Testing the Rank of a Sub-Matrix of Cointegration with a Deterministic Trend

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Abstract

In this paper we consider the test of the rank of the sub-matrix of β, the cointegrating matrix, when the process has a deterministic linear trend. We review the problem of the testing procedure proposed by Kurozumi (2003) and give the alternative test statistic that is asymptotically chi-square distributed. We also propose the test of the rank of the sub-matrix of $\beta_\perp$, the orthogonal matrix to β. Monte Carlo simulations show that our tests proposed in this paper work fairly well in finite samples even when the tests proposed by Kurozumi (2003) perform poorly.

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1. Introduction

A vector error-correction model (VECM) has often been used in the econometric literature as one of the useful models to describe non-stationary time series, and a typical n-dimensional VECM with cointegrating rank r $(0 < r < n)$ is expressed as follows:

$$\triangle x_t = \mu_0 + \mu_1 t + \alpha\beta' x_{t-1} + \sum_{j=1}^{m-1} \Gamma_j \triangle x_{t-j} + \varepsilon_t, \tag{1}$$

for $t = 1, \cdots, T$, where $\triangle = 1 - L$, L is the lag operator, $\{\varepsilon_t\} \sim i.i.d.N(0, \Sigma)$ with Σ being a positive definite matrix, and α and β are $n \times r$ matrices with rank r. The exact condition of the cointegrating relationship is given by Johansen (1991, 1992). The normality assumption on $\{\varepsilon_t\}$ is imposed to obtain the maximum likelihood (ML) estimator and the asymptotic results in this paper will be obtained under weaker conditions as explained in Pesaran and Shin (2002). Since we do not allow a quadratic trend in x_t, we assume that $\mu_1 = \alpha\rho_1$ for a $r \times 1$ vector ρ_1.

Recently, Kurozumi (2003) took notice of the importance of information on the sub-matrix of β and proposed the test of the rank of β_1, the first n_1 rows of β $(0 < n_1 < n)$, for the model with $\mu_1 = 0$. To see the usefulness of information on the rank of β_1, let us consider the triangular representation of the model as used in Phillips (1991) and Saikkonen (1991) among others, where the cointegrating matrix is normalized as $\beta_c = [I_r, \beta_1'^{-1}\beta_2]'$ for $n_1 = r$. To justify this normalization we have to know that β_1 is of full rank, and the Kurozumi's test can be applied to check this condition. Information on the rank is also useful for the Granger non-causality test as explained by Chigira and Yamamoto (2003) and for the long-run Granger non-causality test proposed by Yamamoto and Kurozumi (2003). For the latter test, we also use information on the rank of the sub-matrix of $\beta_\perp$, which is an $n \times (n - r)$ full column rank matrix such that $\beta'\beta_\perp = 0$.

The rank of β_1 can be tested by investigating the eigenvalues of the quadratic form of some non-singular transformation of the estimator of β_1, but the test statistic must be constructed differently according to the specification of the deterministic term, because the limiting distribution of the estimator of β changes depending on the structure of μ_0 and μ_1 as shown by Johansen (1988, 1991, 1994). When μ_0 is specified as 0 or $\rho_0\alpha$ for a suitable matrix ρ_0, that is, x_t has no linear trend, the Kurozumi's test has an asymptotic chi-square distribution. On the other hand, the limiting distribution depends on a nuisance parameter when there are no restrictions on μ_0, that is, x_t has a linear trend. For the latter case, Kurozumi (2003) proposed two testing procedures, one of which is conservative and the other requires the pretest of the structure of the trend parameter. However, by Monte Carlo simulations, both of them are shown to be too conservative in some cases and perform poorly under the alternative hypothesis. Similarly, the test of the rank of the sub-matrix of $\beta_\perp$ is asymptotically conservative when no restrictions are imposed on μ_0.

In this paper we propose the alternative test statistics when x_t has a linear trend. The advantage of the new tests is that they have an asymptotic chi-square distribution, so that neither the conservative test nor the pretest is required. In addition, these tests do not depend on the true value of μ_0 and μ_1 (ρ_1), so that we can use the tests when x_t is stochastically cointegrated ($\mu_1 \neq 0$) as well as when x_t is deterministically cointegrated ($\mu_1 = 0$).

The paper is organized as follows. In Section 2 we briefly explain the problem found in Kurozumi (2003) that arises when testing the rank of β_1 and propose the alternative test statistics. The tests of the rank of the sub-matrix of $\beta_\perp$ are proposed in Section 3, and the finite sample property is investigated in Section 4. Section 5 gives concluding remarks.

The following notation is used throughout the paper. We use $vec(A)$ to stack the *rows* of a matrix A into a column vector, $[x]$ to denote the largest integer $\leq x$, $\bar{a} = a(a'a)^{-1}$ for a full column rank matrix a. $\xrightarrow{p}$ and $\xrightarrow{d}$ signify convergence in probability and convergence in distribution. $A > 0$ implies A is positive definite when $>$ is used for a matrix. We denote the rank of A by $\mathrm{rk}(A)$ and the column space of A by $\mathrm{sp}(A)$. We write integrals like $\int_0^1 X(s)dY(s)'$ simply as $\int XdY'$ to achieve notational economy, and all integrals are from 0 to 1 except where otherwise noted.

2. Alternative tests of $\mathrm{rk}(\beta_1)$

We first consider the model (1) when $\mu_1 = 0$,

$$\triangle x_t = \mu_0 + \alpha\beta' x_{t-1} + \sum_{j=1}^{m-1} \Gamma_j \triangle x_{t-j} + \varepsilon_t. \tag{2}$$

This model is also expressed as

$$x_t = C\sum_{i=1}^{t}\varepsilon_i + \kappa + \tau t + C_1(L)\varepsilon_t + x_0^*, \tag{3}$$

where $C = \beta_\perp(\alpha_\perp'\Gamma\beta_\perp)^{-1}\alpha_\perp'$, $\Gamma = I_n - \sum_{i=1}^{m-1}\Gamma_i$, $\tau = C\mu_0$, $|C_1(z)| = 0$ has roots outside the unit circle, and x_0^* is a stochastic component such that $\beta' x_0^* = 0$. See Johansen (1991, 1995) for details. We partition β into $[\beta_1', \beta_2']'$, where β_1 is an $n_1 \times r$ matrix with $0 < n_1 < n$. We also decompose $\beta_\perp$ and τ into $[\beta_{\perp,1}', \beta_{\perp,2}]'$ and $\tau = [\tau_1', \tau_2']'$ conformably with β. Note that $\beta_1'\beta_{\perp,1}$ does not necessarily equal to zero while $\beta'\beta_\perp = \beta_1'\beta_{\perp,1} + \beta_2'\beta_{\perp,2} = 0$. That is, in general, $\beta_{\perp,1}$ is *not* an orthogonal complement to β_1.

The model (2) can be estimated by the ML method and the ML estimators are denoted by $\hat\alpha$, $\hat\beta$, $\hat\Gamma_i$ and $\hat\Sigma$. We define the (infeasible) normalized estimator of the cointegrating matrix as $\tilde\beta = \hat\beta(\bar\beta'\hat\beta)^{-1}$. Similarly, $\tilde\beta_1$ is defined as the first n_1 rows of $\tilde\beta$.

Kurozumi (2003) investigated the test of $\mathrm{rk}(\beta_1)$ in the same way as Robin and Smith (2000). The testing problem is

$$H_0:\ \mathrm{rk}(\beta_1) = f \quad \text{v.s.} \quad H_1:\ \mathrm{rk}(\beta_1) > f, \tag{4}$$

for a given f where $0 \le f < \min(n_1, r)$.

To test the hypothesis (4), we consider the following determinant equation,

$$|\hat\beta_1\hat\Psi\hat\beta_1{}' - \hat\lambda\hat\Phi| = 0, \tag{5}$$

where $\hat\Psi$ and $\hat\Phi$ are $r \times r$ and $n_1 \times n_1$ matrices such that $\hat\Psi \xrightarrow{p} \Psi > 0$ and $\hat\Phi \xrightarrow{d} \Phi > 0$ almost surely (a.s.). The precise definition of them is given below. These matrices are defined differently depending on the specification of the deterministic term and are selected so that the determinant equation (5) is invariant to the normalizations of $\hat\alpha$ and $\hat\beta$ and that the limiting distribution of the test statistic described below does not depend on a nuisance parameter. Then, since $\tilde\beta_1$ is obtained by the non-singular transformation of $\hat\beta_1$, we can consider the determinant equation (5) with $\tilde\beta_1$ instead of $\hat\beta_1$.

Let $\hat\lambda_1 \ge \hat\lambda_2 \ge \cdots \ge \hat\lambda_{n_1}$ be the ordered eigenvalues of (5). Since $\hat\lambda_{f+1}, \cdots, \hat\lambda_{n_1}$ are shown to converge to zero in probability under the null hypothesis while the others are bounded from zero in probability under the alternative, the test statistic proposed is

$$\mathcal{L}_T = T^2 \sum_{i=f+1}^{n_1} \hat{\lambda}_i.$$

We can also consider $T^2\hat{\lambda}_{f+1}$ as a test statistic, but we will not investigate it because the performance of $T^2\hat{\lambda}_{f+1}$ is similar to that of $\mathcal{L}_T$.

The asymptotic behavior of the test statistic apparently depends on the limiting distribution of $\tilde{\beta}_1$. Using (13.1) of Johansen (1995), $\tilde{\beta}_1$ can be expressed as

$$\tilde{\beta}_1 = \beta_1 + \gamma_1(\gamma'\gamma)^{-1}U_{1T} + \frac{1}{T^{1/2}}\tau_1(\tau'\tau)^{-1}U_{2T}, \tag{6}$$

where the $n \times (n-r-1)$ matrix γ is an orthogonal complement to τ in $\mathrm{sp}(\beta_\perp)$, γ_1 is the first n_1 rows of γ, and $[TU'_{1T}, TU'_{2T}]$ converges in distribution to, say, $[U'_1, U'_2]$. See Section 3 of Kurozumi (2003). The problem we have here is that, if an $n_1 \times 1$ vector $\underline{\tau}$ exists such that $\underline{\tau}'\beta_1 = 0$ and $\underline{\tau}'\gamma_1 = 0$, the third term in (6) dominates in the limit when $\tilde{\beta}_1$ is premultiplied by $\underline{\tau}'$, while only the second term in (6) dominates asymptotically if $\underline{\tau}$ does not exist. From Proposition 1 of Kurozumi (2003) the vector $\underline{\tau}$ exists if and only if τ_2 is equal to zero, and if we know whether it is equal to zero or not, we can appropriately chose $\hat{\Phi}$ and $\hat{\Psi}$ so that the test statistic has an asymptotic chi-square distribution. However, we do not know in practice whether τ_2 is equal to zero or not and then we cannot choose appropriate $\hat{\Phi}$ and $\hat{\Psi}$ without information on τ_2. In other words, we can say that the limiting distribution of the test statistic depends on a nuisance parameter τ_2.

Kurozumi (2003) proposed two testing procedures to cope with this problem, one of which takes advantage of the asymptotic property of the smallest eigenvalue of (5), and the other method is to pretest whether τ_2 is zero or not. However, it was shown by Monte Carlo simulations that these two procedures suffer from a so-called pretest bias in some situations so that they are too conservative to reject the hypothesis under the alternative.

To circumvent this problem, we need to use the estimator of β_1 whose limiting distribution does not depend on the structure of the orthogonal space to β_1, unlike $\tilde{\beta}_1$ in (6). The first candidate proposed here is the estimator that is obtained when we estimate the model (2) assuming that x_t is stochastically cointegrated. More precisely, we estimate the following model:

$$\triangle x_t = \mu_0 + \alpha\beta^{*\prime}x_{t-1}^* + \sum_{j=1}^{m-1}\Gamma_j\triangle x_{t-j} + \varepsilon_t, \tag{7}$$

where $\beta^* = [\beta', \rho_1]'$ with the true value of ρ_1 equal to zero and $x_{t-1}^* = [x_{t-1}', t]'$. Then, the model is estimated by the reduced rank regression (RRR) of R_{0t}^* on R_{1t}^* where R_{0t}^* and R_{1t}^* are regression residuals of $\triangle x_t$ and x_{t-1}^* on a constant and $\triangle x_{t-1}, \cdots, \triangle x_{t-m+1}$. Let $\tilde{\beta}^*$ be an infeasible normalized estimator of β^* defined like $\tilde{\beta}$. Then, in the same way as Lemma 13.2 of Johansen (1995), we can show that

$$\tilde{\beta}^* = \beta^* + \bar{\beta}_\perp^* U_T^*, \tag{8}$$

where $\beta_\perp^* = \text{diag}\{\beta_\perp, 1\}$, $TU_T^* \xrightarrow{d} (\int G^*G^{*\prime}ds)^{-1}\int G^*dV'$, $G^*(r) = G(r) - \int Gds$ with $G(r) = [(\bar{\beta}_\perp' CW(r))', r]'$, $W(r)$ is an n-dimensional Wiener process with a variance matrix Σ, $V(r) = (\alpha'\Sigma^{-1}\alpha)^{-1}\alpha'\Sigma^{-1}W(r)$, and $G^*(r)$ and $V(r)$ are independent. Since the estimator of β_1 is the first n_1 rows of (8), it is apparent from (8) that

$$\tilde{\beta}_1^* = \beta_1 + \beta_{\perp,1}(\beta_\perp'\beta_\perp)^{-1}L'U_T^*,$$

where L is an $(n-r+1)\times(n-r)$ matrix defined by $L = [I_{n-r}, 0]'$. Since there is an $n_1 \times (n_1 - f)$ matrix δ under H_0 whose columns span the orthogonal space to $\text{sp}(\beta_1)$, we can see that

$$T\delta'(\tilde{\beta}_1^* - \beta_1) \xrightarrow{d} \delta'\beta_{\perp,1}(\beta_\perp'\beta_\perp)^{-1}L'\left(\int G^*G^{*\prime}ds\right)^{-1}\int G^*dV' = X', \text{ say.} \tag{9}$$

Then, unlike $\tilde{\beta}_1$, β_1^* is independent of the true value of τ_2.

To construct the test statistic, we use $\hat{\Psi} = \hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}$ and

$$\hat{\Phi} = \hat{\beta}_1^*(\hat{\beta}_n^{*\prime}\hat{\beta}_n^*)^{-1}\hat{\beta}_1^{*\prime} + \hat{\beta}_{\perp,1}^*(\hat{\beta}_{\perp n}^{*\prime}\hat{\beta}_{\perp n}^*)^{-1}L'\left(\Upsilon_T' S_{11}^*\Upsilon_T\right)^{-1}L(\hat{\beta}_{\perp n}^{*\prime}\hat{\beta}_{\perp n}^*)^{-1}\hat{\beta}_{\perp,1}^{*\prime}, \tag{10}$$

where $S_{11}^* = T^{-1}\sum_{i=1}^T R_{1t}^*R_{1t}^{*\prime}$, $\Upsilon_T = \text{diag}\{T^{-1/2}\bar{\hat{\beta}}_\perp^*, 1\}$ and $\hat{\beta}_n^*$ and $\hat{\beta}_{\perp n}^*$ are the first n rows of $\hat{\beta}^*$ and $\hat{\beta}_\perp^*$. The test statistic is denoted by $\mathcal{L}_T^*$.

The other estimator that has the similar property as $\tilde{\beta}_1^*$ is obtained when we estimate the model (2) augmented with t as Perron and Campbell (1993). In this case, the model is estimated by the RRR of R_{0t}^+ on R_{1t}^+ where R_{0t}^+ and R_{1t}^+ are regression residuals of $\triangle x_t$

and x_{t-1} on a constant, a linear trend and $\triangle x_{t-1}, \cdots, \triangle x_{t-m+1}$. Denoting a normalized estimator by $\tilde{\beta}^+$, we have

$$\tilde{\beta}^+ = \beta + \bar{\beta}_\perp U_T^+,$$

where $TU_T^+ \stackrel{d}{\longrightarrow} (\int G^+ G^{+\prime} ds)^{-1} \int G^+ dV'$ and $G^+(\cdot)$ is a projection residual in $L_2[0,1]$ of $\bar{\beta}' CW(r)$ on the space generated by 1 and r. Then, since the estimator of β_1 is the first n_1 rows of $\tilde{\beta}^+$, we have, under the null hypothesis,

$$T\delta'(\tilde{\beta}_1^+ - \beta_1) \stackrel{d}{\longrightarrow} \delta' \beta_{\perp,1} (\beta_\perp' \beta_\perp)^{-1} \left(\int G^+ G^{+\prime} ds \right)^{-1} \int G^+ dV'. \qquad (11)$$

Then, this estimator is also of order T^{-1} and independent of the true value of τ_2.

In this case, the test statistic $\mathcal{L}_T^+$ is constructed using $\hat{\Psi} = \hat{\alpha}' \hat{\Sigma}^{-1} \hat{\alpha}$ and

$$\hat{\Phi} = \hat{\beta}_1^+ (\hat{\beta}^{+\prime} \hat{\beta}^+)^{-1} \hat{\beta}_1^{+\prime} + \hat{\beta}_{\perp,1}^+ \left(\frac{1}{T} \hat{\beta}_\perp^{+\prime} S_{11}^+ \hat{\beta}_\perp^+ \right)^{-1} \hat{\beta}_{\perp,1}^{+\prime}, \qquad (12)$$

where $S_{11}^+ = T^{-1} \sum_{i=1}^T R_{1t}^+ R_{1t}^{+\prime}$.

Theorem 1 *Suppose $f < \min(n_1, r)$ and $\mu_1 = \alpha\rho_1$. Under the null hypothesis, $\mathcal{L}_T^*$, $\mathcal{L}_T^+ \stackrel{d}{\longrightarrow} \chi^2_{(n_1-f)(r-f)}$.*

Remark 1: We can easily see that both test statistics are invariant to the true values of μ_0 and $\mu_1 = \alpha\rho_1$. Then, they can be applied to the stochastic cointegration model ($\rho_1 \neq 0$) as well as the deterministic cointegration model ($\rho_1 = 0$).

Remark 2: If we do not impose any restrictions on μ_1 and the true process has a quadratic trend, the test statistics do not converge to a chi-square distribution. In this case, we will encounter the similar problem as explained in the earlier part of this section and Section 3 of Kurozumi (2003). We will be able to deal with this problem by the RRR of $\triangle x_t$ on $[x_{t-1}, t^2]$ corrected for $[1, t]'$ or by the RRR of $\triangle x_t$ on x_{t-1} corrected for $[1, t, t^2]$. We do not pursue these RRRs in details because our interest is in the case where the process has a linear trend.

$\mathcal{L}_T^*$ and $\mathcal{L}_T^+$ have an advantage over the test proposed in Kurozumi (2003) in that they do not have to rely on information on the existence of $\underline{\tau}$ and they have an exact asymptotic chi-square distribution. On the other hand, they might be inferior in view of power because we inserted an additional regressor to estimate the model. This will be investigated by simulations in the later section.

3. Alternative tests of $\mathrm{rk}(\beta_{\perp,1})$

In this section, we consider the test of the rank of $\beta_{\perp,1}$ when the process is trending. The test statistic is constructed exactly in the same way as the test of $\mathrm{rk}(\beta_1)$. We consider the null hypothesis of $\mathrm{rk}(\beta_{\perp,1}) = g$ against the alternative of $\mathrm{rk}(\beta_{\perp,1}) > g$ where $g < \min(n_1, n-r)$ and the determinant equation

$$|\hat{\beta}_{\perp,1}\hat{\Psi}\hat{\beta}_{\perp,1}' - \hat{\nu}\hat{\Phi}| = 0, \tag{13}$$

where, in this case, $\hat{\Psi}$ and $\hat{\Phi}$ are $(n-r)\times(n-r)$ and $n_1 \times n_1$ matrices such that $\hat{\Psi} \stackrel{d}{\longrightarrow} \Psi > 0$ (a.s.) and $\hat{\Phi} \stackrel{p}{\longrightarrow} \Phi > 0$. The test statistic is

$$\mathcal{L}_{\perp T} = \sum_{i=g+1}^{n_1} \hat{\nu}_i$$

where $\hat{\nu}_i$ for $i = 1, \cdots, n_1$ are the ordered eigenvalues of (13). Using the estimator of the RRR in (2) and choosing $\hat{\Psi}$ and $\hat{\Phi}$ appropriately, Kurozumi (2003) showed that the test statistic converges in distribution to a random variable that is bounded above by a chi-square distribution. Although this test is applicable without investigating τ_2, it sometimes becomes too conservative in finite samples so that, as shown in the next section, it can hardly reject the hypothesis under H_1.

The reason why $\mathcal{L}_{\perp T}$ becomes conservative is that the convergence rate of $\tilde{\beta}$ is different depending on the premultiplying matrices. To see this, let us define $\tilde{\beta}_{\perp} = \beta_{\perp} - \beta(\tilde{\beta}'\beta)^{-1}\tilde{\beta}'\beta_{\perp}$ and

$$\tilde{\beta}_{\perp,1} = \beta_{\perp,1} - \beta_1(\tilde{\beta}'\beta)^{-1}\tilde{\beta}'\beta_{\perp}. \tag{14}$$

We also define an $n_1 \times (n_1 - g)$ full column rank matrix η such that η is orthogonal to $\mathrm{sp}(\beta_{\perp,1})$. Noting that $T\gamma'\tilde{\beta}$ converges in distribution while $\tau'\tilde{\beta}$ is of order $T^{-3/2}$ as proved by Johansen (1991, 1995), we have

$$\begin{aligned} T\eta'(\tilde{\beta}_{\perp,1} - \beta_{\perp,1}) &= -\eta'\beta_1(\tilde{\beta}'\beta)^{-1}[(\tilde{\beta}-\beta)'\gamma T, (\tilde{\beta}-\beta)'\tau T] \\ &= -\eta'\beta_1(\beta'\beta)^{-1}[O_p(1), 0] + o_p(1). \end{aligned}$$

Since the last column of $T\eta'(\tilde{\beta}_{\perp,1} - \beta_{\perp,1})$ converges to zero in probability, we can easily deduce that the limiting distribution has a degenerate conditional variance matrix, which is the reason why the test proposed is conservative.

We can circumvent this degeneracy by using the estimator considered in the previous section. If we estimate the model (7) by the RRR, the limiting distribution of $\hat{\beta}^*_{\perp,1}$ becomes

$$T\eta'(\tilde{\beta}^*_{\perp,1} - \beta_{\perp,1}) \xrightarrow{d} -\eta'\beta_1(\beta'\beta)^{-1}\int dVG^{*\prime}\left(\int G^*G^{*\prime}\right)^{-1}L, \tag{15}$$

which is obtained by adding the superscript * to each matrix in the equation (14) and using the expression (8). In this case, the test statistic $\mathcal{L}^*_{\perp T}$ is constructed using $\hat{\Psi} = \{L'(\Upsilon_T' S^*_{11}\Upsilon_T)^{-1}L\}^{-1}$ and

$$\hat{\Phi} = \hat{\beta}^*_{\perp,1}(\hat{\beta}^{*\prime}_{\perp n}\hat{\beta}^*_{\perp n})^{-1}\hat{\beta}^{*\prime}_{\perp,1} + \hat{\beta}^*_1(\hat{\beta}^{*\prime}_n\hat{\beta}^*_n)^{-1}(\hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha})^{-1}(\hat{\beta}^{*\prime}_n\hat{\beta}^*_n)^{-1}\hat{\beta}^{*\prime}_1. \tag{16}$$

If we use the estimator obtained by the Perron and Campbell's method, we have the same asymptotic result as (15) with L and $G^*(\cdot)$ replaced by I_{n-r} and $G^+(\cdot)$. In this case, the test statistic, which is denoted by $\mathcal{L}^+_{\perp T}$, is constructed using $\hat{\Psi} = T^{-1}\bar{\hat{\beta}}^{+\prime}_\perp S^+_{11}\bar{\hat{\beta}}^+_\perp$ and the same $\hat{\Phi}$ as (16) with the superscript * replaced by $^+$.

Theorem 2 *Suppose $g < \min(n_1, n-r)$ and $\mu_1 = \alpha\rho_1$. Under the null hypothesis, $\mathcal{L}^*_{\perp T}$, $\mathcal{L}^+_{\perp T}$ $\xrightarrow{d} \chi^2_{(n_1-g)(n-r-g)}$.*

The advantage of the above two tests is that they have an asymptotic chi-square distribution so that, at least asymptotically, they are size-controllable tests. We also note that the test statistics are invariant to the true value of μ_0 and μ_1 as explained in Remark 1 and then we can also apply these tests to the stochastic cointegration model as in the previous section.

4. Finite sample simulations

In this section, we investigate the finite sample property of the tests proposed in the previous sections and compare their performance with that of the testing procedures proposed by Kurozumi (2003). The data generating process is a four-dimensional VECM of order one as follows.

$$\triangle x_t = \mu_0 + \alpha\beta' x_{t-1} + \varepsilon_t,$$

where $\{\varepsilon_t\} \sim i.i.d.(0, I_4)$. The settings of parameters are the same as Kurozumi (2003) except for μ_0. Let

$$a_1 = \begin{bmatrix} 0.3 \\ -0.3 \\ -0.8 \\ 0.8 \end{bmatrix}, \ a_2 = \begin{bmatrix} -0.5 \\ 0 \\ -0.3 \\ -0.5 \end{bmatrix}, \ a_3 = \begin{bmatrix} 0.8 \\ 1 \\ -0.5 \\ -0.5 \end{bmatrix}, \ a_4 = \begin{bmatrix} -0.5 \\ -0.8 \\ 0 \\ -0.5 \end{bmatrix},$$

$$b_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -0.5 \end{bmatrix}, \ b_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \ b_3 = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 1 \end{bmatrix}, \ b_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ d_1 = \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0.5 \end{bmatrix}, \ d_2 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}.$$

We consider the following settings of parameters.

parameter sets for the test of $rk(b_1)$				parameter sets for the test of $rk(b_{\perp,1})$			
	α	β	$b_\perp$		α	β	$b_\perp$
DGP1	a_1	b_1	$[b_2, b_3, b_4]$	DGP1o	a_2	b_2	$[b_1, b_3, b_4]$
DGP2	$[a_1, a_2]$	$[b_1, b_2]$	$[b_3, b_4]$	DGP2o	$[a_1, a_2]$	$[b_1, b_2]$	$[b_3, b_4]$
DGP3	$[a_1, a_2, a_3]$	$[b_1, b_2, b_3]$	b_4	DGP3o	$[a_1, a_2, a_4]$	$[b_1, b_2, b_4]$	b_3

We set the $(2,1)$ element of β as c_1, which takes values of 0, 0.01, 0.025, 0.05, 0.075 and 0.1, and consider the test of the rank of the first two rows of β. The case where $c_1 = 0$ corresponds to the null hypothesis under which the rank of β_1 is 0, 1 and 1 for DGP1, 2 and 3, while it is 1, 2 and 2 when $c_1 \neq 0$, which corresponds to the alternative. μ_0 is set to be d_1 or d_2, which corresponds to the case where $\tau_2 \neq [0,0]'$ or $= [0,0]$ with τ_2 being the last two rows of τ.

Similarly, we set the $(2,1)$ element of $\beta_\perp$ as c_2 and consider the test of the rank of the first two rows of $\beta_\perp$. In this case, $c_2 = 0$ implies that the rank of $\beta_{\perp,1}$ is 1, 1 and 0 for DGPo1, o2 and o3, respectively, while it is 2, 2, 1 under the alternatives of $c_2 \neq 0$.

We set $x_0 = 0$ and discard the first 100 observations in all experiments. The number of replication is 5,000, and the level of significance is set equal to 0.05.

For the purpose of comparison of the tests, we also investigate the two testing procedures "TEST1" and "TEST2" proposed by Kurozumi (2003). To conduct the former procedure, we first estimate the model (2) by the RRR and then construct the test statistic $\mathcal{L}_T$ by letting $\hat{\Psi} = \hat{\alpha}'\hat{\Sigma}^{-1}\hat{\alpha}$ and $\hat{\Phi} = \hat{\beta}_1(\hat{\beta}'\hat{\beta})^{-1}\hat{\beta}_1' + \hat{\gamma}_1(\hat{\gamma}'\hat{\gamma})^{-1}(T\hat{\gamma}'S_{11}^{-1}\hat{\gamma})(\hat{\gamma}'\hat{\gamma})^{-1}\hat{\gamma}_1' + 12\hat{\tau}_1(\hat{\tau}'\hat{\tau})^{-2}\hat{\tau}_1'$. The null hypothesis is rejected when $\mathcal{L}_T > c_{(n_1-f)(r-f)}$ while it is accepted when $\mathcal{L}_T < c_{(n_1-f-1)(r-f)}$, where c_k is a critical value of χ^2_k. For the case where $c_{(n_1-f-1)(r-f)} \leq \mathcal{L}_T \leq c_{(n_1-f)(r-f)}$, we calculate T^2 times the smallest non-zero eigenvalue of (5) and the null hypothesis is rejected when it is less than a critical value tabulated in Table 1 of Kurozumi (2003).

On the other hand, TEST2 requires the pretest before the construction of $\mathcal{L}_T$. We fist investigate whether or not $\tau_2 = 0$ using the t-statistic for each element of τ_2. If τ_2 is judged not to be different from zero, we construct $\mathcal{L}_T$ using the same $\hat{\Psi}$ as TEST1 and $\hat{\Phi} = \hat{\beta}_1(\hat{\beta}'\hat{\beta})^{-1}\hat{\beta}_1' + \hat{\gamma}_1(\hat{\gamma}'\hat{\gamma})^{-1}(T\hat{\gamma}'S_{11}^{-1}\hat{\gamma})(\hat{\gamma}'\hat{\gamma})^{-1}\hat{\gamma}_1'$, and compare it with $c_{(n_1-f-1)(r-f)}$. Otherwise, we construct the same statistic $\mathcal{L}_T$ as TEST1 and compare it with $c_{(n_1-f)(r-f)}$.

Table 1 reports the empirical sizes and powers of the tests of $\text{rk}(\beta_1)$. $\mathcal{S}_{1T}(d_1)$ and $\mathcal{S}_{2T}(d_1)$ denote the testing procedures TEST1 and TEST2 when $\mu_0 = d_1$ while $\mathcal{S}_{1T}(d_2)$ and $\mathcal{S}_{2T}(d_2)$ correspond to the case where $\mu_0 = d_2$. $\mathcal{L}_T^*$ and $\mathcal{L}_T^+$ have only one column because they are invariant to μ_0. When the cointegrating rank r is 1, all the tests except for $\mathcal{S}_{1T}(d_2)$ tend to overly reject the null hypothesis ($c_1 = 0$), while $\mathcal{S}_{1T}(d_2)$ is conservative. On the other hand, under the alternative hypothesis ($c_1 > 0$), $\mathcal{S}_{1T}$ and $\mathcal{S}_{2T}$ seem to be slightly more powerful than $\mathcal{L}_T^*$ and $\mathcal{L}_T^+$.

When the cointegrating rank is 2 and μ_0 is d_1, which corresponds to the case where $\tau_2 \neq 0$, $\mathcal{S}_{1T}$ and $\mathcal{S}_{2T}$ have a reasonable size and power, while in the case of $\mu_0 = d_2$, both testing procedures are too conservative and have very low power. On the other hand, both $\mathcal{L}_T^*$ and $\mathcal{L}_T^+$ perform fairly well both under the null and the alternative, although they are slightly less powerful than $\mathcal{S}_{1T}(d_1)$ and $\mathcal{S}_{2T}(d_1)$.

When $r = 3$, we do not have to rely on either $\mathcal{S}_{1T}$ or $\mathcal{S}_{2T}$ but we can use $T \times \mathcal{L}_T$ as the test statistic that is asymptotically chi-square distributed as explained in Kurozumi (2003). All the tests perform well except that $\mathcal{L}_T^*$ and $\mathcal{L}_T^+$ have slightly large size distortions when $T = 100$.

The finite sample performance of the tests of $\text{rk}(\beta_{\perp,1})$ is summarized in Table 2. When $r = 1$ and 2, although $\mathcal{L}_{\perp T}$ performs well for $\mu_0 = d_2$, it has very low power when $\mu_0 = d_1$. On the other hand, $\mathcal{L}_{\perp T}^*$ and $\mathcal{L}_{\perp T}^+$ are more powerful than $\mathcal{L}_{\perp T}$ but also tend to overly reject the null hypothesis. When $r = 3$, we use $T \times \mathcal{L}_{\perp T}$ as the test statistic as explained in Kurozumi (2003), which is not conservative but has an exact asymptotic chi-square distribution. From the table, we can see that $\mathcal{L}_{\perp T}$ performs better than $\mathcal{L}_{\perp T}^*$ and $\mathcal{L}_{\perp T}^+$.

The other interesting feature we can see from Tables 1 and 2 is that the finite sample performance of the tests based on the RRR of (7) are almost the same as that base on the Perron and Campbell's estimation. We also note that they are invariant to μ_0 and ρ_1, so that entries corresponding to these statistics in the tables can be seen as the finite sample performance of them when x_t is stochastically cointegrated, in which case they are the only method to test $\mathrm{rk}(\beta_1)$ and $\mathrm{rk}(\beta_{\perp,1})$.

5. Concluding remarks

In this paper we proposed the test statistics for $\mathrm{rk}(\beta_1)$ and $\mathrm{rk}(\beta_{\perp,1})$ when the process has a linear trend. The advantage of these statistics is that they can be applied to both the deterministic and stochastic cointegration models, and they are asymptotically chi-square distributed, so that we do not have to rely on a pretest or a conservative test.

From Monte Carlo simulations, it is found that none of the test statistics dominates others when $r < n - 1$ and then they should be used to complement each other in practice. For the test of the rank of β_1, if we have a strong confidence from some a priori information on whether τ_2 is equal to zero or not, we recommend using the testing procedures proposed by Kurozumi (2003); otherwise the two test statistics proposed in this paper perform better. On the other hand, for the test of the rank of $\beta_{\perp,1}$, we should carefully use $\mathcal{L}_T$ when we believe $\tau_2 \neq 0$, while the new tests have relatively steady performance in any specification of the deterministic term. Finally, it should be mentioned that the tests proposed in this paper are applicable both to the deterministic and stochastic cointegration models.

Appendix

Proof of Theorem 1: First we consider the limiting distribution of $\mathcal{L}_T^*$. Since the columns of δ span the orthogonal space to $\mathrm{sp}(\beta_1)$ under the null hypothesis, an $n_1 \times f$ matrix $\delta_\perp$ spans the same column space as $\mathrm{sp}(\beta_1)$. Letting $H = [\delta_\perp, \delta]$, we can see that the determinant equation (5) is equivalent to

$$|H'||\tilde{\beta}_1^* \tilde{\Psi} \tilde{\beta}_1^{*\prime} - \hat{\lambda}\tilde{\Phi}||H| = 0, \tag{17}$$

where $\tilde{\Psi}$ and $\tilde{\Phi}$ are defined in the same way as $\hat{\Psi}$ and $\hat{\Phi}$ using the (infeasible) normalized estimator. The equation (17) holds because H is non-singular and the determinant equation is invariant to the normalization of the estimators of β and $\beta_\perp$. Since $\tilde{\beta}^*$, $\tilde{\alpha}$ and $\hat{\Sigma}$ are consistent, we have, using (9),

$$\begin{aligned} H'\tilde{\beta}_1^*\tilde{\Psi}\tilde{\beta}_1^{*\prime}H &= \begin{bmatrix} \delta_\perp'\tilde{\beta}_1^*\tilde{\Psi}\tilde{\beta}_1^{*\prime}\delta_\perp & \delta_\perp\tilde{\beta}_1^*\tilde{\Psi}(\tilde{\beta}_1^{*\prime}\delta T) \\ (T\delta'\tilde{\beta}_1^*)\tilde{\Psi}\tilde{\beta}_1^{*\prime}\delta_\perp & (T\delta'\tilde{\beta}_1^*)\tilde{\Psi}(\tilde{\beta}_1^{*\prime}\delta T) \end{bmatrix} \\ &\xrightarrow{d} \begin{bmatrix} \delta_\perp'\beta_1\Psi\beta_1'\delta_\perp & \delta_\perp'\beta_1\Psi X \\ X'\Psi\beta_1'\delta_\perp & X'\Psi X \end{bmatrix}, \end{aligned}$$

Similarly, since $\tilde{\Upsilon}_T' S_{11}^* \tilde{\Upsilon}_T$ converges in distribution to $\int G^* G^{*\prime} ds$, $\hat{\lambda} H'\tilde{\Phi} H$ becomes

$$T^2\hat{\lambda}\begin{bmatrix} 0 & 0 \\ 0 & \delta'\beta_{\perp,1}(\beta_\perp'\beta_\perp)^{-1}L'(\int G^*G^{*\prime}ds)^{-1}L(\beta_\perp'\beta_\perp)^{-1}\beta_{\perp,1}'\delta \end{bmatrix} + o_p(T^2).$$

Then, the determinant equation (17) is asymptotically equal to

$$\begin{aligned} &\left|\begin{bmatrix} \delta_\perp'\beta_1\Psi\beta_1'\delta_\perp & \delta_\perp'\beta_1\Psi X \\ X'\Psi\beta_1'\delta_\perp & X'\Psi X \end{bmatrix} - T^2\hat{\lambda}\begin{bmatrix} 0 & 0 \\ 0 & \delta'\beta_{\perp,1}(\beta_\perp'\beta_\perp)^{-1}L'(\int G^*G^{*\prime}ds)^{-1}L(\beta_\perp'\beta_\perp)^{-1}\beta_{\perp,1}'\delta \end{bmatrix}\right| \\ &= \left|\delta_\perp'\beta_1\Psi\beta_1'\delta_\perp\right| \times \left| X'\left\{\Psi - \Psi\beta_1'\delta_\perp(\delta_\perp'\beta_1\Psi\beta_1'\delta_\perp)^{-1}\delta_\perp'\beta_1\Psi\right\}X \right. \\ &\quad \left. - T^2\hat{\lambda}\begin{bmatrix} 0 & 0 \\ 0 & \delta'\beta_{\perp,1}(\beta_\perp'\beta_\perp)^{-1}L'(\int G^*G^{*\prime}ds)^{-1}L(\beta_\perp'\beta_\perp)^{-1}\beta_{\perp,1}'\delta \end{bmatrix}\right| = 0. \end{aligned} \tag{18}$$

Therefore, the eigenvalues $\hat{\lambda}_{f+1}, \cdots, \hat{\lambda}_p$ converge in probability to zeros and are of order T^{-2}.

Here, notice that $\delta_\perp'\beta_1$ is of full row rank $n_1 - f$ because $\mathrm{sp}(\delta_\perp) = \mathrm{sp}(\beta_1)$ and $\delta_\perp$ is of full column rank. Then, in the same way as Johansen (1988, p.246), we can find a $r \times (r - f)$ matrix J with rank $(r - f)$ such that

$$JJ' = \Psi - \Psi\beta_1'\delta_\perp(\delta_\perp'\beta_1\Psi\beta_1'\delta_\perp)^{-1}\delta_\perp'\beta_1\Psi, \tag{19}$$

with $J'(\beta_1'\delta_\perp) = 0$ and $J'\Psi^{-1}J = I_{r-f}$, implying that $J'(\alpha'\Sigma^{-1}\alpha)^{-1}J = I_{r-f}$ because $\Psi = \alpha'\Sigma^{-1}\alpha$. Then, (18) becomes

$$\left| X'JJ'X - T^2\hat{\lambda}\delta'\beta_{\perp,1}(\beta_\perp'\beta_\perp)^{-1}L'\left(\int G^*G^{*\prime}ds\right)^{-1}L(\beta_\perp'\beta_\perp)^{-1}\beta_{\perp,1}'\delta\right| = 0. \tag{20}$$

The variance matrix of $X'J$ conditioned on $G^*(\cdot)$ is given by

$$\delta'\beta_{\perp,1}(\beta_\perp'\beta_\perp)^{-1}L'\left(\int G^*G^{*\prime}ds\right)^{-1}L(\beta_\perp'\beta_\perp)^{-1}\beta_{\perp,1}'\delta \otimes I_{r-f}. \tag{21}$$

Since $\delta'\beta_{\perp,1}$ is of full row rank, we can see that the conditional variance is of full rank (a.s.). Then, by multiplying the square root of the left-hand side of (21) from both sides of (20), the determinant equation asymptotically becomes

$$|X_0'X_0 - \lambda I_{n_1-f}| = 0,$$

where $vec(X_0) \sim N(0, I_{(n_1-f)(r-f)})$. Then, $\mathcal{L}_T^*$ converges in distribution to $\chi^2_{(n_1-f)(r-f)}$.

Exactly in the same way, we can show the convergence of $\mathcal{L}_T^+$ using (11).□

Proof of Theorem 2: This is proved exactly in the same way as Theorem 1 by letting $H = [\eta_\perp, \eta]$ where $\eta_\perp$ is an $n_1 \times (n-r-g)$ full column rank matrix such that $\eta_\perp'\eta$.□

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Table 1. Rejection frequencies of the tests of $rk(\beta_1)$

$r=1$	c_1	$\mathcal{S}_{1T}(d_1)$	$\mathcal{S}_{2T}(d_1)$	$\mathcal{S}_{1T}(d_2)$	$\mathcal{S}_{2T}(d_2)$	$\mathcal{L}_T^*$	$\mathcal{L}_T^+$
$T=100$	0	0.103	0.111	0.040	0.100	0.126	0.125
	0.01	0.116	0.124	0.041	0.113	0.136	0.136
	0.025	0.167	0.181	0.066	0.179	0.186	0.184
	0.05	0.349	0.368	0.185	0.383	0.343	0.342
	0.075	0.562	0.581	0.413	0.675	0.534	0.533
	0.1	0.748	0.759	0.711	0.902	0.712	0.712
$T=200$	0	0.073	0.077	0.028	0.081	0.086	0.086
	0.01	0.113	0.120	0.042	0.128	0.124	0.124
	0.025	0.317	0.323	0.142	0.347	0.290	0.290
	0.05	0.739	0.744	0.668	0.898	0.699	0.698
	0.075	0.944	0.945	0.984	0.999	0.919	0.919
	0.1	0.988	0.989	1.000	1.000	0.982	0.982
$r=2$	c_1	$\mathcal{S}_{1T}(d_1)$	$\mathcal{S}_{2T}(d_1)$	$\mathcal{S}_{1T}(d_2)$	$\mathcal{S}_{2T}(d_2)$	$\mathcal{L}_T^*$	$\mathcal{L}_T^+$
$T=100$	0	0.060	0.086	0.000	0.009	0.098	0.098
	0.01	0.070	0.099	0.001	0.015	0.107	0.108
	0.025	0.115	0.164	0.001	0.038	0.145	0.145
	0.05	0.257	0.356	0.006	0.078	0.250	0.251
	0.075	0.466	0.559	0.064	0.108	0.394	0.393
	0.1	0.652	0.714	0.324	0.133	0.561	0.562
$T=200$	0	0.046	0.071	0.000	0.007	0.075	0.075
	0.01	0.080	0.128	0.000	0.033	0.094	0.094
	0.025	0.289	0.404	0.001	0.075	0.227	0.226
	0.05	0.730	0.799	0.190	0.099	0.550	0.549
	0.075	0.936	0.952	0.932	0.125	0.807	0.806
	0.1	0.985	0.985	0.999	0.157	0.931	0.931
$r=3$	c_1	$\mathcal{S}_{1T}(d_1)$	$\mathcal{S}_{2T}(d_1)$	$\mathcal{S}_{1T}(d_2)$	$\mathcal{S}_{2T}(d_2)$	$\mathcal{L}_T^*$	$\mathcal{L}_T^+$
$T=100$	0	0.096	-	0.056	-	0.110	0.110
	0.01	0.925	-	0.420	-	0.318	0.314
	0.025	0.986	-	0.724	-	0.794	0.792
	0.05	0.994	-	0.862	-	0.979	0.979
	0.075	0.996	-	0.901	-	0.998	0.998
	0.1	0.997	-	0.924	-	1.000	1.000
$T=200$	0	0.077	-	0.047	-	0.066	0.066
	0.01	0.999	-	0.758	-	0.729	0.727
	0.025	1.000	-	0.909	-	0.994	0.994
	0.05	1.000	-	0.952	-	1.000	1.000
	0.075	1.000	-	0.967	-	1.000	1.000
	0.1	1.000	-	0.974	-	1.000	1.000

Table 2. Rejection frequencies of the tests of $rk(\beta_{\perp,1})$

$r=1$	c_2	$\mathcal{L}_{\perp T}(d_1)$	$\mathcal{L}_{\perp T}(d_2)$	$\mathcal{L}^*_{\perp T}$	$\mathcal{L}^+_{\perp T}$
	0	0.044	0.088	0.142	0.143
	0.01	0.044	0.096	0.151	0.149
$T=100$	0.025	0.050	0.136	0.178	0.178
	0.05	0.069	0.283	0.286	0.286
	0.075	0.108	0.492	0.430	0.431
	0.1	0.187	0.666	0.587	0.584
	0	0.029	0.066	0.091	0.091
	0.01	0.031	0.096	0.111	0.110
$T=200$	0.025	0.043	0.285	0.233	0.232
	0.05	0.119	0.712	0.573	0.570
	0.075	0.420	0.923	0.830	0.828
	0.1	0.830	0.983	0.940	0.938
$r=2$	c_2	$\mathcal{L}_{\perp T}(d_1)$	$\mathcal{L}_{\perp T}(d_2)$	$\mathcal{L}^*_{\perp T}$	$\mathcal{L}^+_{\perp T}$
	0	0.000	0.062	0.110	0.109
	0.01	0.000	0.067	0.112	0.112
$T=100$	0.025	0.001	0.118	0.152	0.151
	0.05	0.002	0.292	0.276	0.271
	0.075	0.008	0.512	0.445	0.443
	0.1	0.054	0.703	0.612	0.609
	0	0.000	0.049	0.077	0.078
	0.01	0.000	0.090	0.101	0.102
$T=200$	0.025	0.000	0.307	0.248	0.245
	0.05	0.008	0.753	0.610	0.608
	0.075	0.304	0.941	0.858	0.858
	0.1	0.805	0.986	0.954	0.953
$r=3$	c_2	$\mathcal{L}_{\perp T}(d_1)$	$\mathcal{L}_{\perp T}(d_2)$	$\mathcal{L}^*_{\perp T}$	$\mathcal{L}^+_{\perp T}$
	0	0.077	0.074	0.111	0.111
	0.01	0.299	0.201	0.120	0.119
$T=100$	0.025	0.866	0.682	0.172	0.172
	0.05	0.994	0.952	0.343	0.342
	0.075	0.999	0.986	0.553	0.552
	0.1	0.999	0.992	0.741	0.739
	0	0.065	0.066	0.074	0.074
	0.01	0.942	0.779	0.111	0.111
$T=200$	0.025	1.000	0.999	0.288	0.287
	0.05	1.000	1.000	0.713	0.711
	0.075	1.000	1.000	0.923	0.922
	0.1	1.000	1.000	0.986	0.985