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# Shill-proof rules in object allocation problems with money

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# Shill-proof rules in object allocation problems with money*

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**Abstract**

We consider the object allocation problem with money. The seller owns multiple units of an object, and is only interested in her revenue from an allocation. Each buyer receives at most one unit of the object, and has a quasi-linear utility function with private valuations. We study incentives of the seller to increase her revenue by introducing false-name buyers, i.e., *shill bidding*. An (*allocation*) *rule* is *shill-proof* if the seller never benefits from introducing false-name buyers. A rule is a *binary posted prices rule* if there is a profile of posted prices such that whenever a buyer receives the object, she pays either her posted price or zero, and her payment is equal to zero when she does not receive the object. We show that if a rule satisfies *shill-proofness*, *strategy-proofness*, and *non-imposition*, then it is a binary posted prices rule. This result shows that the cost of preventing the seller from shill bidding is equivalent to the rigidity of the payment of each buyer, which highlights the difficulty in preventing the seller from shill bidding. It extends to a model of non-quasi-linear utility functions with interdependent valuations.

**JEL Classification Numbers.** D44, D47, D71, D82

**Keywords.** Shill-proofness, Shill bidding, Strategy-proofness, Posted prices rule, Binary posted prices rule, Multi-unit auctions

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# 1 Introduction

## 1.1 Purpose

In recent years, a countless number of goods are traded via online platforms. Internet auctions are one of the most typical electric commerce, and the market size of internet auctions is huge. Indeed, in 2022, the Gross Merchandise Value of eBay, one of the biggest auction houses in the world, amounted to 73.9 million dollars.[1]

A characteristic of internet auctions is that buyers and sellers are able to trade goods anonymously, i.e., they do not need to reveal their identity in transactions. This anonymity enables a seller to make additional buyer accounts to drive up a hammer price. Such a bidding by a seller is called *shill bidding*. Shill bidding is undesirable from the point of view of buyers as it unfairly raises the prices that they pay. Indeed, most auction houses announce the policy that shill bidding is prohibited, and sellers whose shill bidding is detected are punished by the auction houses or the law.[2] Nevertheless, shill bidding is still pervasive in real-life internet auctions because of the ease of conducting it for a seller and the difficulty of detecting for an auction house. Several empirical as well as experimental results support that shill bidding is pervasive in real-life auctions (Kosmopoulou and De silva, 2007; Engelberg and Willaims, 2009; McCannon and Minuci, 2020; Carlson and Wu, 2022, etc.). For example, Engelberg and Williams (2009) estimated that in Event Ticket Auctions in eBay that ended between September 8, 2004 and September 23, 2004, 1.39 percent of all bids are particular shill bids by sellers.

There are at least two possible approaches to preventing the seller from shill bidding: one is to develop the technology of detecting shill bids, and the other is to design a rule that does not give a seller an incentive to do shill bidding.[3] We take the latter approach, and investigate the class of rules that prevent a seller from shill bidding in the object allocation problem with money.

[1]See the following: https://investors.ebayinc.com/investor-news/press-release-details/2023/eBay-Inc.-Reports-Better-Than-Expected-Fourth-Quarter-2022-Results/default.aspx.

[2]For example, the following is the announcement of the policy for shill bidding by eBay: https://www.ebay.com/help/policies/selling-policies/selling-practices-policy/shill-bidding-policy?id=4353.

[3]Computer scientists have recently taken the former approach and developed detection techniques of shill bidding. See Majadi et al. (2017) for a comprehensive survey on this line of research.

## 1.2 Main result

We study the object allocation problem with money. The seller owns $m \geq 1$ units of an object. A (*consumption*) *bundle* of a buyer is a pair consisting of a consumption of the object and a payment. Each buyer can receive at most one unit of the object, and has a quasi-linear utility function with private valuations over the set of consumption bundles. Note that a quasi-linear utility function is identified with a *valuation* of the object. We consider a model where the set of buyers can vary, and a *domain* is a set of valuations of all the potential buyers. An *economy* is a pair consisting of a set of buyers and a valuation profile for the set of buyers. An *allocation* for a given economy specifies a bundle for each buyer in the economy. We assume that the seller is only interested in her revenue from an allocation, and so she does not care who receives the object. An (*allocation*) *rule* on a domain is a mapping which associates an allocation with each economy.

We introduce a new property of rules which prevents the seller from shill bidding. A rule satisfies *shill-proofness* if, for a given economy, the seller is never able to increase her revenue by introducing false-name buyers with arbitrary valuations (i.e., by shill bidding). We introduce the two standard properties of rules in the literature. A rule satisfies *strategy-proofness* if, for a given economy, no buyer in the economy ever benefits from misrepresenting her valuations. It satisfies *non-imposition* if, for a given economy, a buyer in the economy who views the object as valueless enjoys the utility of zero. Note that *non-imposition* is so mild property that almost all standard rules satisfy it. We regard *shill-proofness* together with *strategy-proofness* and *non-imposition* as our basic desiderata, and investigate the class of rules satisfying those.

We introduce a new class of rules which extends a *posted prices rules* under which each buyer who receives the object pays her posted price, and each buyer who does not receive it pays nothing.[4] A rule is a *binary posted prices rule* if there is a profile of posted prices such that for each economy, if a buyer in the economy receives the object, then she pays either her posted price or zero, and otherwise, she pays nothing. Note that the difference between a binary posted prices rule and a posted prices rule lies in the flexibility in the payment of each buyer: a binary posted prices rule allows the payment of each buyer who receives the object to be either her posted price or zero, while a posted prices rule requires

[4]Recently, several authors have studied the particular subclass of posted prices rules called the *priority rules* (Klaus and Nichifor, 2020, 2021; Shinozaki, 2022; Kawasaki et al., 2023). See Section 2.4 for the detailed discussion.

the payment of each buyer who receives the object to be her posted price.

A domain is said to be *rich* if the class of valuations of each buyer is some interval whose lower bound is zero. The main result of this paper is as follows. We establish that *if a rule on a rich domain satisfies shill-proofness, strategy-proofness, and non-imposition, then it is a binary posted prices rule.* (Theorem 1).

Note that a binary posted prices rule gives little flexibility to each buyer's payment as it must take one of the two values (i.e., it must be either her posted price or zero). Thus, Theorem 1 shows that the cost of preventing the seller from shill bidding is equivalent to the rigidity of the payment of each buyer, which highlights the difficulty in preventing a seller from shill bidding.

We assume that each buyer has a quasi-linear utility function with private valuations in our model, and so in Theorem 1 as well. This assumption is only for the expositional simplicity, and Theorem 1 carries over to a model of non-quasi-linear utility functions with interdependent valuations (Proposition 7). In particular, it holds for a model of quasi-linear utility functions with private, common, or interdependent valuations and that of non-quasi-linear utility functions with private or interdependent valuations.

### 1.3 Organization

The rest of this paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the model. Section 4 introduces the binary posted price rules. Section 5 presents the main result. Section 6 discusses the several extensions of the main result. Section 7 concludes. All the proofs are relegated to the Appendix.

## 2 Related literature

In this section, we review the related literature, and discuss how this paper contributes to it.

### 2.1 Shill bidding by the seller

There are several papers that study the seller's shill bidding as in this paper. Graham et al. (1990) study the equilibrium of an English auction in a private valuations model, and show that the seller may benefit from shill bidding in an equilibrium. Chakraborty and Kosmopoulou (2004) identify the equilibrium of an English auction in a common valuation model, and show that the seller's equilibrium expected revenue under an English auction may get worse off by the possibility of shill bidding. Lamy (2009) shows that in an interdependent valuations model, the seller's equilibrium expected revenue under a second-price auction may get worse off with the possibility of shill bidding than without shill bidding. Levin and Peck (2023) study shill bidding by the seller in a common valuations model, and compare the revenue from optimal shill bidding behavior under an English auction and that under a Sophi auction.

The main difference between this paper and the previous papers that study shill bidding by the seller lies in the attitude toward shill bidding. The previous papers allow the possibility of shill bidding, and study the equilibrium properties or the seller's optimal behavior in the presence of shill bidding under their specific rules. In contrast, we regard shill bidding as a fraud that should be prevented, and study the class of rules preventing the seller from shill bidding. Our main result (Theorem 1) contributes to this line of research by identifying the cost of preventing the seller from shill bidding: the rigidity of the payment of each buyer.

## 2.2 Shill bidding by the buyers

In this paper, we study shill bidding by the seller. In contrast, some papers study rules that prevent the *buyers* from shill bidding (a shill bid by a buyer is called a *false-name bid* in the literature). A rule is *false-name-proof* (Yokoo et al., 2004) if no buyer ever benefits from introducing false-name buyers. Yokoo et al. (2004) identifies a sufficient condition for a Vickrey rule (Vickrey, 1961) to satisfy *false-name-proofness* in a heterogeneous objects model. Sher (2012) identifies the optimal shill bidding of a buyer under a Vickrey rule in a heterogeneous objects model.

The result by Yokoo et al. (2004) implies that in our setting of multiple units of identical objects and unit-demand buyers, a Vickrey rule satisfies *false-name-proofness* as well as *strategy-proofness*, *non-imposition*, and *efficiency*. In contrast, our main result (Theorem 1) shows the difficulty in preventing the seller from shill bidding, and implies the non-existence of a rule satisfying *shill-proofness*, *strategy-proofness*, *efficiency*, and *non-imposition* (Corollary 1). Our result contributes to this line of research by suggesting that preventing the seller from shill bidding is more difficult than preventing the buyers from it.

### 2.3 Other frauds by the seller

Shill bidding is one of the possible frauds by the seller. The classes of rules that prevent the seller from frauds different from shill bidding have been studied. Muto and Shirata (2017) study the seller's incentive to destruct the objects to increase her utility, and show that no rule satisfies *efficiency* and *strategy-proofness* that simultaneously prevents the seller from destructing the objects. Akbarpour and Li (2020) study the seller's incentive to change outcome allocation of a rule without being detected by the buyers, and characterize the first-price rules as the unique static rules preventing such a fraud and satisfying *Bayesian incentive compatibility* and *efficiency*, and the English rules as the unique dynamic rules preventing it and satisfying *strategy-proofness* and *efficiency*. Shinozaki (2024b) studies the seller's incentive to collude with buyers to shut out other buyers, and shows that a Vickrey rule prevents such collusion if and only if the substitutes condition of Kelso and Crawford (1982) holds. This paper complements this line of research by studying the class of rules that prevent the seller from a different fraud from theirs: shill bidding.

### 2.4 Posted prices rules

Recently, several papers have characterized the priority rules in object allocation problems with money (Klaus and Nichifor, 2020, 2021; Shinozaki, 2022; Kawasaki et al., 2023).[5] Note that the class of priority rules is a subset of that of posted price rules, which is in turn a subset of that of binary posted prices rules. Klaus and Nichifor (2020; 2021) establish that in a model with unit-demand buyers, the priority rules are the only rules satisfying *consistency*, *strategy-proofness*, and *non-imposition* together with the other auxiliary properties.[6] Kawasaki et al. (2023) extend the results by Klaus and Nichifor (2020; 2021) to a model with multi-demand buyers. Shinozaki (2022) shows that in a single object model, the priority rules are the only rules satisfying *pairwise strategy-proofness*

[5] A rule is a *priority rule* if there are a priority over buyers and a profile of posted prices such that a buyer with the highest priority among the ones who are willing to pay their posted prices for the object receives the object and pays her posted price, a buyer with the second highest priority receives the object and pays her posted price, and the process continues until no buyer would like to receive the object with her posted price or the objects are exhausted.

[6] A rule satisfies *consistency* if whenever some buyers leave an economy with their bundles, the outcome bundles of the rule for all remaining buyers remain the same in the reduced economy.

and *non-imposition*.[7] Note that *consistency* implies *shill-proofness*, but the converse is not necessarily true. Note also that in general, *pairwise strategy-proofness* does not imply *shill-proofness*, and vice versa. Thus, the results in this paper neither imply nor are implied by the results of these papers. This paper contributes to this line of research by proposing a new variant of a posted prices rule (i.e., a binary posted prices rule) and providing a new foundation of a (binary) posted prices by *shill-proofness*.

# 3 Model

We study the object allocation problem with money. We consider a model where the set of buyers can vary. The set of potential buyers is $\mathbb{N}$. Let $\mathcal{N} = \{N \subseteq \mathbb{N} : 0 < |N| < \infty\}$ denote the family of all non-empty and finite subsets of the set of potential buyers. The single seller owns $m \geq 1$ units of an object. Each buyer can receive at most one unit of the object. Let $M = \{0, 1\}$. The amount of a payment made by a buyer $i \in \mathbb{N}$ is $t_i \in \mathbb{R}$. The **consumption set** of a buyer $i \in \mathbb{N}$ is $M \times \mathbb{R}$. A (**consumption**) **bundle** of a buyer $i \in \mathbb{N}$ is an element of her consumption set, i.e., it is a pair $z_i = (x_i, t_i) \in M \times \mathbb{R}$.

A **valuation** of a buyer $i \in \mathbb{N}$ is $v_i \in \mathbb{R}_+$. Our generic notation for a class of valuations of a buyer $i \in \mathbb{N}$ is $\mathcal{V}_i \subseteq \mathbb{R}_+$. Note that a valuation of a buyer is private information only known for her. Each buyer $i \in \mathbb{N}$ has a *quasi-linear* utility function $u_i : M \times \mathbb{R} \to \mathbb{R}$ such that for some valuation $v_i \in \mathcal{V}_i$, we have that for each $z_i = (x_i, t_i) \in M \times \mathbb{R}$, $u_i(z_i; v_i) = v_i x_i - t_i$. The assumptions of quasi-linear utility functions and private valuations are only for the simplicity of exposition, and the main result of this paper will extend to the case of non-quasi-linear utility functions with interdependent valuations.[8]

Note that under the assumption of quasi-linear utility functions with private valuations, a utility function is equivalent to a valuation. Thus, we identify a class of utility functions as that of valuation functions.

Given $N \in \mathcal{N}$, let $\mathcal{V}_N = \times_{i \in N} \mathcal{V}_i$. A **valuation profile for** $N \in \mathcal{N}$ is a tuple $v_N = (v_i)_{i \in N} \in \mathcal{V}_N$. Let $\mathcal{V}_{\mathbb{N}} = \times_{i \in \mathbb{N}} \mathcal{V}_i$. We call $\mathcal{V}_{\mathbb{N}}$ a **domain**.

Given a domain $\mathcal{V}_{\mathbb{N}}$, an **economy on** $\mathcal{V}_{\mathbb{N}}$ is a pair $e = (N, v_N)$ such that $N \in \mathcal{N}$ and $v_N \in \mathcal{V}_N$. Let $\mathcal{E}(\mathcal{V}_{\mathbb{N}})$ denote the class of economies on $\mathcal{V}_{\mathbb{N}}$. Note that the total number

[7] A rule satisfies *pairwise strategy-proofness* if no pair of agents ever benefits from misrepresenting their valuations.

[8] For the detailed discussion, see Section 6.5.

of units of the object that the seller owns is fixed at $m \geq 1$ throughout the paper, and is omitted in an economy.

Given $N \in \mathcal{N}$, a (**feasible**) **object allocation for** $N$ is a tuple $x_N = (x_i)_{i \in N} \in M^N$ such that $\sum_{i \in N} x_i \leq m$. Note that we focus only on deterministic object allocations. Let $X_N$ denote the set of object allocations for a given $N \in \mathcal{N}$. Given $N \in \mathcal{N}$, a (**feasible**) **allocation for** $N$ is a tuple $z_N = (z_i)_{i \in N} = (x_i, t_i)_{i \in N} \in (M \times \mathbb{R})^N$ such that $(x_i)_{i \in N} \in X_N$. Let $Z_N$ denote the set of allocations for a given $N \in \mathcal{N}$.

The seller is only interested in her revenue from an allocation. Thus, she has the (publicly known) quasi-linear utility function $u_0 : \cup_{N \in \mathcal{N}} Z_N \to \mathbb{R}$ such that for each $N \in \mathcal{N}$ and each $z_N = (x_i, t_i)_{i \in N} \in Z_N$, $u_0(z_N) = \sum_{i \in N} t_i$.

Given a domain $\mathcal{V}_{\mathbb{N}}$, an (**allocation**) **rule on** $\mathcal{V}_{\mathbb{N}}$ is a mapping $f : \mathcal{E}(\mathcal{V}_{\mathbb{N}}) \to \cup_{N \in \mathcal{N}} Z_N$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, $f(e) \in Z_N$.

We introduce the properties of rules. The following is a new property of rules which requires that the seller never benefit from introducing false-name buyers (i.e., shill bidding).

**Shill-proofness.** For each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, there exist no $N' \in \mathcal{N}$ with $N \cap N' = \varnothing$ and $v_{N'} \in \mathcal{V}_{N'}$ such that $\sum_{i \in N} t_i^f(N \cup N', v_{N \cup N'}) > \sum_{i \in N} t_i^f(e)$.

Note that the set of buyers $N'$ in the above definition consists of false-name buyers introduced by the seller, and their payments made by the seller are received by herself. Thus, the payments of the buyers in $N'$ do not affect the seller's revenue, and the seller's revenue after introducing the false-name buyers $N'$ is $\sum_{i \in N} t_i^f(N \cup N', v_{N \cup N'})$, not $\sum_{i \in N \cup N'} t_i^f(N \cup N', v_{N \cup N'})$.

Next, we introduce the two standard properties of rules in the literature. The next property requires that no buyer ever benefit from misrepresenting her valuations.

**Strategy-proofness.** For each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, each $i \in N$, and each $v_i' \in \mathcal{V}_i$, we have $u_i(f_i(e); v_i) \geq u_i(f_i(N, (v_i', v_{N \setminus \{i\}})); v_i)$.

The next property requires that if a buyer is not interested in the object (i.e., if her valuation is zero), then her utility from an outcome bundle of a rule be equal to zero.

**Non-imposition.** For each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and each $i \in N$, if $v_i = 0$, then $u_i(f_i(e); v_i) = 0$.

Note that *non-imposition* is a mild property that almost all standard rules satisfy.

We regard *shill-proofness*, *strategy-proofness*, and *non-imposition* as our basic desiderata, and study the class of rules satisfying the three properties.

In addition, we introduce the three standard properties.

Given $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, an allocation $z_N = (z_i)_{i \in N} \in Z_N$ for $N$ is **(Pareto) efficient for** $e$ if there is no other allocation $z'_N = (z'_i)_{i \in N} \in Z_N$ for $N$ such that (i) for each $i \in N$, $u_i(z'_i; v_i) \geq u_i(z_i; v_i)$, (ii) $u_0(z'_N) \geq u_0(z_N)$, and (iii) at least one of the inequalities in (i) and (ii) is strict. The following remark states that under the assumption of quasi-linear utility functions with private valuations, efficiency of an allocation is equivalent to the maximization of the sum of valuations.

**Remark 1.** Let $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and $z_N = (x_i, t_i)_{i \in N} \in Z_N$. Then, $z_N$ is efficient for $e$ if and only if we have

$$x_N = (x_i)_{i \in N} \in \underset{x'_N \in X_N}{\arg\max} \sum_{i \in N} v_i x_i.$$

First, the next property requires that a rule select an efficient allocation for each economy.

**Efficiency.** For each $e \in \mathcal{E}$, $f(e)$ is efficient for $e$.

Second, the next property requires that no buyer ever get worse off than the non-participation under which she enjoys the utility of zero.

**Individual rationality.** For each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and each $i \in N$, $u_i(f_i(e); v_i) \geq 0$.

Third, the next property requires that the payment of each buyer be non-negative.

**No subsidy.** For each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and each $i \in N$, $t_i^f(e) \geq 0$.

The next remark states that the combination of *individual rationality* and *no subsidy* implies *non-imposition*.

**Remark 2.** Let $\mathcal{V}_{\mathbb{N}}$ be a domain. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying *individual rationality* and *no subsidy*. Then, it satisfies *non-imposition*.

# 4 Binary posted prices rule

In this section, we introduce the new class of rules that we call the binary posted prices rules.

First, as a benchmark, we introduce the posted prices rules.

**Definition 1.** A rule $f$ on $\mathcal{V}_{\mathbb{N}}$ is a **posted prices rule** if there is a profile of posted prices $(p_i^f)_{i \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and each $i \in N$, the following hold.

- If $x_i^f(e) = 1$, then $t_i^f(e) = p_i^f$ and $v_i \geq t_i^f(e)$.
- If $x_i^f(e) = 0$, then $t_i^f(e) = 0$.

The first condition above states that if a buyer receives the object, then she pays her posted price and is willing to pay her posted price. The second one states that if a buyer does not receive the object, then she pays nothing. Note that a posted price may depend on a rule and the identity of a buyer, but does not depend on a valuation profile and the population in an economy.

Then, we introduce a new variant of a posted prices rule that allows a buyer to pay her posted price or zero for the object.

**Definition 2.** A rule $f$ on $\mathcal{V}_{\mathbb{N}}$ is a **binary posted prices rule** if there is a profile of posted prices $(p_i^f)_{i \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and each $i \in N$, the following hold.

- If $x_i^f(e) = 1$, then $t_i^f(e) \in \{p_i^f, 0\}$ and $v_i \geq t_i^f(e)$.
- If $x_i^f(e) = 0$, then $t_i^f(e) = 0$.

Clearly, a posted prices rule is a binary posted prices rule, but the converse is not necessarily true. Thus, the class of posted prices rules is a proper subset of that of binary posted prices rules.

We give several examples of binary posted prices rules in the following example.

**Example 1 (Binary posted prices rules).** The following rules belong to the class of binary posted prices rules.

- A rule $f$ on $\mathcal{V}_{\mathbb{N}}$ is the **no trade rule** if for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and each $i \in N$, $f_i(e) = (0, 0)$.

- A rule $f$ on $\mathcal{V}_{\mathbb{N}}$ is an **oligarchical rule** if there is a set of buyers $\hat{N}^f \in \mathcal{N}$ such that $|\hat{N}^f| \leq m$, and for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, we have that for each $i \in \hat{N}^f \cap N$, $f_i(e) = (1, 0)$, and for each $i \in N \backslash \hat{N}^f$, $f_i(e) = (0, 0)$. An oligarchical rule with $|N^f| = 1$ is said to be a **dictatorial rule**.

- Given an economy $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and a profile of prices $p_N = (p_i)_{i \in N} \in \mathbb{R}^N_+$, let $N^+(e, p_N) = \{i \in N : v_i \geq p_i\}$ denote the set of buyers in $N$ whose valuations are no smaller than their own prices. A rule $f$ on $\mathcal{V}_{\mathbb{N}}$ is a **priority rule** if there are a priority $\succ^f$ over $\mathbb{N}$ and a profile of posted prices $(p^f_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}_+$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, the following hold.

  - The buyer $i_1 \in N$ with the highest priority according to $\succ^f$ among $N^+(e, p^f_N)$ receives the object and pays $p^f_{i_1}$.
  - The buyer $i_2 \in N$ with the second highest priority according to $\succ^f$ among $N^+(e, p^f_N)$ receives the object and pays $p^f_{i_2}$.

    . . .
  - The buyer $i_k \in N$ with the $k$-th highest priority according to $\succ^f$ among $N^+(e, p^f_N)$ receives the object and pays $p^f_{i_k}$, where $k = |N^+(e, p^f_N)|$.
  - Each buyer in $N \backslash N^+(e, p^f_N)$ receives $(0, 0)$.

- Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, the following hold.

  - If $1 \in N$, $v_1 \geq 1$, and $\max_{i \in N \backslash \{1\}} v_i \leq 1$, then $f_1(e) = (1, 1)$.
  - If $1 \in N$, $v_1 < 1$, and $\max_{i \in N \backslash \{1\}} v_i \leq 1$, then $f_1(e) = (0, 0)$.
  - If $1 \in N$ and $\max_{i \in N \backslash \{1\}} v_i > 1$, then $f_1(e) = (1, 0)$.
  - For each $i \in N \backslash \{1\}$, $f_i(e) = (0, 0)$.

□

The last rule in the above example shows that the payment of a buyer who receives the object under a binary posted prices rule may depend on the other buyers' valuations unlike a posted prices rule.

# 5 Main result

In this section, we provide the main result of this paper.

First, we introduce the following standard domain richness condition.

**Definition 3.** A domain $\mathcal{V}_{\mathbb{N}}$ is **rich** if for each $i \in \mathbb{N}$, there is $\overline{v}_i \in \mathbb{R}_{++} \cup \{\infty\}$ such that either $\mathcal{V}_i = [0, \overline{v}_i)$ or $\mathcal{V}_i = [0, \overline{v}_i]$.

The following is the main result of this paper which states that on any rich domain, if a rule satisfies *shill-proofness*, *strategy-proofness*, and *non-imposition*, then it is a binary posted prices rule.

**Theorem 1.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying shill-proofness, strategy-proofness, and non-imposition. Then, it is a binary posted prices rule.*

Theorem 1 states that under *strategy-proofness* and *non-imposition*, a rule satisfies *shill-proofness* only if the payment of a buyer is either her posted price or zero. Thus, it shows that the cost of preventing the seller from shill bidding is equivalent to the rigidity of the payment of each buyer.

An impossibility result for the existence of a rule on a rich domain satisfying *shill-proofness*, *efficiency*, *strategy-proofness*, and *non-imposition* follows from Theorem 1.

**Corollary 1.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. No rule on $\mathcal{V}_{\mathbb{N}}$ satisfies shill-proofness, efficiency, strategy-proofness, and non-imposition.*

In the following three examples, we show the independence of the properties in Theorem 1, i.e., we show that if we drop one of the three properties in Theorem 1, then there is a rule that satisfies the other properties and is different from a binary posted prices rule. In the following examples, let $\mathcal{V}_{\mathbb{N}}$ be an arbitrary rich domain.

**Example 2 (Dropping shill-proofness).** Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, the following hold.

- If $1 \in N$, then [9]

$$f_1(e) = \begin{cases} (1, \max_{i \in N \setminus \{1\}} v_i) & \text{if } v_1 \geq \max_{i \in N \setminus \{1\}} v_i, \\ (0,0) & \text{otherwise.} \end{cases}$$

- For each $i \in N \setminus \{1\}$, $f_i(e) = (0, 0)$.

It is not a binary posted prices rule. It satisfies *strategy-proofness* and *non-imposition*, but violates *shill-proofness*. □

**Example 3 (Dropping strategy-proofness).** Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, (i) $f(e)$ is efficient for $e$, and (ii) for each $i \in N$,

$$t_i^f(e) = \begin{cases} v_i & \text{if } x_i^f(e) = 1, \\ 0 & \text{if } x_i^f(e) = 0. \end{cases}$$

It is not a binary posted prices rule. It satisfies *shill-proofness* and *non-imposition*, but violates *strategy-proofness*. □

**Example 4 (Dropping non-imposition).** Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and each $i \in N$, $f_i(e) = (0, -1)$. It is not a binary posted prices rule. It satisfies *shill-proofness* and *strategy-proofness*, but violates *non-imposition*. □

# 6 Extensions

In this section, we discuss several extensions of the main result (Theorem 1).

## 6.1 Restricting shill bidding behavior

In this subsection, we study the consequence of restricting the seller's shill bidding behavior.

Recall that in our definition of *shill-proofness*, we allow the seller to introduce an arbitrary number of false-name buyers. In practice, however, the seller may avoid introducing many false-name buyers in fear of detection of such buyers. The next property weakens *shill-proofness* so that the seller may introduce only a single false-name buyer.

[9] If $N = \{1\}$, then let $\max_{i \in N \setminus \{1\}} v_i = 0$.

**Single shill-proofness.** For each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, there exist no $i \in \mathbb{N} \backslash N$ and $v_i \in \mathcal{V}_i$ such that $\sum_{j \in N} t_j^f(N \cup \{i\}, v_{N \cup \{i\}}) > \sum_{j \in N} t_j^f(e)$.

An interesting question is whether Theorem 1 is still valid even if we weaken *shill-proofness* to *single shill-proofness*. The next example gives a negative answer to this question, i.e., it shows that the binary posted prices rules are not the only rules satisfying *single shill-proofness*, *strategy-proofness*, and *non-imposition*.

**Example 5.** Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. For simplicity, for each $i \in \mathbb{N}$, let $\overline{v}_i = 2$. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, the following hold.

- Suppose that $N = \{i\}$ for some $i \in \{2, 3\}$. If $v_i \geq 1$, then $f_i(e) = (1, 1)$, and otherwise, $f_i(e) = (0, 0)$.
- Suppose that $N = \{i\}$ for some $i \in \mathbb{N} \backslash \{2, 3\}$. Then, $f_i(e) = (0, 0)$.
- Suppose $N = \{1, i\}$ for some $i \in \{2, 3\}$. Then, $f_1(e) = (0, 0)$. If $v_i \geq 1$, then $f_i(e) = (1, 1)$, and otherwise, $f_i(e) = (0, 0)$.
- Suppose $N = \{2, 3\}$. If $v_2 \geq 1$, then $f_2(e) = (1, 1)$ and $f_3(e) = (0, 0)$. If $v_2 < 1$ and $v_3 \geq 1$, then $f_2(e) = (0, 0)$ and $f_3(e) = (1, 1)$. If $v_2 < 1$ and $v_3 < 1$, then $f_2(e) = f_3(e) = (0, 0)$.
- Suppose $N = \{1, 2, 3\}$. If $v_2 \geq 1$, $v_3 \geq 1$, and $v_2 + v_3 < 3$, then

  $$f_1(e) = \begin{cases} (1, \frac{1}{2}) & \text{if } v_1 \geq \frac{1}{2}, \\ (0, 0) & \text{if } v_1 < \frac{1}{2}, \end{cases}$$

  and $f_2(e) = f_3(e) = (0, 0)$. If $v_2 \geq 1$, $v_3 \geq 1$, and $v_2 + v_3 \geq 3$, then

  $$f_1(e) = \begin{cases} (1, \frac{1}{3}) & \text{if } v_1 \geq \frac{1}{3}, \\ (0, 0) & \text{if } v_1 < \frac{1}{3}, \end{cases}$$

  and $f_2(e) = f_3(e) = (0, 0)$. If $v_2 < 1$ or $v_3 < 1$, then $f_1(e) = f_2(e) = f_3(e) = (0, 0)$.
- In any other case, $f_i(e) = (0, 0)$ for each $i \in N$.

It is not a binary posted prices rule as the payment of buyer 1 can be 0, $\frac{1}{3}$, or $\frac{1}{2}$. It satisfies *single shill-proofness*, *strategy-proofness*, and *non-imposition*. Note that it violates *shill-proofness*. □

In contrast with Theorem 1, the next proposition states that Corollary 1 remains valid even if we weaken *shill-proofness* to *single shill-proofness*.

**Proposition 1.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. No rule on $\mathcal{V}_{\mathbb{N}}$ satisfies single shill-proofness, efficiency, strategy-proofness, and non-imposition.*

## 6.2 Reserve price

In practice, the seller often uses a reserve price in order to avoid a too low revenue. In this subsection, we incorporate the seller's reserve price into our model, and extend the main result (Theorem 1) to a model with a reserve price.

First, we introduce a property that respects the seller's reserve price $r \in \mathbb{R}_+$. The following property requires that a buyer receive the object only when her payment is no smaller than a reserve price $r$.

**Respecting reserve price $r$.** Given $r \in \mathbb{R}_+$, for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and each $i \in N$, if $x_i^f(e) = 1$, then $t_i^f(e) \geq r$.

Second, we introduce a variant of a posted prices rule (not a binary posted prices rule) that takes a reserve price $r$ into account. Given a reserve price $r \in \mathbb{R}_+$, a rule $f$ on $\mathcal{V}_{\mathbb{N}}$ is a **posted prices rule with reserve price** $r$ if there is a profile of posted prices $(p_i^f)_{i \in \mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and each $i \in N$, the following hold.

- If $x_i^f(e) = 1$, then $t_i^f(e) = \max\{p_i^f, r\}$ and $v_i \geq t_i^f(e)$.
- If $x_i^f(e) = 0$, then $t_i^f(e) = 0$.

The following result extends Theorem 1 to a model with a reserve price $r \in \mathbb{R}_{++}$.

**Proposition 2.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $r \in \mathbb{R}_{++}$. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying shill-proofness, respecting reserve price $r$, strategy-proofness, and non-imposition. Then, it is a posted prices rule with reserve price $r$.*

Proposition 2 states that the resulting class of rules from *shill-proofness*, *respecting reserve price* $r \in \mathbb{R}_{++}$, *strategy-proofness*, and *non-imposition* is that of posted prices rules with reserve price $r$, not that of binary posted prices rules with reserve prices $r$. This contrasts with Theorem 1 which states that the resulting class of rules from *shill-proofness*, *strategy-proofness*, and *non-imposition* is that of binary posted prices rules.

In Proposition 2, we assume that a reserve price $r$ is positive. If $r = 0$, then any binary posted prices rule *respects reserve price* $r$. Thus, Theorem 1 does not change even if we require a rule satisfy *respecting reserve price* $r = 0$, i.e., if a rule on a rich domain satisfies *shill-proofness*, *respecting reserve price* $r = 0$, *strategy-proofness*, and *non-imposition*, then it is a binary posted prices rules.

## 6.3 Bayesian incentive compatibility

In this subsection, we study how the main result (Theorem 1) changes when we weaken *strategy-proofness* to *Bayesian incentive compatibility*.

First, we introduce *Bayesian incentive compatibility*. To do so, we need to introduce the distribution of valuations. Given $N \in \mathcal{N}$, the valuation profile $v_N$ for $N$ is distributed according to a distribution function $F_N$ with a density function $f_N$ such that for each $v_N \in \mathcal{V}_N$, $f_N(v_N) > 0$.

A distribution is said to be **independent** if for each $i \in \mathbb{N}$, the valuation $v_i$ of buyer $i$ is distributed according to $F_i$ with a density function $f_i$, and for each $N \in \mathcal{N}$ and each $v_N \in \mathcal{V}_N$, $F_N(v_N) = \times_{i \in N} F_i(v_i)$ and $f_N(v_N) = \times_{i \in N} f_i(v_i)$. Further, an independent distribution is said to be **symmetric** if for each pair $i, j \in \mathbb{N}$, $\mathcal{V}_i = \mathcal{V}_j$ and $F_i = F_j$.

Given a rule $f$ on $\mathcal{V}_{\mathbb{N}}$, $N \in \mathcal{N}$, a buyer $i \in N$, and a pair $v_i, v_i' \in \mathcal{V}_i$, let

$$\begin{aligned}
&\mathbb{E}_{v_{N\setminus\{i\}}}\left[u_i(f_i(N,(v_i', v_{N\setminus\{i\}})); v_i)\middle| v_i\right] \\
&= \int_{v_{N\setminus\{i\}} \in \mathcal{V}_{N\setminus\{i\}}} u_i(f_i(N,(v_i', v_{N\setminus\{i\}})); v_i) f_{N\setminus\{i\}}(v_{N\setminus\{i\}}|v_i) dv_{N\setminus\{i\}}
\end{aligned}$$

denote the expected utility of buyer $i$ when her valuation is $v_i$ and her report is $v_i'$, where $f_{N\setminus\{i\}}(\cdot|v_i)$ is a conditional density function of $v_{N\setminus\{i\}}$ given $v_i$.

The next property requires that the truth-telling be a Bayesian Nash equilibrium in a direct revelation game associated with a rule.

**Bayesian incentive compatibility.** For each $N \in \mathcal{N}$, each $i \in N$, and each pair $v_i, v_i' \in \mathcal{V}_i$, we have

$$\mathbb{E}_{v_{N\setminus\{i\}}}\left[u_i(f_i(N,(v_i,v_{N\setminus\{i\}}));v_i)\middle|v_i\right] \geq \mathbb{E}_{v_{N\setminus\{i\}}}\left[u_i(f_i(N,(v_i',v_{N\setminus\{i\}}));v_i)\middle|v_i\right].$$

Note that *strategy-proofness* implies *Bayesian incentive compatibility*, but the converse is not necessarily true. Note also that in the definition of *Bayesian incentive compatibility*, we allow a distribution to be dependent or independent but asymmetric.

The following example shows that if we weaken *strategy-proofness* to *Bayesian incentive compatibility*, then Theorem 1 is no longer valid, i.e., it shows that a rule satisfying *shill-proofness*, *Bayesian incentive compatibility*, and *non-imposition* is not necessarily a binary posted prices rule.

**Example 6.** Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain such that for each $i \in \mathbb{N}$, $\mathcal{V}_i = [0,1]$. Suppose that the distribution is independent and symmetric, and for each $i \in \mathbb{N}$, $F_i$ is a uniform distribution on $\mathcal{V}_i = [0,1]$. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ such that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, the following hold.

- Suppose $N = \{1\}$. If $v_1 \geq \frac{1}{2}$, then $f_1(e) = (1, \frac{1}{2})$, and otherwise, $f_1(e) = (0,0)$.
- Suppose $N = \{1,2\}$. If $v_1 \geq \frac{1}{2}$ and $v_2 \geq \frac{1}{2}$, then $f_1(e) = (1, \frac{1}{4})$, and otherwise, $f_1(e) = (0,0)$.
- Suppose $1 \in N$ and $N \in \mathcal{N}\setminus\{\{1\},\{1,2\}\}$. Then, $f_1(e) = (0,0)$.
- For each $i \in N\setminus\{1\}$, $f_i(e) = (0,0)$.

It is not a binary posted prices rule. It satisfies *shill-proofness*, *Bayesian incentive compatibility*, and *non-imposition*. Note that it violates *strategy-proofness*. □

The next proposition states that even if we weaken *strategy-proofness* to *Bayesian incentive compatibility*, an impossibility result in Proposition 1 remains true.

**Proposition 3.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. No rule on $\mathcal{V}_{\mathbb{N}}$ satisfies single shill-proofness, Bayesian incentive compatibility, efficiency, and non-imposition.*

Since *shill-proofness* implies *single shill-proofness*, we obtain the following.

**Corollary 2.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. No rule on $\mathcal{V}_{\mathbb{N}}$ satisfies shill-proofness, Bayesian incentive compatibility, efficiency, and non-imposition.*

Note that in Proposition 3 and Corollary 2, we do not require that a distribution be independent or symmetric. Thus, they hold not only for independent and symmetric distributions but also for dependent or asymmetric distributions.

Almost all standard rules (e.g., a first-price rule, a third price rule, and an all-pay rule) satisfy *Bayesian incentive compatibility*, *efficiency*, and *non-imposition* in a single object setting. Thus, Proposition 3 and Corollary 2 imply that such rules violate *single shill-proofness* and *shill-proofness*.

Since *Bayesian incentive compatibility* is weaker than *strategy-proofness* (i.e., the latter implies the former), Proposition 1 follows from Proposition 3. Thus, we only prove Proposition 3, and do not give the proof of Proposition 1.

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## 6.4 Characterization

Theorem 1 states that only the binary posted prices rules satisfy *shill-proofness*, *strategy-proofness*, and *non-imposition*. In this subsection, we study whether the converse of Theorem 1 (i.e., all the binary posted prices rules satisfy the three properties) is also true.

First, the next proposition states that all the binary posted prices rules satisfy *individual rationality* and *no subsidy*, and so by Remark 2, they do *non-imposition* as well. We omit a straightforward proof.

**Proposition 4.** *Let $\mathcal{V}_{\mathbb{N}}$ be a domain. Any binary posted prices rule on $\mathcal{V}_{\mathbb{N}}$ satisfies individual rationality, no subsidy, and non-imposition.*

The next two examples show that some binary posted prices rules violate *shill-proofness* and *strategy-proofness*. Thus, the converse of Theorem 1 is not necessarily true.

**Example 7 (Binary posted prices rule violates shill-proofness).** Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. For simplicity, let $\overline{v}_i = 2$ for each $i \in \mathbb{N}$. Let $f$ be a binary posted prices rule on $\mathcal{V}_{\mathbb{N}}$ associated with a profile of posted prices $(p_i^f)_{i \in \mathbb{N}}$ such that $p_i^f = 1$ for each $i \in \mathbb{N}$, and for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, the following hold.

- If $|N| = 1$, then for $i \in N$, $f_i(e) = (0, 0)$.

- If $|N| \geq 2$, then $f(e)$ is equivalent to the outcome of a priority rule for $e$ associated with a priority $\succ^f$ over $\mathbb{N}$ such that $1 \succ^f 2$.

We show that $f$ violates *shill-proofness*. By richness, we can choose $v_{\{1,2\}} \in \mathcal{V}_{\{1,2\}}$ such that $v_1 > 1$ and $v_2 > 1$. Let $e = (\{1\}, v_1) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$. Then, $f_1(e) = (0,0)$. By $1 \succ^f 2$ and $v_1 > 1 = p_1^f$, $f_1(\{1,2\}, v_{\{1,2\}}) = (1,1)$, and so $t_1^f(\{1,2\}, v_{\{1,2\}}) = 1$. Thus, $t_1^f(\{1,2\}, v_{\{1,2\}}) = 1 > 0 = t_1^f(e)$, and $f$ violates *shill-proofness*. □

**Example 8 (Binary posted prices rule violates strategy-proofness).** Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. For simplicity, let $\overline{v}_i = 3$ for each $i \in \mathbb{N}$. Let $f$ be a binary posted prices rule on $\mathcal{V}_{\mathbb{N}}$ associated with a profile of posted prices $(p_i^f)_{i \in \mathbb{N}}$ such that $p_i^f = 1$ for each $i \in \mathbb{N}$, and for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, the following hold.

- If $1 \in N$, then

$$f_1(e) = \begin{cases} (1,1) & \text{if } v_1 \geq 2, \\ (0,0) & \text{otherwise.} \end{cases}$$

- For each $i \in N \backslash \{1\}$, $f_i(e) = (0,0)$.

We show that $f$ violates *strategy-proofness*. By richness, we can choose $v_1 \in \mathcal{V}_1$ such that $v_1 = 1.5$. Let $e = (\{1\}, v_1) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$. Then, $f_1(e) = (0,0)$. Again, by richness, we can choose $v_1' \in \mathcal{V}_1$ such that $v_1' = 2$. Then, $f_1(\{1\}, v_1') = (1,1)$. By $v_1 = 1.5 > 1$, we have $u_1(f_1(\{1\}, v_1'); v_1) > u_1(f_1(e); v_1)$. Thus, $f$ violates *strategy-proofness*. □

Given the above examples, in the remaining part of this subsection, we identify a condition under which a binary posted prices rule satisfies *shill-proofness* and *strategy-proofness*.

The next proposition gives a necessary and sufficient condition for a binary posted prices rule to satisfy *shill-proofness* and *strategy-proofness*. Note that it holds on any domain that is not necessarily rich. We omit a straightforward proof.

**Proposition 5.** *Let $\mathcal{V}_{\mathbb{N}}$ be a domain. A binary posted prices rule on $\mathcal{V}_{\mathbb{N}}$ satisfies shill-proofness and strategy-proofness if and only if for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, the following hold.*

- *For each $N' \in \mathcal{N}$ with $N \cap N' = \varnothing$ and each $v_{N'} \in \mathcal{V}_{N'}$, we have*

$$\sum_{i \in N} t_i^f(N \cup N', v_{N \cup N'}) \leq \sum_{i \in N} t_i^f(e).$$

- *If* $x_i^f(e) = 1$, *then for each* $v_i' \in \mathcal{V}_i$ *with* $v_i' > t_i^f(e)$, $x_i^f(N, (v_i', v_{N\setminus\{i\}})) = 1$.
- *If* $x_i^f(e) = 1$, *then for each* $v_i' \in \mathcal{V}_i$ *with* $x_i^f(N, (v_i', v_{N\setminus\{i\}})) = 1$, $t_i^f(e) = t_i^f(N, (v_i', v_{N\setminus\{i\}}))$.

Note that the first condition in Proposition 5 is a restatement of the definition of *shill-proofness*. In fact, the class of binary posted prices rule leaves room for the choice of posted prices $(p_i^f)_{i\in\mathbb{N}}$ and flexibility in defining an object allocation rule, which makes it difficult to identify a closed-form necessary and sufficient condition for a binary posted prices rule to satisfy *shill-proofness*. The second and the third conditions are concerned with *strategy-proofness* of a binary posted prices rule. The second one prevents profitable manipulation of a rule such that a buyer who receives no object gets better off by receiving the object as a result of a false report of a valuation. Note that the rule in Example 8 violates *strategy-proofness* because of this type of manipulation. The third one prevents profitable manipulation such that a buyer can lower her posted price without changing her consumption of the object.

Theorem 1 and Propositions 4 and 5 together enable us to obtain a characterization of the class of rules satisfying *shill-proofness*, *strategy-proofness*, and *non-imposition*.

**Proposition 6.** *Let* $\mathcal{V}_{\mathbb{N}}$ *be a rich domain. A rule on* $\mathcal{V}_{\mathbb{N}}$ *satisfies shill-proofnes, strategy-proofnes, and non-imposition if and only if it is a binary posted prices rule such that for each* $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, *the following hold.*

- *For each* $N' \in \mathcal{N}$ *with* $N \cap N' = \varnothing$ *and each* $v_{N'} \in \mathcal{V}_{N'}$, *we have*

$$\sum_{i\in N} t_i^f(N \cup N', v_{N\cup N'}) \leq \sum_{i\in N} t_i^f(e).$$

- *If* $x_i^f(e) = 1$, *then for each* $v_i' \in \mathcal{V}_i$ *with* $v_i' > t_i^f(e)$, $x_i^f(N, (v_i', v_{N\setminus\{i\}})) = 1$.
- *If* $x_i^f(e) = 1$, *then for each* $v_i' \in \mathcal{V}_i$ *with* $x_i^f(N, (v_i', v_{N\setminus\{i\}})) = 1$, $t_i^f(e) = t_i^f(N, (v_i', v_{N\setminus\{i\}}))$.

Recall that in Example 1 in Section 4, we introduce several examples of binary posted prices rules. It is straightforward to show that these binary posted prices rules satisfy the conditions in Proposition 6. Thus, Proposition 6 implies that these rules satisfy *shill-proofness*, *strategy-proofness*, and *non-imposition*.

## 6.5 Non-quasi-linear utility functions with interdependent valuations

In this subsection, we show that the main result (Theorem 1) extends to a setting of non-quasi-linear utility functions with interdependent valuations.

We briefly introduce a model of non-quasi-linear utility functions with interdependent valuations. Each buyer $i \in \mathbb{N}$ has a type $\theta_i \in \mathbb{R}^{k_i}_+$, where $1 \leq k_i < \infty$. Our generic notation for a class of types of a buyer $i \in \mathbb{N}$ is $\Theta_i \subseteq \mathbb{R}^{k_i}_+$. A type of a buyer is private information only known for her.

Given $N \in \mathcal{N}$, let $\Theta_N = \times_{i \in N} \Theta_i$. Let $\Theta_{\mathbb{N}} = \times_{i \in \mathbb{N}} \Theta_i$. We call $\Theta_{\mathbb{N}}$ a **domain**.

Given $i \in \mathbb{N}$, let $\mathcal{N}_i = \{N \in \mathcal{N} : i \in N\}$ denote the family of sets of buyers that include buyer $i$. Each buyer $i \in \mathbb{N}$ has a utility function $u_i : M \times \mathbb{R} \times \cup_{N \in \mathcal{N}_i} \Theta_N \to \mathbb{R} \cup \{-\infty\}$ such that for some budget $b_i : \cup_{N \in \mathcal{N}_i} \Theta_N \to \mathbb{R}_+ \cup \{\infty\}$, we have that for each $z_i = (x_i, t_i) \in M \times \mathbb{R}$, each $N \in \mathcal{N}_i$, and each $\theta_N \in \Theta_N$, if $t_i \leq b_i(\theta_N)$, then $u_i(z_i, \theta_N) > -\infty$, and if $t_i > b_i(\theta_N)$, then $u_i(z_i, \theta_N) = -\infty$. Note that we allow a utility function of a buyer (i) to exhibit income effects, (ii) to face hard budget constraints, and (iii) to depend on other buyers' types. For each $N \in \mathcal{N}_i$ and each $\theta_N \in \Theta_N$, we normalize that $u_i((0,0), \theta_N) = 0$.

We assume that a utility function of a buyer $i \in \mathbb{N}$ satisfies the following properties.

**Weak desirability of the object.** For each $N \in \mathcal{N}_i$, each $\theta_N \in \Theta_N$, and each $t_i \in \mathbb{R}$ with $t_i \leq b_i(\theta_N)$, we have $u_i((1, t_i), \theta_N) \geq u_i((0, t_i), \theta_N)$.

**Money monotonicity.** For each $N \in \mathcal{N}_i$, each $\theta_N \in \Theta_N$, each $x_i \in M$, and each pair $t_i, t'_i \in \mathbb{R}$ with $t_i < t'_i \leq b_i(\theta_N)$, we have $u_i((x_i, t_i), \theta_N) > u_i((x_i, t'_i), \theta_N)$.

**Finiteness.** For each $N \in \mathcal{N}_i$, each $\theta_N \in \Theta_N$, and each $t_i \in \mathbb{R}$ with $u_i((0, t_i), \theta_N) \geq u_i((1, b_i(\theta_N)), \theta_N)$, there is $w_i(t_i, \theta_N) \in \mathbb{R}$ such that $u_i((1, w_i(t_i, \theta_N)), \theta_N) = u_i((0, t_i), \theta_N)$.

Given $i \in \mathbb{N}$, $N \in \mathcal{N}_i$, and $\theta_N \in \Theta_N$ such that $u_i((0,0), \theta_N) \geq u_i((1, b_i(\theta_N)), \theta_N)$, we call $w_i(\theta_N) \in \mathbb{R}$ with $u_i((1, w_i(\theta_N)), \theta_N) = u_i((0,0), \theta_N)$ the **valuation for** $\theta_N$. Note that such $w_i(\theta_N)$ exists by finiteness. For notational convenience, if $u_i((1, b_i(\theta_N)), \theta_N) > u_i((0,0), \theta_N)$, then let $w_i(\theta_N) = \infty$.

Given $i \in \mathbb{N}$, $N \in \mathcal{N}_i$, and $\theta_N \in \Theta_N$, the **truncated valuation for** $\theta_N$ is $v_i(\theta_N) =$

$\min\{w_i(\theta_N), b_i(\theta_N)\}$. Note that $v_i(\theta_N) \in \mathbb{R}_+$. A truncated valuation $v_i(\theta_N)$ plays the same role as a valuation $v_i$ in a model of quasi-linear utility functions with private valuations that we have studied so far.

Given a domain $\Theta_{\mathbb{N}}$, an **economy on** $\Theta_{\mathbb{N}}$ is a pair $e = (N, \theta_N)$ such that $N \in \mathcal{N}$ and $\theta_N \in \Theta_N$. Let $\mathcal{E}(\Theta_{\mathbb{N}})$ denote the class of economies on $\Theta_{\mathbb{N}}$.

An object allocation and an allocation for $N \in \mathcal{N}$ are defined in the same way as before. The seller's utility function $u_0$ is also defined in the same way as before.

Given a domain $\Theta_{\mathbb{N}}$, an (**allocation**) **rule on** $\Theta_{\mathbb{N}}$ is a mapping $f : \mathcal{E}(\Theta_{\mathbb{N}}) \to \cup_{N \in \mathcal{N}} Z_N$ such that for each $e = (N, \theta_N) \in \mathcal{E}(\Theta_{\mathbb{N}})$, $f(e) \in Z_N$.

We introduce the properties of rules. All the properties that we have introduced so far except for *strategy-proofness* are defined in the same way as before. The following property is a counterpart of *strategy-proofness* for a model with interdependent valuations. It requires that the truth-telling be an ex-post Nash equilibrium in a direct revelation game associated with a rule.

**Ex-post incentive compatibility.** For each $e = (N, \theta_N) \in \mathcal{E}(\Theta_{\mathbb{N}})$, each $i \in N$, and each $\theta'_i \in \Theta_i$, we have $u_i(f_i(e); \theta_N) \geq u_i(f_i(N, (\theta'_i, \theta_{N \setminus \{i\}})); \theta_N)$.

Note that in a model with private valuations, *ex-post incentive compatibility* is equivalent to *strategy-proofness*.

A posted prices rule and a binary posted prices rule are defined by means of a truncated valuation $v_i(\theta)$ instead of a valuation $v_i$ in a model of quasi-linear utility functions with private valuations.

Given a buyer $i \in \mathbb{N}$, $N \in \mathcal{N}_i$, and $\theta_{N \setminus \{i\}}$, let $\mathcal{V}_i(\theta_{N \setminus \{i\}}) = \{v_i(\theta_i, \theta_{N \setminus \{i\}}) : \theta_i \in \Theta_i\}$ denote the set of truncated valuations for $(\theta_i, \theta_{N \setminus \{i\}})$ given $\theta_{N \setminus \{i\}}$. Note that $\mathcal{V}_i(\theta_{N \setminus \{i\}}) \subseteq \mathbb{R}_+$.

We introduce a counterpart of a rich domain for a model of non-quasi-linear utility functions with interdependent valuations. A domain $\Theta_{\mathbb{N}}$ is **rich** if for each $i \in \mathbb{N}$ and each $\theta_{N \setminus \{i\}} \in \Theta_{N \setminus \{i\}}$, there is $\overline{v}_i(\theta_{N \setminus \{i\}}) \in \mathbb{R}_{++}$ such that either $\mathcal{V}_i(\theta_{N \setminus \{i\}}) = [0, \overline{v}_i(\theta_{N \setminus \{i\}}))$ or $\mathcal{V}_i(\theta_{N \setminus \{i\}}) = [0, \overline{v}_i(\theta_{N \setminus \{i\}})]$. Note that the upper bound $\overline{v}_i(\theta_{N \setminus \{i\}})$ of $\mathcal{V}_i(\theta_{N \setminus \{i\}})$ may depend on other buyers' types $\theta_{N \setminus \{i\}}$.

The following classes of utility functions are examples of those with rich domains. Note that almost all classes of utility functions of interest can be expressed as rich domains.

- A utility function $u_i$ of a buyer $i \in \mathbb{N}$ is **quasi-linear with private valuations** (Vickrey, 1961) if for each $N \in \mathcal{N}_i$ and each $\theta_N \in \Theta_N$, (i) $b_i(\theta_N) = \infty$, (ii) $v_i(\theta_N) = v_i(\theta_i)$, and (iii) for each $z_i = (x_i, t_i) \in M \times \mathbb{R}$, we have $u_i(z_i, \theta_N) = v_i(\theta_i)x_i - t_i$. Note that this corresponds to a utility function that we have studied so far.

- A utility function $u_i$ of a buyer $i \in \mathbb{N}$ is **quasi-linear with common valuations** (Wilson, 1969) if there is a probability distribution $G$ over $[\underline{v}, \overline{v}] \times \Theta_{\mathbb{N}}$ with a density function $g$ such that for each $\theta_N \in \Theta_N$, (i) $b_i(\theta_N) = \infty$, (ii) $v_i(\theta_N) = v_i(\theta_i) = \mathbb{E}_{(\tilde{v}, \tilde{\theta}_{N\setminus\{i\}})}[\tilde{v}|\theta_i] = \int_{(\tilde{v}, \tilde{\theta}_{N\setminus\{i\}})} \tilde{v} g_N(\tilde{v}, \tilde{\theta}_N | \tilde{\theta}_i = \theta_i) d(\tilde{v}, \tilde{\theta}_{N\setminus\{i\}})$, where $g_N(\cdot|\theta_i)$ is a conditional density function of $(v, \theta_N)$ given $\theta_i$, and (iii) for each $z_i = (x_i, t_i) \in M \times \mathbb{R}$, we have $u_i(z_i, \theta_N) = v_i(\theta_i)x_i - t_i$. Note that whether a domain is rich for quasi-linear utility functions with common valuations depends on a distribution $G$.[10]

- A utility function $u_i$ of a buyer $i \in \mathbb{N}$ is **quasi-linear with interdependent valuations** (Milgrom and Weber, 1982) if for each $N \in \mathcal{N}_i$ and each $\theta_N \in \Theta_N$, (i) $b_i(\theta_N) = \infty$, and (iii) for each $z_i = (x_i, t_i) \in M \times \mathbb{R}$, we have $u_i(z_i, \theta_N) = v_i(\theta_N)x_i - t_i$.

- A utility function $u_i$ of a buyer $i \in \mathbb{N}$ is **quasi-linear with soft budgets** (Saitoh and Serizawa, 2008) if for each $N \in \mathcal{N}_i$ and each $\theta_N \in \Theta_N$, (i) $b_i(\theta_N) = \infty$, and (ii) there are an income $I_i(\theta_N) \in \mathbb{R}_+ \cup \{\infty\}$, an interest rate $r \in \mathbb{R}_+$, and $\tilde{v}_i(\theta_N) \in \mathbb{R}_+$ such that for each $z_i = (x_i, t_i) \in M \times \mathbb{R}$, we have

$$u_i((z_i, t_i), \theta_N) = \begin{cases} \tilde{v}_i(\theta_N)x_i - t_i & \text{if } t_i \leq I_i(\theta_N), \\ \tilde{v}_i(\theta_N)x_i - I_i(\theta_N) - (1+r)(t_i - I_i(\theta_N)) & \text{if } t_i > I_i(\theta_N). \end{cases}$$

  Note that $\tilde{v}_i(\theta_N)$ is in general different from $v_i(\theta_N)$.[11]

- A utility function $u_i$ of a buyer $i \in \mathbb{N}$ is **quasi-linear with public soft budgets** if for each $N \in \mathcal{N}_i$ and each $\theta_N \in \Theta_N$, (i) $b_i(\theta_N) = \infty$, and (ii) there are an income $I_i \in \mathbb{R}_+ \cup \{\infty\}$, an interest rate $r \in \mathbb{R}_+$, and $\tilde{v}_i(\theta_N) \in \mathbb{R}_+$ such that for each $z_i =$

[10]For example, suppose that $\underline{v} = 0$ and $\overline{v} = 1$, and $\Theta_i = [0, 1]$ for each $i \in \mathbb{N}$. If $G$ is a uniform distribution (i.e., for each $N \in \mathcal{N}$ and each $(v, \theta_N) \in [0, 1]^{|N|+1}$, $g_N(v, \theta_N) = 1$), then a domain is not rich. In contrast, if $G$ has a density function $g$ such that for each $N \in \mathcal{N}$ and each $(v, \theta_N) \in [0, 1]^{|N|+1}$, $g(v, \theta_N) = v \times (\times_{i \in N} \theta_i)$, then a domain is rich.

[11]Indeed, if $\tilde{v}_i(\theta_N) \leq I_i(\theta_N)$, then $v_i(\theta_N) = \tilde{v}_i(\theta_N)$, and otherwise, $v_i(\theta_N) = \frac{\tilde{v}_i(\theta_N) + rI_i(\theta_N)}{1+r}$.

$(x_i, t_i) \in M \times \mathbb{R}$ we have,

$$u_i((z_i, t_i), \theta_N) = \begin{cases} \tilde{v}_i(\theta_N)x_i - t_i & \text{if } t_i \leq I_i, \\ \tilde{v}_i(\theta_N)x_i - I_i - (1+r)(t_i - I_i) & \text{if } t_i > I_i. \end{cases}$$

- A utility function $u_i$ of a buyer $i \in \mathbb{N}$ is **quasi-linear with hard budegets** (Dobzinski et al., 2012; Hafalir et al., 2012, etc.) if for each $N \in \mathcal{N}_i$, each $\theta_N \in \Theta_N$, and each $z_i = (x_i, t_i) \in M \times \mathbb{R}$, we have

$$u_i(z_i, \theta_N) = \begin{cases} v_i(\theta_N)x_i - t_i & \text{if } t_i \leq b_i(\theta_N), \\ -\infty & \text{if } t_i > b_i(\theta_N), \end{cases}$$

- A utility function $u_i$ of a buyer $i \in \mathbb{N}$ is **quasi-linear with public hard budegets** (Dobzinski et al., 2012, Lavi and May, 2012, etc.) if (i) there is $b_i \in \mathbb{R}_+ \cup \{\infty\}$ such that for each $N \in \mathcal{N}_i$ and each $\theta_N \in \Theta_N$, $b_i(\theta_N) = b_i$, and (ii) for each $N \in \mathcal{N}_i$, each $\theta_N \in \Theta_N$, and each $z_i = (x_i, t_i) \in M \times \mathbb{R}$, we have

$$u_i(z_i, \theta_N) = \begin{cases} v_i(\theta_N)x_i - t_i & \text{if } t_i \leq b_i, \\ -\infty & \text{if } t_i > b_i, \end{cases}$$

- A utility function $u_i$ of a buyer $i \in \mathbb{N}$ exhibits **income effects with private valuations** (Saitoh and Serizawa, 2008; Sakai, 2008, etc.) if for each $N \in \mathcal{N}_i$ and each $\theta_N \in \Theta_N$, $v_i(\theta_N) = v_i(\theta_i)$.

- A utility function $u_i$ of a buyer $i \in \mathbb{N}$ exhibits **positive income effects** (Baisa, 2020, Malik and Mishra, 2021, etc.) if for each $N \in \mathcal{N}_i$, each $\theta_N \in \Theta_N$, and each pair $t_i, t'_i \in \mathbb{R}$ such that $t_i < t'_i$ and $u_i((0, t'_i), \theta_N) \geq u_i((1, b_i(\theta_N)), \theta_N)$, we have $w_i(t_i, \theta_N) - t_i > w_i(t'_i, \theta_N) - t'_i$.

- A utility function $u_i$ of a buyer $i \in \mathbb{N}$ exhibits **negative income effects** (Shinozaki et al., 2022) if for each $N \in \mathcal{N}_i$, each $\theta_N \in \Theta_N$, and each pair $t_i, t'_i \in \mathbb{R}$ such that $t_i < t'_i$ and $u_i((0, t'_i), \theta_N) \geq u_i((1, b_i(\theta_N)), \theta_N)$, we have $w_i(t_i, \theta_N) - t_i < w_i(t'_i, \theta_N) - t'_i$.

Now, we are ready to introduce the result in a model of non-quasi-linear utility functions with interdependent valuations. The next proposition states that Theorem 1 carries over to a model of non-quasi-linear utility functions with interdependent valuations. Since its proof is essentially same as that of Theorem 1, we omit it.

**Proposition 7.** *Let $\Theta_{\mathbb{N}}$ be a rich domain. Let $f$ be a rule on $\Theta_{\mathbb{N}}$ satisfying shill-proofness, ex-post incentive compatibility, and non-imposition. Then, it is a binary fixed prices rule.*

Recall that we introduced several examples of the classes of utility functions with rich domains. Proposition 7 holds for the classes of utility functions such as quasi-linear utility functions with independent, common, or interdependent valuations and non-quasi-linear utility functions with independent or interdependent valuations.

## 7 Conclusion

In this paper, we have studied the class of rules that prevent the seller from raising her revenue from introducing false-name buyers (i.e., shill bidding) in the object allocation problem with money. The main result of this paper (Theorem 1) shows that only the binary posted prices rules satisfy *shill-proofness*, *strategy-proofness*, and *non-imposition*. Thus, if the planner would like to prevent the seller from shill bidding, then she must accept the rigidity of the payment of each buyer.

Finally, we discuss a possible direction of future research given our main result. Our result highlights the difficulty in preventing the seller's shill bidding, and so suggests the importance of technology that detects the seller's shill bidding. To study how the seller does shill bidding under several rules of importance will help the researchers develop effective detection technology of shill bidding. In our companion paper (Shinozaki, 2024a), we study how the seller optimally does shill bidding under a Vickrey rule and a pay as bid rule with private valuations. To further study the seller's shill bidding behavior under various rules and modeling assumptions will be an important topic of future research.

# Appendix

## A Proof of Theorem 1

In this section, we prove Theorem 1. The proof is in a series of lemmata.

The following lemma shows that the combination of *strategy-proofness* and *non-imposition* implies the combination of *individual rationality* and *no subsidy*.[12]

**Lemma 1.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $f$ be a rule on $\mathcal{V}$ satisfying strategy-proofness and non-imposition. Then, it satisfies individual rationality and no subsidy.*

*Proof.* First, we show that $f$ satisfies *individual rationality*. Let $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and $i \in N$. By contradiction, suppose $0 > u_i(f_i(e); v_i)$. By richness, we can choose $v'_i \in \mathcal{V}$ such that $v'_i = 0$. By *non-imposition*, $u_i(f_i(N, (v'_i, v_{N\setminus\{i\}})); v'_i) = 0$. Thus, by $v'_i = 0$, $f_i(N, (v'_i, v_{N\setminus\{i\}})) \in \{(1, 0), (0, 0)\}$. Thus,

$$u_i(f_i(N, (v'_i, v_{N\setminus\{i\}})); v_i) \geq 0 > u_i(f_i(e); v_i),$$

where the first inequality follows from $v_i \geq 0$. However, this contradicts *strategy-proofness*.

Second, we show that $f$ satisfies *no subsidy*. Let $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and $i \in N$. By contradiction, suppose $t^f_i(e) < 0$. By richness, there is $v'_i \in \mathcal{V}$ such that $v'_i = 0$. Then, by *non-imposition*, $u_i(f_i(N, (v'_i, v_{N\setminus\{i\}})); v'_i) = 0$. By $t^f_i(e) < 0$ and $v'_i = 0$, we have

$$u_i(f_i(e); v'_i) > 0 = u_i(f_i(N, (v'_i, v_{N\setminus\{i\}})); v'_i),$$

which contradicts *strategy-proofness*. □

The following lemma shows that under a rule satisfying *strategy-proofness* and *non-imposition*, if a buyer does not receive the object, then she makes no payment.

**Lemma 2.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying strategy-proofness and non-imposition. Let $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and $i \in N$. If $x^f_i(e) = 0$, then $t^f_i(e) = 0$.*

*Proof.* Suppose $x^f_i(e) = 0$. By Lemma 1, $f$ satisfies *no subsidy*. Thus, $t^f_i(e) \geq 0$. Also, by Lemma 1, $f$ satisfies *individual rationality*. Thus, $u_i(f_i(e)) \geq 0$. By $x^f_i(e) = 0$, $t^f_i(e) \leq 0$. Thus, $t^f_i(e) = 0$. □

The following lemma shows that under a rule satisfying *strategy-proofness* and *non-imposition*, if a buyer receives the object, then her valuation is no smaller than her payment.

[12] Klaus and Nichifor (2020) show in a model with non-negative payments that the combination of *strategy-proofness* and *non-imposition* implies *individual rationality* (see their Lemma 1). Note that they a priori assume that the model includes only non-negative payments. We generalize their lemma by showing that the combination also implies *no subsidy* in a model where the payments are not necessarily non-negative.

**Lemma 3.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying strategy-proofness and non-imposition. Let $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and $i \in N$. If $x_i^f(e) = 1$, then $v_i \geq t_i^f(e)$.*

*Proof.* Suppose by contradiction that $x_i^f(e) = 1$, but $v_i < t_i^f(e)$. Then, $0 > u_i(f_i(e); v_i)$. By Lemma 1, $f$ satisfies *individual rationality*. Thus, $u_i(f_i(e); v_i) \geq 0$, a contradiction. □

The following lemma states that under a rule satisfying *strategy-proofness* and *non-imposition*, if a buyer receives the object in an economy that includes only her, then her payment is independent of her own valuation.

**Lemma 4.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying strategy-proofness and non-imposition. Let $i \in \mathbb{N}$. For each pair $v_i, v_i' \in \mathcal{V}_i$ with $x_i^f(\{i\}, v_i) = x_i^f(\{i\}, v_i') = 1$, we have $t_i^f(\{i\}, v_i) = t_i^f(\{i\}, v_i')$.*

*Proof.* By contradiction, suppose that there is a pair $v_i, v_i' \in \mathcal{V}_i$ such that $x_i^f(\{i\}, v_i) = x_i^f(\{i\}, v_i') = 1$, but $t_i^f(\{i\}, v_i) \neq t_i^f(\{i\}, v_i')$. Without loss of generality, let $t_i^f(\{i\}, v_i') < t_i^f(\{i\}, v_i)$. Then, by $x_i^f(\{i\}, v_i) = x_i^f(\{i\}, v_i') = 1$,

$$u_i(f_i(\{i\}, v_i'); v_i) > u_i(f_i(\{i\}, v_i); v_i),$$

which contradicts *strategy-proofness*. □

In the remaining part of this section, let $\mathcal{V}_{\mathbb{N}}$ be a rich domain, and $f$ a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying *shill-proofness*, *individual rationality*, and *non-imposition*.

For each $i \in \mathbb{N}$, let $p_i^f \in \mathbb{R}$ be such that if there is $v_i \in \mathcal{V}_i$ such that $x_i^f(\{i\}, v_i) = 1$, then $p_i^f = t_i^f(\{i\}, v_i)$, and otherwise, $p_i^f = 0$. By Lemma 4, $p_i^f$ is well-defined. By Lemma 1, $f$ satisfies *no subsidy*, so that $p_i^f \in \mathbb{R}_+$.

The following lemma states that if a buyer receives the object and her payment is positive, then she also receives the object in an economy that only includes her.

**Lemma 5.** *Let $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and $i \in N$. If $x_i^f(e) = 1$ and $t_i^f(e) > 0$, then $x_i^f(\{i\}, v_i) = 1$.*

*Proof.* Suppose by contradiction that $x_i^f(e) = 1$ and $t_i^f(e) > 0$, but $x_i^f(\{i\}, v_i) = 0$. Then, by Lemma 2, $t_i^f(\{i\}, v_i) = 0$. Then, $\{i\} \subsetneq N$, and

$$t_i^f(e) > 0 = t_i^f(\{i\}, v_i),$$

which contradicts *shill-proofness*. $\square$

The following lemma states that if a buyer receives the object and her payment is positive, then her payment in an economy that only includes her is greater than zero and no greater than her maximal possible valuation.

**Lemma 6.** *Let* $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ *and* $i \in N$. *If* $x_i^f(e) = 1$ *and* $t_i^f(e) > 0$, *then* $0 < p_i^f \leq \overline{v}_i$.

*Proof.* Suppose $x_i^f(e) = 1$ and $t_i^f(e) > 0$. By Lemma 5, $x_i^f(\{i\}, v_i) = 1$. Thus, by Lemma 4, $t_i^f(\{i\}, v_i) = p_i^f$. By Lemma 1, $f$ satisfies *individual rationality*. Thus, $p_i^f \leq v_i$. By $v_i \in \mathcal{V}_i \subseteq [0, \overline{v}_i]$, $v_i \leq \overline{v}_i$. Thus, $p_i^f \leq \overline{v}_i$. Further, if $p_i^f = 0$, then $\{i\} \subsetneq N$, and

$$t_i^f(e) > 0 = p_i^f = t_i^f(\{i\}, v_i),$$

which contradicts *shill-proofness*. Thus, we have $p_i^f > 0$. $\square$

The following lemma states that if a buyer receives the object and her payment is positive in an economy, then her payment is equivalent to her payment in an economy that only includes her.

**Lemma 7.** *Let* $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ *and* $i \in N$. *If* $x_i^f(e) = 1$ *and* $t_i^f(e) > 0$, *then* $t_i^f(e) = p_i^f$.

*Proof.* Suppose by contradiction that $x_i^f(e) = 1$ and $t_i^f(e) > 0$, but $t_i^f(e) \neq p_i^f$. By the definition of $p_i^f$, $\{i\} \subsetneq N$.

First, suppose $t_i^f(e) < p_i^f$. By Lemma 6, $0 < p_i^f \leq \overline{v}_i$. Thus, by richness, we can choose $v_i' \in \mathcal{V}_i$ such that $t_i^f(e) < v_i' < p_i^f$. Then, by $x_i^f(e) = 1$, *strategy-proofness*, and Lemma 2, $f_i(N, (v_i', v_{N\setminus\{i\}})) = f_i(e)$. By *individual rationality* (which follows from Lemma 1) and Lemma 4, $x_i^f(\{i\}, v_i') = 0$. Thus, by Lemma 2, $t_i^f(\{i\}, v_i') = 0$. By $t_i^f(e) > 0$, we have

$$t_i^f(N, (v_i', v_{N\setminus\{i\}})) = t_i^f(e) > 0 = t_i^f(\{i\}, v_i'),$$

which contradicts *shill-proofness*.

Second, suppose $t_i^f(e) > p_i^f$. By $p_i^f \geq 0$, $t_i^f(e) > 0$. By $x_i^f(e) = 1$ and $t_i^f(e) > 0$, Lemma 5 implies $x_i^f(\{i\}, v_i) = 1$. Thus, by Lemma 4, $t_i^f(\{i\}, v_i) = p_i^f$. By $t_i^f(e) > p_i^f$, we have

$$t_i^f(e) > p_i^f = t_i^f(\{i\}, v_i),$$

which contradicts *shill-proofness*. □

Now, we are in a position to complete the proof of Theorem 1. Let $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and $i \in N$. If $x_i^f(e) = 1$, then by Lemma 7, either $t_i^f(e) = p_i^f$ or $t_i^f(e) = 0$, and by Lemma 3, $v_i \geq t_i^f(e)$. If $x_i^f(e) = 0$, then by Lemma 2, $t_i^f(e) = 0$. Thus, $f$ is a binary posted prices rule. ■

# B Proof of Proposition 2

In this section, we prove Proposition 2. Let $r \in \mathbb{R}_{++}$. Let $f$ be a rule satisfying *shill-proofness*, *respecting reserve price* $r$, *strategy-proofness*, and *non-imposition*. Then, by Theorem 1, it is a binary posted prices rule. Let $(p_i^f)_{i \in \mathbb{N}}$ be a profile of posted prices associated with $f$. Let $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and $i \in N$ be such that $x_i^f(e) = 1$. Then, $t_i^f(e) \in \{p_i^f, 0\}$. Since $f$ respects reserve price $r$, $t_i^f(e) \geq r > 0$. Thus, $t_i^f(e) = p_i^f$. By $t_i^f(e) \geq r$, $p_i^f \geq r$. Thus, $t_i^f(e) = p_i^f = \max\{p_i^f, r\}$. Thus, $f$ is a posted prices rule with reserve price $r$. ■

# C Proof of Proposition 3

In this section, we prove Proposition 3. We begin with the four lemmata concerning the properties of *Bayesian incentive compatible* rules.

The following lemma is an analogue of Lemma 1 for a rule satisfying *Bayesian incentive compatibility* and *non-imposition* in an economy that consists of only a single buyer.

**Lemma 8.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying Bayesian incentive compatibility and non-imposition. Let $i \in \mathbb{N}$ and $v_i \in \mathcal{V}_i$. We have (i) $u_i(f_i(\{i\}, v_i); v_i) \geq 0$ and (ii) $t_i^f(\{i\}, v_i) \geq 0$.*

*Proof.* Note that in an economy with a single buyer, *Bayesian incentive compatibility* is equivalent to *strategy-proofness* : both require that for each $e = (\{i\}, v_i) \in \mathcal{E}$ and each $v_i' \in \mathcal{V}_i$, $u_i(f_i(e); v_i) \geq u_i(f_i(\{i\}, v_i'); v_i)$. Thus, by the same proof as in Lemma 1, we have that for each $i \in \mathbb{N}$ and each $v_i \in \mathcal{V}_i$, $u_i(f_i(\{i\}, v_i); v_i) \geq 0$ and $t_i^f(\{i\}, v_i) \geq 0$. □

The next lemma is an analogue of Lemma 2 for a rule satisfying *Bayesian incentive compatibility* and *non-imposition* in an economy that consists of only a single buyer. Since its proof is same as that of Lemma 2 given Lemma 8, we omit it.

**Lemma 9.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying Bayesian incentive compatibility and non-imposition. Let $i \in \mathbb{N}$ and $v_i \in \mathcal{V}_i$. If $x_i^f(\{i\}, v_i) = 0$, then $t_i^f(\{i\}, v_i) = 0$.*

The next lemma states that under a rule satisfying *Bayesian incentive compatibility*, *efficiency*, and *non-imposition*, if a buyer receives the object in an economy that includes only her, then her payment is equal to zero.

**Lemma 10.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying Bayesian incentive compatibility, efficiency, and non-imposition. Let $i \in \mathbb{N}$ and $v_i \in \mathcal{V}_i$. If $x_i^f(\{i\}, v_i) = 1$, then $t_i^f(\{i\}, v_i) = 0$.*

*Proof.* By contradiction, suppose $x_i^f(\{i\}, v_i) = 1$, but $t_i^f(\{i\}, v_i) > 0$. Then, by richness, we can choose $v_i' \in \mathcal{V}_i$ such that $0 < v_i' < t_i^f(\{i\}, v_i)$. We show that $x_i^f(\{i\}, v_i') = 0$. Suppose by contradiction that $x_i^f(\{i\}, v_i') = 1$. Then, by Lemma 8, $t_i^f(\{i\}; v_i') \leq v_i'$. By $v_i' < t_i^f(\{i\}, v_i)$, $t_i^f(\{i\}, v_i') < t_i^f(\{i\}, v_i)$. By $x_i^f(\{i\}, v_i') = x_i^f(\{i\}, v_i) = 1$, we have

$$u_i(f_i(\{i\}; v_i'); v_i) > u_i(f_i(\{i\}, v_i); v_i),$$

which contradicts *Bayesian incentive compatibility*. Thus, $x_i^f(\{i\}, v_i') = 0$. However, by $v_i' > 0$, *efficiency* implies that $x_i^f(\{i\}, v_i') = 1$, which contradicts $x_i^f(\{i\}, v_i') = 0$. □

The following lemma states that under a rule satisfying *single shill-proofness*, *Bayesian incentive compatibility*, *efficiency*, and *non-imposition*, the revenue of the seller is always non-positive.

**Lemma 11.** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $f$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying single shill-proofness Bayesian incentive compatibility, efficiency, and non-imposition. For each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ and each $i \in N$, $\sum_{i \in N} t_i^f(e) \leq 0$.*

*Proof.* We do the proof by induction on the number of buyers in an economy.

INDUCTION BASE. For each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ with $|N| = 1$ and $i \in N$, by Lemmata 9 and 10, $t_i^f(e) = 0$.

Induction hypothesis. Suppose that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ with $|N| = n \geq 1$, $\sum_{i \in N} t_i^f(e) \leq 0$.

Induction step. Let $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$ be such that $|N| = n + 1$, where $n \geq 1$. For each $i \in N$, by *single shill-proofness* and the induction hypothesis, we have

$$\sum_{j \in N \setminus \{i\}} t_j^f(e) \leq \sum_{j \in N \setminus \{i\}} t_j^f(N \setminus \{i\}, v_{N \setminus \{i\}}) \leq 0.$$

Summing up these inequalities over all the buyers in $N$, we have

$$(|N| - 1) \sum_{i \in N} t_i^f(e) = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} t_j^f(e) \leq 0.$$

By $|N| \geq 2$, we get

$$\sum_{i \in N} t_i^f(e) \leq 0,$$

as desired. □

Now, we invoke the following two facts.

First, the following is a version of the revenue equivalence theorem by Myerson (1981).

**Fact 1 (Myerson, 1981).** *Let $\mathcal{V}_{\mathbb{N}}$ be a rich domain. Let $f, g$ be a rule on $\mathcal{V}_{\mathbb{N}}$ satisfying Bayesian incentive compatibility, efficiency, and non-imposition. For each $N \in \mathcal{N}$,*

$$\mathbb{E}_{v_N}\Big[\sum_{i \in N} t_i^f(N, v_N)\Big] = \mathbb{E}_{v_N}\Big[\sum_{i \in N} t_i^g(N, v_N)\Big].$$

Given an economy $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, let $v^k(e)$ denote the $k$-th highest valuation among $v_N$. For notational convenience, let $v^k(e) = 0$ if $k > |N|$. A rule $f$ on $\mathcal{V}_{\mathbb{N}}$ is a **Vickrey rule** (Vickrey, 1961) if for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, (i) $f(e)$ is efficient for $e$, and (ii) for each $i \in N$, $t_i^f(e) = x_i^f(e) v^{m+1}(e)$. Note that for each $e = (N, v_N) \in \mathcal{E}(\mathcal{V}_{\mathbb{N}})$, the revenue of the seller for $e$ under a Vickrey rule $f$ on $\mathcal{E}(\mathcal{V}_{\mathbb{N}})$ is

$$\sum_{i \in N} t_i^f(e) = m v^{m+1}(e).$$

Second, the following fact states that a Vickrey rule satisfies *Bayesian incentive compatibility*, *efficiency*, and *non-imposition*.[13]

**Fact 2 (Vickrey, 1961).** *Let $\mathcal{V}_{\mathbb{N}}$ be a domain. A Vickrey rule on $\mathcal{V}_{\mathbb{N}}$ satisfies* Bayesian incentive compatibility*,* efficiency*, and* non-imposition*.*

We complete the proof of Proposition 3. Suppose by contradiction that there is a rule $f$ on a rich domain $\mathcal{V}_{\mathbb{N}}$ satisfying *single shill-proofness*, *Bayesian incentive compatibility*, *efficiency*, and *non-imposition*. Let $N \in \mathcal{N}$ be such that $|N| > m$. Let $g$ be a Vickrey rule on $\mathcal{V}_{\mathbb{N}}$. By our assumption, $f$ satisfies *Bayesian incentive compatibility*, *efficiency*, and *non-imposition*, and by Fact 2, $g$ does so as well. Thus, by Fact 1, we have

$$\mathbb{E}_{v_N}\Big[\sum_{i\in N} t_i^f(N, v_N)\Big] = \mathbb{E}_{v_N}\Big[\sum_{i\in N} t_i^g(N, v_N)\Big] = m\mathbb{E}_{v_N}[v^{m+1}(N, v_N)] > 0, \tag{1}$$

where the inequality follows from $|N| > m$. In contrast, Lemma 11 implies that

$$\mathbb{E}_{v_N}\Big[\sum_{i\in N} t_i^f(N, v_N)\Big] \leq 0,$$

which contradicts (1). ■

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