frac{(k-\hat{k}_{W,1})^2}{(T-\hat{k}_{W,1})^4}-\frac{(T-k_2^0)^2}{(T-\hat{k}_{W,1})^4}\right)\left(\sum_{t=k+1}^{k_2^0}X_t\right)^2\\
&+\left(\frac{(T-k)^2}{(T-\hat{k}_{W,1})^4}-\frac{(T-k_2^0)^2}{(T-\hat{k}_{W,1})^4}\right)\left(\sum_{t=\hat{k}_{W,1}+1}^{k}X_t\right)^2\\
&+2\left(\frac{(k-\hat{k}_{W,1})^2}{(T-\hat{k}_{W,1})^4}+\frac{(k_2^0-\hat{k}_{W,1})(T-k_2^0)}{(T-\hat{k}_{W,1})^4}\right)\left(\sum_{t=k_2^0+1}^{T}X_t\right)\left(\sum_{t=k+1}^{k_2^0}X_t\right)\\
&+2\left(-\frac{(k-\hat{k}_{W,1})(T-k)}{(T-\hat{k}_{W,1})^4}-\frac{(T-k_2^0)^2}{(T-\hat{k}_{W,1})^4}\right)\left(\sum_{t=k+1}^{k_2^0}X_t\right)\left(\sum_{t=\hat{k}_{W,1}+1}^{k}X_t\right)\\
&+2\left(-\frac{(k-\hat{k}_{W,1})(T-k)}{(T-\hat{k}_{W,1})^4}+\frac{(k_2^0-\hat{k}_{W,1})(T-k_2^0)}{(T-\hat{k}_{W,1})^4}\right)\left(\sum_{t=k_2^0+1}^{T}X_t\right)\left(\sum_{t=\hat{k}_{W,1}+1}^{k}X_t\right)\\
=&O_p\left(\lambda_T^{-2}T^{-2}\right)+O_p\left(\lambda_T^{-1}T^{-3/2}\right). \qquad (80)
\end{aligned}
$$

This suggests that (80) multiplied by T convergences to zero in probability.

The difference between the two first terms on the right-hand side of (78) and (79) multiplied by T becomes

$$
\begin{aligned}
&T\left\{\frac{(k-\hat{k}_{W,1})^2(T-k)^2}{(T-\hat{k}_{W,1})^4}\left(\frac{T-k_2^0}{T-k}(\mu_3-\mu_2)\right)^2-\frac{(k_2^0-\hat{k}_{W,1})^2(T-k_2^0)^2}{(T-\hat{k}_{W,1})^4}(\mu_3-\mu_2)^2\right\}\\
=&\frac{T(T-k_2^0)^2}{(T-\hat{k}_{W,1})^4}\left((k-\hat{k})^2-(k_2^0-\hat{k})^2\right)(\mu_3-\mu_2)^2\\
=&\frac{T(T-k_2^0)^2}{(T-\hat{k}_{W,1})^4}(s^2\lambda_T^{-4}+2s\lambda_T^{-2}(k_2^0-\hat{k}_{W,1})\lambda_T^2)(\delta_3-\delta_2)^2\lambda_T^2\\
\xrightarrow{p}&2\frac{(1-\tau_2^0)^2(\tau_2^0-\tau_1^0)}{(1-\tau_1^0)^4}s(\delta_3-\delta_2)^2. \qquad (81)
\end{aligned}
$$

The difference between the two second terms on the right-hand side of (78) and (79) multiplied by T

becomes

$$
\begin{aligned}
&T\left\{2\frac{(k-\hat{k}_{W,1})^2(T-k)^2}{(T-\hat{k}_{W,1})^4}\left(\frac{T-k_2^0}{T-k}(\mu_3-\mu_2)\right)\left(\frac{1}{T-k}\sum_{t=k+1}^{T}X_t-\frac{1}{k-\hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k}X_t\right)\right.\\
&\quad\left.-2\frac{(k_2^0-\hat{k}_{W,1})^2(T-k_2^0)^2}{(T-\hat{k}_{W,1})^4}(\mu_3-\mu_2)\left(\frac{1}{T-k_2^0}\sum_{t=k_2^0+1}^{T}X_t-\frac{1}{k_2^0-\hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k_2^0}X_t\right)\right\}\\
=&2\frac{T(T-k_2^0)\lambda_T}{(T-\hat{k}_{W,1})^4}(\delta_3-\delta_2)\left\{\left((k-\hat{k}_{W,1})^2-(k_2^0-k)^2\right)\sum_{t=k_2^0+1}^{T}X_t\right.\\
&+\left((k-\hat{k}_{W,1})^2+(k_2^0-\hat{k}_{W,1})(T-k_2^0)\right)\sum_{t=k+1}^{k_2^0+1}X_t\\
&\left.-\left((k-\hat{k}_{W,1})(T-k)+(k_2^0-\hat{k}_{W,1})(T-k_2^0)\right)\sum_{t=\hat{k}_{W,1}+1}^{k}X_t\right\}\\
\Rightarrow&\frac{2(1-\tau_2^0)}{(1-\tau_1^0)^4}(\delta_3-\delta_2)\left((\tau_2^0-\tau_1^0)^2+(\tau_2^0-\tau_1^0)(1-\tau_2^0)\right)\sigma a(1)B_1(|s|),
\end{aligned}
\tag{82}
$$

where $B_1(\cdot)$ is a standard Brownian motion on $[0,\infty)$.

For $s>0$, we have

$$
\begin{aligned}
Q_{T'}^2(k/T)=&\frac{(k-\hat{k}_{W,1})^2(T-k)^2}{(T-\hat{k}_{W,1})^4}\left(\frac{k_2^0-\hat{k}_{W,1}}{k-\hat{k}_{W,1}}(\mu_3-\mu_2)\right)^2\\
&+2\frac{(k-\hat{k}_{W,1})^2(T-k)^2}{(T-\hat{k}_{W,1})^4}\left(\frac{k_2^0-\hat{k}_{W,1}}{k-\hat{k}_{W,1}}(\mu_3-\mu_2)\right)\left(\frac{1}{T-k}\sum_{t=k+1}^{T}X_t-\frac{1}{k-\hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k}X_t\right)\\
&+\frac{(k-\hat{k}_{W,1})^2(T-k)^2}{(T-\hat{k}_{W,1})^4}\left(\frac{1}{T-k}\sum_{t=k+1}^{T}X_t-\frac{1}{k-\hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k}X_t\right)^2,
\end{aligned}
\tag{83}
$$

while $Q_{T'}^2(k_2^0/T)$ is given by (79). The difference between the two third terms on the right-hand side of (83) and (79) becomes $o_p(T^{-1})$ in the same manner as the case where $k\le k_2^0$. The difference between the two first terms on the right-hand side of (83) and (79) multiplied by T becomes

$$
\begin{aligned}
&T\left(\frac{(k-\hat{k}_{W,1})^2(T-k)^2}{(T-\hat{k}_{W,1})^4}\left(\frac{k_2^0-\hat{k}_{W,1}}{k-\hat{k}_{W,1}}(\mu_3-\mu_2)\right)^2-\frac{(k_2^0-\hat{k}_{W,1})^2(T-k_2^0)^2}{(T-\hat{k}_{W,1})^4}(\mu_3-\mu_2)^2\right)\\
=&T\frac{(k_2^0-\hat{k}_{W,1})^2}{(T-\hat{k}_{W,1})^4}\left((T-k)^2-(T-k_2^0)^2\right)(\mu_3-\mu_2)^2\\
\xrightarrow{p}&\frac{(\tau_2^0-\tau_1^0)^2}{(1-\tau_1^0)^4}(-2(1-\tau_2^0)s)(\delta_3-\delta_2)^2.
\end{aligned}
\tag{84}
$$

The difference between the two second terms on the right-hand side of (83) and (79) multiplied by T becomes

$$
\begin{aligned}
& T\left\{2\frac{(k-\hat{k}_{W,1})^2(T-k)^2}{(T-\hat{k}_{W,1})^4}\left(\frac{k_2^0-\hat{k}_{W,1}}{k-\hat{k}_{W,1}}(\mu_3-\mu_2)\right)\left(\frac{1}{T-k}\sum_{t=k+1}^{T}X_t-\frac{1}{k-\hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k}X_t\right)\right.\\
&\left.-2\frac{(k_2^0-\hat{k}_{W,1})^2(T-k_2^0)^2}{(T-\hat{k}_{W,1})^4}(\mu_3-\mu_2)\left(\frac{1}{T-k_2^0}\sum_{t=k_2^0+1}^{T}X_t-\frac{1}{k_2^0-\hat{k}_{W,1}}\sum_{t=\hat{k}_{W,1}+1}^{k_2^0}X_t\right)\right\}\\
=&2T\frac{(k_2^0-\hat{k}_{W,1})}{(T-\hat{k}_{W,1})^4}\lambda_T(\delta_3-\delta_2)\left((k-\hat{k}_{W,1})(T-k)-(k_2^0-\hat{k}_{W,1})(T-k_2^0)\right)\sum_{t=k+1}^{T}X_t\\
&+2T\frac{(k_2^0-\hat{k}_{W,1})}{(T-\hat{k}_{W,1})^4}\lambda_T(\delta_3-\delta_2)\left(-(T-k)^2-(k_2^0-\hat{k}_{W,1})(T-k_2^0)\right)\sum_{t=k_2^0+1}^{k}X_t\\
&+2T\frac{(k_2^0-\hat{k}_{W,1})}{(T-\hat{k}_{W,1})^4}\lambda_T(\delta_3-\delta_2)\left(-(T-k)^2+(T-k_2^0)^2\right)\sum_{t=\hat{k}_{W,1}+1}^{k_2^0}X_t\\
\Rightarrow&-2\frac{(\tau_2^0-\tau_1^0)}{(1-\tau_1^0)^4}(\delta_3-\delta_2)\left((1-\tau_2^0)^2+(1-\tau_2^0)(\tau_2^0-\tau_1^0)\right)\sigma a(1)B_2(s),
\end{aligned}
\tag{85}
$$

where $B_2(\cdot)$ is a standard Brownian motion on $[0,\infty)$ independent of $B_1(\cdot)$.

Therefore, we have, by (80)–(85),

$$
T\left(Q_{T'}^2(k/T)-Q_{T'}^2(k_2^0/T)\right)\Rightarrow W(s),
\tag{86}
$$

where $W(s)=W_1(s)$ when $k\le k_2^0$ and $W_2(s)$ when $k>k_2^0$ with

$$
\begin{aligned}
W_1(s)=&2\frac{(1-\tau_2^0)^2(\tau_2^0-\tau_1^0)}{(1-\tau_1^0)^4}s(\delta_3-\delta_2)^2+\frac{2(1-\tau_2^0)}{(1-\tau_1^0)^4}(\delta_3-\delta_2)\left((\tau_2^0-\tau_1^0)^2+(\tau_2^0-\tau_1^0)(1-\tau_2^0)\right)\sigma a(1)B_1(|s|),\\
W_2(s)=&-2\frac{(\tau_2^0-\tau_1^0)^2(1-\tau_2^0)}{(1-\tau_1^0)^4}s(\delta_3-\delta_2)^2-2\frac{(\tau_2^0-\tau_1^0)}{(1-\tau_1^0)^4}(\delta_3-\delta_2)\left((1-\tau_2^0)^2+(1-\tau_2^0)(\tau_2^0-\tau_1^0)\right)\sigma a(1)B_2(s).
\end{aligned}
$$

Similarly, it can be proved that we have the same limiting distribution when $\hat{k}_{W,1}\le k_1^0$. By the change of variables with $s=\frac{(1-\tau_1^0)^2}{(\delta_3-\delta_2)^2}\sigma^2a^2(1)u$ and the CMT to (86), we obtain

$$
\lambda_T^2(\hat{k}_{W,2}-k_2^0)\Rightarrow\left(\frac{1-\tau_1^0}{\delta_3-\delta_2}\right)^2\sigma^2a^2(1)\underset{u\in(-\infty,\infty)}{\arg\max}\{\Gamma(u)\},
$$

where

$$
\Gamma(u)=\begin{cases}B_1(|u|)-(1-\tau_2^0)|u| & \text{if } u\le 0\\ B_2(u)-(\tau_2^0-\tau_1^0)u & \text{if } u>0\end{cases}.
$$

■

The proof of Theorem 3 proceeds similarly to the proof of Proposition 3 in Bai (1997a). Define

$$\hat{k}_i^* = \begin{cases} \hat{k}_1^* = \arg\max_{1\le k\le k_0^*} Q_T^2(k/T) \\ \hat{k}_2^* = \arg\max_{k_0^*<k\le T-1} Q_T^2(k/T) \end{cases}$$

where $k_0^* = (k_1^0 + k_2^0)/2$. We demonstrate that $\hat{\tau}_i^* = \hat{k}_i^*/T$ are consistent for $\hat{k}_i^*$ for $i = 1$ and 2, implying that $\hat{\tau}_{W,1}$ converges to either τ_1^0 or τ_2^0. Therefore, we prove that the probability of $Q_T^2(\hat{k}_1^*/T) > Q_T^2(\hat{k}_2^*/T)$ approaches 1/2, which implies the theorem.

To prove the consistency of $\hat{k}_i^*$ for $i = 1$ and 2, we need the following lemma.

Lemma 7 *Under Assumptions 1, 2, 3(i), and 5, there exists a $C > 0$ such that*

$$\mathbb{E}\left[Q_T^2(k_1^0/T)\right] - \mathbb{E}\left[Q_T^2(k/T)\right] \ge C\frac{|k-k_1^0|}{T}, \quad \text{if } k \le k_0^*,$$
$$\mathbb{E}\left[Q_T^2(k_2^0/T)\right] - \mathbb{E}\left[Q_T^2(k/T)\right] \ge C\frac{|k-k_2^0|}{T}, \quad \text{if } k_0^* \le k.$$

Proof of Lemma 7: For $k \le k_1^0$, by Lemma 2, we have

$$\begin{aligned}
&\mathbb{E}\left[Q_T^2(k_1^0/T)\right] - \mathbb{E}\left[Q_T^2(k/T)\right] \\
=&\frac{{k_1^0}^2 - k^2}{T^4}\left((T-k_2^0)(\mu_3-\mu_2) + (T-k_1^0)(\mu_2-\mu_1)\right)^2 + \mathbb{E}\left[R_T^2(k_1^0)\right] - \mathbb{E}\left[R_T^2(k)\right] \\
\ge&\frac{k_1^0 - k}{T}C - \frac{k-k_1^0}{T}O(T^{-1}).
\end{aligned}$$

Similarly, for $k_1^0 < k \le k_0^*$, we have

$$\begin{aligned}
&\mathbb{E}\left[Q_T^2(k_1^0/T)\right] - \mathbb{E}\left[Q_T^2(k/T)\right] \\
=&\frac{1}{T^4}\left((T-k_2^0)k_1^0(\delta_3-\delta_2) + (T-k_1^0)k_1^0(\delta_2-\delta_1)\right)^2 - \frac{1}{T^4}\left((T-k_2^0)k(\delta_3-\delta_2) + (T-k)k_1^0(\delta_2-\delta_1)\right)^2 \\
&+\mathbb{E}\left[R_T^2(k_1^0)\right] - \mathbb{E}\left[R_T^2(k)\right].
\end{aligned} \tag{87}$$

The first and second terms on the right-hand side of (87) are, for a $C > 0$,

$$\begin{aligned}
&\frac{1}{T^4}\left[(T-k_2^0)k_1^0(\delta_3-\delta_2) + (T-k_1^0)k_1^0(\delta_2-\delta_1)\right]^2 - \frac{1}{T^4}\left[(T-k_2^0)k(\delta_3-\delta_2) + (T-k)k_1^0(\delta_2-\delta_1)\right]^2 \\
=&\frac{1}{T^4}\left[(T-k_2^0)(k_1^0+k)(\delta_3-\delta_2) + (2T-k_1^0-k)k_1^0(\delta_2-\delta_1)\right] \\
&\quad\times\left[(T-k_2^0)(k_1^0-k)(\delta_3-\delta_2) + (k-k_1^0)k_1^0(\delta_2-\delta_1)\right] \\
\ge&\frac{|k-k_1^0|}{T}C,
\end{aligned} \tag{88}$$

because Assumption 5 ensures that $Q^2(\tau)$ is strictly convex in $\tau \in [\tau_1^0, \tau_2^0]$.

Substituting (88) into (87), it can be shown that, by Lemma 2,

$$\mathbb{E}\left[Q_T^2(k_1^0/T)\right] - \mathbb{E}\left[Q_T^2(k/T)\right] \geq \frac{|k-k_1^0|}{T}C - \frac{|k-k_1^0|}{T}O(T^{-1}). \tag{89}$$

Similarly, for $k_0^* < k \leq k_2^0$, we have

$$\begin{aligned}
&\mathbb{E}\left[Q_T^2(k_2^0/T)\right] - \mathbb{E}\left[Q_T^2(k/T)\right] \\
=&\frac{1}{T^4}\left((T-k_2^0)k_2^0(\mu_3-\mu_2)+(T-k_2^0)k_1^0(\mu_2-\mu_1)\right)^2 - \frac{1}{T^4}\left((T-k_2^0)k(\mu_3-\mu_2)+(T-k)k_1^0(\mu_2-\mu_1)\right)^2 \\
&+\mathbb{E}\left[R_T^2(k_2^0)\right] - \mathbb{E}\left[R_T^2(k)\right] \\
\geq&\frac{k_2^0-k}{T}C - \frac{|k-k_2^0|}{T}O(T^{-1}),
\end{aligned}$$

where the inequality of the difference in the expectations is obtained by changing $R_T^2(k_1^0)$ to $R_T^2(k_2^0)$ in Lemma 2.

For $k_2^0 < k$, we have

$$\begin{aligned}
&\mathbb{E}\left[Q_T^2(k_2^0/T)\right] - \mathbb{E}\left[Q_T^2(k/T)\right] \\
=&\frac{(T-k_2^0)^2-(T-k)^2}{T^4}\left(k_2^0(\mu_3-\mu_2)+k_1^0(\mu_2-\mu_1)\right)^2 + \mathbb{E}\left[R_T^2(k_2^0)\right] - \mathbb{E}\left[R_T^2(k)\right] \\
\geq&\frac{k-k_2^0}{T}C - \frac{|k-k_2^0|}{T}O(T^{-1}).
\end{aligned}$$

■

Lemma 8 *Under Assumptions 1, 2, 3(i) and 5,*

$$\hat{\tau}_1^* - \tau_1^0 = O_p\left(T^{-1/2}\right) \quad \text{and} \quad \hat{\tau}_2^* - \tau_2^0 = O_p\left(T^{-1/2}\right).$$

Proof of Lemma 8: We first consider the case where $k \leq k_0^*$. As $Q_T^2(\hat{k}_1^*/T) - Q_T^2(k_1^0/T) \geq 0$, (41) implies (with k restricted in $[1, k_0^*]$),

$$\mathbb{E}[Q_T^2(k_1^0/T)] - \mathbb{E}[Q_T^2(\hat{k}_1^*)] \leq 2 \sup_{1\leq k\leq k_0^*}\left|Q_T^2(k/T) - \mathbb{E}[Q_T^2(k/T)]\right|. \tag{90}$$

Then, by Lemma 4 with $\lambda_T = 1$ and Lemma 7, we have

$$\frac{|\hat{k}_1^* - k_1^0|}{T} \leq C^{-1}O_p(T^{-1/2}),$$

which indicates that $|\hat{\tau}_1^* - \tau| = O_p(T^{-1/2})$.

We can prove it analogously for the case of $k_0^* < k$.

The consistency of $\hat{\tau}_2^*$ is proved similarly and we omit the proof. ■

Lemma 9 *Under Assumptions 1, 2, 3(i) and 5, for every $\epsilon > 0$ there exists an $M > 0$ such that*

$$P\left(\min_{k\in D^{(i)}_{T,M^c}} \left\{Q_T^2(k/T) - Q_T^2(k_i^0)\right\} \le 0\right) < \epsilon, \quad i = 1, 2, \tag{91}$$

where

$$D^{(1)}_{T,M^c} = \left\{k; T\eta \le k \le k_0^*, |k - k_1^0| > M\right\}, \tag{92}$$

$$D^{(2)}_{T,M^c} = \left\{k; k_0^* + 1 \le k \le T(1-\eta), |k - k_2^0| > M\right\}. \tag{93}$$

Proof of Lemma 9: It is sufficient to show that, by the same method as Lemma 5, for any $\eta > 0$ and $\epsilon > 0$,

$$P\left(\sup_{k\in D^{(1)}_{T,M^c}} \left\{T\left|R^2_{1,T}(k_1^0) - R^2_{i,T}(k)\right|/|k - k_1^0|\right\} > \eta\right) < \epsilon, \quad i = 1, 2, \tag{94}$$

(95)

and

$$P\left(\sup_{k\in D^{(2)}_{T,M^c}} \left\{T\left|R^2_{2,T}(k_2^0) - R^2_{j,T}(k)\right|/|k - k_2^0|\right\} > \eta\right) < \epsilon, \quad j = 2, 3, \tag{96}$$

where $R^2_{1,T}(k)$ and $R^2_{2,T}(k)$ are defined by (43) and (44), and

$$R^2_{3,T}(k) = 2w_T^2(k)c_T(k)(\bar{X}_k^* - \bar{X}_k) - R_T^2(k). \tag{97}$$

This proof proceeds similarly to the proof of Lemma 5, and the details are omitted. ■

Lemma 10 *Under Assumptions 1, 2, 3(i), and 5, for all $\epsilon > 0$ there exists an $M < \infty$ such that*

$$P\left(T|\hat{\tau}_i^* - \tau_i^0| > M\right) < \epsilon, \quad \textit{for } i = 1 \textit{ and } 2.$$

Proof of Lemma 10: Using Lemmas 8 and 9, we can prove the statement analogously to Proposition 2. ■

Proof of Theorem 3: As $\hat{\tau}_{W,1}$ equals either $\hat{\tau}_1^*$ or $\hat{\tau}_2^*$, which is T consistent for τ_1^0 or τ_2^0 by Lemma 10, it is sufficient to show that, as $T \to \infty$,

$$P\left(Q_T^2(\hat{k}_1^*/T) - Q_T^2(\hat{k}_2^*/T) > 0\right) = P\left(\sqrt{T}\left(Q_T^2(\hat{k}_1^*/T) - Q_T^2(\hat{k}_2^*/T)\right) > 0\right) \to \frac{1}{2}.$$

For $\hat{k}_1^* \leq k_1^0$, we have

$$\begin{aligned}
Q_T^2(\hat{k}_1^*/T) - Q_T^2(k_1^0/T) =& w_T^2(\hat{k}_1^*)a_T^2(\hat{k}_1^*) + 2w_T^2(\hat{k}_1^*)a_T(\hat{k}_1^*)(\bar{X}_{\hat{k}_1^*}^* - \bar{X}_{\hat{k}_1^*}) + R_T^2(\hat{k}_1^*) \\
&- (w_T^2(k_1^0)a_T^2(k_1^0) + 2w_T^2(k_1^0)a_T(k_1^0)(\bar{X}_{k_1^0}^* - \bar{X}_{k_1^0}) + R_T^2(k_1^0)) \\
=& \frac{\hat{k}_1^{*2} - k_1^{0\,2}}{T^4}\left\{(T - k_2^0)(\mu_3 - \mu_2) + (T - k_1^0)(\mu_2 - \mu_1)\right\}^2 \\
&+ 2\frac{\hat{k}_1^{*2}}{T^4}(T - \hat{k}_1^*)\left((T - k_2^0)(\mu_3 - \mu_2) + (T - k_1^0)(\mu_2 - \mu_1)\right)(\bar{X}_{\hat{k}_1^*}^* - \bar{X}_{\hat{k}_1^*}) \\
&- 2\frac{k_1^{0\,2}}{T^4}(T - k_1^0)\left((T - k_2^0)(\mu_3 - \mu_2) + (T - k_1^0)(\mu_2 - \mu_1)\right)(\bar{X}_{k_1^0}^* - \bar{X}_{k_1^0}) \\
&+ \left\{R_T^2(\hat{k}_1^*) - R_T^2(k_1^0)\right\}. \qquad (98)
\end{aligned}$$

The first term on the right-hand side of (98) is

$$\frac{\hat{k}_1^{*2} - k_1^{0\,2}}{T^4}\left\{(T - k_2^0)(\mu_3 - \mu_2) + (T - k_1^0)(\mu_2 - \mu_1)\right\}^2 = O_p(T^{-1}),$$

because $\hat{k}_1^{*2} - k_1^{0\,2} = (\hat{k}_1^* - k_1^0)(\hat{k}_1^* + k_1^0) = O_p(T)$ in view of Lemma 10. The second and third terms on the right-hand side of (98) become

$$\begin{aligned}
&2\frac{\hat{k}_1^{*2}}{T^4}(T - \hat{k}_1^*)\left((T - k_2^0)(\mu_3 - \mu_2) + (T - k_1^0)(\mu_2 - \mu_1)\right)(\bar{X}_{\hat{k}_1^*}^* - \bar{X}_{\hat{k}_1^*}) \\
&- 2\frac{k_1^{0\,2}}{T^4}(T - k_1^0)\left((T - k_2^0)(\mu_3 - \mu_2) + (T - k_1^0)(\mu_2 - \mu_1)\right)(\bar{X}_{k_1^0}^* - \bar{X}_{k_1^0}) \\
=& \frac{2}{T^4}\left((T - k_2^0)(\mu_3 - \mu_2) + (T - k_1^0)(\mu_2 - \mu_1)\right) \\
&\times \left\{\left(\hat{k}_1^{*2}(T - \hat{k}_1^*) - k_1^{0\,2}(T - k_1^0)\right)(\bar{X}_{\hat{k}_1^*}^* - \bar{X}_{\hat{k}_1^*}) + k_1^{0\,2}(T - k_1^0)\left((\bar{X}_{\hat{k}_1^*}^* - \bar{X}_{\hat{k}_1^*}) - (\bar{X}_{k_1^0}^* - \bar{X}_{k_1^0})\right)\right\}.
\end{aligned}$$

Notably,

$$\begin{aligned}
\left(\hat{k}_1^{*2}(T - \hat{k}_1^*) - k_1^{0\,2}(T - k_1^0)\right)(\bar{X}_{\hat{k}_1^*}^* - \bar{X}_{\hat{k}_1^*}) =& (\hat{k}_1^* - k_1^0)(T\hat{k}_1^* + Tk_1^0 - \hat{k}_1^{*2} - \hat{k}_1^* k_1^0 - k_1^{0\,2})(\bar{X}_{\hat{k}_1^*}^* - \bar{X}_{\hat{k}_1^*}) \\
=& O(T^{3/2}), \qquad (99)
\end{aligned}$$

and

$$
\begin{aligned}
&{k_1^0}^2(T-k_1^0)\left((\bar{X}^*_{\hat{k}_1^*}-\bar{X}_{\hat{k}_1^*})-(\bar{X}^*_{k_1^0}-\bar{X}_{k_1^0})\right)\\
=&{k_1^0}^2(T-k_1^0)\left(\frac{k_1^*-\hat{k}_1^0}{(T-\hat{k}_1^*)(T-k_1^0)}\sum_{t=k_1^0+1}^{T}X_t+\frac{T+k_1^0-\hat{k}_1^*}{k_1^0(T-\hat{k}_1^*)}\sum_{t=k_1^*+1}^{k_1^0}X_t-\frac{k_1^0-\hat{k}_1^*}{\hat{k}_1^*k_1^0}\sum_{t=1}^{\hat{k}_1^*}X_t\right)\\
=&o_p(T^{5/2}).
\end{aligned}
\tag{100}
$$

By (99) and (100), the second and third terms on the right-hand side of (98) is $o_p(T^{-1/2})$. On the contrary, the fourth term on the right-hand side of (98) is $O_p(T^{-1})$. Therefore, we have $Q_T^2(\hat{k}_1^*/T)-Q_T^2(k_1^0/T)=o_p(T^{-1/2})$. The same result is obtained for $k_1^0<\hat{k}_1^*$.

In exactly the same manner, we can demonstrate that $Q_T^2(\hat{k}_2^*/T)-Q_T^2(k_2^0/T)=o_p(T^{-1/2})$, and thus, we obtain

$$
P\left(Q_T^2(\hat{k}_1^*/T)-Q_T^2(\hat{k}_2^*/T)>0\right)=P\left(Q_T^2(k_1^0/T)-Q_T^2(k_2^0/T)+o_p(T^{-1/2})>0\right).
$$

Therefore, it is sufficient to show that

$$
P\left(\sqrt{T}\left(Q_T^2(k_1^0/T)-Q_T^2(k_2^0/T)\right)>0\right)\to\frac{1}{2}. \tag{101}
$$

From the definitions of $Q_T^2(k_1^0/T)$ and $Q_T^2(k_2^0/T)$, we observe that

$$
Q_T^2(k_1^0/T)=w_T^2(k_1^0)a_T^2(k_1^0)+2w_T(k_1^0)a_T(k_1^0)R_T(k_1^0)+R_T^2(k_1^0), \tag{102}
$$

$$
Q_T^2(k_2^0/T)=w_T^2(k_2^0)b_T^2(k_2^0)+2w_T(k_2^0)b_T(k_2^0)R_T(k_2^0)+R_T^2(k_2^0). \tag{103}
$$

The difference between the two first terms on the right-hand side of (102) and (103) is 0 by Assumption 5. The difference between the two third terms on the right-hand side of (102) and (103) is $O_p(T^{-1})$ by the FCLT. On the contrary, the difference between the two second terms on the right-hand side of

(102) and (103) becomes

$$
\begin{aligned}
&\sqrt{T}\left(2w_T(k_1^0)a_T(k_1^0)R_T(k_1^0) - 2w_T(k_2^0)b_T(k_2^0)R_T(k_2^0)\right)\\
=&2\sqrt{T}\left\{\frac{{k_1^0}^2(T-k_1^0)}{T^4}\left[(T-k_2^0)(\mu_3-\mu_2)+(T-k_1^0)(\mu_2-\mu_1)\right](\bar{X}^*_{k_1^0}-\bar{X}_{k_1^0})\right\}\\
&-2\sqrt{T}\left\{\frac{k_2^0(T-k_2^0)}{T^4}\left[(T-k_2^0)k_2^0(\mu_3-\mu_2)+(T-k_2^0)k_1^0(\mu_2-\mu_1)\right](\bar{X}^*_{k_2^0}-\bar{X}_{k_2^0})\right\}\\
\Rightarrow&2{\tau_1^0}^2(1-\tau_1^0)\left[(1-\tau_2^0)(\delta_3-\delta_2)+(1-\tau_1^0)(\delta_2-\delta_1)\right]\left[\frac{1}{1-\tau_1^0}(B(1)-B(\tau_1^0))-\frac{1}{\tau_1^0}B(\tau_1^0)\right]\\
&-2\tau_2^0(1-\tau_2^0)\left[(1-\tau_2^0)\tau_2^0(\delta_3-\delta_2)+(1-\tau_2^0)\tau_1^0(\delta_2-\delta_1)\right]\left[\frac{1}{1-\tau_2^0}(B(1)-B(\tau_2^0))-\frac{1}{\tau_2^0}B(\tau_2^0)\right],
\end{aligned}
\tag{104}
$$

where $B(\cdot)$ is a standard Brownian motion on $[0,1]$. This result indicates that (104) is normally distributed with mean 0, which implies (101). ■

Proof of Theorem 4: First, we note that

$$
\hat{k}_{W,1} = \underset{1\le k\le T-1}{\arg\max}\left\{Q_T^2(k/T)\right\} = \underset{1\le k\le T-1}{\arg\max}\left\{\left(\sqrt{T}Q_T(k/T)\right)^2\right\}.
$$

For $k=[T\tau]\le k_1^0$, we have, by the FCLT,

$$
\begin{aligned}
\sqrt{T}Q_T(k/T) =&\sqrt{T}\frac{k(T-k)}{T^2}(\bar{Y}_k^*-\bar{Y}_k)\\
=&\frac{k}{T}\frac{1}{\sqrt{T}}\sum_{t=k+1}^{T}X_t-\frac{T-k}{T}\frac{1}{\sqrt{T}}\sum_{t=1}^{k}X_t\\
&+\sqrt{T}\frac{k}{T^2}(T-k_2^0)\delta_3\sqrt{h}+\sqrt{T}\frac{k}{T^2}(k_2^0-k_1^0)\delta_2\sqrt{h}\\
&+\sqrt{T}\frac{k}{T^2}(k_1^0-k)\delta_1\sqrt{h}-\sqrt{T}\frac{T-k}{T^2}k\delta_1\sqrt{h}\\
\Rightarrow&a(1)\sigma\left(-B(\tau)+\tau B(1)\right)+\tau(1-\tau_2^0)(\delta_3-\delta_2)+\tau(1-\tau_1^0)(\delta_2-\delta_1).
\end{aligned}
$$

Similarly, for $k_1^0<k=[T\tau]\le k_2^0$, we have

$$
\sqrt{T}Q_T(k/T) \Rightarrow a(1)\sigma\left(-B(\tau)+\tau B(1)\right)+\tau(1-\tau_2^0)(\delta_3-\delta_2)+\tau_1^0(1-\tau)(\delta_2-\delta_1),
$$

and, for $k_2^0<k=[T\tau]$,

$$
\sqrt{T}Q_T(k/T) \Rightarrow a(1)\sigma\left(-B(\tau)+\tau B(1)\right)+\tau_2^0(1-\tau)(\delta_3-\delta_2)+\tau_1^0(1-\tau)(\delta_2-\delta_1).
$$

Applying the CMT for the argument of the maximum, the theorem is obtained. ■

References

Aue, A. and Horváth, L. (2013). Structural breaks in time series. *Journal of Time Series Analysis*, 34:1–16.

Baek, Y. (2023). Estimation of a structural break point in linear regression models. *Journal of Business & Economic Statistics*. https://doi.org/10.1080/07350015.2022.2154777.

Bai, J. (1994). Least squares estimation of a shift in linear processes. *Journal of Time Series Analysis*, 15(5):453–472.

Bai, J. (1997a). Estimating multiple breaks one at a time. *Econometric Theory*, 13(3):315–352.

Bai, J. (1997b). Estimation of a change point in multiple regression models. *The Review of Economics and Statistics*, 79(4):551–563.

Bai, J., Chen, H., Chong, T. T. L., and Wang, S. X. (2008). Generic consistency of the break-point estimators under specification errors in a multiple-break model. *Econometrics Journal*, 11:287–307.

Bai, J. and Perron, P. (1998). Estimating and testing linear models with multiple structural changes. *Econometrica*, 66:47–78.

Casini, A. and Perron, P. (2019). Structural breaks in time series. In Hamilton, J. H., editor, *Oxford Research Encyclopedia of Economics and Finance*. Oxford University Press, New York.

Casini, A. and Perron, P. (2021a). Continuous record asymptotics for Change-Points models. arXiv preprint arXiv:1803.10881.

Casini, A. and Perron, P. (2021b). Continuous record laplace-based inference about the break date in structural change models. *Journal of Econometrics*, 224(1):3–21.

Casini, A. and Perron, P. (2022). Generalized Laplace inference in multiple change-points models. *Econometric Theory*, 38(1):35–65.

Chong, T. T. L. (1995). Partial parameter consistency in a misspecified structural change model. *Economics Letters*, 49(4):351–357.

Hinkley, D. V. (1970). Inference about the change-point in a sequence of random variables. *Biometrika*, 57(1):1–17.

Jiang, L., Wang, X., and Yu, J. (2018). New distribution theory for the estimation of structural break point in mean. *Journal of Econometrics*, 205(1):156–176.

Jiang, L., Wang, X., and Yu, J. (2020). In-fill asymptotic theory for structural break point in autoregressions. *Econometric Reviews*, 40(4):359–386.

Perron, P. (2006). Dealing with structural breaks. In Mills, T. C. and Patterson, K., editors, *Palgrave Handbook of Econometrics, Vol.1: Econometric Theory*. Palgrave Macmillan, New York.

Qu, Z. and Perron, P. (2007). Estimating and testing structural changes in multivariate regressions. *Econometrica*, 75:459–502.

Tayanagi, T. and Kurozumi, E. (2023). In-fill asymptotic distribution of the change point estimator when estimating breaks one at a time. *Journal of Time Series Econometrics*. https://doi.org/10.1515/jtse-2022-0013.

Yao, Y.-C. (1987). Approximating the distribution of the maximum likelihood estimate of the change-point in a sequence of independent random variables. *Annals of Statistics*, 15(3):1321–1328.

8 Figure

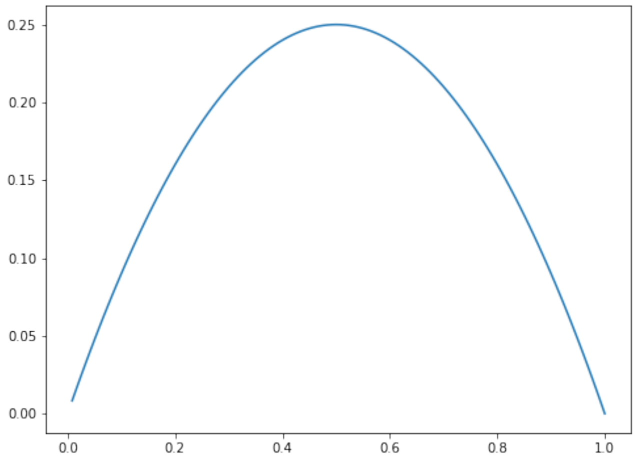

Figure 1: The weight function w_T

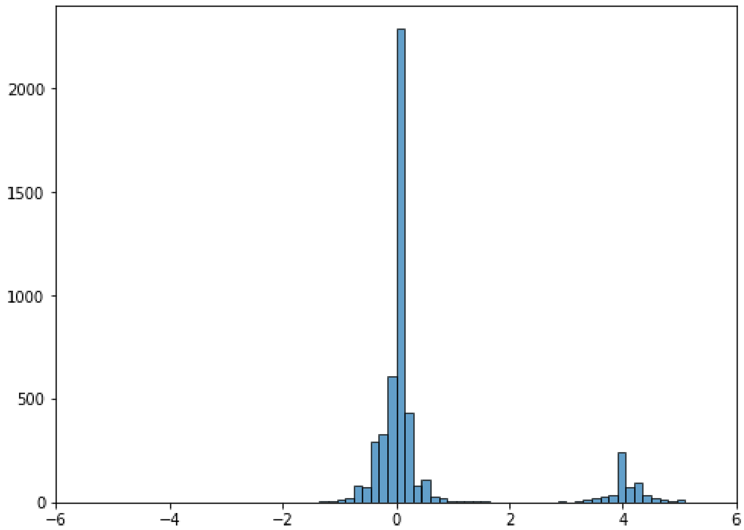

Figure 2: The finite sample distribution of $\lambda_T^2(\hat{k}_{W,1} - k_1^0)$ in the shrinking shift case

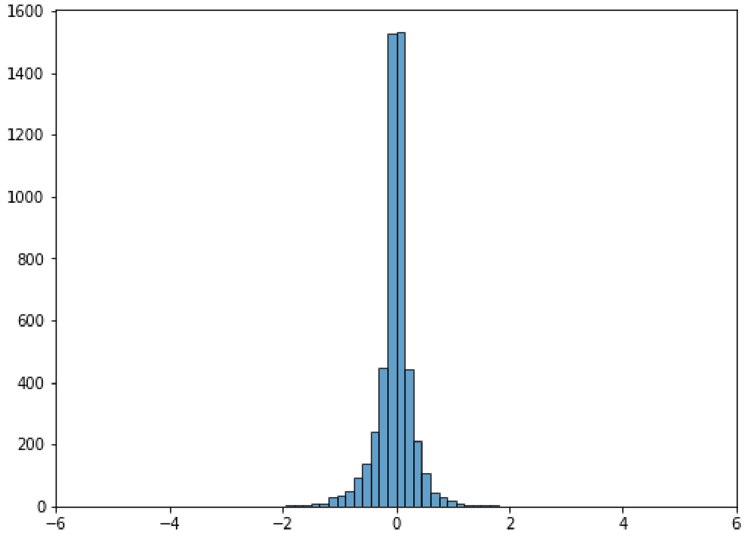

Figure 3: The long-span asymptotic distribution of $\lambda_T^2(\hat{k}_{W,1} - k_1^0)$ in the shrinking shift case

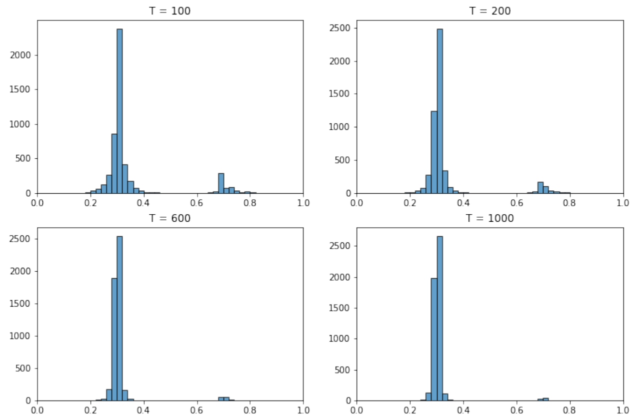

Figure 4: The finite sample distribution of $\hat{\tau}_{W,1}$ in the shrinking shift case ($T = \{100, 200, 600, 1000\}$)

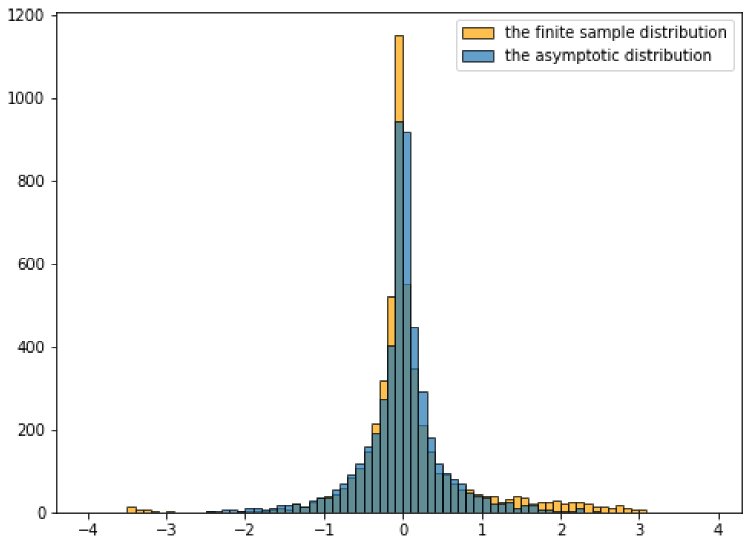

Figure 5: The finite sample and asymptotic distributions of $\lambda_T^2(\hat{k}_{W,2} - k_2^0)$ in the shrinking shift case with $\tau_1^0 = 0.33, \tau_2^0 = 0.67$

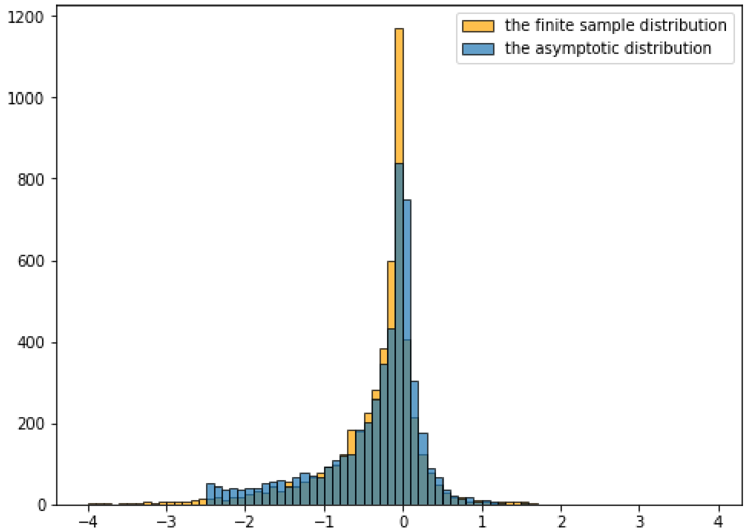

Figure 6: The finite sample and asymptotic distributions of $\lambda_T^2(\hat{k}_{W,2} - k_2^0)$ in the shrinking shift case with $\tau_1^0 = 0.33, \tau_2^0 = 0.80$

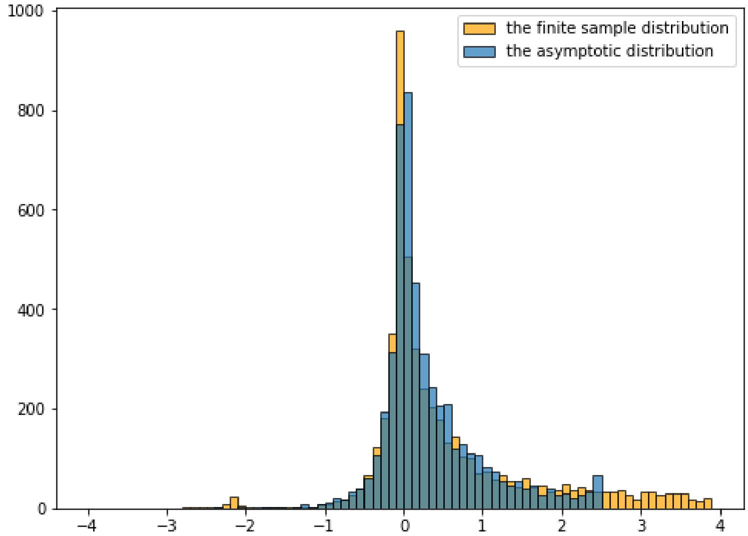

Figure 7: The finite sample and asymptotic distributions of $\lambda_T^2(\hat{k}_{W,2} - k_2^0)$ in the shrinking shift case with $\tau_1^0 = 0.33, \tau_2^0 = 0.54$

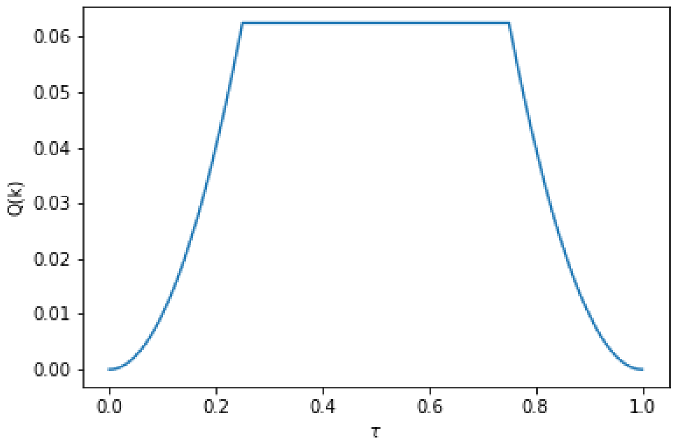

Figure 8: The plot of $Q^2(\tau)$ when (21) holds without (11)

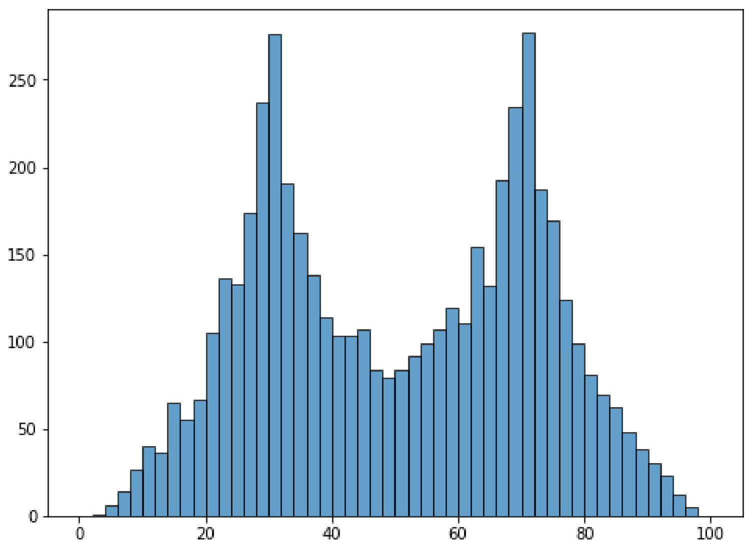

Figure 9: The finite sample distribution of $\hat{k}_{W,1}$ when $Q^2(\tau_1^0) = Q^2(\tau_2^0)$ $(T = 100)$

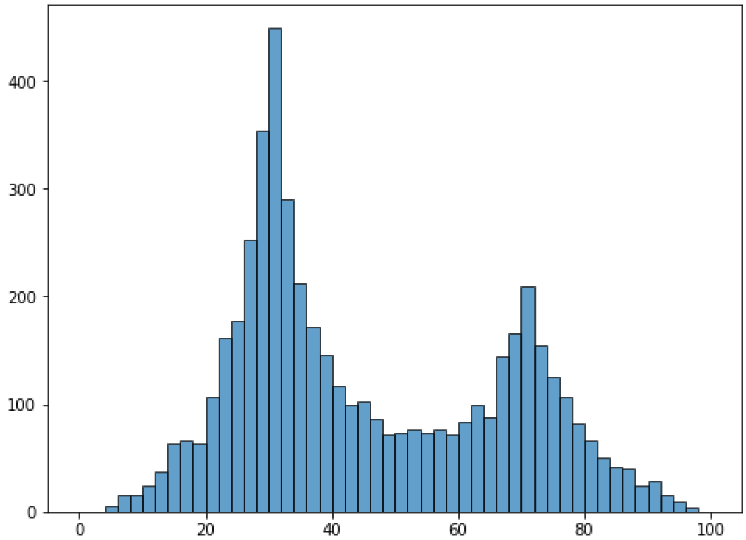

Figure 10: The finite sample distribution of $\hat{k}_{W,1}$ when the breaks are small ($T = 100$)

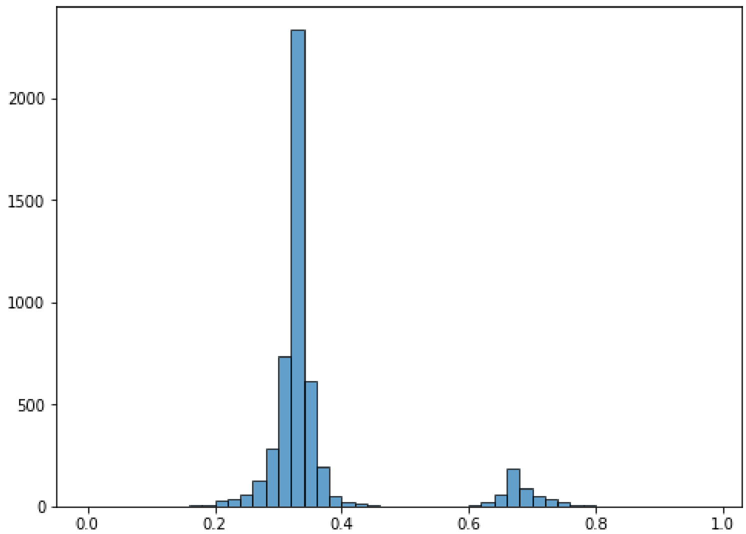

Figure 11: The in-fill asymptotic distribution of $\hat{\tau}_{W,1}$

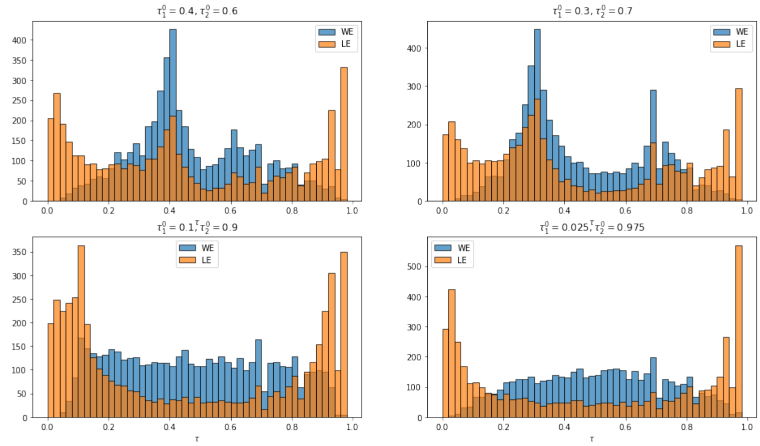

Figure 12: The finite sample distributions of $\hat{\tau}_{W,1}$ (WE) and $\hat{\tau}_{LS,1}$ (LE) with $\mu_0 = 0.0, \delta_1 = 0.0, \delta_2 = 0.4, \delta_3 = 0.1, \lambda_T = 1, T = 100$ and $(\tau_1, \tau_2) = \{(0.4, 0.6), (0.3, 0.7), (0.1, 0.9), (0.025, 0.975)\}$.

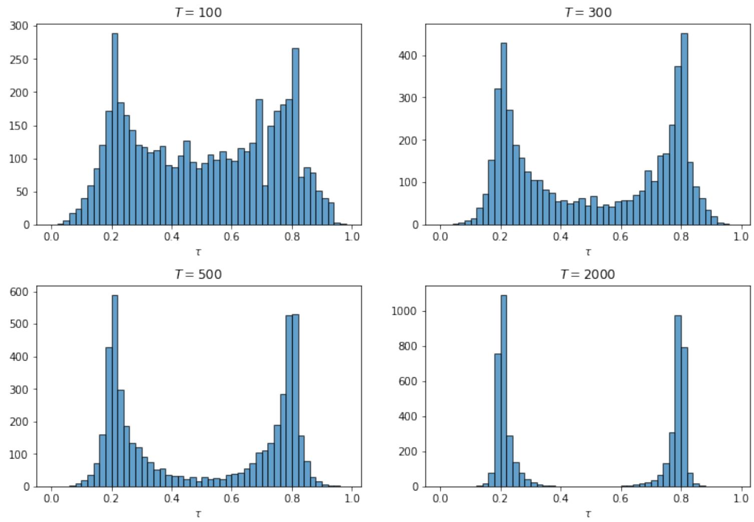

Figure 13: The finite sample distribution of $\hat{\tau}_{W,1}$ in the case where $Q(k_1^0) = Q(k_2^0)$ with $\tau_1^0 = 0.2, \tau_2^0 = 0.8, \mu_0 = 0.0, \delta_1 = 0.1, \delta_2 = 0.4, \delta_3 = 0.1, \lambda_T = 1, T = \{100, 300, 500, 2000\}$.

Table 1: RMSE, bias and SE of $\hat{\tau}_{W,1}$ and $\hat{\tau}_{LS,1}$ with $T = 100$ (the fixed shift case; $\lambda_T = 1$)

		RMSE		Bias		SE	
$(\delta_1, \delta_2, \delta_3)$	(τ_1^0, τ_2^0)	WE	LE	WE	LE	WE	LE
(0.0, 0.4, 0.5)	(0.1, 0.7)	0.418	0.463	0.352	0.307	0.225	0.347
	(0.3, 0.7)	0.206	0.289	0.123	0.111	0.165	0.266
	(0.5, 0.7)	0.124	0.234	0.014	0.019	0.123	0.233
	(0.1, 0.9)	0.421	0.473	0.351	0.310	0.232	0.357
	(0.3, 0.9)	0.205	0.304	0.114	0.113	0.170	0.282
	(0.5, 0.9)	0.133	0.254	0.006	0.016	0.133	0.254
(0.0, 0.6, 0.3)	(0.1, 0.7)	0.441	0.469	0.366	0.308	0.246	0.353
	(0.3, 0.7)	0.171	0.258	0.070	0.056	0.156	0.252
	(0.5, 0.7)	0.125	0.246	-0.023	-0.039	0.123	0.243
	(0.1, 0.9)	0.398	0.442	0.309	0.262	0.250	0.355
	(0.3, 0.9)	0.140	0.225	0.059	0.045	0.127	0.220
	(0.5, 0.9)	0.095	0.192	-0.010	-0.023	0.095	0.191
(0.0, 0.9, 0.5)	(0.1, 0.7)	0.410	0.379	0.315	0.206	0.262	0.319
	(0.3, 0.7)	0.088	0.140	0.022	0.004	0.085	0.140
	(0.5, 0.7)	0.068	0.141	-0.020	-0.032	0.065	0.137
	(0.1, 0.9)	0.328	0.334	0.224	0.149	0.240	0.299
	(0.3, 0.9)	0.071	0.102	0.026	0.003	0.066	0.102
	(0.5, 0.9)	0.051	0.091	-0.008	-0.015	0.050	0.090

Table 2: RMSE, bias and SE of $\hat{\tau}_{W,1}$ and $\hat{\tau}_{LS,1}$ with $T = 300$ (the fixed shift case; $\lambda_T = 1$)

		RMSE		Bias		SE	
$(\delta_1, \delta_2, \delta_3)$	(τ_1^0, τ_2^0)	WE	LE	WE	LE	WE	LE
(0.0, 0.4, 0.5)	(0.1, 0.7)	0.381	0.366	0.311	0.198	0.220	0.307
	(0.3, 0.7)	0.147	0.180	0.085	0.061	0.119	0.169
	(0.5, 0.7)	0.073	0.124	0.015	0.024	0.071	0.122
	(0.1, 0.9)	0.363	0.380	0.281	0.196	0.229	0.326
	(0.3, 0.9)	0.132	0.196	0.067	0.055	0.113	0.189
	(0.5, 0.9)	0.073	0.151	0.006	0.020	0.073	0.149
(0.0, 0.6, 0.3)	(0.1, 0.7)	0.436	0.375	0.341	0.202	0.270	0.316
	(0.3, 0.7)	0.077	0.121	0.017	-0.001	0.075	0.121
	(0.5, 0.7)	0.058	0.126	-0.019	-0.031	0.054	0.122
	(0.1, 0.9)	0.312	0.311	0.199	0.123	0.240	0.286
	(0.3, 0.9)	0.058	0.086	0.022	0.002	0.054	0.086
	(0.5, 0.9)	0.041	0.070	-0.005	-0.010	0.040	0.069
(0.0, 0.9, 0.5)	(0.1, 0.7)	0.374	0.241	0.250	0.085	0.278	0.225
	(0.3, 0.7)	0.021	0.029	0.005	-0.007	0.021	0.028
	(0.5, 0.7)	0.026	0.038	-0.009	-0.013	0.025	0.036
	(0.1, 0.9)	0.181	0.145	0.091	0.026	0.157	0.143
	(0.3, 0.9)	0.028	0.023	0.011	-0.001	0.026	0.023
	(0.5, 0.9)	0.020	0.025	-0.003	-0.003	0.020	0.025

Table 3: RMSE, bias and SE of $\hat{\tau}_{W,1}$ and $\hat{\tau}_{LS,1}$ with $T = 100$ (the shrinking shift case; $\lambda_T = T^{-1/4}$)

		RMSE		Bias		SE	
$(\delta_1, \delta_2, \delta_3)$	(τ_1^0, τ_2^0)	WE	LE	WE	LE	WE	LE
(0.0, 0.4, 0.5)	(0.1, 0.7)	0.454	0.537	0.397	0.392	0.220	0.366
	(0.3, 0.7)	0.285	0.401	0.190	0.185	0.213	0.355
	(0.5, 0.7)	0.205	0.350	0.004	0.006	0.205	0.350
	(0.1, 0.9)	0.456	0.536	0.398	0.391	0.222	0.367
	(0.3, 0.9)	0.288	0.404	0.191	0.186	0.215	0.358
	(0.5, 0.9)	0.208	0.353	0.003	0.003	0.208	0.353
(0.0, 0.6, 0.3)	(0.1, 0.7)	0.456	0.537	0.398	0.393	0.223	0.365
	(0.3, 0.7)	0.280	0.400	0.179	0.182	0.215	0.356
	(0.5, 0.7)	0.207	0.354	-0.004	-0.006	0.207	0.354
	(0.1, 0.9)	0.453	0.534	0.393	0.387	0.225	0.369
	(0.3, 0.9)	0.274	0.393	0.174	0.177	0.212	0.351
	(0.5, 0.9)	0.200	0.344	-0.001	-0.005	0.200	0.344
(0.0, 0.9, 0.5)	(0.1, 0.7)	0.449	0.522	0.388	0.373	0.227	0.366
	(0.3, 0.7)	0.258	0.374	0.153	0.152	0.208	0.341
	(0.5, 0.7)	0.190	0.335	-0.010	-0.014	0.190	0.334
	(0.1, 0.9)	0.443	0.518	0.379	0.364	0.230	0.369
	(0.3, 0.9)	0.244	0.360	0.142	0.143	0.199	0.330
	(0.5, 0.9)	0.177	0.313	-0.006	-0.011	0.177	0.313

Table 4: RMSE, bias and SE of $\hat{\tau}_{W,1}$ and $\hat{\tau}_{LS,1}$ with $T = 300$ (the shrinking shift case; $\lambda_T = T^{-1/4}$)

		RMSE		Bias		SE	
$(\delta_1, \delta_2, \delta_3)$	(τ_1^0, τ_2^0)	WE	LE	WE	LE	WE	LE
(0.0, 0.4, 0.5)	(0.1, 0.7)	0.447	0.532	0.388	0.371	0.222	0.381
	(0.3, 0.7)	0.263	0.397	0.173	0.179	0.198	0.354
	(0.5, 0.7)	0.178	0.338	0.013	0.015	0.177	0.338
	(0.1, 0.9)	0.447	0.536	0.387	0.374	0.225	0.384
	(0.3, 0.9)	0.266	0.404	0.172	0.179	0.202	0.362
	(0.5, 0.9)	0.183	0.349	0.007	0.014	0.183	0.348
(0.0, 0.6, 0.3)	(0.1, 0.7)	0.449	0.528	0.387	0.364	0.228	0.382
	(0.3, 0.7)	0.249	0.387	0.146	0.151	0.202	0.356
	(0.5, 0.7)	0.181	0.349	-0.008	-0.010	0.181	0.349
	(0.1, 0.9)	0.438	0.519	0.371	0.348	0.233	0.385
	(0.3, 0.9)	0.238	0.368	0.140	0.141	0.192	0.340
	(0.5, 0.9)	0.163	0.323	-0.001	-0.006	0.163	0.323
(0.0, 0.9, 0.5)	(0.1, 0.7)	0.435	0.491	0.363	0.317	0.240	0.375
	(0.3, 0.7)	0.200	0.324	0.097	0.099	0.174	0.309
	(0.5, 0.7)	0.143	0.298	-0.014	-0.021	0.142	0.297
	(0.1, 0.9)	0.414	0.476	0.335	0.293	0.243	0.375
	(0.3, 0.9)	0.180	0.293	0.090	0.084	0.156	0.280
	(0.5, 0.9)	0.120	0.249	-0.007	-0.014	0.120	0.249

Table 5: RMSE, bias and SE of $\hat{\tau}_{LS,2}$ and $\hat{\tau}_{W,2}$ with $T = 100$ (the fixed shift case; $\lambda_T = 1$)

			RMSE		Bias		SE	
$(\delta_1, \delta_2, \delta_3)$	(τ_1^0, τ_2^0)	# of exclusion	WE	LE	WE	LE	WE	LE
(0.0, 0.4, 0.5)	(0.1, 0.7)	153	0.200	0.308	0.015	-0.034	0.199	0.307
	(0.3, 0.7)	87	0.172	0.269	0.013	-0.021	0.172	0.268
	(0.5, 0.7)	60	0.157	0.224	0.077	0.050	0.137	0.219
	(0.1, 0.9)	147	0.279	0.392	-0.195	-0.238	0.200	0.311
	(0.3, 0.9)	105	0.262	0.353	-0.194	-0.222	0.175	0.275
	(0.5, 0.9)	90	0.193	0.289	-0.134	-0.168	0.140	0.235
(0.0, 0.6, 0.3)	(0.1, 0.7)	130	0.191	0.277	0.031	-0.011	0.189	0.277
	(0.3, 0.7)	97	0.145	0.248	-0.013	-0.029	0.145	0.247
	(0.5, 0.7)	99	0.131	0.231	0.039	0.003	0.125	0.231
	(0.1, 0.9)	136	0.285	0.375	-0.197	-0.220	0.206	0.304
	(0.3, 0.9)	57	0.252	0.328	-0.192	-0.207	0.162	0.255
	(0.5, 0.9)	60	0.172	0.249	-0.118	-0.143	0.124	0.204
(0.0, 0.9, 0.5)	(0.1, 0.7)	52	0.177	0.256	0.012	-0.018	0.177	0.256
	(0.3, 0.7)	19	0.113	0.206	-0.026	-0.020	0.110	0.205
	(0.5, 0.7)	28	0.096	0.172	0.030	0.026	0.092	0.170
	(0.1, 0.9)	51	0.304	0.362	-0.226	-0.206	0.204	0.298
	(0.3, 0.9)	9	0.242	0.288	-0.190	-0.170	0.149	0.232
	(0.5, 0.9)	8	0.158	0.194	-0.116	-0.105	0.106	0.164

Table 6: RMSE, bias and SE of $\hat{\tau}_{LS,2}$ and $\hat{\tau}_{W,2}$ with $T = 300$ (the fixed shift case; $\lambda_T = 1$)

			RMSE		Bias		SE	
$(\delta_1, \delta_2, \delta_3)$	(τ_1^0, τ_2^0)	# of exclusion	WE	LE	WE	LE	WE	LE
(0.0, 0.4, 0.5)	(0.1, 0.7)	46	0.208	0.342	-0.023	-0.072	0.207	0.334
	(0.3, 0.7)	8	0.156	0.268	-0.000	-0.020	0.156	0.267
	(0.5, 0.7)	6	0.132	0.205	0.071	0.064	0.111	0.195
	(0.1, 0.9)	59	0.316	0.441	-0.239	-0.278	0.207	0.343
	(0.3, 0.9)	14	0.268	0.362	-0.215	-0.235	0.161	0.275
	(0.5, 0.9)	9	0.177	0.252	-0.133	-0.147	0.116	0.204
(0.0, 0.6, 0.3)	(0.1, 0.7)	33	0.165	0.236	0.039	-0.005	0.160	0.236
	(0.3, 0.7)	8	0.099	0.195	-0.025	-0.027	0.096	0.193
	(0.5, 0.7)	11	0.085	0.171	0.024	0.013	0.082	0.171
	(0.1, 0.9)	29	0.297	0.369	-0.216	-0.200	0.204	0.310
	(0.3, 0.9)	6	0.232	0.287	-0.178	-0.158	0.149	0.240
	(0.5, 0.9)	0	0.147	0.201	-0.104	-0.104	0.103	0.172
(0.0, 0.9, 0.5)	(0.1, 0.7)	2	0.139	0.173	0.025	-0.005	0.136	0.173
	(0.3, 0.7)	0	0.073	0.135	-0.020	-0.011	0.070	0.134
	(0.5, 0.7)	0	0.064	0.122	0.018	0.021	0.062	0.120
	(0.1, 0.9)	3	0.288	0.300	-0.214	-0.139	0.192	0.266
	(0.3, 0.9)	0	0.210	0.226	-0.153	-0.103	0.143	0.201
	(0.5, 0.9)	0	0.129	0.154	-0.087	-0.064	0.095	0.140

Table 7: RMSE, bias and SE of $\hat{\tau}_{LS,2}$ and $\hat{\tau}_{W,2}$ with $T = 100$ (the shrinking shift case; $\lambda_T = T^{-1/4}$)

			RMSE		Bias		SE	
$(\delta_1, \delta_2, \delta_3)$	(τ_1^0, τ_2^0)	# of exclusion	WE	LE	WE	LE	WE	LE
(0.0, 0.4, 0.5)	(0.1, 0.7)	264	0.183	0.310	0.021	-0.015	0.182	0.310
	(0.3, 0.7)	251	0.181	0.302	0.020	-0.012	0.180	0.302
	(0.5, 0.7)	202	0.179	0.296	0.032	-0.003	0.176	0.296
	(0.1, 0.9)	251	0.260	0.381	-0.182	-0.222	0.185	0.310
	(0.3, 0.9)	229	0.257	0.375	-0.184	-0.219	0.180	0.305
	(0.5, 0.9)	211	0.244	0.361	-0.169	-0.202	0.175	0.300
(0.0, 0.6, 0.3)	(0.1, 0.7)	216	0.184	0.308	0.022	-0.017	0.182	0.307
	(0.3, 0.7)	236	0.179	0.298	0.012	-0.010	0.178	0.297
	(0.5, 0.7)	221	0.177	0.300	0.022	-0.010	0.175	0.300
	(0.1, 0.9)	230	0.260	0.374	-0.181	-0.214	0.186	0.307
	(0.3, 0.9)	210	0.256	0.366	-0.182	-0.210	0.180	0.299
	(0.5, 0.9)	223	0.237	0.352	-0.167	-0.199	0.169	0.291
(0.0, 0.9, 0.5)	(0.1, 0.7)	216	0.187	0.307	0.015	-0.019	0.187	0.307
	(0.3, 0.7)	197	0.172	0.296	0.006	-0.023	0.172	0.295
	(0.5, 0.7)	194	0.163	0.282	0.030	0.005	0.160	0.282
	(0.1, 0.9)	211	0.260	0.378	-0.182	-0.220	0.186	0.308
	(0.3, 0.9)	175	0.260	0.359	-0.193	-0.212	0.174	0.290
	(0.5, 0.9)	177	0.222	0.330	-0.157	-0.188	0.156	0.272

Table 8: RMSE, bias and SE of $\hat{\tau}_{LS,2}$ and $\hat{\tau}_{W,2}$ with $T = 300$ (the shrinking shift case; $\lambda_T = T^{-1/4}$)

			RMSE		Bias		SE	
$(\delta_1, \delta_2, \delta_3)$	(τ_1^0, τ_2^0)	# of exclusion	WE	LE	WE	LE	WE	LE
(0.0, 0.4, 0.5)	(0.1, 0.7)	152	0.191	0.337	0.010	-0.024	0.191	0.336
	(0.3, 0.7)	152	0.178	0.327	0.006	-0.039	0.178	0.324
	(0.5, 0.7)	111	0.166	0.302	0.033	-0.001	0.163	0.302
	(0.1, 0.9)	164	0.269	0.418	-0.192	-0.239	0.189	0.343
	(0.3, 0.9)	148	0.263	0.392	-0.193	-0.224	0.178	0.321
	(0.5, 0.9)	141	0.240	0.371	-0.172	-0.209	0.167	0.306
(0.0, 0.6, 0.3)	(0.1, 0.7)	171	0.184	0.325	0.019	-0.021	0.183	0.324
	(0.3, 0.7)	136	0.171	0.315	-0.005	-0.034	0.171	0.313
	(0.5, 0.7)	161	0.162	0.312	0.020	-0.026	0.161	0.311
	(0.1, 0.9)	165	0.273	0.400	-0.193	-0.221	0.193	0.333
	(0.3, 0.9)	135	0.261	0.391	-0.195	-0.229	0.174	0.317
	(0.5, 0.9)	121	0.221	0.354	-0.158	-0.202	0.154	0.291
(0.0, 0.9, 0.5)	(0.1, 0.7)	135	0.185	0.321	0.008	-0.028	0.184	0.320
	(0.3, 0.7)	90	0.155	0.300	-0.015	-0.043	0.154	0.297
	(0.5, 0.7)	115	0.139	0.279	0.025	-0.014	0.137	0.278
	(0.1, 0.9)	112	0.290	0.421	-0.209	-0.247	0.201	0.341
	(0.3, 0.9)	67	0.267	0.380	-0.211	-0.236	0.163	0.297
	(0.5, 0.9)	83	0.193	0.313	-0.142	-0.182	0.131	0.255