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The Multi-Threshold Generalized Sufficientarianism and Level-Oligarchy*

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Abstract

This paper investigates a class of social welfare orderings that satisfy the standard and acceptable axioms in the literature: anonymity, strong Pareto, separability, and Pigou-Dalton transfer (or, convexity). Due to the lack of continuity, we show that the class of social welfare orderings typically has some thresholds satisfying the following property, which we call *level-oligarchy*: individuals whose utility is less than the value are prioritized over the other individuals whose utility is greater than the value. First, we provide the novel reduced form characterization that a social welfare ordering satisfies anonymity, strong Pareto, separability, and convexity must be either the weak generalized utilitarian or level-oligarchy. Next, by dropping convexity and instead requiring Pigou-Dalton transfer and a mild continuity axiom, we characterize the new class of social welfare orderings, the *multi-threshold generalized sufficientarian orderings*, which subsumes the leximin, generalized utilitarian, and critical-level sufficientarian social welfare orderings as special cases. Therefore, we can provide a unified characterization for the important class of social welfare orderings only by the permissible axioms. In particular, although the social judgment from both classes of orderings seems quite different, our result implies that the difference between the utilitarian and leximin orderings just comes from the degree of continuity.

JEL classification: D31, D63, D70.

Keywords: Social welfare ordering, utilitarian, leximin, sufficientarianism, distributive justice.

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1 Introduction

1.1 Motivation

A sufficientarian principle is one of the major normative principles that has been rapidly attracting attention in economic theory in recent years. Its essential requirement is that, in the context of distributive justice, the central problem is not relative inequality of well-being or resources itself among individuals, but the existence of individuals who do not have enough (Frankfurt, 1987; Crisp, 2003). Mathematically, the sufficientitarian rule is equivalent to a social welfare ordering that gives absolute priority to the aggregate welfare or number of individuals whose well-being is below a certain threshold (Alcantud et al., 2022; Bossert et al., 2022, 2023; Chambers and Ye, 2023).[1] Since the idea of sufficientarianism is incorporated into various modern social systems, analyzing what kind of axiomatic system can characterize it is an important step to understanding the nature and significance of sufficientatianism.[2]

There are not many reasonable axioms on which there is a consensus in the context of measuring social welfare. Axioms that are still valuable today include anonymity (symmetry), strong Pareto (monotonicity), separability, Pigou-Dalton transfer (or its variant, convexity), and continuity. Some scholars have recommended the separability condition from the viewpoint of time-consistency or independence of the existence of the dead (Blackorby et al., 2005, Ch.8; Adler and Holtug, 2019), but others do not agree with it due to considerations for relative inequality or some types of the repugnant conclusions (Sakamoto, 2023). Among them, it is well-known that anonymity, strong Pareto, separability, and continuity characterize the generalized utilitarian rule (Blackorby et al., 2005, Theorem 4.7). Since sufficientitarianism does not satisfy continuity, it is an interesting endeavor to characterize what kind of acceptable social welfare orderings can survive if continuity is dropped in the system of standard axioms.

The purpose of this study is to show how standard axioms that seem to have sufficiently valid normative implications restrict a class of separable social welfare orderings and to jointly characterize the weak generalized utilitarian, multi-threshold generalized sufficientarian, and leximin rules in a reduced form. Our first main result shows that a social welfare ordering sat-

[1]Brown (2005) and Hirose (2016) also consider mathematical formulations of the idea of sufficientarianism in a framework of social choice theory although they do not intend to provide axiomatic characterizations for the proposed orderings. See also Adler (2019).

[2]As Alcantud et al. (2022) discuss, the sufficientarian idea can be found in social welfare programs such as minimum income compensation for social insurance and pensions.

isfying anonymity, strong Pareto, separability, and convexity must be either weak generalized utilitarian or level-oligarchy. The level-oligarchy requires that, for a specific threshold, even very small improvement for poor individuals whose well-being is below the threshold should overwhelm any sacrifices of the remaining wealthy individuals, no matter how great sacrifices are made to the rich group, as long as their well-being is above the threshold.[3] Therefore, the level-oligarchy is an interesting property that is inherited by both the multi-threshold generalized sufficientarian and leximin rules. This fact shows that feasible options for a class of separable social welfare orderings that satisfies standard axioms are quite limited, in the sense that, essentially, they must be either refinements of generalized utilitarian, multi-threshold generalized sufficientarian, or leximin rules.

By dropping convexity and instead requiring Pigou-Dalton transfer and weak continuity only on the threshold intervals, our second main result shows that a class of acceptable social welfare orderings must be either the generalized utilitarian, multi-threshold generalized sufficientarian, or leximin rules. In fact, if the set of thresholds is empty, then the multi-threshold generalized sufficientarian ordering is equivalent to the generalized utilitarian rule. On the contrary, if the set of thresholds is a countable dense set of real numbers (e.g., rational numbers), then the multi-threshold generalized sufficientarian ordering is equivalent to the leximin rule. Since the leximin rule is generally interpreted as a normative principle that ultimately takes into account relative inequality from an egalitarian perspective, it is a remarkable fact that the leximin rule is derived as the limit of multi-threshold generalized sufficientarianism that considers only absolute levels of well-being and ignores any relative inequality.[4]

In sum, this paper has three main contributions. First, we show that a class of separable social welfare orderings that satisfy the above-mentioned standard axioms is severely limited in the sense that it must result in weak generalized utilitarianism or level-oligarchy. Second, we succeed in jointly characterizing the generalized utilitarian, multi-threshold generalized sufficientarian, and leximin rules in a unified manner. Third, by showing that multi-threshold

[3]Note that the level-oligarchy differs from the usual oligarchy of Arrowian social choice theory in two ways. First, in the Arrowian oligarchy, a decisive group is the unique set of individuals, whereas, in our level-oligarchy, the group must be variable, that is, the set of individuals with below-threshold well-being. Second, in the Arrowian oligarchy, each member of the decisive group has a veto, whereas in the level-oligarchy, each member does not have a veto. We use the term "level-oligarchy" in contrast to rank-dictatorship and its generalization.

[4]See Sakamoto (2021) for theoretical relationships among the ideas of egalitarianism, prioritarianism, and sufficientarianism.

generalized sufficientarianism is equivalent to the leximin rule as a special case where the set of thresholds is the universal set of real numbers, irrational numbers, or rational numbers, we clarify theoretical relationships among the generalized utilitarian, multi-threshold sufficientarian, and leximin rules, which essentially belong to the same class of acceptable social welfare orderings.

1.2 Related literature

Our paper is closely related to the recent development of sufficientarian principles in axiomatic studies of social welfare orderings. Previous studies have axiomatically characterized sufficientarianism by using axioms that explicitly assume the existence of a threshold (Bossert et al., 2022, 2023), or by using axioms that imply the existence of a threshold (Alcantud et al., 2022). Bossert et al. (2022, 2023) characterize the generalized sufficientarian rule in the setting of variable populations by using axioms that explicitly assume the existence of a threshold, which makes it easier to characterize the rule. In the setting of a fixed population, Alcantud et al. (2022) also characterize the basic sufficientarian rule (that only relies on the number of individuals whose well-being is below a threshold) by using axioms that explicitly assume the existence of a threshold. Exceptionally, Chambers and Ye (2023) also characterize the basic sufficientarian rule in the setting of a fixed population but multidimensional commodity space without explicitly assuming the threshold itself. They use an axiom called sufficientarian judgment to axiomatize sufficientarianism. However, their axiom uses the non-increasing property of thresholds, that is, the situation where everyone is at a threshold level is socially better than the situation where one's well-being is slightly below the threshold and another's well-being enormously increases while the others remain the same well-being. In this sense, they implicitly assume the existence of thresholds.[5] As noted above, the axiomatic characterizations of sufficientarianism in the literature have always been based on the system of axioms that (1) explicitly assumes the existence of a threshold, or (2) implicitly incorporates something close to the threshold into an axiom. Contrastingly, we succeeded in characterizing a broader class of

[5]Another essential difference is that sufficientarian judgment is incompatible with the strong Pareto principle. In our notation defined in Section 2, the modified version of sufficientarian judgment is defined as follows: for any $a, b, c \in \mathbb{R}$, if $n * b \succ \big(a, (n-1) * b\big)$, then $\big(a, (n-1) * b\big) \succsim \big(a, c, (n-2) * b\big)$. As we can see from this definition, sufficientarian judgment is incompatible with the strong Pareto principle. Moreover, since Chambers and Ye (2023) characterize sufficientarianism by relying on the number of individuals with below-threshold well-being, this sufficientarianism gives absolute priority to the number of individuals with well-being below b if b is a threshold.

acceptable social welfare orderings, including the generalized utilitarian, multi-threshold generalized sufficientarian, and the leximin rules in a reduced form, without explicitly or implicitly assuming the existence of thresholds.

Our results can also be understood as an extension of Deschamps and Gevers's (1978) celebrated joint characterization of the weak utilitarian, leximin, and leximax rules in a reduced form. They show that a class of separable social welfare orderings that satisfy anonymity, strong Pareto, and scale invariance to positive affine transformations must be either the weak utilitarian, leximin, or leximax rules. Sakamoto (2024) also generalizes their result by imposing rank separability instead of separability and finds a new class of social welfare orderings called the generalized leximin rule that generalizes the rank-weighted utilitarian rule. However, it should be noted that scale invariance, which requires invariance for all specific types of monotone transformations, is neither a necessary nor sufficient condition for interpersonal comparisons of well-being and has no normative significance (Moreau and Weymark, 2016).[6] Therefore, this study does not impose any scale invariance condition on social welfare orderings and shows that a class of separable social welfare orderings must be a refinement of either the generalized sufficientarian, or its special forms, such as the generalized utilitarian and leximin rules. It is worth emphasizing that the important steps for proving this result do not require any advanced techniques, and it shows the proof tools developed by Sakamoto (2024) to be very useful.

The rest of this paper is organized as follows. Section 2 explains the basic notation and definitions in this paper. Section 3 proves our reduced form characterization of the weak generalized utilitarian and level-oligarchy. In Section 4, we provide our main characterization of the multi-threshold generalized sufficientarianism. We provide some discussions in Section 5. Finally, Section 6 has some concluding remarks and summarizes the results of the paper.

[6]Recently, it turns out that the invariance conditions for some types of monotonic transformations, which have been interpreted as interpersonal comparability of utilities in the traditional welfare economics and social choice theory, are merely requirements of scale invariance, which significantly restrict the functional forms in the representations of social welfare orderings and do not have any normative implications. For details, see Morreau and Weymark (2016) and Sakamoto (2021, 2024).

2 Social welfare orderings

2.1 Basic setup and axioms

Let $N = \{1, \ldots, n\}$ with $n \geq 3$ be the set of individuals. For any $M \subseteq N$, the cardinality of M is denoted by the lowercase letter m. We assume that each individual's utility or well-being is defined by a real number $u_i \in \mathbb{R}$. Let $u_N = (u_i)_{i \in N} \in \mathbb{R}^n$ be a utility profile. For any $M \subseteq N$ and $u_N \in \mathbb{R}^n$, we write $u_M = (u_i)_{i \in M} \in \mathbb{R}^m$ and $u_{-M} = (u_i)_{i \in N \setminus M} \in \mathbb{R}^{n-m}$ as sub-utility profiles of u_N. For the notational convenience, we also write $u_{-i} = u_{-\{i\}}$ and $u_{-i,j} = u_{-\{i,j\}}$ for some i, j. For any $a \in \mathbb{R}$ and $m \in \mathbb{N}$, let us denote $m * a = (a, \ldots, a) \in \mathbb{R}^m$.

For a subset of utility profiles, let $\mathcal{F} = \left\{ u \in \mathbb{R}^n \middle| u = n * a, a \in \mathbb{R} \right\}$ be the set of utility profiles on (n-dimensional) 45-degree line. For each $u_N \in \mathbb{R}^n$, we write $u_{[i]}$ is the i-th lowest ranked value in u_N, where $u_{[1]} \leq \ldots \leq u_{[n]}$. For each $u_N \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$, let $L(u_N, \theta) = \{i \in N | u_i < \theta\}$, $H(u_N, \theta) = \{i \in N | u_i > \theta\}$, and $I(u_N, \theta) = \{i \in N | u_i = \theta\}$ be the set of individuals whose utility in u_N is less than, greater than, and equal to θ, respectively. For any $u_N, v_N \in \mathbb{R}^n$, we write $u_N \geq v_N$ if $u_i \geq v_i$ for any $i \in N$, and $u_N > v_N$ if $u_N \geq v_N$ and $u \neq v$. For any $u_N \in \mathbb{R}^n$ and permutation π on N, we write $\pi \circ u_N = (u_{\pi(i)})$. Let Π be the set of all permutations on N.

We consider a social welfare ordering $\succsim$, that is a complete, transitive, and reflexive binary relation on $\mathbb{R}^n$. Let $\succ$ and $\sim$ be the asymmetric and symmetric parts of $\succsim$, respectively. For any $u_N \in \mathbb{R}^n$, let $U_{\succsim}(u_N) = \{v_N \in \mathbb{R}^n | v_N \succsim u_N\}$ and $L_{\succsim}(u_N) = \{v_N \in \mathbb{R}^n | u_N \succsim v_N\}$ be the upper and lower contour set of u_N, respectively. Similarly, let $SU_{\succsim}(u_N) = \mathbb{R}^n \setminus L_{\succsim}(u_N)$, $SL_{\succsim}(u_N) = \mathbb{R}^n \setminus U_{\succsim}(u_N)$, and $I_{\succsim}(u_N) = U_{\succsim}(u_N) \cap L_{\succsim}(u_N)$ be the strict upper contour set, the strict lower contour set, and the indifferent set of u_N, respectively.

We consider the following standard axioms in the literature to define normatively appealing and acceptable social welfare orderings.

Axiom 1 (Anonymity)**.** *For any $u_N \in \mathbb{R}^n$ and any $\pi \in \Pi$, $u_N \sim \pi \circ u_N$.*

Axiom 2 (Strong Pareto)**.** *For any $u_N, v_N \in \mathbb{R}^n$, $u_N \succsim v_N$ if $u_N \geq v_N$, and $u_N \succ v_N$ if $u_N > v_N$.*

Axiom 3 (Separability)**.** *For any $M \subseteq N$, $u_M, v_M \in \mathbb{R}^m$, and $w_{-M}, w'_{-M} \in \mathbb{R}^{n-m}$,*

$$(u_M, w_{-M}) \succsim (v_M, w_{-M}) \Leftrightarrow (u_M, w'_{-M}) \succsim (v_M, w'_{-M}).$$

Axiom 4 (Pigou-Dalton transfer)**.** *For any $u_N, v_N \in \mathbb{R}^n$ such that $u_i = v_i + \delta \leq v_j - \delta = u_j$ for some i, j and $\delta > 0$, and $u_k = v_k$ for any $k \neq i, j$, we have $u_N \succsim v_N$.*

Axiom 5 (Convexity). *For any* $u_N, v_N \in \mathbb{R}^n$ *with* $u_N \succsim v_N$ *and* $\alpha \in [0,1]$, $\alpha u_N + (1-\alpha) v_N \succsim v_N$.

Axiom 6 (Continuity). *For any* $u_N \in \mathbb{R}^n$, *both* $U_{\succsim}(u_N)$ *and* $L_{\succsim}(u_N)$ *are closed.*

Anonymity requires that a social welfare ordering should treat each individual's well-being impartially. Strong Pareto is an efficiency axiom, which demands that social welfare should increase whenever no one's utility decreases and at least one individual's utility increases. Separability is a kind of invariance axiom that requires social welfare orderings to ignore the information of individuals whose utility levels are the same in the two profiles. Pigou-Dalton transfer and convexity are considered equity axioms. Note that convexity is equivalent to stating that, for any u_N, the upper contour set $U_{\succsim}(u_N)$ is convex. Moreover, it is well-known that, under anonymity, convexity implies Pigou-Dalton transfer. Continuity is a robustness axiom for a social ordering against small perturbations of utility profiles.

2.2 Multi-threshold generalized sufficientarianism

We consider some social welfare orderings that satisfy standard axioms stated in Subsection 2.1. The following two classes of social orderings are well-known.

Definition 1. A social welfare ordering $\succsim$ is the L-leximin if, for any $u_N, v_N \in \mathbb{R}^n$, $u_N \succsim v_N$ if and only if there exists $l \in \{1, \ldots, L-1\}$ such that

$$u_{[i]} = v_{[i]} \text{ for all } i \leq l, \text{ and } u_{[l+1]} > v_{[l+1]},$$

or

$$u_{[i]} = v_{[i]} \text{ for all } i \leq L.$$

If $L = n$, then it is called leximin.

Definition 2. A social welfare ordering $\succsim$ is a generalized utilitarian if there exists a continuous and strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that, for any $u_N, v_N \in \mathbb{R}$, $u_N \succsim v_N$ if and only if

$$\sum_{i \in N} g(u_i) \geq \sum_{i \in N} g(v_i).$$

In Definition 2, if we only require that $u_N \succ v_N$ if $\sum_{i \in N} g(u_i) > \sum_{i \in N} g(v_i)$, it is called weak generalized utilitarian. For a special case, if g is the identity function, i.e., (pure) utilitarian, it is called weak utilitarian (Deschamps and Gevers, 1978). In this case, the social welfare ordering $\succsim$ must be a refinement of the utilitarian social welfare ordering.[7]

[7]We say that a social welfare ordering $\succsim$ is a refinement of $\succsim'$ if, for any $u_N, v_N \in \mathbb{R}^n$, $u_N \succ' v_N \Rightarrow u_N \succ v_N$.

The leximin ordering satisfies all the axioms except for continuity, but it satisfies a stronger equity condition than Pigou-Dalton transfer, called Hammond equity (Hammond, 1976).[8] Hammond (1976) shows that the social welfare ordering satisfies anonymity, strong Pareto, and Hammond equity if and only if it is the leximin ordering. In contrast, a generalized utilitarian ordering satisfies all the axioms if g is concave. The characterization holds due to the well-known result of Debreu's (1959) additively separable representation theorem.

Recently, Bossert et al. (2022) proposed a new social welfare ordering, called critical-level sufficientarian ordering, that inherits both the philosophical idea of sufficientarianism principles in distributive justice (Frankfurt, 1987; Crisp, 2003) and the critical-level generalized-utilitarian population principles (Blackorby and Donaldson, 1984).[9]

Definition 3. A social welfare ordering $\succsim$ is a critical-level sufficientarian if there is a continuous and strictly increasing function $g : \mathbb{R} \to \mathbb{R}$, and $\theta \in \mathbb{R}$ such that, for any $u_N, v_N \in \mathbb{R}^n$, $u_N \succsim v_N$ if and only if

$$\sum_{i \in L(u_N,\theta)} \big(g(u_i) - g(\theta)\big) > \sum_{i \in L(v_N,\theta)} \big(g(v_i) - g(\theta)\big) \text{ or,}$$

$$\sum_{i \in L(u_N,\theta)} \big(g(u_i)-g(\theta)\big) = \sum_{i \in L(v_N,\theta)} \big(g(v_i)-g(\theta)\big) \text{ and } \sum_{i \in H(u_N,\theta)} \big(g(u_i)-g(\theta)\big) \geq \sum_{i \in H(v_N,\theta)} \big(g(v_i)-g(\theta)\big).$$

The class of rules evaluates the social welfare of utility profile u_N by the following two-step procedure. First, it divides individuals into two groups, $L(u_N, \theta)$ and $H(u_N, \theta)$, and calculates the representative social welfare for both groups by the (critical-level) utilitarian rule. The value θ is called the threshold. Next, it evaluates the two-dimensional vectors of representative social welfare of each group in a lexicographic manner. Therefore, the individuals whose utility is less than the threshold are absolutely prioritized, and the welfare comparison for the individuals whose utility is greater than the threshold is taken into account as a tie-break. This ordering

[8] Hammond equity states that, for any $u_N, v_N \in \mathbb{R}^n$ such that $v_i < u_i \leq u_j < v_j$ for some i, j, and $u_k = v_k$ for any $k \neq i, j$, we have $u_N \succsim v_N$.

[9] Alcantud et al. (2022) consider the following slightly different social welfare ordering, called (core) sufficientarian social welfare ordering: For any $u_N, v_N \in \mathbb{R}^n$, $u_N \succsim v_N$ if and only if

$$|H(u_N, \theta) \cup I(u_N, \theta)| \geq |H(v_N, \theta) \cup I(v_N, \theta)|.$$

Note that, contrasting to the critical-level sufficientarian social welfare orderings, the sufficientarian social welfare ordering count $|I(u_N, \theta)|$ for a welfare level of u_N. Hence, this ordering is not a subclass of critical-level sufficientarian orderings though those names are quite similar. Nonetheless, if we allow discontinuity of g, we can treat both classes of orderings in a unified class.

can be seen as the composition of a generalized utilitarian and the 2-leximin orderings. Note that if $\theta \to \infty$ and g is bounded, this ordering coincides with a generalized utilitarian ordering.

As the above observation shows, both utilitarian and leximin orderings might be considered special classes of particular social welfare orderings. To treat all the above classes of orderings in a unified manner, we propose the following new class of social welfare ordering, referred to as the *multi-threshold generalized sufficientarian* social welfare orderings.

Definition 4. A social welfare ordering $\succsim$ is a multi-threshold generalized sufficientarian if there are countable real numbers $\{\theta_k\}_{k=1}^K \subseteq \mathbb{R}$ with $-\infty < \theta_1 < \ldots < \theta_K$ and an upper semi-continuous function $g : \mathbb{R} \to \mathbb{R}$ that is strictly increasing and concave within each interval $(-\infty, \theta_1], (\theta_1, \theta_2], \ldots, (\theta_{K-1}, \theta_K], [\theta_K, \infty)$ such that, for any $u_N, v_N \in \mathbb{R}^n$, $u_N \succsim v_N$ if and only if there is $k^* \leq K$ such that

$$\sum_{i \in N_k(u_N)} \big(g(u_i) - g(\theta_k)\big) = \sum_{i \in N_k(v_N)} \big(g(v_i) - g(\theta_k)\big) \text{ for all } k \leq k^* \text{ and,}$$

$$\sum_{i \in N_{k^*+1}(u_N)} \big(g(u_i) - g(\theta_{k^*+1})\big) > \sum_{i \in N_{k^*+1}(v_N)} \big(g(v_i) - g(\theta_{k^*+1})\big),$$

or

$$\sum_{i \in N_k(u_N)} \big(g(u_i) - g(\theta_k)\big) = \sum_{i \in N_k(v_N)} \big(g(v_i) - g(\theta_k)\big) \text{ for all } k \leq K, \text{ and,}$$

$$\sum_{i \in N_{K+1}(u_N)} \big(g(u_i) - g(\theta_K)\big) \geq \sum_{i \in N_{K+1}(v_N)} \big(g(v_i) - g(\theta_K)\big),$$

where $\theta_0 = -\infty$ and

$$N_k(u_N) = \begin{cases} \big\{i \in N \big| u_i \in (\theta_{k-1}, \theta_k]\big\} \text{ if } k = 1, \ldots, K, \\ \big\{i \in N \big| u_i \in [\theta_K, \infty)\big\} \text{ if } k = K+1. \end{cases}$$

Note that it allows $\{\theta_k\}_{k=1}^K$ to be countably infinite. In this case, we regard $\theta_K = \infty$ if the sequence $\{\theta_k\}_{k=1}^\infty$ diverges, and $\theta_K = \theta^* < \infty$ if $\{\theta_k\}_{k=1}^\infty$ converges to some real value θ^*. Note also that we can use g with $\lim_{x \to \infty} g(x) < \infty$, so that the class of orderings is well-defined even for $\theta_K = \infty$. Since $\mathbb{Q}$ is a countable dense in $\mathbb{R}$, observe that this class of orderings indeed includes the above-mentioned social orderings as subclasses in the following sense.[10]

[10]One might consider a remembrance between this observation and the well-known fact that both utilitarian and maximin orderings are extreme cases of Atkinson's social welfare orderings in terms of the limit of the elasticity. However, in our class of social welfare orderings, both utilitarian and leximin orderings are extreme cases in terms of the number of thresholds, which is not related to the elasticity. More technically, our class does not include the maximin social welfare ordering.

Proposition 1. *Let $\succsim$ be a multi-threshold generalized sufficientarian. Then, the following results hold.*

(i) *If $\{\theta_k\}_{k=1}^{\infty}$ is a countable dense set of $\mathbb{R}$ and $g(x) = x$ for any $x \in \mathbb{R}$, then it coincides with the leximin ordering.*

(ii) *If $K = 1$, $\theta_1 \to \infty$ and $\lim_{x\to\infty} g(x) < \infty$, then it coincides with a generalized utilitarian ordering with a concave g.*

(iii) *If $K = 1$ and $\theta_1 < \infty$, then it coincides with a critical-level sufficientarian ordering.*

In Section 4, we provide an axiomatic foundation for the multi-threshold generalized sufficientarian orderings by the standard axioms in Subsection 2.1. Therefore, by Proposition 1, we can provide a unified characterization for the leximin, generalized utilitarian, and critical-level sufficientarian social welfare orderings as special cases.

3 A unified reduced form characterization

Suppose that the social welfare ordering $\succsim$ is the leximin and consider a utility profile $n * \theta \in \mathcal{F}$. Then, by definition, for any $\Delta > 0$ and $\varepsilon > 0$, we have $n * \theta \succ \big(\theta - \varepsilon, \theta + \Delta, (n-2) * \theta\big)$. This implies that an *arbitrarily small* amount of improvement in well-being for the individual less than θ is preferred to an *arbitrarily large* amount of expense in well-being for the individual greater than θ, as long as the utility is still greater than θ after the expense. For the critical-level sufficientarian orderings, the threshold utility profile $n * \theta$ satisfies the above property.

If a social welfare ordering $\succsim$ is not continuous, it may have many such utility profiles. Let

$$\Theta = \left\{\theta \in \mathbb{R} \middle| n * \theta \succ \big(\theta - \varepsilon, \theta + \Delta, (n-2) * \theta\big) \text{ for any } \Delta > 0 \text{ and } \varepsilon > 0\right\}$$

be the set of all *thresholds* of $\succsim$. By definition, to identify whether $\theta \in \mathbb{R}$ is a threshold or not, we must consider all the perturbations around θ with any $\Delta > 0$ and $\varepsilon > 0$. However, if $\succsim$ satisfies strong Pareto and convexity, we can focus on only the neighborhood of θ in the following sense.

Lemma 1. *Suppose that $\succsim$ satisfies strong Pareto and convexity. For any $\theta \in \mathbb{R}$, if $n * \theta \succ \big(\theta - \varepsilon, \theta + \Delta, (n-2) * \theta\big)$ for some $\Delta > 0$ and for any $\varepsilon > 0$, then $\theta \in \Theta$.*

Proof. Take any $\theta \in \mathbb{R}$ that satisfies the condition. Suppose that there exists $\Delta' > 0$ such that $(\theta - \varepsilon, \theta + \Delta', (n-2) * \theta) \succsim n * \theta$. By strong Pareto, $\Delta' > \Delta$. Then, by convexity, for any $\alpha \in [0, 1]$, we have

$$(\theta - \alpha\varepsilon, \theta + \alpha\Delta', (n-2) * \theta) = (\alpha(\theta - \varepsilon) + (1-\alpha)\theta, \alpha(\theta + \Delta') + (1-\alpha)\theta, (n-2) * \theta) \succsim n * \theta.$$

By choosing $\alpha \leq \frac{\Delta}{\Delta'} < 1$, strong Pareto implies that

$$(\theta - \alpha\varepsilon, \theta + \Delta, (n-2) * \theta) \succsim (\theta - \alpha\varepsilon, \theta + \alpha\Delta', (n-2) * \theta) \succsim n * \theta,$$

which contradicts transitivity. □

Under the standard axioms we are interested in, the threshold values can be characterized as the following important property.

Lemma 2. *Suppose that a social welfare ordering $\succsim$ satisfies anonymity, strong Pareto, separability, and Pigou-Dalton transfer. Then, for any $\theta \in \mathbb{R}$, $\theta \in \Theta$ if and only if, for any $u_N \in \mathbb{R}^n$ such that $u_i < \theta < u_j$ for some i, j, we have $(u_i + \varepsilon, \theta, u_{-i,j}) \succ u_N$ for any $\varepsilon > 0$.*

Proof. First, we show the if direction. Take any $\theta \in \mathbb{R}$ satisfying the condition and any $u_N \in \mathbb{R}^n$ with $u_i < \theta < u_j$. Let $\varepsilon = \theta - u_i$ and $\Delta = u_j - \theta$ so that $u_i = \theta - \varepsilon$ and $u_j = \theta + \Delta$. Then, by our assumption, $(\theta, \theta, u_{-i,j}) \succ (\theta - \varepsilon, \theta + \Delta, u_{-i,j}) = u_N$. By separability, we have $n * \theta \succ (\theta - \varepsilon, \theta + \Delta, (n-2) * \theta)$. Since $\Delta > 0$ and $\varepsilon > 0$ can be arbitrary chosen, we have $\theta \in \Theta$.

We next show the only if direction. Seeking a contradiction, suppose that there exists $\theta \in \Theta$, $u_N \in \mathbb{R}^n$ with $u_i < \theta < u_j$ for some i, j, and $\varepsilon > 0$ such that $u_N \succsim (u_i + \varepsilon, \theta, u_{-i,j})$. By separability, it is equivalent to $(u_i, u_j, (n-2) * \theta) \succsim (u_i + \varepsilon, \theta, (n-2) * \theta)$. Then, by Pigou-Dalton transfer, we have $(u_i + \varepsilon, u_j, \theta - \varepsilon, (n-3) * \theta) \succsim (u_i, u_j, (n-2) * \theta)$. Hence, by transitivity and anonymity, we have $(\theta - \varepsilon, u_j, u_i + \varepsilon, (n-3) * \theta) \succsim (\theta, \theta, u_i + \varepsilon, (n-3) * \theta)$. However, by separability, this is equivalent to $(\theta - \varepsilon, u_j, (n-2) * \theta) \succsim n * \theta$, which contradicts $\theta \in \Theta$. □

By Lemma 2, we can deduce the following reduced form characterization of a class of the social welfare orderings that satisfy strong Pareto, anonymity, separability, and Pigou-Dalton transfer, which we call *level-oligarchy*.

Theorem 1. *Suppose that a social welfare ordering $\succsim$ satisfies anonymity, strong Pareto, separability, and Pigou-Dalton transfer. Then, for any $\theta \in \Theta$, $m \in \mathbb{N}$ with $m \leq n$, $\Delta > 0$, $\varepsilon > 0$, and $u < \theta$, we have*

$$\big((n-m)*\theta, m*u\big) \succ \big((n-m)*(\theta+\Delta), m*(u-\varepsilon)\big).$$

Proof. By Lemma 2 and strong Pareto, for any $\theta \in \Theta$, $m \in \mathbb{N}$ with $m \leq n$, $\Delta > 0$, $\varepsilon > 0$, and $u < \theta$, we have

$$\begin{aligned}
\big((n-m)*\theta, m*u\big) &\succ \Big((n-m-1)*\theta, \theta+\Delta, m*\big(u-\frac{(m+1)}{n}\varepsilon\big)\Big)\\
&\vdots\\
&\succ \Big((n-m-k)*\theta, k*(\theta+\Delta), m*\big(u-\frac{(m+k)}{n}\varepsilon\big)\Big)\\
&\vdots\\
&\succ \big((n-m)*(\theta+\Delta), m*(u-\varepsilon)\big),
\end{aligned}$$

for any $k = 1, \ldots, n-m$. Then, transitivity implies the desired relation. □

Next, by strengthening Pigou-Dalton transfer to convexity, we can obtain the following reduced form characterization of weak generalized utilitarian orderings.

Theorem 2. *Suppose that a social welfare ordering $\succsim$ satisfies anonymity, strong Pareto, separability, and convexity. Then, if $\Theta = \emptyset$, $\succsim$ must be a weak generalized utilitarian with a concave function $g : \mathbb{R} \to \mathbb{R}$.*

Proof. Let $\succsim$ be a welfare social ordering which satisfies all the axioms and $\Theta = \emptyset$. Our proof is divided into the following five steps. From the first to third steps, we rely on the modified arguments for that of Theorem 1 in Sakamoto (2024).

Step 1: Preparations.

Take any $u_N \in \mathbb{R}^n$ and $M \subseteq N$, and let

$$W_M(u_N) = \inf\Big\{\alpha \max_{i\in M} u_i + (1-\alpha)\min_{i\in M} u_i \Big| \Big(m*\big(\alpha \max_{i\in M} u_i + (1-\alpha)\min_{i\in M} u_i\big), u_{-M}\Big) \succ u_N, \alpha \in [0,1]\Big\}.$$

By strong Pareto, it is well-defined. Moreover, by separability, the set in the righthand side

$$\Big\{\alpha \max_{i\in M} u_i + (1-\alpha)\min_{i\in M} u_i \Big| \Big(m*\big(\alpha \max_{i\in M} u_i + (1-\alpha)\min_{i\in M} u_i\big), u_{-M}\Big) \succ u_N, \alpha \in [0,1]\Big\}$$

does not depend on u_{-M}, so that $W_M(u_N)$ is independent of the choice of u_{-M}. We can regard $W(u_N) \equiv W_N(u_N)$ as *pseudo-welfare* level of u_N. Note that, even if $u_N > v_N$, it may happen that $W(u_N) = W(v_N)$. By this preparation, we define the following social welfare ordering $\hat{\succsim}$: For any $u_N, v_N \in \mathbb{R}^n$,

$$\begin{cases} u_N \hat{\succ} v_N \text{ if } W(u_N) > W(v_N) \text{ or } u_N > v_N, \\ u_N \hat{\sim} v_N \text{ otherwise.} \end{cases}$$

By construction, $\hat{\succsim}$ is well-defined and it satisfies strong Pareto. Moreover, since $\succsim$ satisfies anonymity, $\hat{\succsim}$ satisfies anonymity as well. In the following, we show that $\succsim$ is a refinement of $\hat{\succsim}$, and it satisfies separability, convexity, and continuity.

Step 2: For any $u_N, v_N \in \mathbb{R}^n$ with $u_{-M} = v_{-M}$ for some $M \subseteq N$, if $W_M(u_N) > W_M(v_N)$, then $u_N \succ v_N$. Hence, $\succsim$ is a refinement of $\hat{\succsim}$.

Since $W_M(u_N) > W_M(v_N)$, by construction,

$$u_N \succ \Big(m * (\frac{1}{2}(W_M(u_N) + W_M(v_N))), u_{-M}\Big) = \Big(m * (\frac{1}{2}(W_M(u_N) + W_M(v_N))), v_{-M}\Big) \succ v_N,$$

so that $u_N \succ v_N$ by transitivity. By this fact, we have shown that, for any $u_N, v_N \in \mathbb{R}^n$, $u_N \hat{\succ} v_N \Leftrightarrow W(u_N) > W(v_N)$ or $u_N > v_N \Rightarrow u_N \succ v_N$, so that $\succsim$ is a refinement of $\hat{\succsim}$.

Step 3: For any $u_N, v_N \in \mathbb{R}^n$ with $u_{-M} = v_{-M}$ for some $M \subseteq N$, if $W_M(u_N) = W_M(v_N)$, then $W(u_N) = W(v_N)$.

By construction, we can observe that

$$W_M(u_N) = W_M\Big(\big(m * W_M(u_N), u_{-M}\big)\Big).$$

We show that $W\Big(\big(m * W_M(u_N), u_{-M}\big)\Big) = W(u_N)$. If this equality holds, we have

$$\begin{aligned} W(u_N) &= W\Big(\big(m * W_M(u_N), u_{-M}\big)\Big) \\ &= W\Big(\big(m * W_M(v_N), v_{-M}\big)\Big) \\ &= W(v_N), \end{aligned}$$

as desired.

Seeking a contradiction, suppose that $W\Big((m*W_M(u_N), u_{-M})\Big) = W(u_N)+\mu$ for some $\mu > 0$. Let λ be such that $W\Big((m*\lambda, u_{-M})\Big) = W(u_N)$. If $\lambda \geq W_M(u_N)$, then

$$\begin{aligned} W\Big((m*\lambda, u_{-M})\Big) &\geq W\Big((m*W_M(u_N), u_{-M})\Big) \\ &= W(u_N)+\mu \\ &> W(u_N) \\ &= W\Big((m*\lambda, u_{-M})\Big), \end{aligned}$$

so that we must have $\lambda < W_M(u_N)$.

For any $\mu' \in (0,\mu)$, let λ' be such that $W\Big((m*\lambda', u_{-M})\Big) = W(u_N)+\mu'$. By the above discussion, $\lambda' \in \big(\lambda, W_M(u_N)\big)$. Then, since $\lambda' < W_M(u_N)$, by definition of $W_M(u_M)$, we must have $u_N \gtrsim \big(m*\lambda', u_{-M}\big)$. Therefore, by Step 1,

$$W(u_N) \geq W\Big((m*\lambda', u_{-M})\Big).$$

On the other hand, since $\mu' > 0$, we have

$$\begin{aligned} W\Big((m*\lambda', u_{-M})\Big) &= W(u_N)+\mu' \\ &> W(u_N), \end{aligned}$$

which is a contradiction. Since the symmetric argument can be applied to the case with $\mu < 0$, we can conclude that $W\Big((m*W_M(u_N), u_{-M})\Big) = W(u_N)$.

Step 4: $\hat{\gtrsim}$ is separable.

By Step 2 and Step 3 together with strong Pareto of $\hat{\gtrsim}$, we can verify that $\hat{\gtrsim}$ is separable. Take any $M \subseteq N$, $u_M, v_M \in \mathbb{R}^m$, and $w_{-M} \in \mathbb{R}^{n-m}$. It is enough to show that $(u_M, w_{-M})\hat{\succ}(v_M, w_{-M})$ implies $(u_M, w'_{-M})\hat{\succ}(v_M, w'_{-M})$ for any $w'_{-M} \in \mathbb{R}^{n-m}$. By definition of $\hat{\gtrsim}$, either (i) $W(u_N) > W(v_N)$, or (ii) $(u_M, w_{-M}) > (v_M, w_{-M})$. If (ii) holds, by strong Pareto of $\hat{\gtrsim}$, we have $(u_M, w'_{-M})\hat{\succ}(v_M, w'_{-M})$ for any $w'_{-M} \in \mathbb{R}^{n-m}$. If (i) holds, since

$$(u_M, w_{-M})\hat{\sim} n*W(u_M, w_{-M})\hat{\sim}\Big(m*\big(W_M(u_M, w_{-M})\big), w_{-M})\Big)$$

and

$$(v_M, w_{-M})\hat{\sim} n*W(v_M, w_{-M})\hat{\sim}\Big(m*\big(W_M(v_M, w_{-M})\big), w_{-M})\Big)$$

by Step 3, it must be that $W_M(u_M, w_{-M}) > W_M(v_M, w_{-M})$ by strong Pareto of $\hat{\succsim}$. Since $W_M(u_M, w_{-M}) = W_M(u_M, w'_{-M})$ and $W_M(v_M, w_{-M}) = W_M(v_M, w'_{-M})$ for any $w'_{-M} \in \mathbb{R}^{n-m}$, we have

$$\Big(m * \big(W_M(u_M, w'_{-M})\big), w'_{-M})\Big) > \Big(m * \big(W_M(u_M, w'_{-M})\big), w'_{-M})\Big).$$

Therefore, by strong Pareto of $\hat{\succsim}$ and Step 3, we have $(u_M, w'_{-M}) \hat{\succ} (v_M, w'_{-M})$.

Step 5: $\hat{\succsim}$ is convex and continuous.

To see the convexity of $\hat{\succsim}$, take any $u_N, v_N \in \mathbb{R}^n$ with $u_N \succsim v_N$ and $\alpha \in [0, 1]$. It is enough to show that $W : \mathbb{R}^n \to \mathbb{R}$ is quasi-concave. By convexity of $\succsim$, we have $\alpha u_N + (1-\alpha)v_N \succsim v_N$. Hence, by Step 2, we have $W(\alpha u_N + (1-\alpha)v_N) \geq W(v_N) = \min\{W(u_N), W(v_N)\}$, which implies that W is quasi-concave.

To see the continuity of $\hat{\succsim}$, take any $c \in \mathbb{R}$ and $\Delta > 0$. Then, by our assumption that $\Theta = \emptyset$ and $\succsim$ satisfies convexity, by Lemma 1, there exits $\varepsilon > 0$ such that $\big(c-\varepsilon, c+\Delta, (n-2)*c\big) \succsim n*c$. Moreover, by convexity and anonymity, if $\varepsilon > \Delta$, we must have $n*c \succ \big(c-\varepsilon, c+\Delta, (n-2)*c\big)$. Therefore,

$$\varepsilon(\Delta, c) = \sup\Big\{\varepsilon > 0 \Big| \big(c-\varepsilon, c+\Delta, (n-2)*c\big) \succsim n*c\Big\}$$

is well-defined and $\varepsilon(\Delta, c) \leq \Delta$. We also set $\varepsilon(0, c) = 0$ by strong Pareto of $\succsim$. Then, by strong Pareto of $\hat{\succsim}$ and the fact that $\succsim$ is a refinment of $\hat{\succsim}$, we have $\big(c - \varepsilon(\Delta, c), c + \Delta, (n-2)*c\big) \hat{\sim} n*c$. By separability of $\hat{\succsim}$, for any $u_{-i,j} \in \mathbb{R}^{n-2}$, if $(c, c, u_{-i,j}) \hat{\sim} n*c$, then $\big(c - \varepsilon(\Delta, c), c+\Delta, u_{-i,j}\big) \hat{\sim} (c, c, u_{-i,j}) \hat{\sim} n*c$. Hence,

$$I_{\hat{\succsim}}(n*c) = \bigcup_{\pi\in\Pi} \bigcup_{u_{-i,j}\in\mathbb{R}^{n-2}, (c,c,u_{-i,j})\hat{\sim} n*c} \bigcup_{\Delta>0} \Big\{v \in \mathbb{R}^n \Big| v = \pi \circ \big(c - \varepsilon(\Delta, c), c+\Delta, u_{-i,j}\big)\Big\} \cup \{n*c\}.$$

Therefore, by strong Pareto of $\hat{\succsim}$, we have

$$SU_{\hat{\succsim}}(n*c) = \bigcup_{\pi\in\Pi} \bigcup_{u_{-i,j}\in\mathbb{R}^{n-2}, (c,c,u_{-i,j})\hat{\sim} n*c} \bigcup_{\Delta\geq 0} \Big\{v \in \mathbb{R}^n \Big| v > \pi \circ \big(c - \varepsilon(\Delta, c), c+\Delta, u_{-i,j}\big)\Big\},$$

which is an open set. The symmetric argument shows that $SL_{\hat{\succsim}}(n*c)$ is also an open set. Since c is arbitrary taken, for any $u_N \in \mathbb{R}^n$, both $SU_{\hat{\succsim}}(u_N)$ and $SL_{\hat{\succsim}}(u_N)$ are open set by applying the argument for $c = W(u_N)$, which implies that $\hat{\succsim}$ is continuous.

Finally, by the above steps, we have shown that $\hat{\succsim}$ satisfies anonymity, strong Pareto, separability, convexity, and continuity. Therefore, by Debreu's (1959) additively separable representation theorem, there exists a continuous, strictly increasing, and concave function $g : \mathbb{R} \to \mathbb{R}$ such that, for any $u_N, v_N \in \mathbb{R}^n$, $u_N \hat{\succsim} v_N$ if and only if

$$\sum_{i \in N} g(u_i) \geq \sum_{i \in N} g(v_i).$$[11]

Since $\succsim$ is a refinement of $\hat{\succsim}$, we can conclude that $\succsim$ must be a weak generalized utilitarian with a concave function $g : \mathbb{R} \to \mathbb{R}$, which completes the proof. □

By Theorem 1 and Theorem 2, we can obtain our first main result about a reduced form characterization of the social welfare ordering that satisfies anonymity, strong Pareto, separability, and convexity.

Corollary 2. *Suppose that a social welfare ordering $\succsim$ satisfies anonymity, strong Pareto, separability, and convexity. Then, $\succsim$ is either level-oligarchy or a weak generalized utilitarian with a concave function $g : \mathbb{R} \to \mathbb{R}$.*

It should be noted that Theorem 2 does not hold if we impose Pigou-Dalton transfer instead of convexity. For example, let us consider the following aggregation rule: for any $u_N, v_N \in \mathbb{R}^n$,

$$u_N \succsim v_N \Leftrightarrow \Big(\sum_{i \in N} g_1(u_i), \sum_{i \in N; u_i \in [0,10]} g_2(u_i)\Big) \succsim_L \Big(\sum_{i \in N} g_1(v_i), \sum_{i \in N; v_i \in [0,10]} g_2(v_i)\Big),$$

where $g_1 : \mathbb{R} \to \mathbb{R}$ is a concave and strictly increasing function except for the closed interval $[0, 10]$ and $g_2 : [0, 10] \to \mathbb{R}$ is concave and strictly increasing function, and $\succsim_L$ is a lexicographic ordering on $\mathbb{R}^2$. This ordering aggregates a lexicographic combination of the monotonic function g_1 that does not increase only on a certain closed interval $[0, 10]$, and the increasing function g_2 that is defined only on this closed interval $[0, 10]$. Although such an ordering cannot be represented by any real-valued functions in general and is not any refinement of generalized utilitarianism (since g_1 is not globally monotone), it satisfies anonymity, strong Pareto, separability, and Pigou-Dalton transfer.

[11] Since $\hat{\succsim}$ satisfies anonymity and convexity, it also satisfies Pigou-Dalton transfer, which implies that g satisfies mid-point concavity, i.e, $g_k\left(\frac{x+y}{2}\right) \geq \frac{g(x)+g(y)}{2}$ for any $x, y \in \mathbb{R}$. Then, by continuity and mid-point concavity, g is concave. This fact is also used in the proof of Theorem 3.

4 Axiomatic foundation for the multi-threshold generalized sufficientarian

As for the leximin ordering, the critical-level sufficientarian ordering, and hence the multi-threshold generalized sufficientarian ordering, does not satisfy continuity. In particular, it has countably many thresholds. Nonetheless, it is continuous within the region between two adjacent thresholds. Therefore, we consider the following weaker continuity axiom.

Axiom 7 (Restricted continuity)**.** *For any* $-\infty \leq \theta < \theta' \leq \infty$ *with* $[\theta, \theta'] \cap \big(\Theta \cup \{\pm\infty\}\big) = \{\theta, \theta'\}$ *and* $u_N \in R^n(\theta, \theta')$*, both* $U_{\succsim}(u_N)$ *and* $L_{\succsim}(u_N)$ *are closed in* $R^n(\theta, \theta')$*, where*

$$R^n(\theta, \theta') = \begin{cases} (-\infty, \theta']^n \text{ if } \theta = -\infty, \\ [\theta, \theta']^n \setminus \{n * \theta\} \text{ if } -\infty < \theta, \theta' < \infty, \\ [\theta, \infty)^n \text{ if } \theta' = \infty. \end{cases}$$

Restricted continuity only requires continuity for $\succsim$ in a small cube $R^n(\theta, \theta')$ constructed by the two adjacent thresholds. Importantly, note that Θ are not exogenously given and it is endogenously determined by $\succsim$. Moreover, arbitrary discontinuity for $\succsim$ is allowed as a primitive. As a consequence, if there are no such adjacent thresholds, this axiom does not require anything on $\succsim$. This is contrastive to the related analysis of sufficinetarian orderings in the literature that the threshold θ is exogenously given (Bossert et al. 2022, 2023; Alcantud et al. 2022) or the existence of the threshold implicitly assumed (Chambers and Ye, 2023). We are ready to state our second main result.

Theorem 3. *A social welfare ordering* $\succsim$ *satisfies anonymity, strong Pareto, separability, Pigou-Dalton transfer, and restricted continuity if and only if it is a multi-threshold generalized sufficientarian.*

Proof. Since necessity is obvious, it is enough to show sufficiency. Let $\succsim$ be a social welfare ordering which satisfies all the axioms. By anonymity, we only consider a utility profile $u_N \in \mathbb{R}$ such that $u_1 \leq \ldots \leq u_n$ without loss of generality. The proof is divided into the following five steps.

Step 1: Construction of thresholds.

We first construct thresholds from Θ for a multi-threshold generalized sufficientarian social ordering. If Θ does not include any interval, then we choose $\{\theta_k\}_{k=1}^K = \Theta$. Otherwise, suppose that there exists an interval $[\theta, \theta'] \subset \Theta$. Take any $u_N \in [\theta, \infty]^n \setminus (\theta', \infty)^n$ and any $v_N \in \mathbb{R}^n$. If $u_i = v_i$ for any $i \leq l - 1$ for some $l \leq n$ and $u_l > v_l$ with $u_l, v_l \in [\theta, \theta']$, by Lemma 2, we must have $u_N \succ v_N$. Therefore, for such u_N and $v_N \in \mathbb{R}^n$, $u_N \succsim v_N$ if and only if $u_i = v_i$ for any $i \leq l - 1$ and $u_l > r > v_l$ for some $r \in \mathbb{Q} \cap [\theta, \theta']$, by denseness of $\mathbb{Q} \cap [\theta, \theta']$ in $[\theta, \theta']$. Hence, by discretizing Θ, let us define thresholds $\{\theta_k\}_{k=1}^K$ such that

$$\{\theta_k\}_{k=1}^K = \Theta \cap \mathbb{Q},$$

where each element is enumerated to $-\infty < \theta_1 < \ldots < \theta_K$. Note that, when $\{\theta_k\}_{k=1}^K$ is countably infinite, we regard $\theta_K = \infty$ if the sequence $\{\theta_k\}_{k=1}^\infty$ diverges, and $\theta_K = \theta^* < \infty$ if $\{\theta_k\}_{k=1}^\infty$ converges to some real value θ^*. For notational convenience, we set $\theta_0 = -\infty$ and $\theta_{K+1} = \infty$.

Step 2: Construction of the function $g : \mathbb{R} \to \mathbb{R}$.

For any $k = 1, \ldots, K + 1$, let $\Omega_k^n \subseteq \mathbb{R}^n$ be such that

$$\Omega_k^n = R^n(\theta_{k-1}, \theta_k) = \begin{cases} (-\infty, \theta_k]^n \text{ if } k = 1, \\ [\theta_{k-1}, \theta_k]^n \setminus \{n * \theta_{k-1}\} \text{ if } k = 2, \ldots, K, \text{ and } K \geq 2, \\ [\theta_K, \infty)^n \text{ if } k = K + 1. \end{cases}$$

That is, Ω_k^n is the sub-domains of utility profiles, each of which is a n-dimensional cube whose extreme points are threshold values constructed in Step 1. By restricting $\succsim$ on each Ω_k^n with $k = 1, \ldots, K + 1$, anonymity, strong Pareto, separability, Pigou-Dalton transfer, and restricted continuity imply that there exist concave, strictly increasing, and continuous functions $g_k : \Omega_k^1 \to \mathbb{R}$ such that

$$\begin{aligned} u_N \succsim v_N \quad &\Leftrightarrow \quad \sum_{i \in N} \big(g_k(u_i) - g_k(\theta_k)\big) \geq \sum_{i \in N} \big(g_k(v_i) - g_k(\theta_k)\big) \\ &\Leftrightarrow \quad \sum_{i \in N_k(u_N)} \big(g_k(u_i) - g_k(\theta_k)\big) \geq \sum_{i \in N_k(v_N)} \big(g_k(v_i) - g_k(\theta_k)\big) \end{aligned}$$

for any $u_N, v_N \in \Omega_k^n$ and for any $k \leq K$, and

$$u_N \succsim v_N \quad \Leftrightarrow \quad \sum_{i \in N} \big(g_{K+1}(u_i) - g_{K+1}(\theta_K)\big) \geq \sum_{i \in N} \big(g_{K+1}(v_i) - g_{K+1}(\theta_K)\big)$$

for any $u_N, v_N \in \Omega^n_{K+1}$, after normalization.

Then, let us define $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = \begin{cases} g_k(x) - g_k(\theta_k) \text{ if } x \in \Omega^1_k \text{ and } k = 1, \dots, K, \\ g_{K+1}(x) - g_{K+1}(\theta_K) \text{ if } x \in \Omega^1_{K+1}. \end{cases}$$

Note that g is well-defined and upper semi-continuous, where discontinuity happens only on the threshold values $\theta_1, \dots, \theta_{K-1}$.

For any $u_N \in \mathbb{R}^n$, we write $u_N = (u^1, u^2, \dots, u^{K+1})$ where $u^k = (u_i)_{i \in N_k(u_N)}$ for any $k = 1, 2, \dots, K+1$. Let $C(u^k) \in \Omega^1_k$ be the value such that $C(u^k) = g^{-1}\Big(\frac{1}{|N_k(u_N)|} \sum_{i \in N_k(u_N)} g(u_i)\Big)$. That is, $C(u^k)$ is the *per-capita utility level* for individuals to obtain the representative welfare level of u_N within $\Omega^{|N_k(u_N)|}_k$.

Step 3: For any $u_N \in \mathbb{R}^n$, we have

$$u_N \sim \Big(|N_1(u_N)| * C(u^1), \dots, |N_{K+1}(u_N)| * C(u^{K+1})\Big).$$

Let $v_N = \Big(u^1, \big(n - |N_1(u_N)|\big) * \theta_1\Big) \in \Omega^n_1$. By construction and Step 2, we must have

$$v_N \sim \Big(|N_1(u_N)| * C(u^1), \big(n - |N_1(u_N)|\big) * \theta_1\Big) \in \Omega^n_1.$$

Then, by separability, this is equivalent to

$$u_N = (u^1, u^2, \dots, u^{K+1}) \sim \Big(|N_1(u_N)| * C(u^1), u^2, \dots, u^{K+1}\Big).$$

Similarly, suppose that

$$\begin{aligned} &\Big(|N_1(u_N)| * C(u^1), \dots, |N_{k-1}(u_N)| * C(u^{k-1}), u^k, u^{k+1}, \dots, u^{K+1}\Big) \\ \sim \; &\Big(|N_1(u_N)| * C(u^1), \dots, |N_{k-1}(u_N)| * C(u^{k-1}), |N_k(u_N)| * C(u^k), u^{k+1}, \dots, u^{K+1}\Big) \end{aligned}$$

for some $k \le K$. Then, since

$$\Big(u^{k+1}, \big(n - |N_{k+1}(u_N)|\big) * \theta_{k+1}\Big) \sim \Big(|N_{k+1}(u_N)| * C(u^{k+1}), \big(n - |N_{k+1}(u_N)|\big) * \theta_{k+1}\Big)$$

by construction and Step 2, we must have

$$\begin{aligned} &\Big(|N_1(u_N)| * C(u^1), \dots, |N_k(u_N)| * C(u^k), u^{k+1}, u^{k+2}, \dots, u^{K+1}\Big) \\ \sim \; &\Big(|N_1(u_N)| * C(u^1), \dots, |N_k(u_N)| * C(u^k), |N_{k+1}(u_N)| * C(u^{k+1}), u^{k+2}, \dots, u^{K+1}\Big) \end{aligned}$$

by separability. Hence, by induction, we obtain

$$\begin{aligned} u_N &\sim \Big(|N_1(u_N)| * C(u^1), u^2, \ldots, u^{K+1}\Big) \\ &\vdots \\ &\sim \Big(|N_1(u_N)| * C(u^1), \ldots, |N_k(u_N)| * C(u^k), |N_{k+1}(u_N)| * C(u^{k+1}), u^{k+2}, \ldots, u^{K+1}\Big) \\ &\vdots \\ &\sim \Big(|N_1(u_N)| * C(u^1), \ldots, |N_{K+1}(u_N)| * C(u^{K+1})\Big). \end{aligned}$$

for any $k = 1, \ldots, K$. Finally, transitivity implies the desired relation.

Step 4: For any $u_N, v_N \in \mathbb{R}^n$, suppose that $C(u^k) = C(v^k)$ for any $1 \leq k \leq k^* \leq K$. Then,

$$u_N \succsim v_N \Leftrightarrow \Big(| \cup_{k=1}^{k^*} N_k(u_N)| * \theta_{k^*}, u_{-\cup_{k=1}^{k^*} N_k(u_N)}\Big) \succsim \Big(| \cup_{k=1}^{k^*} N_k(v_N)| * \theta_{k^*}, u_{-\cup_{k=1}^{k^*} N_k(v_N)}\Big).$$

Suppose that $C(u^1) = C(v^1)$. Then, by Step 3, we have

$$u_N \sim u'_N = \Big(|N_1(u_N)| * C(u^1), u_{-N_1(u_N)}\Big), v_N \sim v' = \Big(|N_1(v_N)| * C(v^1), u_{-N_1(v_N)}\Big).$$

Suppose that $|N_1(u_N)| \geq |N_1(v_N)|$. Let

$$u''_N = \Big(|N_1(v_N)| * C(v^1), \big(|N_1(u_N)| - |N_1(v_N)|\big) * \theta_1, u_{-N_1(u_N)}\Big).$$

Then, since $C(u^1) = C(v^1)$, by separability and Step 2, we have $u'_N \sim u''_N$. Moreover, by separability and transitivity, we have

$$u_N \sim u''_N \succsim v'_N \sim v_N \Leftrightarrow \Big(\big(n - |N_1(u_N)|\big) * \theta_1, u_{-N_1(u_N)}\Big) \succsim \Big(\big(n - |N_1(v_N)|\big) * \theta_1, u_{-N_1(v_N)}\Big).$$

If $|N_1(u_N)| < |N_1(v_N)|$, by the symmetric argument, we also have the same conclusion. By repeating this argument, we obtain

$$\begin{aligned} u_N \succsim v_N &\Leftrightarrow \Big(\big(n - |N_1(u_N)|\big) * \theta_1, u_{-N_1(u_N)}\Big) \succsim \Big(\big(n - |N_1(v_N)|\big) * \theta_1, u_{-N_1(v_N)}\Big) \\ &\vdots \\ &\Leftrightarrow \Big(\big(n - | \cup_{k=1}^{l} N_k(u_N)|\big) * \theta_l, u_{-\cup_{k=1}^{l} N_k(u_N)}\Big) \succsim \Big(\big(n - | \cup_{k=1}^{l} N_k(v_N)|\big) * \theta_l, u_{-\cup_{k=1}^{l} N_k(v_N)}\Big) \\ &\vdots \\ &\Leftrightarrow \Big(| \cup_{k=1}^{k^*} N_k(u_N)| * \theta_{k^*}, u_{-\cup_{k=1}^{k^*} N_k(u_N)}\Big) \succsim \Big(| \cup_{k=1}^{k^*} N_k(v_N)| * \theta_{k^*}, u_{-\cup_{k=1}^{k^*} N_k(v_N)}\Big). \end{aligned}$$

as desired.

Step 5: For any $u_N, v_N \in \mathbb{R}^n$, (i) suppose that $C(u^k) = C(v^k)$ for any $1 \leq k \leq k^* \leq K$ and $C(u^{k^*+1}) > C(v^{k^*+1})$. Then, $u_N \succ v_N$. (ii) Suppose that $C(u^k) = C(v^k)$ for any $k \leq K+1$. Then, $u_N \sim v_N$.

First, suppose that $C(u^k) = C(v^k)$ for any $1 \leq k \leq k^* \leq K-1$ and $C(u^{k^*+1}) > C(v^{k^*+1})$. By Step 4, it is enough to show that

$$\left(|\cup_{k=1}^{k^*} N_k(u_N)| * \theta_{k^*}, u_{-\cup_{k=1}^{k^*} N_k(u_N)}\right) \succ \left(|\cup_{k=1}^{k^*} N_k(v_N)| * \theta_{k^*}, u_{-\cup_{k=1}^{k^*} N_k(v_N)}\right).$$

By Step 2, since g_{k^*+1} is strictly increasing, there exists $\delta > 0$ such that

$$\left((n - |N_{k^*+1*}(u_N)|) * \theta_{k^*}, |N_{k^*+1}(u_N)| * C(u^{k^*+1})\right) \sim n * \left(C(u^{k^*+1}) + \delta\right).$$

Then, by taking $\varepsilon < \left(C(u^{k^*+1}) - C(v^{k^*+1})\right) + \delta$, we can also find $\varepsilon' > 0$ such that

$$n * \left(C(u^{k^*+1}) + \varepsilon\right) \sim \left(|N_{k^*+1}(v_N)| * \left(C(v^{k^*+1}) + \varepsilon'\right), (n - |N_{k^*+1}(v_N)|) * \theta_{k^*+1}\right).$$

By the above construction, we obtain

$$\begin{aligned}
\left(|\cup_{k=1}^{k^*} N_k(u_N)| * \theta_{k^*}, u_{-\cup_{k=1}^{k^*} N_k(u_N)}\right) &\succ \left((n - |N_{k^*+1*}(u_N)|) * \theta_{k^*}, |N_{k^*+1}(u_N)| * C(u^{k^*+1})\right) \\
&\sim n * \left(C(u^{k^*+1}) + \delta\right) \\
&\succ n * \left(C(v^{k^*+1}) + \varepsilon\right) \\
&\sim \left(|N_{k^*+1}(v_N)| * \left(C(v^{k^*+1}) + \varepsilon'\right), (n - |N_{k^*+1}(v_N)|) * \theta_{k^*+1}\right) \\
&\succ \left(|\cup_{k=1}^{k^*} N_k(v_N)| * \theta_{k^*}, u_{-\cup_{k=1}^{k^*} N_k(v_N)}\right),
\end{aligned}$$

where the first relation is held by strong Pareto and the last relation is held by the repeated use of Lemma 2.

Next, suppose that $C(u^k) = C(v^k)$ for any $k \leq K$. Then, by Steps 2 and 4, we can say that $u_N \succsim v_N$ if and only if $\sum_{i \in N_{K+1}(u_N)} \left(g_{K+1}(u_i) - g_{K+1}(c_K)\right) \geq \sum_{i \in N_{K+1}(v_N)} \left(g_{K+1}(v_i) - g_{K+1}(c_K)\right)$, which is equivalent to $C(u^{K+1}) \geq C(v^{K+1})$.

Finally, by Step 5, we have shown that, for any $u_N, v_N \in \mathbb{R}$, $u_N \succsim v_N$ if and only if there exists $k^* \leq K$ such that $C(u^k) = C(v^k)$ for any $k \leq k^*$ and $C(u^{k^*+1}) > C(v^{k^*+1})$, or $C(u^k) = C(v^k)$ for any $k \leq K$ and $C(u^{K+1}) \geq C(v^{K+1})$. By definition, this is the multi-threshold generalized sufficientarian ordering with thresholds $\{\theta_k\}_{k=1}^K$ defined in Step 1 and $g : \mathbb{R} \to \mathbb{R}$ defined in Step 2, and thus, we complete the proof. □

Corollary 2 and Theorem 3 imply that, for a class of separable social welfare orderings, if we regard standard axioms we discussed as normatively appealing, the available options are quite limited: basically, they must be either refinements of generalized utilitarian, multi-threshold generalized sufficientarian, or leximin rules. This result can be understood as an extension of Deschamps and Gevers's (1978) joint characterization that a class of separable social welfare orderings that satisfy anonymity, strong Pareto, and scale invariance to positive affine transformations must be either the weak utilitarian, leximin, or leximax rules. Sakamoto (2024) also generalizes their results by imposing rank-separability instead of separability and finds a new class of social welfare orderings called the generalized leximin rule. We elaborate on the implications of Corollary 2 and Theorem 3 and the relationship among our results, Deschamps and Gevers (1978), and Sakamoto (2024) in Section 5.

5 Discussions

In this section, we discuss several issues related to our results and future topics.

Strong vs. weak Pareto

Even if we require weak Pareto instead of strong Pareto, the essence of our results can still survive. Since separability cannot distinguish one situation where every person's well-being increases from another situation where just a single person's well-being increases while the rest of individual well-being remains the same, aggregated welfare must be the same as the case of strong Pareto. Of course, the strong Pareto principle is a proper axiom of acceptable social welfare orderings and, we believe, weak Pareto is a just subsidiary condition for strengthening impossibility results. Our results are based on the possibility results, and theoretical implications of weaker versions of any standard axioms are not helpful to construct a class of practical and acceptable social welfare orderings.

Role of convexity in Theorem 2

As we have seen after Theorem 2, if we require neither convexity nor continuity, we can obtain many peculiar forms of social welfare orderings. Note that, however, the main part of the results holds for Pigou-Dalton transfer alone: If Pigou-Dalton transfer instead of concavity is required, then the aggregation principle that never be represented by real functions as above can survive in addition to the weak generalized utilitarianism and level-oligarchy. In addition, by imposing some very weak trade-off conditions among individual well-being instead of concavity, it is possible to rule out such an eccentric ordering. In any case, although such an ordering is mathematically interesting as a counterexample to the representability of binary relations, it seems to be insignificant for constructing a class of practical and acceptable social evaluation methods. Hence, if we prefer the simplicity of the theoretical implications, concavity should be required.

Meaning of thresholds

Thresholds of sufficientarianism usually have specific meanings, where their meaning varies depending upon the underlying contexts. Once we specify a context, our characterization of multi-threshold generalized sufficientarian social welfare orderings can be interpreted as a foundation for real-life applications of indexes incorporating some thresholds in the specific context, such as poverty indexes initiated by Sen (1976). According to the World Bank, in September 2022, the international poverty line (IPL), a global absolute minimum, was revised to 2.15 USD per day (based on 2017 PPPs).[12] The lower and upper middle-income poverty lines are 3.65 and 6.85 USD per day, respectively. By imposing specific axioms to guarantee the existence of thresholds, we can consider these three poverty lines as three thresholds and a corresponding multi-threshold generalized sufficientarian social welfare ordering may serve as a new poverty index. Nonetheless, thresholds of the class of orderings do not have such specific meaning a priori and we do not intend to give particular interpretations; our primary concern for the characterization, Theorem 3, is to identify the theoretical possibilities implied by the standard axioms.

[12]https://www.worldbank.org/en/news/factsheet/2022/05/02/fact-sheet-an-adjustment-to-global-poverty-lines (accessed March 8, 2024).

Available options from standard axioms

By Corollary 2, if we accept anonymity, strong Pareto, separability, and convexity, then the available options are either level-oligarchy or a weak generalized utilitarian. Since level-oligarchy is the property for each threshold and we can obtain a weak generalized utilitarian ordering within the region between two adjacent thresholds by applying Theorem 2, we can construct the following social welfare ordering $\succsim$. As in the Step 1 of the proof of Theorem 3, let $\{\theta_k\}_{k=1}^K$ be the thresholds. Then, by applying Theorem 2 for each Ω_k^n, we can obtain weak generalized utilitarian orderings $\{\succsim_k^*\}_{k=1}^{K+1}$ for each region, each of which is a refinement of a generalized utilitarian ordering $\succsim_k$. For notational abbreviation, we interpret $u_N \succsim_k^* v_N$ as $\left((u_i)_{i\in N_k(u_N)}, (n - |N_k(u_N)|) * \theta_k\right) \succsim_k^* \left((v_i)_{i\in N_k(v_N)}, (n - |N_k(v_N)|) * \theta_k\right)$. Then, the remaining steps of the proof of Theorem 3, or level-oligarchy suggests that, for any $u_N, v_N \in \mathbb{R}^n$, $u_N \succsim v_N$ if and only if there exists $k^* \leq K$ such that $u_N \succsim_k^* v_N$ for any $k \leq k^*$ and $u_N \succ_{k^*+1}^* v_N$, or $u_N \succsim_k^* v_N$ for any $k = 1, \ldots, K+1$. Notice that, since each $\succsim_k^*$ is a weak generalized utilitarian, $\succsim$ is not necessarily a refinement of a multi-threshold generalized sufficientarian: it may happen that $u_N \succ v_N$ even if $u_N \sim_k v_N$ and $v_N \succ_{k+1} u_N$ for some k. In this sense, $\succsim$ is a "region-wise" refinement of a multi-threshold generalized sufficientarian. For instance, we can use the utilitarian-first and leximin-second rules (Kamaga, 2018) for each region. Our results suggest that such a class of social ordering can be compatible with standard axioms.

Sakamoto (2024) characterized a similar class of rules, the generalized leximin orderings, where weak generalized utilitarian orderings are replaced with rank-weighted utilitarian in the above specification, by weakening Separability to Rank-separability. Note that both results by Deschamps and Gevers (1978) and Sakamoto (2024) use a scale-invariance axiom, which is sometimes criticized by normative viewpoints. Contrastingly, our result does not rely on any scale-invariance property.

Variable population

Bossert et al. (2022) originally considered the critical-level sufficientarian orderings in the variable population model. We adopted their model in the fixed and finite population model since our primary purpose is to investigate a class of social welfare ordering with standard axioms in the setting. Our results are easily extended to the setting of variable populations. Indeed, suppose we require the existence of weak critical-levels and the utility independence axiom (a stronger condition of our separability axiom). In that case, the level-oligarchy theorem for variable populations is easily obtained. However, it should be noted that a class of the critical-level generalized utilitarian (a generalization of generalized utilitarian rules in the setting of variable populations) and multi-threshold generalized sufficientarian rules for variable populations must face severe problems called "the weak repugnant conclusion" (Greaves, 2017) and "the reverse repugnant conclusion with a threshold" (Sakamoto, 2023).[13] Therefore, as long as we intend to keep the separability-flavored axiom, we must accept the very undesirable conclusions for some situations.

6 Concluding remarks

This paper investigates the theoretical implications of standard axioms in the setting of separable social welfare orderings. Surprisingly, standard axioms severely restrict a class of acceptable social welfare orderings to the only sufficientarian-flavored one in the sense that, essentially, only the class of refinements of the generalized utilitarian, multi-threshold generalized sufficientarian, and leximin rules can survive as long as it satisfies separability. Also, we find the novel property "level-oligarchy" which contrasts with the famous "rank-dictatorship". These results show that if we would like to commit to separability, which ignores relative inequality for all situations in measuring social welfare, the only acceptable social welfare orderings must be generalized utilitarian, sufficientarian, or leximin. We shall not argue that separability should be a proper candidate that a social planner admires as one of the normative virtues in measuring social welfare. However, although separability is indeed strong and the class of rule is limited together with other standard axioms, we believe, our result can be interpreted as a possibility result for justifications of acceptable social welfare orderings.

[13] Bossert et al. (2023) show that their class of social welfare orderings, a generalization of Definition 3, can avoid both repugnant and sadistic conclusions by differentiating the exogenous threshold and critical-level in the critical-level sufficientarian ordering. As Greaves (2017) shows, the weak repugnant conclusion is very similar to the original repugnant conclusion, provided the critical level is set to be sufficiently low. On the contrary, the reverse repugnant conclusion with a threshold, which is proposed by Sakamoto (2023), is very repugnant if the critical level is set to be sufficiently high. In both cases, any refinements of critical-level generalized utilitarianism and sufficientarianism cannot escape from these dilemmas.

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