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Extended Social Ordering Functions for
Rationalizing Fair Game Forms à la Rawls and Sen

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Extended Social Ordering Functions for Rationalizing Fair Game Forms à la Rawls and Sen

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Abstract

We examine the possibility of constructing social ordering functions, each of which associates a social ordering over the feasible pairs of allocations and allocation rules with each simple production economy. Three axioms on the admissible class of social ordering functions are introduced, which embody the values of procedural fairness, non-welfaristic egalitarianism, and welfaristic consequentialism, respectively. The logical compatibility of these axioms and their lexicographic combinations subject to constraints are examined. Two social ordering functions which give priority to procedural values rather than to consequential values are identified, which can uniformly rationalize a nice allocation rule in terms of the values of procedural fairness, non-welfaristic egalitarianism, and Pareto efficiency.

JEL Classification Numbers: D63, D71, I31

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1 Introduction

Even in the developed market economies with matured market mechanisms and assured basic liberties, the issue of providing each individual with equitable standard of living without undue sacrifice of social efficiency and/or individual autonomy still remains largely unresolved. Even when there is a wide social consensus that the uncompromising pursuit of economic efficiency, individual autonomy, and the equitable provision of decent living standard is a target which is hardly sustainable, there remains a further issue of forming a social agreement on the priority to be assigned to the plural moral principles pursuing, respectively, social efficiency, individual autonomy, and equitable provision of decent living standard. This paper is devoted to the problem of combining these plural moral principles lexicographically and conditionally without falling into the impasse of logical inconsistency. In this context, recollect that John Rawls (1971, 1993) made an interesting proposal to combine plural moral principles lexicographically, but his proposal had to be confronted with a criticism by Amartya Sen and Bernard Williams (1982) who pointed out that the lexicographic combination of plural moral principles may be logically inconsistent. Our analysis represents an attempt to circumscribe the conditions under which the logical coherence of philosophical scenario articulated by Rawls and Sen can be rigorously ascertained.

To lend concreteness to the problem at hand, we focus on the following three moral principles in the context of defining a fair allocation rule as a game form in a class of simple production economies. The first moral principle is procedural in nature, and it requires that all individuals in the society should be assured of the minimal extent of autonomy in choosing his contribution to cooperative production. The second moral principle is consequential in nature, and it requires the Pareto efficiency of equilibrium social outcomes. The third moral principle is meant to capture an aspect of non-welfaristic egalitarianism along the line of Rawls and Sen, which is formally articulated in terms of the maximin assignment of individual capabilities rather than in terms of individual utilities. The logical coherence of one lexicographic combination or the other of moral principles, with or without further constraints on their applicability, can be verified by examining the existence, or the lack thereof, of a fair allocation rule as a game form thereby defined.

Two rather novel features of our analysis may deserve further clarifications. The first novel features is the capability maximin rule which is meant
to give substance to our conception of equitable provision of decent living standard. Instead of using Rawls's own formulation of the difference principle articulated in terms of what he christened the social primary goods, we are identifying the least advantage individual by means of what Sen (1980, 1985) christened capabilities, which are meant to capture the freedom individuals can enjoy in pursuit of their own lives they have reasons to choose. The gist of this approach is to shift the focus of our attention from the subjective happiness or satisfaction enjoyed by individuals to the objective opportunities in the functioning space to which individuals can rightfully access. The second novel feature is the use we make of the extended social ordering function which associates a social ordering over the pairs of feasible resource allocations and allocation rules as game forms with each economic environment. It is this analytical device that enables us to treat a procedural moral principle requiring individual autonomy, a welfaristic moral principle requiring the Pareto efficiency of social outcomes, and a non-welfaristic but consequential moral principle in the form of capability maximin rules simultaneously within a unified framework. It is also this analytical device which allows us to talk sensibly about rationality and uniform rationality of allocation rules as game forms.

Apart from this introduction, this paper consists of four sections and an appendix. In Section 2, we introduce a class of simple production economies, allocation rules as game forms, and extended social ordering functions. In Section 3, we formulate three basic axioms of fair allocation rules, and examine the existence of an allocation rule which is qualified to be fair in terms of these axioms. Section 4 defines the three basic axioms on extended social ordering functions and presents our possibility theorems. Section 5 concludes this paper with several final remarks. All the involved proofs are relegated into the Appendix at the end of the paper for the sake of simplicity of exposition.

2 The Basic Framework

2.1 Economic Environments and Allocation Rules

Consider an economy with the population $N = \{1, 2, \ldots, n\}$, where $2 \leq n < +\infty$. One good $y \in \mathbb{R}_+$ is produced from the vector of labor inputs $x = \ldots$.
The production process of this economy is described by the production function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$, which maps each $\mathbf{x} \in \mathbb{R}^n_+$ into $y = f(\mathbf{x}) \in \mathbb{R}_+$. It is assumed that $f$ satisfies continuity, strict increasingness, concavity, and $f(0) = 0$.

All individuals are assumed to have the common upper bound $\bar{x}$ of labor-leisure time, where $0 < \bar{x} < +\infty$. For each individual $i \in N$, his consumption vector is denoted by $z_i = (x_i, y_i) \in [0, \bar{x}] \times \mathbb{R}_+$, where $x_i$ is his labor time and $y_i$ is his share of output. Each $i \in N$ is characterized by his preference ordering on $[0, \bar{x}] \times \mathbb{R}_+$, which can be represented by a utility function $u_i : [0, \bar{x}] \times \mathbb{R}_+ \to \mathbb{R}$. We assume that $u_i$ is strictly monotonic (decreasing in labor time and increasing in the share of output) on $[0, \bar{x}] \times \mathbb{R}_+$, continuous and quasi-concave on $[0, \bar{x}] \times \mathbb{R}^+_+$. It is also assumed that $u_i(z_i) > u_i(x_i, 0)$ for all $z_i \in [0, \bar{x}] \times \mathbb{R}^+_+$ and all $x_i \in [0, \bar{x}]$. We denote the class of utility functions satisfying these assumptions by $\mathcal{U}$.

Since the production function $f$ is fixed throughout this paper, we may identify one economy simply by $\mathbf{u} \in \mathcal{U}^n$, where $\mathbf{u} = (u_1, \ldots, u_n)$. A feasible allocation in our economy is a vector $\mathbf{z} = (z_i)_{i \in N} = (x_i, y_i)_{i \in N} \in ([0, \bar{x}] \times \mathbb{R}_+)^n$ such that $f(\mathbf{x}) \geq \sum_N y_i$, where $\mathbf{x} = (x_1, \ldots, x_n)$. Let $Z$ be the set of all feasible allocations.

To complete the description of how our simple economy functions, what remains is to specify an allocation rule which assigns, to each $i \in N$, how many hours he/she works, and how much share of output he/she receives in return. In this paper, an allocation rule is modelled as a game form which is a pair $\gamma = (M, g)$, where $M = M_1 \times \cdots \times M_n$ is the set of admissible profiles of individual strategies, and $g$ is the outcome function which maps each strategy profile $\mathbf{m} \in M$ into a unique outcome $g(\mathbf{m}) \in Z$. For each $\mathbf{m} \in M$, $g(\mathbf{m}) = (g_i(\mathbf{m}))_{i \in N}$, where $g_i(\mathbf{m}) = (g_{i1}(\mathbf{m}), g_{i2}(\mathbf{m})) \in [0, \bar{x}] \times \mathbb{R}_+$ for each $i \in N$, represents a feasible allocation resulting from the strategic interactions among individuals represented by the strategy profile $\mathbf{m}$. Let $\Gamma$ be the set of all game forms representing allocation rules of our economy. Given an allocation rule $\gamma = (M, g) \in \Gamma$ and an economy $\mathbf{u} \in \mathcal{U}^n$, we obtain a fully-fledged specification of a non-cooperative game $(N, \gamma, \mathbf{u})$. Since the set of players $N$ is fixed throughout this paper, we may omit $N$ and describe a game as $(\gamma, \mathbf{u}) \in \Gamma \times \mathcal{U}^n$ without ambiguity.

An important juncture in our analysis of the performance of game forms

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1 In what follows, $\mathbb{R}_+$, $\mathbb{R}_+^n$ and $\mathbb{R}_+^n$ denote, respectively, the set of non-negative real numbers, the non-negative orthant and the positive orthant in the Euclidean $n$-space.
as social decision-making rules is the specification of the equilibrium concept. Throughout this paper, we will focus on the Nash equilibrium concept. To describe an equilibrium outcome of a game \((\gamma, u)\), where \(\gamma = (M, g)\), define \(m_{-i} = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n)\) for each \(m \in M\) and \(i \in N\), which is an element of a set \(M_i := \times_{j \in N \setminus \{i\}} M_j\). Given an \(m_{-i} \in M_{-i}\) and an \(m'_i \in M_i\), \((m'_i; m_{-i})\) may be construed as an admissible strategy profile obtained from \(m\) by replacing \(m_i\) with \(m'_i\). Given a game \((\gamma, u) \in \Gamma \times \mathcal{U}^n\), an admissible strategy profile \(m^* \in M\) is a pure strategy Nash equilibrium if \(u_i(g_i(m^*)) \geq u_i(g_i(m_i; m^*_{-i}))\) holds for each \(i \in N\) and each \(m_i \in M_i\). The set of all pure strategy Nash equilibria of the game \((\gamma, u)\) is denoted by \(NE(\gamma, u)\). A feasible allocation \(z^* \in Z\) is a pure strategy Nash equilibrium allocation of the game \((\gamma, u)\) if \(z^* = g(m^*)\) holds for some \(m^* \in NE(\gamma, u)\). The set of all pure strategy Nash equilibrium allocations of the game \((\gamma, u)\) is denoted by \(A_{NE}(\gamma, u)\).

2.2 Extended Social Ordering Functions

In our comprehensive framework of analysis for social choice of allocation rules as game forms, a crucial role is played by the concept of extended social ordering functions, which are defined over the set of extended social alternatives, viz., pairs of feasible allocations and allocation rules. The intended interpretation of an extended social alternative, viz., a pair \((z, \gamma) \in Z \times \Gamma\), is that a feasible allocation \(z\) is attained through an allocation rule \(\gamma\).

As we explained in section 1, the concept of extended social ordering functions enables us to treat the principle of individual autonomy, the Pareto principle on resource allocations, and the principle of equal provision of individual objective well-being in a unified framework. Indeed, the definition of social ordering by means of the standard binary relation over the set of allocations would not enable us to treat the axiom of individual autonomy in choice procedure appropriately, since the information of such an aspect is not contained in the description of allocations. Likewise, if we adopt the definition of social ordering by means of the binary relation over the set of game forms, such a framework would not provide us with the informational basis for discussing the Pareto principle as well as the principle of equal opportunity.

\(^2\)The concept of an extended social alternative was introduced by Pattanaik and Suzumura (1994; 1996) capitalizing on the thought-provoking suggestion by Arrow (1951, pp.89-91). See, also, Suzumura (1996; 1999; 2000).
Note that a feasible allocation \( z \in Z \) may or may not be realizable through an allocation rule \( \gamma \in \Gamma \). Indeed, an extended social alternative \((z, \gamma) \in Z \times \Gamma\) is realizable only when an economy \( u \in \mathcal{U}^n \) is given and \( z \in A_{NE}(\gamma, u) \) holds. Let \( \mathcal{R}(u) \) denote the set of realizable extended social alternatives under \( u \in \mathcal{U}^n \).

What we call an extended social ordering function (ESOF) is a function \( Q: \mathcal{U}^n \to (Z \times \Gamma)^2 \) such that \( Q(u) \) is an ordering on \( \mathcal{R}(u) \) for every \( u \in \mathcal{U}^n \).\(^3\)

The intended interpretation of the ordering \( Q(u) \) is that, for any extended social alternatives \((z^1, \gamma^1), (z^2, \gamma^2) \in \mathcal{R}(u)\), \([(z^1, \gamma^1), (z^2, \gamma^2)] \in Q(u)\) holds if and only if attaining a feasible allocation \( z^1 \) through an allocation rule \( \gamma^1 \) is at least as good as attaining a feasible allocation \( z^2 \) through an allocation rule \( \gamma^2 \) according to the social judgments embodied in \( Q(u) \).\(^4\) The asymmetric part and the symmetric part of \( Q(u) \) will be denoted by \( P(Q(u)) \) and \( I(Q(u)) \), respectively. The set of all ESOFs will be denoted by \( \mathcal{Q} \).

Once an ESOF \( Q \in \mathcal{Q} \) is specified, the set of best extended social alternatives is given, for each \( u \in \mathcal{U}^n \), by

\[
B(u: Q) \equiv \{(z, \gamma) \in \mathcal{R}(u) \mid \forall (z', \gamma') \in \mathcal{R}(u) : ((z, \gamma), (z', \gamma')) \in Q(u)\}.
\]

The set of socially chosen allocation rules is then given by

\[
D(u: Q) \equiv \{\gamma \in \Gamma \mid \exists z \in Z : (z, \gamma) \in B(u: Q)\}.
\]

We say that an allocation rule \( \gamma \in \Gamma \) is uniformly rationalizable\(^5\) via the ESOF \( Q \in \mathcal{Q} \) if and only if

\[
\gamma \in \bigcap_{u \in \mathcal{U}^n} D(u: Q)
\]

\(^3\)A binary relation \( R \) on a universal set \( X \) is a quasi-ordering if it satisfies reflexivity and transitivity. An ordering is a quasi-ordering satisfying completeness as well.

\(^4\)Note that this concept of an ESOF enables us to accommodate both consequential values and procedural values in the social evaluation of feasible allocations and allocation rules. If an ESOF \( Q_e \) is such that, for each \( u \in \mathcal{U}^n \), \([(z, \gamma^1), (z, \gamma^2)] \in I(Q_e(u)) \) holds for all \((z, \gamma^1), (z, \gamma^2) \in \mathcal{R}(u)\), it represents a social evaluation that cares only about consequential outcomes of resource allocations. In this sense, \( Q_e \) may be christened the purely consequential ESOF. In contrast, an ESOF \( Q_p \) such that, for each \( u \in \mathcal{U}^n \), \([(z^1, \gamma), (z^2, \gamma)] \in I(Q_p(u)) \) holds for all \((z^1, \gamma), (z^2, \gamma) \in \mathcal{R}(u)\) embodies a social evaluation that cares only about procedural features of resource allocations. In this sense, \( Q_p \) may be christened the purely procedural ESOF. In between these two polar extreme cases, there is a wide range of ESOFs which embody both consequential values and procedural values.

\(^5\)Recollect that a pair \((z, \gamma) \in B(u: Q)\) is said to be rationalizable by an ordering \( Q(u) \) on \( \mathcal{R}(u) \) if and only if \((z, \gamma) \) is judged to be at least as good as any other pair in \( \mathcal{R}(u) \) in terms of the ordering \( Q(u) \). By a slight abuse of terminology, we may say in this case
holds. By definition, such an allocation rule $\gamma$ applies uniformly to each and every $u \in U^n$ without violating the values embodied in the ESOF $Q$. This implies that once $\gamma \in \bigcap_{u \in U^n} D(u : Q)$ is chosen, it will prevail no matter how frivolously the profile $u$ undergoes a change. Since the allocation rule as a game form is nothing other than the formal method of specifying the legal-structure prevailing in the society prior to the realization of the profile of individual utility functions, it seems desirable, if at all possible, to design ESOF $Q \in Q$ satisfying the requirement of uniform rationalizability.

3 Fair Allocation Rules as Game Forms

In this section, we will discuss what properties qualify a game form to be a “fair” allocation rule, and examine the existence of such a rule. Let us begin with three conditions for allocation rules as game forms, which embody a value of individual autonomy, a value of economic efficiency, and a value of equal opportunity for individual objective well-being, respectively.

First, we introduce a condition for individual autonomy which every individual can enjoy in playing an economic game:

**Definition 1** [Kranich (1994)]: An allocation rule $\gamma = (M, g) \in \Gamma$ is labor-sovereign if, for all $i \in N$ and all $x_i \in [0, \bar{x}]$, there exists $m_i \in M_i$ such that, for all $m_{-i} \in M_{-i}$, $g_i(m_i, m_{-i}) = x_i$.

Let $\Gamma_{LS}$ denote the subclass of $\Gamma$ which consists solely of allocation rules satisfying labor sovereignty.

The next condition requires the Pareto efficiency of equilibrium outcomes. That is, the Nash equilibrium allocations of the games defined by fair allocation rules as game forms should be Pareto efficient. For each $u \in U^n$, let\(^6\)

$$PO(u) \equiv \{z \in Z \mid \forall z' \in Z, \exists i \in N : u_i(z_i) \geq u_i(z'_i)\}.$$ 

**Definition 2**: An allocation rule $\gamma = (M, g) \in \Gamma$ is efficient if, for any $u \in U^n$, $z \in A_{NE}(\gamma, u)$ implies $z \in PO(u)$.

that $\gamma \in D(u : Q)$ is rationalizable by $Q(u)$. An allocation rule $\gamma \in \bigcap_{u \in U^n} D(u : Q)$ may be said to be uniformly rationalizable, as it is rationalizable by $Q(u)$ no matter which $u \in U^n$ may materialize.

\(^6\)Note that $PO(u)$ is the set of Pareto efficient allocations by the strict monotonicity of every utility function.
Let $\Gamma_{PE}$ denote the subclass of $\Gamma$ which consists solely of efficient allocation rules.

### 3.1 Functioning and Capability

Our next condition for a fair allocation rule is meant to capture an aspect of non-welfaristic egalitarianism. It hinges on an objective concept of individual well-being in the spirit of the functioning and capability approach proposed by Sen (1980, 1985).

Suppose that there are $m$ basic functionings in the economy, which are relevant for all individuals in describing their objective well-beings, such as being healthy and free from diseases, having enough longevity, being well-informed, being able to participate in community life. These functionings are attainable by means of consumption vectors. We assume that these $m$ functionings can be measured by means of adequate non-negative real numbers. Thus, an achievement of functioning $k$, where $k = 1, 2, \ldots, m$, by individual $i$ is denoted by $b_{ik} \in \mathbb{R}_+^m$. Individual $i$’s achievement of basic functionings is given by listing $b_{ik}$: $b_i = (b_{i1}, \ldots, b_{im}) \in \mathbb{R}_+^m$. For each $i \in N$, $i$’s capability correspondence is defined as $c_i : [0, \mathcal{F}] \times \mathbb{R}_+ \to \mathbb{R}_+^m$ which associates with each $z_i \in [0, \mathcal{F}] \times \mathbb{R}_+$ a non-empty subset $c_i(z_i)$ of $\mathbb{R}_+^m$. This $c_i(z_i)$ is called $i$’s capability at his consumption vector $z_i$, which is the opportunity set of basic functionings when his consumption vector is $z_i$.

In what follows, we assume that the capability correspondences satisfy the following requirements:

(a) For all $z_i = (x_i, y_i)$, $z_i' = (x_i', y_i') \in [0, \mathcal{F}] \times \mathbb{R}_+$ such that $x_i = x_i'$ and $y_i \leq y_i'$ (resp. $y_i < y_i'$), $c_i(z_i) \subseteq c_i(z_i')$ (resp. $c_i(z_i) \subset c_i(z_i')$) hold, where $c_i(z_i)$ stands for the interior of $c_i(z_i)$.\footnote{For all vectors $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_p) \in \mathbb{R}_p$, $a \geq b$ if and only if $a_i \geq b_i$ $(i = 1, \ldots, p)$; $a > b$ if and only if $a \geq b$ and $a \neq b$; $a \gg b$ if and only if $a_i > b_i$ $(i = 1, \ldots, p)$.}

(b) For all $z_i \in [0, \mathcal{F}] \times \mathbb{R}_+$, $c_i(z_i)$ is compact and comprehensive in $\mathbb{R}_+^m$; and

(c) $c_i$ is continuous on $[0, \mathcal{F}] \times \mathbb{R}_+$.

Let us denote the universal class of capability correspondences which meet the above three requirements by $\mathcal{C}$. Given a profile of individual capabilities $c \in \mathcal{C}^n$ and for each $z = (z_i)_{i \in N} \in Z$, $c(z) = (c_i(z_i))_{i \in N}$ denotes a feasible assignment of individual capabilities.
3.2 J-Based Capability Maximin Allocations

We are now ready to introduce the third condition for a fair allocation rule as a game form, which requires that every Nash equilibrium allocation of the game defined by the allocation rule should guarantee a maximal level of capability to the least advantaged individual, where the identification of the least advantaged individual is made in terms of individual capabilities.

To be more precise, we introduce an ordering over capabilities, which represents an evaluation on the impersonal well-being of capabilities, in order to identify who is the least advantaged in terms of capability assignments. Let the universal set of capabilities be

\[ \mathcal{K} \equiv \{ C \subseteq \mathbb{R}_+^k \mid \exists c \in \mathcal{C} \text{ and } \exists z \in [0, \overline{z}] \times \mathbb{R}_+ : c(z) = C \}. \]

Suppose that the society has an evaluation on the impersonal well-being of capabilities, which is represented by an ordering relation \( J \subseteq \mathcal{K} \times \mathcal{K} \) satisfying completeness: [for all \( C, C' \in \mathcal{K} \), \( (C, C') \in J \) or \( (C', C) \in J \) and transitivity: [for all \( C, C', C'' \in \mathcal{K} \), if \( (C, C') \in J \) & \( (C', C'') \in J \), then \( (C, C'') \in J \). \( P(J) \) and \( I(J) \) denote, respectively, the asymmetric part and symmetric part of \( J \).

At this juncture, let us introduce an appropriate topology into the space \( \mathcal{K} \) in terms of the Hausdorff metric.\(^8\) Equipped with this topology, we suppose that the ordering \( J \) satisfies the following intuitively plausible axioms:

(3.1.1) Monotonicity: For all \( C, C' \in \mathcal{K} \), if \( C \supseteq C' \) then \( (C, C') \in J \), and if \( \text{int } C \supseteq C' \), then \( (C, C') \in P(J) \).

(3.1.2) Dominance: For all \( C, C', C'' \in \mathcal{K} \), if \( [(C, C') \in P(J) \) and \( (C, C'') \in P(J) \)], then \( (C, C' \cup C'') \in P(J) \).

(3.1.3) Continuity: For all \( C \in \mathcal{K} \), and a sequence of capabilities \( \{C^r\}_{r=1}^\infty \) such that \( C^r \in \mathcal{K} \) for all \( r \) and \( C^* = \lim_{r \to \infty} C^r \in \mathcal{K} \), if \( (C^r, C) \in J \) for all \( r \), then \( (C^*, C) \in J \).

By Xu (2003), we have the following characterization of the ordering \( J \subseteq \mathcal{K} \times \mathcal{K} \).

\(^8\)For any compact sets \( C, C' \subseteq \mathbb{R}^m \), the Hausdorff metric between \( C \) and \( C' \) is defined by

\[ d(C, C') \equiv \max\{\max\{\delta(b, C) \mid b \in C'\}, \max\{\delta(b, C') \mid b \in C\}\}, \]

where \( \delta(b, C) \equiv \min_{b' \in C} \|b - b'\| \), and \( \|b - b'\| \) is the Euclidean distance between \( b \) and \( b' \).
Proposition 1 [Xu (2003)]: If the ordering $J \subseteq K \times K$ satisfies Monotonicity, Dominance, and Continuity, then there exists a continuous and increasing function $g : \mathbb{R}_+^n \to \mathbb{R}$ such that for all $C, C' \in K$,

$$(C, C') \in J \iff \left[ \max_{b \in C} g(b) \geq \max_{b' \in C'} g(b') \right].$$

Denote the admissible class of evaluations which meet all of (3.1.1), (3.1.2), and (3.1.3) by $J$.

Given the evaluation $J \in J$ and the profile of capability correspondences $C \in \mathcal{C}$, let us define the subset $C_{\text{min}}^J(z) \subseteq K$, for each feasible allocation $z = (z_1, \ldots, z_n) \in Z$, by: $c_i(z_i) \in C_{\text{min}}^J(z)$ if and only if $(c_j(z_j), c_i(z_i)) \in J$ for all $j \in N$. For any $J \in J$ and $x \in [0, \bar{x}]^n$, let us define:

$$Z(x; J) \equiv \{(x, y) \in Z \mid \forall (x, y') \in Z : (C_{\text{min}}^J(x, y), C_{\text{min}}^J(x, y')) \in J\}.$$ 

The set $Z(x; J)$ consists of feasible allocations which are maximal in terms of the evaluation $J$ for the given $x \in [0, \bar{x}]^n$. In this sense, this set may be construed to consist of $J$-reference capability maximin allocations for the given $x \in [0, \bar{x}]^n$. Then, the third condition may be stated as follows:

Definition 3: An allocation rule $\gamma = (M, g) \in \Gamma$ is called the $J$-reference capability maximin rule if, for any $u \in U^n$, $z \in A_{NE}(\gamma, u)$ implies $z \in Z(x; J)$, where $z = (x, y)$.\(^{10}\)

Let $\Gamma_{\text{JCM}}$ denote the subclass of $\Gamma$ which consists solely of $J$-reference capability maximin allocation rules.

3.3 Existence of Fair Allocation Rules as Game Forms

We are now ready to discuss the existence of a fair allocation rule as a game form which is labor sovereign, Pareto efficient, and $J$-reference capability maximin. A game form $\gamma = (M, g)$ is said to be Nash-solvable if $A_{NE}(\gamma, u) \neq \emptyset$ for each and every $u \in U^n$. Denote the set of Nash solvable game forms by $\Gamma_{\text{NS}}$.

\(^{9}\)The expression $(C_{\text{min}}^J(x), C_{\text{min}}^J(x')) \in J$ in this definition is slightly abusive. The rigorous expression should go as follows: for any $c_i(z_i) \in C_{\text{min}}^J(x)$ and any $c_j(z'_j) \in C_{\text{min}}^J(x')$, $(c_i(z_i), c_j(z'_j)) \in J$.

\(^{10}\)This type of allocation rule is originated in Gotoh and Yoshihara (1999, 2003).
**Assumption 1**: The utility function $u_i$ of each and every agent has the following property: $\forall z_i \in [0,\bar{x}] \times \mathbb{R}^{++}, u_i(z_i) > 0$, and $u_i(\bar{x},0) = 0$.

**Assumption 2**: The production function $f$ is continuously differentiable.

**Theorem 1**: Under Assumption 1 and Assumption 2, and for any given evaluation $J \in \mathcal{J}$, there exists an allocation rule $\gamma^* \in \Gamma_{LS} \cap \Gamma_{PE} \cap \Gamma_{JCM} \cap \Gamma_{NS}$.

In this theorem, Assumption 1 can be weakened to the claim that $u_i$ is bounded from below. Moreover, Assumption 2 is not essential: it is introduced just to simplify the argument. Indeed, we can construct an allocation rule having the property of Theorem 1 even without Assumption 2, although the construction of such an allocation rule will be more complicated than the current method.

Although Gotoh and Yoshihara (1999, 2003) proposed and characterized the class of allocation rules $\Gamma_{LS} \cap \Gamma_{JCM}$, it was unclear whether or not there exists an element of this class which also belongs to $\Gamma_{PE}$. Here, however, Theorem 1 shows that if the method of ranking capability sets is constrained by the three plausible conditions (3.1.1), (3.1.2), and (3.1.3) discussed by Xu (2002, 2003), we can successfully find an allocation rule in $\Gamma_{LS} \cap \Gamma_{PE} \cap \Gamma_{JCM} \cap \Gamma_{NS}$.

### 4 ESOFs for Rationalizing Fair Allocation Rules

#### 4.1 Three Basic Axioms on ESOFs and Their Compatibility

In what follows, we will examine the possibility of an ESOF embodying the three distinct values of individual autonomy, economic efficiency, and equal opportunity for individual objective well-being, along with uniform rationalizability of the allocation rule $\gamma^*$. To begin with, let us formulate three basic axioms of ESOFs, which go as follows:

**Labor Sovereignty (LS):** For all $u \in U^n$ and all $(z,\gamma),(z',\gamma') \in R(u)$, if $\gamma \in \Gamma_{LS}$ and $\gamma' \in \Gamma \setminus \Gamma_{LS}$, then $((z,\gamma),(z',\gamma')) \in P(Q(u))$.

**Respect of the J-Reference Least Advantaged (J-LA):** For all $u \in U^n$ and all $(z,\gamma),(z',\gamma') \in R(u)$, if $z = (x,y)$, $z' = (x',y')$ and $x = x'$, then:

$$((z,\gamma),(z',\gamma')) \in Q(u) \iff (C^J_{\min}(z),C^J_{\min}(z')) \in J,$$
\((\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in P(Q(\mathbf{u})) \Leftrightarrow (C^J_{\min} (\mathbf{z}), C^J_{\min} (\mathbf{z}')) \in P (J)\). 

**Pareto in Allocations (PA):** For all \(\mathbf{u} \in \mathcal{U}^n\) and all \((\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})\), if \(u_i(z_i) > u_i(z'_i)\) for all \(i \in N\), then \(((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in P(Q(\mathbf{u}))\), and if \(u_i(z_i) = u_i(z'_i)\) for all \(i \in N\), then \(((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in I(Q(\mathbf{u}))\).

Among the above three axioms, **LS** is a requirement of purely procedural fairness, as it imposes some constraints on the admissible class of ESOFs without having recourse to the nature of consequential outcomes. **J-LA** is a requirement of non-welfaristic egalitarianism, which attempts to enable the least advantaged individual, where the least advantaged individual is identified in terms of his capability by the social value judgements \(J\), to secure as large capability as possible by choosing a feasible allocation appropriately. Finally, **PA**, being a variant of Paretianism, is based squarely on welfaristic consequentialism.

It may well be asked why **J-LA** imposes the premise \(x = x'\). The reason is twofold. First, the choice of individual labor hours should be regarded as a matter to be left to individual responsibility. Social value judgements should keep silence on a matter of this nature. Second, since \(J\) is a complete ordering, if the requirement of **J-LA** is applied to ESOFs without the premise \(x = x'\), then it gives us a complete ordering by this axiom only, leaving no room for applying the Paretian axiom at all.

It is of little surprise that for any \(J \in \mathcal{J}\), there exists no social ordering function which satisfies any two of the basic **LS**, **J-LA**, and **PA**. When two or more basic principles irrevocably conflict with each other and yet we do not want to discard any one of these principles altogether, a natural step to follow is to introduce a priority rule among these principles so as to define their lexicographic combinations. This idea has been explored repeatedly in the literature of normative economics, the most recent example being Tadenuma (2003).

To explore this intuitive idea systematically in our present context, take any distinct \(\alpha, \beta, \gamma \in \{\text{LS, J-LA, PA}\}\) and define a subclass \(Q^{\alpha; \beta; \gamma}\) of ESOFs as follows: any \(Q \in Q^{\alpha; \beta; \gamma}\) implies that for any \((\mathbf{z}, \gamma), (\mathbf{z}', \gamma') \in \mathcal{R}(\mathbf{u})\), \(((\mathbf{z}, \gamma), (\mathbf{z}', \gamma')) \in Q(\mathbf{u})\) (resp. \(P(Q(\mathbf{u}))\)) if (1) the axiom \(\alpha\) requires it; (2) the axiom \(\beta\) requires it, given that the axiom \(\alpha\) keeps silence; or (3) the axiom \(\gamma\) requires it, given that the axioms \(\alpha\) and \(\beta\) keep silence.\(^{11}\) Let

---

\(^{11}\)Recollect that **LS**, **J-LA**, and **PA** are expressed in the conditional form of “if (A), then (B)” style. Thus, whenever the condition (A) is not satisfied for **Axiom \(\alpha\)**, where
us define $Q^{lex} = \bigcup_{\alpha, \beta, \gamma \in \{LS, J-LA, PA\}} Q^{\alpha+\beta-\gamma}$, in which each element applies the three basic axioms lexicographically.

Even these lexicographic combinations of the basic three axioms are incompatible, as the following proposition holds.

**Proposition 2:** $Q^{lex}$ is empty.

This impossibility result comes from incompatibility between $J-LA$ and $PA$ even in the form of lexicographic combination. To verify this negative assertion, we have only to check the following:

**Example 1:** Let there be two types of relevant functionings, and let $N = \{1, 2\}$ and $\overline{x} = 3$. The production function is given by $f(x_1, x_2) = x_1 + x_2$ for all $(x_1, x_2) \in \mathbb{R}_+^2$. Individuals have the same capability correspondence $c$ which is defined as follows: For any $z \in [0, \overline{x}] \times \mathbb{R}_+$, $c(z) \equiv \{(b_1, b_2) \in \mathbb{R}_+^2 \mid \exists z^1, z^2 \in [0, \overline{x}] \times \mathbb{R}_+, z^1 + z^2 \leq z, b_k = a_k(z^k) (k = 1, 2)\}$, where $a_1(x, y) = (\overline{x} - x)^2 \cdot y^2$ and $a_2(x, y) = (\overline{x} - x)^2 \cdot y^2$ for any $(x, y) \in [0, \overline{x}] \times \mathbb{R}_+$. Note that the mapping $a_k(\cdot)$ assigns to each consumption vector an achievement of functioning $k$. Thus, $b_k = a_k(z^k)$ implies that if the consumption vector $z^k$ is utilized for functioning $k$, then the functioning $k$ is attained at the level of $b_k$.

Consider two feasible allocations $z^* = ((1, 1), (1, 1))$ and $z^{**} = ((2, 2), (2, 2))$. For some $\alpha \in (0, 1)$, let $z^*(\alpha) = ((1, 1 + \alpha), (1, 1 - \alpha))$ and $z^{**}(\alpha) = ((2, 2 - \alpha), (2, 2 + \alpha))$. It is easy to check that, for any $J \in J$, $(C^J_{\min}(z^*), C^J_{\min}(z^{**}(\alpha))) \in P(J)$ and $(C^J_{\min}(z^{**}), C^J_{\min}(z^{**}(\alpha))) \in P(J)$, since $J$ satisfies Monotonicity. Individual 1’s utility function $u_1$ is defined for all $(x, y) \in [0, \overline{x}] \times \mathbb{R}_+$ by

$$u_1(x, y) = (1 - \alpha) \cdot (\overline{x} - x) + y,$$

whereas individual 2’s utility function $u_2$ is defined for all $(x, y) \in [0, \overline{x}] \times \mathbb{R}_+$ by

$$u_2(x, y) = \begin{cases} 
(1 - \alpha) \cdot (\overline{x} - x) + y & \text{if } x \in [0, 1) \\
(1 + \alpha) \cdot (\overline{x} - x) + y & \text{if } x \in [1, \overline{x}]
\end{cases}.$$
This situation is described in the consumption space and in the functioning space in Figure 1.

Let $\gamma^*, \gamma^*(\alpha), \gamma^{**}$, and $\gamma^{**}(\alpha)$ be the allocation rules in $\Gamma \setminus \Gamma_{LS}$ which generate the realizable allocations $z^*, z^*(\alpha), z^{**}$, and $z^{**}(\alpha)$, respectively, when the economy is defined by $u = (u_1, u_2) \in \mathcal{U}^2$.

Take any ESOF $Q \in Q^{lex}$. Then, compare $(z^*, \gamma^*)$ with $(z^*(\alpha), \gamma^*(\alpha))$, and compare $(z^{**}, \gamma^{**})$ with $(z^{**}(\alpha), \gamma^{**}(\alpha))$. Since $\gamma^*, \gamma^*(\alpha), \gamma^{**}, \gamma^{**}(\alpha) \in \Gamma \setminus \Gamma_{LS}$, and $z^*$ (resp. $z^{**}$) and $z^*(\alpha)$ (resp. $z^{**}(\alpha)$) are Pareto non-comparable, LS and PA keep silence for any $Q \in Q^{lex}$. In contrast, by $J$-LA, we have for any $Q \in Q^{lex}$,

$((z^*, \gamma^*), (z^*(\alpha), \gamma^*(\alpha))) \in P(Q(u)), ((z^{**}, \gamma^{**}), (z^{**}(\alpha), \gamma^{**}(\alpha))) \in P(Q(u))$.

Next, compare $(z^{**}, \gamma^{**})$ with $(z^*(\alpha), \gamma^*(\alpha))$, and $(z^{**}(\alpha), \gamma^{**}(\alpha))$ with $(z^*, \gamma^*)$. Here, not only LS but also $J$-LA keep silence, since $x^{**} \neq x^*(\alpha)$ and $x^{**}(\alpha) \neq x^*$. In contrast, by PA, we have for any $Q \in Q^{lex}$,

$((z^*(\alpha), \gamma^*(\alpha)), (z^{**}(\alpha), \gamma^{**}(\alpha))), ((z^*, \gamma^*), (z^{**}, \gamma^{**})) \in I(Q(u))$, since $u_1(z^*(\alpha)) = u_1(z^{**}(\alpha)) = 3 - \alpha, u_2(z^*(\alpha)) = u_2(z^{**}(\alpha)) = 3 + \alpha, u_1(z^{**}(\alpha)) = 3 - 2\alpha, u_2(z^{**}(\alpha)) = 3 + 2\alpha$. Thus, any $Q \in Q^{lex}$ is not a consistent binary relation, hence it cannot be an ordering. ■

4.2 Existence of ESOFs Uniformly Rationalizing $\gamma^*$

To secure the existence of a compatible lexicographic combination of our basic axioms, further concession seems to be indispensable in view of Proposition 2. For each given $J \in J$, let us introduce the following conditional variants of $J$-LA and PA, respectively:

\footnote{A finite subset $\{x^1, \ldots, x^t\}$ of a universal set $X$, where $2 \leq t < +\infty$, satisfying $(x^1, x^2) \in P(R)$, $(x^2, x^3) \in R$, $\cdots$, $(x^t, x^1) \in R$ is called an incoherent cycle of order $t$ of a binary relation $R$ on $X$. $R$ is said to be consistent if there exists no incoherent cycle of any order. A binary relation $R^*$ is called an extension of $R$ if and only if $R \subseteq R^*$ and $P(R) \subseteq P(R^*)$. It is shown in Suzumura (1983, Theorem A(5)) that there exists an ordering extension of $R$ if and only if $R$ is consistent.}
**Theorem 2:** For all $u \in U^n$ and all $(z, \gamma, (z', \gamma')) \in R(u)$ with $z, z' \in PO(u)$, if $z = (x, y)$, $z' = (x', y')$, and $x = x'$, then:

\[
((z, \gamma), (z', \gamma')) \in Q(u) \iff (C_{\min}^J(z), C_{\min}^J(z')) \in J, \\
((z, \gamma), (z', \gamma')) \in P(Q(u)) \iff (C_{\min}^J(z), C_{\min}^J(z')) \in P(J).
\]

**PA \cap Z(J):** For all $u \in U^n$ and all $(z, \gamma, (z', \gamma')) \in R(u)$ such that $z = (x, y) \in Z(x; J)$ and $z' = (x', y') \in Z(x'; J)$, if $u_i(z_i) > u_i(z_i')$ for all $i \in N$, then $((z, \gamma), (z', \gamma')) \in P(Q(u))$, and if $u_i(z_i) = u_i(z_i')$ for all $i \in N$, then $((z, \gamma), (z', \gamma')) \in I(Q(u))$.

Observe that $z, z' \in PO(u)$ implies that $z$ and $z'$ are Pareto noncomparable. This implies that $J$-$LA \cap PO$ requires the applicability of $J$-$LA$ in a special case where $PA$ keeps silence. Also, observe that $z = (x, y) \in Z(x; J)$ and $z' = (x', y') \in Z(x'; J)$ imply $x \neq x'$ as we will show below. This implies that $PA \cap Z(J)$ requires the applicability of $PA$ in a special case where $J$-$LA$ keeps silence.

Now, for each given $J \in \mathcal{J}$, let us consider two subclasses of ESOFs, viz., $Q_{LS}^{J}$-$J$-$LA$-$PA\cap Z(J)$ and $Q_{LS}^{PA}$-$J$-$LA$-$PO$. Note that $Q_{LS}^{J}$-$J$-$LA$-$PA \subseteq Q_{LS}^{J}$-$J$-$LA$-$PA\cap Z(J)$ and $Q_{LS}^{PA}$-$J$-$LA \subseteq Q_{LS}^{PA}$-$J$-$LA$-$PO$. This is because $Q \in Q_{LS}^{J}$-$J$-$LA$-$PA\cap Z(J)$ implies that for any $(z, \gamma), (z', \gamma') \in R(u)$, $((z, \gamma), (z', \gamma')) \in Q(u)$ (resp. $P(Q(u))$) holds if (1) $LS$ requires it; or (2) $J$-$LA$ requires it, given that $LS$ keeps silence; or (3) $PA$ requires it, given that not only $LS$ and $J$-$LA$ keep silence, but also $z = (x, y) \in Z(x; J)$ and $z' = (x', y') \in Z(x'; J)$ hold. Also, $Q \in Q_{LS}^{PA}$-$J$-$LA$-$PO$ implies that for any $(z, \gamma), (z', \gamma') \in R(u)$, $((z, \gamma), (z', \gamma')) \in Q(u)$ (resp. $P(Q(u))$) holds if (1) $LS$ requires it; or (2) $PA$ requires it, given that $LS$ keeps silence; or (3) $J$-$LA$ requires it, given that not only $LS$ and $PA$ keep silence, but also $z, z' \in PO(u)$ holds.

Thus, although both $Q_{LS}^{J}$-$J$-$LA$-$PA$ and $Q_{LS}^{PA}$-$J$-$LA$-$PO$ are empty sets as shown in Proposition 2, it may well be expected that $Q_{LS}^{J}$-$J$-$LA$-$PA\cap Z(J)$ and $Q_{LS}^{PA}$-$J$-$LA$-$PO$ might be non-empty. The following theorems show that this is indeed the case.

**Theorem 2:** For any given $J \in \mathcal{J}$, there exists $Q^*_J \in Q_{LS}^{J}$-$J$-$LA$-$PA\cap Z(J)$ such that

\[
\bigcap_{u \in U^n} D(u : Q^*_J) = \Gamma_{LS} \cap \Gamma_{PE} \cap \Gamma_{JCM} \cap \Gamma_{NS}.
\]

**Theorem 3:** For any given $J \in \mathcal{J}$, there exists $Q^{*_J} \in Q_{LS}^{PA}$-$J$-$LA$-$PO$ such that

\[
\bigcap_{u \in U^n} D(u : Q^{*_J}) = \Gamma_{LS} \cap \Gamma_{PE} \cap \Gamma_{JCM} \cap \Gamma_{NS}.
\]
We have thus shown that there exist two ESOFs which, in their respective ways, lexicographically combine the requirement of individual autonomy and the (conditional) requirements of non-welfaristic egalitarianism and welfaristic consequentialism. In general, it might be the case that the difference in the order of lexicographic application of the axioms would lead to different rational choice. In other words, the rationally chosen allocation rule via any \( Q \in Q^{LS,J-\text{LA}^{-}}(PA \cap Z(J)) \) would be a reference capability maximin rule, but it might not implement Pareto efficient allocations in Nash equilibria, whereas the rationally chosen allocation rule via \( Q' \in Q^{LS-\text{PA}^{-}(J-\text{LA}^{-}\text{PO})} \) would implement Pareto efficient allocations in Nash equilibria, but it might not be a reference capability maximin rule. In spite of this general observation, an interesting feature of Theorem 2 and Theorem 3 is that there exist \( Q_J' \in Q^{LS^{J-\text{LA}^{-}}(PA \cap Z(J))} \) and \( Q_J^{**} \in Q^{LS^{-}\text{PA}^{-}(J-\text{LA}^{-}\text{PO})} \) such that both have the common set of “first best” allocation rules including \( \gamma^* \) as their uniformly rational game forms.

5 Conclusion

In the concrete context of a simple production economy, this paper identified three moral principles on the desirability of resource allocation rules as game forms. The first principle requires that all individuals should be assured of the minimal extent of autonomy in choosing his contribution to cooperative production. The second principle requires that the consequential social outcomes should be Pareto efficient. The third principle requires that the consequential social outcome should warrant each and every individual of an equitable provision of decent living standard, which may be formally identified by means of the maximin assignment of individual capabilities in the sense of Sen.

\[ Q_J' \text{ and } Q_J^{**} \text{ commonly confer priority to } LS. \text{ LS being an axiom of procedural fairness, } Q_J' \text{ and } Q_J^{**} \text{ belong to the subclass of ESOFs which give priority to procedural considerations vis-à-vis consequential considerations. We can likewise consider the possible subclass of ESOFs which give priority to consequential considerations vis-à-vis procedural considerations. To be precise, it can be shown that:} \]

**Theorem*: For any given } J \in J, Q^{J-\text{LA}^{-}(PA \cap Z(J))}\text{-LS \ (resp. } Q^{PA^{-}(J-\text{LA}^{-}\text{PO})}\text{-LS)} \neq \emptyset.

However, there is no real parallelism between Theorem 2 and Theorem 3, on the one hand, and Theorem*, on the other. This is because an ESOF } Q \text{ in Theorem* may well fail to assure the non-emptiness of } B(u : Q).
It goes without saying that each one of these principles, in isolation, is a highly appealing moral claim. In combination, however, they represent a moral claim which is logically too demanding to be satisfied. A natural response to this logical impasse in the light of Rawls's (1971, 1993) thoughtful suggestion is to combine these principles lexicographically. In a related but distinct context of the equity-efficiency trade-off, Koichi Tadenuma (2002) successfully exploited this idea and has shown that the equity-first and efficiency-second lexicographic combination of these two moral principles can resolve the equity-efficiency trade-off, whereas the efficiency-first and equity-second lexicographic combination thereof cannot serve as a resolvent of the equity-efficiency trade-off. In our present context, however, we have shown that any one of the possible lexicographic combinations of our three moral principles still represent a moral claim which is logically too demanding to be satisfied. In this sense, Sen and Williams's (1982) acute warning is fully justified in our analytical setting. We are thus led to examine the conditional versions of the component moral principles and their lexicographic combinations. Theorem 2 and Theorem 3, which represent our possibility theorems in this paper, are on the workability of lexicographic combinations of conditional moral principles. It is true that the workable lexicographic combinations of conditional moral principles are far more complex than the simple lexicographic combinations of moral principles à la Rawls. This being the case, it is all the more noteworthy that these conditional moral principles can nevertheless rationalize in common the set of fair allocation rules as game forms, whose existence is established by Theorem 1.

Let us conclude by pointing out an open question. It is one thing to show the existence of a social ordering function which may identify a fair resource allocation rule, and it is quite another to show how such a social ordering function can be generated through democratic social decision-making procedures. However, an analysis to the latter effect requires quite distinct conceptual development, which cannot but be left as one of our future agendas.
6 Appendix

6.1 Proof of Theorem 1

Let the set of feasible assignments of capabilities be defined by

\[ C(Z) \equiv \{ c(z) \in K^n \mid z \in Z \}, \]

which is compact in the Hausdorff topological space \( K^n \). Define a compact-valued and continuous correspondence \( Y : [0, \overline{x}]^n \rightarrow \mathbb{R}_+^n \) by \( Y(x) \equiv \{ y = (y_i)_{i \in N} \in \mathbb{R}_+^n \mid f(x) \geq \sum_{N} y_i \} \) for each \( x \in [0, \overline{x}]^n \). Then, we can show, given \( J \in J \) and the profile of capability correspondences \( c \in C^n \), the following lemmas.

**Lemma 1:** For each \( x \in [0, \overline{x}]^n \), \( Z(x; J) \) is non-empty and compact.

**Proof.** For each \( x \in [0, \overline{x}]^n \), let \( C(\{x\} \times Y(x)) \equiv \{ c(x, y) \in C(Z) \mid y \in Y(x) \} \) and \( MC(x) \equiv \{ c(x, y) \in C(Z) \mid \forall y' \in Y(x) : (C_{\min}^J(x, y), C_{\min}^J(x, y')) \in J \}. \)

Since \( J \) is continuous on \( K \) and \( C(\{x\} \times Y(x)) \) is compact, we are assured that \( MC(x) \) is non-empty and compact. Thus, \( Z(x; J) \) is non-empty and compact. \( \blacksquare \)

**Lemma 2:** For each \( x \in [0, \overline{x}]^n \), \( z \in Z(x; J) \) implies \((c_i(z_i), c_j(z_j)) \in I(J)\) for all \( i, j \in N \).

**Proof.** Suppose there exist \( x \in [0, \overline{x}]^n \) and \( z \in Z(x; J) \) such that \((c_i(z_i), c_j(z_j)) \in P(J)\) for some \( i, j \in N \). Then, \((c_i(z_i), C_{\min}^J(z)) \in P(J)\) holds. Consider an alternative allocation \( z' \in Z \) in which

\[
\begin{align*}
z'_i &= (x_i, y_i - \varepsilon) \text{ for some small enough } \varepsilon > 0, \\
z'_j &= (x_j, y_j + \frac{\varepsilon}{\#N(C_{\min}^J(z))}) \text{ for all } j \in N(C_{\min}^J(z)), \text{ and} \\
z'_h &= (x_h, y_h) \text{ for any other } h \in N \setminus (N(C_{\min}^J(z)) \cup \{i\}), \end{align*}
\]

where \( N(C_{\min}^J(z)) \equiv \{ j \in N \mid c_j(z_j) \in C_{\min}^J(z) \} \). Then, by the conditions (a) and (c) of capability correspondences, we can see that \((c_j(z'_j), c_j(z_j)) \in P(J)\) for all \( j \in N(C_{\min}^J(z)) \), and so that \((C_{\min}^J(z'), C_{\min}^J(z)) \in P(J)\) holds, which is a contradiction. \( \blacksquare \)

**Lemma 3:** For each \( x \in [0, \overline{x}]^n \), \( Z(x; J) \) is singleton.
Proof. Suppose there exist \( x \in [0, \bar{x}]^n \) and \( z, z' \in Z(x; J) \) such that \( z \neq z' \), which implies \( y \neq y' \). Thus, there exists at least two individuals \( i, j \in N \) such that \( y_i > y'_i \) and \( y_j < y'_j \). Then, by the condition (a) of capability correspondences, \( c_i'(z_i) \supset c_i'(z'_i) \) and \( c_j(z_j) \subset c_j(z'_j) \). By these set-inclusion relations, (3.1.1) of J, and Lemma 2, \( (c_i(z_i), c_j(z_j)) \in I(J) \) and \( (c_j(z_j'), c_i(z'_i)) \in P(J) \) should hold. However, since \( z' \in Z(x; J) \), \( (c_i(z_i), c_j(z'_j)) \in I(J) \) should also hold by Lemma 2, which is a contradiction.

Let us define an ordering \( R_J \subseteq Z \times Z \) as follows: for all \( z, z' \in Z \), \( (z, z') \in R_J \) (resp. \( P(R_J) \)) \( \Leftrightarrow \left( \mathcal{O}^J_{\min}(z), \mathcal{O}^J_{\min}(z') \right) \in J \) (resp. \( P(J) \)). Since \( c \) is continuous, \( R_J \) is continuous on \( Z \).

Lemma 4: \( \bigcup_{x \in [0, \bar{x}]^n} Z(x; J) \) has a closed graph in \( Z \).

Proof. Let a sequence \( \{(x^\mu, y^\mu)\}^{+\infty}_{\mu=1} \) be such that \( (x^\mu, y^\mu) \rightarrow (x, y) \) as \( \mu \to +\infty \), and \( (x^\mu, y^\mu) \in Z(x^\mu; J) \) for every \( \mu = 1, \ldots \), ad. inf. Suppose that \( (x, y) \notin Z(x; J) \). Then, there exists \( (x, y') \in Z(x; J) \) such that \( ((x, y'), (x, y)) \in P(R_J) \), because \( Z(x; J) \) is the set of maximal element of \( R_J \) over \( \{x\} \times Y(x) \). Since \( Y \) is l.h.c., there exists a sequence \( \{(x^\mu, y'^\mu)\}^{+\infty}_{\mu=1} \) such that \( (x^\mu, y'^\mu) \in \{x^\mu\} \times Y'(x^\mu) \) for every \( \mu = 1, \ldots \), ad. inf., and \( (x^\mu, y'^\mu) \rightarrow (x, y') \) \( \mu \to +\infty \). Then, for a large enough \( \mu \), \( ((x^\mu, y'^\mu), (x, y')) \in P(R_J) \) by continuity of \( R_J \) on \( \{x\} \times Y(x) \), which is a contradiction. Thus, \( \bigcup_{x \in [0, \bar{x}]^n} Z(x; J) \) has a closed graph in \( Z \).

Proposition 3 [Yoshihara (2000)]\(^{[\text{14}]}\): Let Assumption 1 hold. Let \( h : [0, \bar{x}]^n \to \mathbb{R}^n_+ \) be a continuous function such that, for each \( x \in [0, \bar{x}]^n \), \( h(x) = y \) and \( f(x) = \sum_N y_i \). Then, for any \( u \in U^n \), there exists \( x^* \in [0, \bar{x}]^n \) such that \( (x^*, h(x^*)) \) is a Pareto efficient allocation for \( u \).

Proof. Given \( u \in U^n \), Let \( S(u) \) be the utility possibility set of feasible allocations, and \( \partial S(u) \) be its boundary. Since every utility function is strictly increasing, \( \partial S(u) \) is the set of Pareto efficient utility allocations.

\(^{[\text{14}]}\): Corchón and Puy (1998; Theorem 1) showed the same result under a stronger assumption than that in this proposition.
By Assumption 1, $0 \notin \partial S(\mathbf{u})$. Thus, $\sum u_h > 0$ for every $\mathbf{u} = (u_i)_{i \in N} \in \partial S(\mathbf{u})$, and the mapping

$$\tilde{v} : \partial S(\mathbf{u}) \to \Delta^{n-1}$$

by $\tilde{v}(\mathbf{u}) = \frac{\mathbf{u}}{\sum u_h}$,

is well-defined and continuous on $\partial S(\mathbf{u})$, where $\Delta^{n-1}$ is an $n-1$-dimensional unit simplex. By Arrow and Hahn (1971; Lemma 5.3, p.114), $\tilde{v}$ is a homeomorphism. Denote its inverse by $\hat{u}$. Define a correspondence

$$\hat{W} : \Delta^{n-1} \to Z$$

by $\hat{W}(\hat{u}(\mathbf{v})) \equiv \{ z \in Z \mid u_i(z_i) \geq \hat{u}_i(v_i)(\forall i \in N) \}$.

By Arrow and Hahn (1971; Theorem 4.5, Corollary 5, p.99), $\hat{W}$ is upper hemi-continuous with non-empty compact convex values.

Given a continuous function $h$ and $\mathbf{z} = (x_i, y_i)_{i \in N} \in Z$, let $E_i(\mathbf{x}, y_i) \equiv h_i(\mathbf{x}) - y_i$. Then, we define the following optimization problem:

$$\max_{v \in \Delta^{n-1}} \sum_{i \in N} v_i \cdot E_i(\mathbf{x}, y_i).$$

By Berge’s maximum theorem, we can define an upper hemi-continuous correspondence $\Theta : Z \to \Delta^{n-1}$ by

$$\Theta(\mathbf{z}) \equiv \{ \mathbf{v}^* \in \Delta^{n-1} \mid \mathbf{v}^* \in \arg\max_{v \in \Delta^{n-1}} \sum_{i \in N} v_i \cdot E_i(\mathbf{x}, y_i) \}.$$

Note that $\Theta$ is non-empty compact and convex-valued.

Now, we define a correspondence $\Phi : \Delta^{n-1} \times Z \to \Delta^{n-1} \times Z$ by

$$\Phi(\mathbf{v}, \mathbf{z}) \equiv \Theta(\mathbf{z}) \times \hat{W}(\hat{u}(\mathbf{v})), $$

which is upper hemi-continuous with non-empty compact convex values. By Kakutani’s fixed point theorem,

$$\exists (\mathbf{v}^*, \mathbf{z}^*) \in \Delta^{n-1} \times Z \text{ s.t. } (\mathbf{v}^*, \mathbf{z}^*) \in \Phi(\mathbf{v}^*, \mathbf{z}^*).$$

By definition, $\hat{u}(\mathbf{v}^*) = (u_i(z_i^*))_{i \in N}$, so that $\mathbf{z}^*$ is Pareto efficient for $\mathbf{u}$. Finally, we show that $\mathbf{z}^*$ is Pareto efficient for $\mathbf{u}$. Finally, we show that $\mathbf{z}^* = (x^*, h(x^*))$. To do this, it is sufficient to show $E_i(x^*, y_j^*) = 0$ for all $i \in N$. Assume that there exists $j \in N$ such that $E_j(x^*, y_j^*) > 0$. Then, since $\mathbf{z}^*$ is Pareto efficient, there exists $l \in N$ such that $E_l(x^*, y_l^*) < 0$. To maximize $\sum v_i \cdot E_i(\mathbf{x}, y_i)$, we obtain $v_l^* = 0$. Then, $u_i(z_i^*) = \hat{u}_i(v_i^*) = 0$, so that, by the strict monotonicity of $u_l$ and Assumption 1, we obtain either
(1) \( x^*_i = \overline{v} \) and \( y^*_i > 0 \), or (2) \( x^*_i \leq \overline{v} \) and \( y^*_i = 0 \). First, (1) is impossible because \( z^* \) is Pareto efficient for \( u \). In fact, the vector \((1,0)\) is a unique price which supports \( z^*_i \) of case (1) as an expenditure minimizing consumption. However, \((1,0)\) cannot be consistent with any profit maximizing production except the origin. Second, (2) implies \( E_i(x^*, y^*_i) \geq 0 \), which is a contradiction. Thus, \( E_i(x^*, y^*_i) = 0 \) for all \( i \in N \), as was to be verified. ■

**Lemma 5:** Let Assumption 1 hold. Then, for each \( u \in \mathcal{U}^n \), there exists a Pareto efficient allocation \( z^* \in Z \) such that \( z^* \in \bigcup_{x \in [0,\overline{v}]^n} Z(x; J) \).

**Proof.** Let a correspondence \( h^J : [0,\overline{v}]^n \to Y([0,\overline{v}]^n) \) be such that \( \{x\} \times h^J(x) = Z(x; J) \) for each \( x \in [0,\overline{v}]^n \). Since \( Y([0,\overline{v}]^n) \) is compact, \( h^J \) is u.h.c. by **Lemma 4**. Moreover, for each \( x \in [0,\overline{v}]^n \), \( h^J(x) \) is singleton, since \( Z(x; J) \) is singleton by **Lemma 3**. Thus, \( h^J \) is a continuous function. Then, under Assumption 1, we can obtain the desired result by the application of **Proposition 3**. ■

**Proposition 4** [Yoshihara (2000)]: Let Assumption 1 and Assumption 2 hold. Let \( h : [0,\overline{v}]^n \to \mathbb{R}^+_n \) be a continuous function such that, for each \( x \in [0,\overline{v}]^n \), \( h(x) = y \) and \( f(x) = \sum_N y_i \), and for any \( i, j \in N \) with \( c_i = c_j \), \( x_i = x_j \) implies \( h_i(x) = h_j(x) \). Then, there exists a game form \( \gamma = \left( ([0,\overline{v}] \times \mathbb{R}^+)^n, g \right) \in \Gamma_{LS} \) such that, for any \( u \in \mathcal{U}^n \), \( z \in A_{NE}(\gamma, u) \) holds if and only if \( z = (x, h(x)) \), and it is Pareto efficient.

**Proof.** By **Proposition 3**, the continuous function \( h \) attains some Pareto efficient allocations for each \( u \in \mathcal{U}^n \). In other words, for any \( u \in \mathcal{U}^n \), there exists \( x^* \in [0,\overline{v}]^n \) such that \((x^*, h(x^*))\) is a Pareto efficient allocation for \( u \). Let us denote by \( P(h : u) \) the set of all such Pareto efficient allocations which are attained by \( h \) under \( u \in \mathcal{U}^n \). Note that \(([0,\overline{v}]^n, h) \in \Gamma_{LS} \). For \( h \), we sometimes use notation like \( h_i(x) \), which refers to \( i \)-th component of the vector \( h(x) \).

**Step 1:** For each \( \tilde{z} \in P(h : u) \), we construct an outcome function \( h^{\tilde{z}} : [0,\overline{v}]^n \to \mathbb{R}^+_n \) such that \(([0,\overline{v}]^n, h^{\tilde{z}}) \in \Gamma_{LE} \) and \( \tilde{z} \in A_{NE} \left( ([0,\overline{v}]^n, h^{\tilde{z}}), u \right) \).

By Assumption 2, we can define a continuous function \( f'_i : [0,\overline{v}]^n \to \mathbb{R}_+ \) by \( f'_i(x) \equiv \frac{\partial f_i(x)}{\partial x_i} \) for all \( x \in [0,\overline{v}]^n \). Given \( u \in \mathcal{U}^n \), \( \tilde{z} = (\hat{x}, \hat{y}) \in P(h : u) \), and \( i, j \in N \), let

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15The method of constructing \( \gamma^* \) comes from Yoshihara (2000).
\[
\lambda_i(\bar{x}) \equiv \begin{cases} 
\frac{\bar{y}_i + f'_i(\bar{x}) \cdot (\bar{x}_i - \bar{x}_i) - h_i(\bar{x}, \bar{x}_i)}{(\bar{x}_i - \bar{x}_i)^2} & \text{if } \bar{x}_j \neq \bar{x}_i, \\
0 & \text{if } \bar{x}_j = \bar{x}_i.
\end{cases}
\]

Given \(u \in U^n\), \(\hat{z} \in P(h : u)\), and \(x \in [0, \overline{x}]^n\), define for each \(i \in N\),

\[
\Psi_i(x) = \begin{cases} 
\bar{y}_i + f'_i(\bar{x}) \cdot (x_i - \bar{x}_i) & \text{if } x_i \in (\bar{x}_i - \varepsilon_i(\bar{x}), \bar{x}_i + \varepsilon_i(\bar{x})) \\
\bar{y}_i + f'_i(\bar{x}) \cdot (x_i - \bar{x}_i) - \mu^j_{j^*}(\bar{x}) \cdot (x_i - \bar{x}_i)^2 & \text{otherwise}
\end{cases},
\]

where

\[
\varepsilon_i(\bar{x}) \equiv \min_{j \neq i, x_j \neq \bar{x}_i} \| \bar{x}_j, \bar{x}_i \|, \quad j^*(x_i) = \max_{j \neq i} \left\{ \arg \min_{j \neq i} \| \bar{x}_j, x_i \| \right\},
\]

and

\[
\mu^j_{j^*}(\bar{x}) = \begin{cases} 
\lambda_{j^*}(x_i)(\bar{x}) & \text{if } 0 \leq \lambda_{j^*}(x_i)(\bar{x}) \\
0 & \text{otherwise}
\end{cases}.
\]

By construction of \(\Psi_i(x)\), we have (i) \(\Psi_i(x) = \bar{y}_i\) if \(x_i = \bar{x}_i\); (ii) \(\Psi_i(x) \leq \bar{y}_i + f'_i(\bar{x}) \cdot (x_i - \bar{x}_i)\) if \(x_i \neq \bar{x}_i\); and (iii) \(\Psi_i(x) = \min\{h_i(\bar{x}_i, x_i), \bar{y}_i + f'_i(\bar{x}) \cdot (x_i - \bar{x}_i)\}\) if \(x_i = \bar{x}_j\) for some \(j \neq i\).

For each \(i \in N\), define

\[
\zeta_i(x) = \min \left\{ \max \{0, \Psi_i(x)\}, f(x) \right\}.
\]

Moreover, for each \(i \in N\) and each \(x \in [0, \overline{x}]^n\), define,

\[
n(x, x_i) \equiv \# \{ j \in N \mid x_j = x_i \}.
\]

Then, given \(u \in U^n\) and \(\hat{z} \in P(h : u)\), define a function \(h_{\bar{x}} : [0, \overline{x}]^n \rightarrow \mathbb{R}_+^n\) as follows: for each \(x \in [0, \overline{x}]^n\), and for each \(i \in N\),

\[
h_{\bar{x}}(x) = \begin{cases} 
\zeta_i(x) & \text{if } \forall j \neq i, x_j = \hat{x}_j, \\
[f(x) - n(x, x_j) \cdot \zeta_j(x)] \cdot \frac{1}{n-n(x, x_j)} & \text{if } \exists j \neq i, \forall l \neq j, x_l = \hat{x}_k, \& x_j \neq \hat{x}_i \\
\zeta_j(x) & \text{if } \exists j \neq i, \forall l \neq j, x_l = \hat{x}_l, \& x_j = \hat{x}_i \\
h_i(x) & \text{otherwise}.
\end{cases}
\]
Moreover, given \( Z \) this deviators. The set \( N \) defined by using Step 3:

\[ u \]

for each \( x \) and \( (x, y) \):

\[ \left\{(x', y') : (x', y') \in [0, \overline{\Sigma}] \times \mathbb{R}^n_+ \right\}, \]

Let us introduce two functions \( h_0 \) and \( h^m \) as follows:

1. \( h_0 : [0, \overline{\Sigma}]^n \to \mathbb{R}_+ \) by \( h_0(x) = 0 \) for each \( (x, y) \in [0, \overline{\Sigma}]^n \times \mathbb{R}^n_+ \), and

2. \( h^m : [0, \overline{\Sigma}]^n \times \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) by,

\[ h^m_i(x, y) = \begin{cases} \frac{f(x)}{\#(\text{max } N(y))} & \text{if } i \in N^m(y) \\ 0 & \text{if } k \notin N^m(y) \end{cases}, \]

for each \( (x, y) \in [0, \overline{\Sigma}]^n \times \mathbb{R}^n_+ \), and for all \( i \in N \), where \( N^m(y) = \{ i \in N : \forall j \in N : y_i \geq y_j \} \).

It is clear that \(([0, \overline{\Sigma}]^n, h^0)\), \(([0, \overline{\Sigma}]^n \times \mathbb{R}^n_+), h^m) \in \Gamma_{LS} \). Note that for any \( u \in U^n \), there is no Nash equilibrium for the game defined by \(([0, \overline{\Sigma}]^n \times \mathbb{R}^n_+), h^m) \).

**Step 3:** We construct a game form \( \gamma^* = ([0, \overline{\Sigma}] \times \mathbb{R}_+)^n, g^* \), in which \( g^* \) is defined by using \( \{ h^m \}_{u \in U^n} \), \( h^0 \), and \( h^m \).

Given \( x \in [0, \overline{\Sigma}]^n \) and \( y \in \mathbb{R}^n_+ \), let \( \rho(x, y : h) \equiv \{ u \in U^n : (x, h(x)) \in P(h : u) \& h(x) = y \} \). Let us call \( (x, y) \in Z \) a potential \( P^h \)-allocation if \( \rho(x, y : h) \neq \emptyset \). Given \( x \in [0, \overline{\Sigma}]^n \) and \( y \in \mathbb{R}^n_+ \), let

\[ N(x, y) \equiv \{ l \in N : \exists (x_l', y_l') \neq (x_l, y_l) \in [0, \overline{\Sigma}] \times \mathbb{R}^n_+ : \rho((x_l', x_{-l}), (y_l', y_{-l}) : h) \neq \emptyset \}. \]

The set \( N(x, y) \) will be used in defining \( \gamma^* \) below as the set of potential deviators. That is, if \( \rho(x, y : h) = \emptyset \), and there is some \( j \in N(x, y) \), then this \( j \) may be interpreted as deviating from \( P(h : u) \) for some \( u \in U^n \).

Given \( j \in N \), \( x \in [0, \overline{\Sigma}]^n \), and \( y \in \mathbb{R}^n_+ \), let

\[ X_j(x_{-j}, y_{-j}) \equiv \{ x'_j \in [0, \overline{\Sigma}] : \rho\left( (x'_j, h_j(x'_j, x_{-j})) , (x_{-j}, y_{-j}) : h \right) \neq \emptyset \}, \]

and

\[ Z(x_j, x_{-j}, y_{-j}) \equiv \{ (x'_j, h_j(x'_j, x_{-j})), (x_{-j}, y_{-j}) \in Z : x'_j \in X_j(x_{-j}, y_{-j}) \}. \]

Moreover, given \( j \in N \), \( x \in [0, \overline{\Sigma}]^n \), and \( y \in \mathbb{R}^n_+ \), let

\[ \tilde{z}(x_j, x_{-j}, y_{-j}) \equiv \arg \min_{(x'_j, y'_j) \in Z(x_j, x_{-j}, y_{-j})} \left( f_j(x'_j, x_{-j}) \cdot (x_j - x'_j) \right). \]
Step 4: or applied in order to punish all potential deviators. It is clear that 2-2
viator is inconsistent with any potential
In this Rule 3:
Rule 2:
Rule 1:
Note that the above three notations will be used in defining γ̂ below to
punish a unique potential deviator. If ρ(x, y : h) = ∅ and {j} = N(x, y),
then we can identify j as the unique potential deviator. Then, by definition
of N(x, y), X_i(x_j, y_j) is non-empty, so that Z(x_j, x_j, y_j) is non-empty.
Note that Z(x_j, x_j, y_j) is the set of potential P̃-allocations which would
be implemented if j were not to deviate. Then, by selecting ẑ(x_j, x_j, y_j)
from this set, we will consider the outcome function g* in order to punish j
in such a situation.

Given x ∈ [0, ˘x] n, let N°(x) = {i ∈ N | x_i ∈ [0, ˘x]}, max N°(x) = {i ∈
N°(x) | ∃j ∈ N°(x) s.t. x_j > x_i}, and max N(x) = {i ∈ N | ∃j ∈ N s.t.
x_j > x_i}. Now, let us define a labor sovereign and equal treatment of equals
rule γ̂* = ([0, ˘x] × R+) n, g* in the following way: given a strategy profile
(x, y) ∈ ([0, ˘x] × R+) n,

Rule 1: if ρ(x, y : h) ̸= ∅, then g*(x, y) = (x, ĥ(x)), where ĥ = (x, h(x)).

Rule 2: if ρ(x, y : h) = ∅ and there exists a non-empty N(x, y), then
2-1: if #N(x, y) > 1, then g*(x, y) = (x, h^0(x)),
2-2: if N(x, y) = {j}, then g_j*(x, y) = (x_j, ĥ_j(x_j, x_j, y_j)(x)) and for all i ̸= j,

ĝ_i*(x, y) =
{(x_i, h^0_i(x)) if {j} = max N(x) ∩ max N°(x),
(x_i, ĥ_i(x_j, x_j, y_j)(x)) otherwise.

Rule 3: in all other cases, g*(x, y) = (x, h^m(x, y)).

In this γ̂*, if a strategy profile (x, y) is consistent with a potential P̃-
allocation, then Rule 1 applies, and (x, y) becomes the outcome; if (x, y)
is inconsistent with any potential P̃-allocation, and a unique potential
deviator j is identified, then Rule 2-2 applies and identifies some potential
P̃-allocation ẑ(x_j, x_j, y_j) = (ẑ_j, (x_j, y_j)), which would be the outcome
if j were not to deviate. Thus, j is punished by ĥ_j(x_j, x_j, y_j)(x) under Rule
2-2. If (x, y) corresponds to neither of the above two cases, then h^m or h^0 is
applied in order to punish all potential deviators. It is clear that γ̂* ∈ Γ_LS,
since in every case of strategy profile, the value of g* is that of either h^m, h^m,
or ĥ-types.

Step 4: We show that ANE(γ̂*, u) = P(h : u) for all u ∈ U^n.

(1) First, we show that ANE(γ̂*, u) ⊇ P(h : u) for all u ∈ U^n. Let
z = (x, y) ∈ P(h : u). Then, if a strategy profile of every agent is (x, y) =
(x_i, y_i) \in N \subseteq \mathbb{R}_+^n$, then $g^*(x, y) = (x, h^*(x)) = (x, h(x)) = (x, y)$ by Rule 1. Suppose that individual $j \in N$ deviates from $(x_i, y_i)$ to $(x'_j, y'_j)$. Then, if $j$ induces Rule 2-1, then $g^*_j((x'_j, x_{-j}), (y'_j, y_{-j})) = 0$. If $j$ induces Rule 1, he cannot be better off, since $y'_j = f(x'_j, x_{-j}) - \sum_{i \neq j} y_i$ holds true, and $z$ is Pareto efficient for $u$. If $j$ induces Rule 2-2, then $g^*_j((x'_j, x_{-j}), (y'_j, y_{-j})) = \langle x'_j, h_j(x'_j, x_{-j}, y_{-j}) \rangle (x'_j, x_{-j})$. Since $z \in Z(x'_j, x_{-j}, y_{-j})$ holds true, it follows that if $G(x'_j, x_{-j}, y_{-j}) = ((\tilde{x}_j, x_{-j}), (\tilde{y}_j, y_{-j}))$, then

$$h_j(x'_j, x_{-j}, y_{-j})(x'_j, x_{-j}) \leq \tilde{y}_j + f_j^*(((\tilde{x}_j, x_{-j})) \cdot (x'_j - \tilde{x}_j) \leq y_j + f_j(x) \cdot (x'_j - x_j).$$

This implies that $j$ cannot be better off by this deviation. Note that $j$ cannot induce Rule 3. Thus, $A_{NE}(\gamma^*, u) \supseteq P(h : u)$ holds.

(2) Second, we show that $A_{NE}(\gamma^*, u) \subseteq P(h : u)$. Let $(x, y) = (x_i, y_i) \in \mathbb{R}_+^n$ be a Nash equilibrium of the game $(\gamma^*, u)$. Note that $(x, y)$ cannot correspond to Rule 3. This is because every agent $j$ can get everything in Rule 3 by changing from $y_j$ to large enough $y'_j > \max \{\max \{y_i\}_{i \neq j}, f(x) \}$. Also, $(x, y)$ cannot correspond to Rule 2-2, since, in Rule 2-2, there is an agent $j \in N \setminus N(x, y)$ who can induce Rule 3 by changing from $y_j$ to large enough $y'_j > \max \{\max \{y_i\}_{i \neq j}, f(x) \}$. Finally, $(x, y)$ cannot correspond to Rule 2-1 either, since every agent $l \in N(x, y)$ can induce Rule 2-2 by changing from $y_l$ to $y'_l = f(x) + \varepsilon$, so that $l$ can obtain positive output. Suppose that $(x, y)$ corresponds to Rule 1. Then, for $z = (x, y) = (x, h(x))$, $g^*(x, y) = (x, h^*(x)) = z \in A_{NE}(\gamma^*, u)$. Suppose that $z$ is not Pareto efficient. Then, there is at least one individual $i \in N$ who changes slightly from $x_i$ to $x'_i$, so that $u_i(x_i, h^*_i(x)) < u_i(x'_i, h^*_i(x'_i, x_{-i}))$. Note that if $x'_i = x_i + \varepsilon$ where the value $|\varepsilon|$ is small enough, then $h^*_i(x'_i, x_{-i}) = y_i + f_i(x) \cdot (x'_i - x_i)$, and $\tilde{z}(x'_i, x_{-i}, y_{-i}) = z$ also holds by the concavity of $f$. Thus, by changing from $(x_i, y_i)$ to $(x'_i, y'_i)$, where $y'_i = f(x'_i, x_{-i}) + \varepsilon$, $i$ can induce Rule 2-2 and obtain $g^*_i((x'_i, x_{-i}), (y'_i, y_{-i})) = (x'_i, h^*_i(x'_i, x_{-i}))$. This is a contradiction, since $z \in A_{NE}(\gamma^*, u)$. Thus, $z$ is Pareto efficient for $u$. $\blacksquare$

Proof of Theorem 1: Given $J \in J$, for each $u \in U^n$, let $PM^J(u) \equiv PO(u) \cap \left[ \bigcup_{x \in [0, \mathbb{R}_+^n]} Z(x; J) \right]$. Note that if $z = (x, y) = \text{PM}^J(u)$, then $h^J(x) = y$. Moreover, $([0, \mathbb{R}_+^n], h_J) \in \Gamma_{LS}$ holds true. It follows from the property of $Z(\cdot; J)$ that $\{(x, y)\} = Z(x; J)$, $c_i = c_j$, and $x_i = x_j$ for some $i, j \in N$.
implies $y_i = y_j$. Thus, by Proposition 4, there exists an allocation rule
$
\gamma^* = (\{[0, \bar{x}] \times \mathbb{R}_+\}^n, g^*) \in \Gamma_{LS}

\text{such that } A_{NE}(\gamma^*, u) = PM^J(u) \text{ holds for all } u \in \mathcal{U}^n.

6.2 Proofs of Theorem 2 and Theorem 3

Proof of Theorem 2: Given $u \in \mathcal{U}^n$, let $S(u)$ be the utility possibility set of feasible allocations, and $\partial S(u)$ be its boundary. Since every utility function is strictly increasing, $\partial S(u)$ is the set of Pareto efficient utility allocations.

Now we define an ordering $V(u)$ over $S(u)$ as follows:
1) if $\bar{u}, \bar{u}' \in \partial S(u)$, then $\langle \bar{u}, \bar{u}' \rangle \in I(V(u))$,
2) for any $\bar{u}, \bar{u}' \in S(u)$, there exist $\alpha, \alpha' \in [1, +\infty)$ such that $\alpha \cdot \bar{u}, \alpha' \cdot \bar{u}' \in \partial S(u)$ and $\langle \bar{u}, \bar{u}' \rangle \in V(u)$ if and only if $\alpha \leq \alpha'$. This ordering $V(u)$ is continuous over $S(u)$.

Given $J \in \mathcal{J}$ and $u \in \mathcal{U}^n$, let us define a complete ordering $R_{u,J}$ over
$
\bigcup_{x \in [0, \bar{x}]^n} Z(x; J)

\text{as follows: for any } z, z' \in \bigcup_{x \in [0, \bar{x}]^n} Z(x; J), (z, z') \in R_{u,J} \iff

\langle u(z), u(z') \rangle \in V(u). \text{ This ordering } R_{u,J} \text{ is continuous on } \bigcup_{x \in [0, \bar{x}]^n} Z(x; J),

\text{and its maximal element consists of } z \in PM^J(u), \text{ where } PM^J(u) = PO(u) \cap

\left[ \bigcup_{x \in [0, \bar{x}]^n} Z(x; J) \right], \text{ which is non-empty by Lemma 5. Given } J \in \mathcal{J}, \text{ let } R_J(x)

\text{be the restriction of } R_J \text{ into } ([x] \times Y(x))^2.

Consider a binary relation $R_{u,J} \cup \left[ \bigcup_{x \in [0, \bar{x}]^n} R_J(x) \right]$ over $Z$. It is easy to see that this binary relation is consistent, so that there exists an ordering extension $R^*_{u,J}$ of $R_{u,J} \cup \left[ \bigcup_{x \in [0, \bar{x}]^n} R_J(x) \right]$ by Suzumura’s (1983) extension theorem. Based upon this $R^*_{u,J}$, let us consider an ordering function $Q_J^*$ as follows: for each $u \in \mathcal{U}^n$ and all $(z, \gamma), (z', \gamma') \in R(u)$,
1) if $\gamma \in \Gamma_{LS}$ and $\gamma' \in \Gamma \setminus \Gamma_{LS}$, then $((z, \gamma), (z', \gamma')) \in P(Q_J^*(u));$
2) if either $\gamma, \gamma' \in \Gamma_{LS}$ or $\gamma, \gamma' \in \Gamma \setminus \Gamma_{LS}$, then

\begin{align*}
((z, \gamma), (z', \gamma')) & \in Q_J^*(u) \iff (z, z') \in R^*_{u,J}, \\
((z, \gamma), (z', \gamma')) & \in P(Q_J^*(u)) \iff (z, z') \in P(R^*_{u,J}).
\end{align*}

Note that $Q_J^*(u)$ is complete and transitive, and $Q_J^* \in Q^{LSJ}_{\mathcal{S}^*-J_{LA}^{\bullet}(PA \cap Z(J))}$ by the definition. Finally, we can see that $Q_J^*$ uniformly rationalizes $\gamma^* \in \Gamma_{LS}$ as well as any $\gamma^{**} \in \Gamma_{LS} \cap \Gamma_{NS}$ whose every Nash equilibrium allocation always belongs to $PM^J(u)$ for any $u \in \mathcal{U}^n$.  

26
Proof of Theorem 3: Given \( u \in \mathcal{U}^n \) and \( \bigcup_{x \in [0, \bar{x}]^n} Z(x; J) \), let us define an ordering \( R^0_{u,J} \) over \( PO(u) \) as follows: for all \( z, z' \in PO(u) \),

1) if \( x = x' \), then \((z, z') \in R^0_{u,J} \) (resp. \( P \left( R^0_{u,J} \right) \)) \( \Leftrightarrow \) \( (C^J_{\text{min}}(z), C^J_{\text{min}}(z')) \in J \) (resp. \( P(J) \));

2) if \( x \neq x' \), then \((z, z') \in R^0_{u,J} \) (resp. \( P \left( R^0_{u,J} \right) \)) \( \Leftrightarrow \) \( \frac{\max_{b \in C^J_{\text{min}}(u)} g(b)}{\max_{b \in C^J_{\text{min}}(z(x,J))} g(b)} \geq \frac{\max_{b \in C^J_{\text{min}}(u')} g(b)}{\max_{b \in C^J_{\text{min}}(z(x,J))} g(b)} \).

Note that the set of maximal elements of the ordering \( R^0_{u,J} \) over \( PO(u) \) coincides with \( PM^J(u) \).

Next, given \( u \in \mathcal{U}^n \), let us define the strict Pareto preference relation (resp. the Pareto indifference relation) \( SP_u \subseteq Z \times Z \) (resp. \( IP_u \subseteq Z \times Z \)) by \((z, z') \in SP_u \Leftrightarrow u_i(z_i) > u_i(z'_i) \) for all \( i \in N \) (resp. \( (z, z') \in IP_u \Leftrightarrow u_i(z_i) = u_i(z'_i) \) for all \( i \in N \)). Then, define a quasi-ordering \( P_u \subseteq Z \times Z \) as \( P_u \equiv SP_u \cup IP_u \).

Consider a binary relation \( P_u \cup R^0_{u,J} \) on \( Z \). It is easy to see that this binary relation is consistent, so that there exists an ordering extension \( R^s_{u,J} \) of \( P_u \cup R^0_{u,J} \) by Suzumura’s (1983) extension theorem. Based upon this \( R^s_{u,J} \), let us consider an ordering function \( Q^*_{J} \) as follows: for each \( u \in \mathcal{U}^n \) and all \( (z, \gamma), (z', \gamma') \in R^s_{u,J} \),

1) if \( \gamma \in \Gamma_{LS} \) and \( \gamma' \in \Gamma \setminus \Gamma_{LS} \), then \((z, \gamma), (z', \gamma') \in P(Q^*_{J}(u)) \);

2) if either \( \gamma, \gamma' \in \Gamma_{LS} \) or \( \gamma, \gamma' \in \Gamma \setminus \Gamma_{LS} \), then

\[
((z, \gamma), (z', \gamma')) \in Q^*_{J}(u) \Leftrightarrow (z, z') \in R^s_{u,J},
((z, \gamma), (z', \gamma')) \in P(Q^*_{J}(u)) \Leftrightarrow (z, z') \in P(R^s_{u,J}).
\]

Note that \( Q^*_{J}(u) \) is complete and transitive, and \( Q^*_{J} \in Q_{LS}^{\text{PA}^J \cap \text{LA} \cap \text{PO}} \) by the definition. Finally, we can see that \( Q^*_{J} \) uniformly rationalizes \( \gamma^* \in \Gamma_{LS} \) as well as any \( \gamma^* \in \Gamma_{LS} \cap \Gamma_{NS} \) whose every Nash equilibrium allocation always belongs to \( PM^J(u) \) for any \( u \in \mathcal{U}^n \). \( \blacksquare \)

References


27


Example 1 in the consumption space

Example 1 in the functioning space

Figure 1