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On blocking coalitions: Linking Mas-Colell with Grodal-Schmeidler-Vind

by

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This paper was dedicated to the memory of Birgit Grodal, to whom the authors owe an unlimited debt of gratitude. Her wisdom, guidance and friendship is already so sorely missed. Sadly, during the writing of the paper we lost Karl Vind, whose quiet and towering presence can never be replaced.

Abstract

In this paper we investigate the question of how many coalitions of a given relative size would block a non-Warlasian allocation in large finite economies. It is shown that in finite economies, if a Pareto optimal allocation is bounded away from being Walrasian, then, for any two numbers α, β between 0 and 1, the proportion of blocking coalitions in the set of all coalitions with relative size between α and β, is arbitrarily close to $\frac{1}{2}$, as the number of individuals in the economy becomes large.

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1 Introduction

It is well-known that in a finite exchange economy the set of core allocations contains the set of Walrasian (or competitive) allocations, and as the population of an economy gets large core allocations become arbitrarily close to Walras allocations in an appropriate sense (Debreu and Scarf [5], and Anderson [1]). Thus, for every allocation f, which is not in a neighborhood of Walrasian allocations, there exists a (blocking) coalition S that, by means of exchange, can improve upon the utility of every member of S generated by allocation f. However, various communication, information, transportation, legal and institutional constraints may put restrictions on coalition formation. As Grodal [9], p.581, points out: "In reality the lack of communication restricts the set of coalitions that can be formed." In light of this observation, an important question is, in addition to mere existence of a coalition that blocks a given non-Walrasian allocation, is the number and composition of the set of blocking coalitions.

An issue of a number of blocking coalitions was originally raised by Champsaur and Laroque [4]. A more general treatment of the problem under smoothness assumptions has been provided by Mas-Colell [14], who showed that in large finite pure exchange economies, nearly half of all coalitions block a Pareto optimal allocation which is "bounded away from being Walrasian".[1] Since Pareto optimality of a given allocation rules out the possibility of being blocked by a coalition as well as its complement, the 50% of all coalitions represents an upper bound on the number of blocking coalitions. Thus, the result of Mas-Colell implies that if the number of individuals in an economy rises, the proportion of blocking coalitions within the entire set of coalitions would approach this upper bound.

The important contribution by Mas-Colell [14], however, left open the question of the size of blocking coalitions. The additional barriers to formation of coalitions of various sizes, mentioned above, bring into the focus the question of a number of blocking coalitions of different sizes. One can immediately notice that, due to the property of the binomial distribution, most coalitions consist of roughly half the individuals in the large finite economy, i.e., the proportion of individuals to the entire population in most coalitions is arbitrarily close to one half. In other words, in large economies, under the uniform distribution on the space of all coalitions, it is with probability arbitrarily close to one that a randomly chosen coalition is of relative size close to one half. Therefore, loosely speaking, the size of most coalitions is about one half of the population, and Mas-Colell's result actually states that approximately half of these coalitions block a Pareto optimal allocation which is bounded away from being Walrasian.

The focus of this paper is the study of the number of coalitions that contain a given fraction of the agents and block a given Pareto optimal non-Warlasian allocation. It is natural to address this question in the context of the limit variant of large finite economies, namely those with an atomless measure space of economic agents, introduced by Aumann [2]. In this framework the core coincides with the set of Walrasian allocations (Aumann [2], Vind [19], and Hildenbrand [10], [11]), and thus, for every non-Walrasian allocation there is a blocking

[1] See precise definition below.

coalition. This however still left open the question of the measure (or, the proportional size to the population) of blocking coalitions. This issue has been examined by Grodal [9], Schmeidler [18] and Vind [20]. These papers show that in an economy with an atomless measure space of agents, for any positive ε and any allocation which is not a Walrasian allocation, there exists a blocking coalition whose proportional size to the population is exactly ε. Thus, "in particular, the formation of the coalition of all agents or any 'large' coalition is not needed to ensure the Pareto optimality of the final allocation" [18]. The main goal of this paper is to provide the link between the results obtained within the framework of large finite and atomless economies concerning the existence and the number of blocking coalitions. We show that for a Pareto optimal allocation which is bounded away from being Walrasian, any two numbers α, β representing the proportions to the entire population size of an economy, the number of blocking coalitions with relative size between α and β is arbitrarily close to one half of the total number of coalitions of that relative size.

Our result can be viewed as a generalization of the finite analogue of the Grodal, Schmeidler and Vind contributions, who studied the existence of a blocking coalition whose size is within a given range. We strengthen their results by showing that almost half of the coalitions of a given proportional size will block. More specifically, for small β's and $\alpha = 0$ our result yields a generalization of the finite analogue of Schmeidler's, while, for α close to β, we obtain a considerable strengthening of the finite version of Vind's.

Related literaure

The result of Mas-Colell [14] can be restated in an alternative manner: if one puts the uniform distribution on the space of two coalition partitions of the set of all individuals and chooses such a partition at random, then with probability one it contains a blocking coalition. This conclusion has been generalized by Greenberg and Weber [8] who considered partitions of all individuals into several coalitions. Under the uniform distribution on the set of partitions that contain a given number $J \geq 2$ of coalitions, Greenberg and Weber [8] show that the probability of such a partition to contain a coalition that blocks a given non-Walrasian allocation, is arbitrarily close to one. Another generalization of Mas-Colell's result has been pursued by Yamazaki [21] who examined an extention of the notion of the "distance" of allocations from being Walrasian. By introducing the concept of "Walras degrees" Yamazaki [21] examined a wider class of non-Walrasian allocations and showed that the basic result of Mas-Colell holds within this class as well. Greenberg and Weber [8] and Kirman, Oddou and Weber [13] studied the number of blocking coalitions of a given absolute size and showed that for a given size m, almost a half of all coalitions with m individuals block a given non-Warlasian allocation when the number of agents in the economy increases. Shitovitz [17] initiated the study of the number or the measure of blocking coalitions in atomless exchange economies. To overcome the difficulty of counting the number or the proportion of blocking coalitions in a combinatorial way in economies with an infinity of agents, he analyzed economies with a finite number of types and identified a coalition with its type profile. By considering profiles that represent coalitions with the same proportion of types as in the whole economy, Shitovitz [17] proved a "local" result, that for every equal treatment Pareto optimal allocation which is not Walrasian, there is a ball in the type profile space around the given type profile so that nearly half of the profiles in the ball are blocking. Following the Shitovitz's approach, Graziano [7] investigated this problem in atomless economies with a continuum of commodities by relying on the Rustichini and Yannelis [16] equivalence result between the sets of core and Walrasian allocations in this framework.

The paper is organized as follows. In Section 2 the model and the statement of the main result along with some discussions are given. In Section 3 the statements of auxiliary results are stated, and the proof of the main theorem is given using the auxiliary results, whose proofs are provided in Appendix I. Finally, three lemmata which are used in our proofs are collected in Appendix II.

2 The Model and the Main Result

We use a general equilibrium model with a differentiable framework as in Mas-Colell [15].

Commodities and Agents. There are a finite number ℓ of commodities and a finite set $N = \{1, \ldots, n\}$ of agents. The consumption set X of each agent is the interior of the non-negative orthant of the ℓ-dimensional Euclidean space (i.e., $X = \mathbb{R}^{\ell}_{++}$).

Preferences. For each agent $i \in N$, the preference relation $\succsim_i$ over X can be represented by a utility function u_i, which is twice continuously differentiable, strictly quasi-concave, monotonic (hence, strictly monotonic) and whose upper contour sets do not cut the axis, i.e., for every $x \in X$ the set $\{z \in X|\ u_i(z) > u_i(x)\}$ is closed in $\mathbb{R}^{\ell}_{++}$. The set of all such preferences is denoted by $\mathcal{P}$.

Given $\succsim \in \mathcal{P}$, let $p : X \to \Delta$ denote the continuously differentiable function which associate's with every $x \in X$ the normalized supporting prices of the upper contour set, at x. That is, if we denote the interior of the price simplex by $\Delta = \{p \in \mathbb{R}^{\ell}_{++}|\ \sum_{j=1}^{\ell} p^j\}$ and by $Du(x)$ the vector of marginal utilities at x, i.e.,

$$Du(x) = \left(\frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_\ell}(x) \right),$$

then the function $p : X \to \Delta$ is given by $p(x) = (1/\|Du(x)\|)\, Du(x)$, where u represents $\succsim$. (Note that strict monotonicity implies $Du(x) \gg 0$). We introduce a topology on $\mathcal{P} \times X$ as we need to restrict our treatment to subsets of $\mathcal{P} \times X$. Specifically, we use the topology of uniform convergence on compact sets of the $p(\cdot)$ functions and their first partial derivatives, and the space of consumption characteristics $\mathcal{P} \times X$ is endowed with the product topology.

An *economy* $\mathcal{E}$ is a mapping from the set N into the space $\mathcal{P} \times X$. Every agent $i \in N$ is fully characterized by the pair $\mathcal{E}(i) = (\succsim_i, e_i)$, where $\succsim_i \in \mathcal{P}$ is i's preferences and $e_i \in X$ is i's initial endowment. A *coalition* S is a nonempty subset of N. An allocation f for $\mathcal{E}$ is said to be *blocked* by coalition S if there is an allocation g for S such that $g : S \to X$ such that $\sum_{i \in S} g(i) \le \sum_{i \in S} e_i$ and $g(i) \succ_i f(i)$ for all $i \in S$.

An allocation f for $\mathcal{E}$ is *Pareto optimal* if it is not blocked by the grand coalition N. We denote by $\mathcal{PO}(\mathcal{E})$ the set of Pareto optimal allocations for $\mathcal{E}$. For every $f \in \mathcal{PO}(\mathcal{E})$, the convexity of preferences yields the existence of supporting prices $\pi(f) \in \Delta$ such that $\pi(f) = p(f(i))$ for all $i \in N$. An allocation f is said to be *Walrasian* if $\pi(f) \cdot f(i) = \pi(f) \cdot e_i$ for all i.

Following [14], we define the deviation of f from being Walrasian by

$$\operatorname{dev}(f) := \sqrt{\frac{\sum_{i \in N} \left(\pi(f) \cdot f(i) - \pi(f) \cdot e_i\right)^2}{n}}\,,$$

and for every positive number δ, the set of δ-*Walrasian Pareto optimal allocations*, $\mathcal{WPO}_\delta(\mathcal{E})$, is defined by

$$\mathcal{WPO}_\delta(\mathcal{E}) := \left\{ f \in \mathcal{PO}(\mathcal{E}) \;\middle|\; \operatorname{dev}(f) \leq \frac{\delta}{\sqrt{n}} \right\}.$$

Given a set of agents S, s denotes the *absolute size* (or the cardinality) of the set S, whereas the *relative size* of S is defined to be $\dfrac{s}{n}$. Throughout the rest of the paper we will use upper case letters to denote sets of agents and lower case letters for the absolute size of those sets.

For a given economy $\mathcal{E}$ and an allocation f for $\mathcal{E}$, we define, for any $0 \leq \alpha < \beta \leq 1$,

$$\mathcal{S}_{\alpha\beta}(n) = \{S \subset N |\ \alpha n \leq s \leq \beta n\},$$

the set of all coalitions in the economy $\mathcal{E}$ whose relative size is at least α and does not exceed β,

$$\mathcal{B}_{\alpha\beta}(n; \mathcal{E}, f) = \{S \subset \mathcal{S}_{\alpha\beta}(n) |\ S \text{ blocks } f\},$$

the set of ALL coalitions of relative size between α and β that block the allocation f. Finally, the proportion of such coalitions within the set $\mathcal{S}_{\alpha\beta}(N)$ is denoted by $\rho_{\alpha\beta}(n; \mathcal{E}, f)$, i.e.,

$$\rho_{\alpha\beta}(n; \mathcal{E}, f) = \frac{b_{\alpha\beta}(n; \mathcal{E}, f)}{s_{\alpha\beta}(n)}.$$

As alluded to in the introduction, the latter term is precisely the magnitude that we investigate. We can now state our main result:

Theorem 2.1 *Let a compact set of consumption characteristics $E \subset \mathcal{P} \times X$ and a compact subset of the consumption set $K \subset X$ be given. For any $0 \leq \alpha < \beta \leq 1$ and any $0 < \varepsilon < \frac{1}{2}$, there exists $\delta > 0$ such that for any finite set N with n agents, for any economy $\mathcal{E} : N \to E$ and any Pareto optimal allocation $f : N \to K$, if $f \notin \mathcal{WPO}_\delta(\mathcal{E})$, then the following inequality holds:*

$$\left| \rho_{\alpha\beta}(n; \mathcal{E}, f) - \frac{1}{2} \right| < \varepsilon.$$

That is, if a Pareto optimal allocation $f : N \to K$ is not δ-Walrasian, then the proportion of blocking coalitions of relative size between α and β differs from one half only by an arbitrarily small $\varepsilon > 0$.

The above result shows that for all allocations which are bounded away from being Walrasian and for every $0 \le \alpha < \beta \le 1$, $\rho_{\alpha\beta}(N; \mathcal{E}, f)$ is approximately $\frac{1}{2}$ when the number of individuals in the economy is large.

As we pointed out above, the relative size of most coalitions is close to $\frac{1}{2}$. Therefore, for α, β with $\alpha < \frac{1}{2} < \beta$, Theorem 2.1 can be derived from [14].[2] The main novelty of the present paper is, therefore, the examination of the case of either "small" $(0 \le \alpha < \beta \le \frac{1}{2})$, or "large" coalitions $(\frac{1}{2} \le \alpha < \beta \le 1)$.

3 Proof of the Main Result

In order to prove the Theorem, we need the following three results which will be stated and proved in Appendix I.

Proposition 3.1 *Let E and K be as given in the Theorem. Then for any $0 < \varepsilon < \frac{1}{2}$ and any $0 \le \alpha < \beta \le \frac{1}{2}$, there exists a constant $\delta > 0$ such that for any finite set N, for any economy $\mathcal{E} : N \to K$ and any Pareto optimal allocation $f : N \to K$, $f \notin \mathcal{WPO}_\delta(\mathcal{E})$, the following holds:*

$$\rho_{\alpha\beta}(n; \mathcal{E}, f) > \frac{1}{2} - \varepsilon.$$

Proposition 3.2 *Let E and K be as given in the Theorem. Then for any $0 \le \varepsilon < \frac{1}{2}$ and any $0 \le \alpha < \beta \le \frac{1}{2}$, there exists a constant $\delta > 0$ such that for any finite set N with n agents, for any economy $\mathcal{E} : N \to K$ and any Pareto optimal allocation $f : N \to K$, $f \notin \mathcal{WPO}_\delta(\mathcal{E})$, the following holds :*

$$\rho_{1-\beta,1-\alpha}(n; \mathcal{E}, f) > \frac{1}{2} - \varepsilon.$$

Proposition 3.3 *(i) Let $0 \le \alpha < \beta \le \frac{1}{2}$. Then*

$$\lim_{n\to\infty} \frac{s_{0\alpha}(n)}{s_{0\beta}(n)} = 0.$$

(ii) Let $0 \le \alpha < \beta \le \frac{1}{2}$. Then

$$\lim_{n\to\infty} \frac{s_{1-\beta,1}(n)}{s_{1-\alpha,1}(n)} = 0\,.$$

The proofs of these three results will be given in Appendix I after the completion of the proof of the main result. Thus, assuming the validity of the above claims we proceed to the proof.

[2] For $\alpha = 0$ and $\beta = 1$ it is precisely the Mas-Colell result.

Proof of the Theorem

Let $0 < \varepsilon < \frac{1}{2}$ be given. Let us show first that for every economy $\mathcal{E}$, any Pareto optimal allocation f and for every number $0 \leq \alpha < \beta < \frac{1}{2}$

$$\rho_{\alpha\beta}(n; \mathcal{E}, f) + \rho_{1-\beta,1-\alpha}(n; \mathcal{E}, f) \leq 1. \tag{1}$$

Indeed, let $S \in \mathcal{B}_{\alpha\beta}(n; \mathcal{E}, f)$. Since f is Pareto optimal, this means that $\overline{S} = N \setminus S$, the complement of S does not block f, i.e., $\overline{S} \notin \mathcal{B}_{1-\beta,1-\alpha}(n; \mathcal{E}, f)$, but, of course, $\overline{S} \in \mathcal{S}_{1-\beta,1-\alpha}(n)$. Therefore,

$$b_{\alpha\beta}(n; \mathcal{E}, f) + b_{1-\beta,1-\alpha}(n; \mathcal{E}, f) \leq \max\{s_{\alpha\beta}(n), s_{1-\beta,1-\alpha}(n)\}.$$

Finally, since $s_{\alpha\beta}(n) = s_{1-\beta,1-\alpha}(n)$, the assertion follows.

If $\beta = \frac{1}{2}$, then (1) should be modified to

$$\rho_{\alpha\beta}(n; \mathcal{E}, f) + \rho_{1-\beta,1-\alpha}(n; \mathcal{E}, f) \leq 1 + \frac{s_{\frac{1}{2},\frac{1}{2}}(n)}{s_{\alpha,\frac{1}{2}}(n)}. \tag{2}$$

It is well-known (see e.g. [6], p. 180) that

$$\lim_{n\to\infty} \frac{s_{\frac{1}{2},\frac{1}{2}}(n)}{s_{0,\frac{1}{2}}(n)} = 0.$$

Thus, it follows from Proposition 3.3 that

$$\lim_{n\to\infty} \frac{s_{\frac{1}{2},\frac{1}{2}}(n)}{s_{\alpha,\frac{1}{2}}(n)} = 0.$$

In particular, there exists n_0 such that for all $n > n_0$,

$$\frac{s_{\frac{1}{2},\frac{1}{2}}(n)}{s_{\alpha,\frac{1}{2}}(n)} < \frac{\varepsilon}{2}\,.$$

In order to complete the proof of the Theorem, we distinguish between the following two cases:

Case 1: Either $0 \leq \alpha < \beta \leq \frac{1}{2}$ or $\frac{1}{2} \leq \alpha < \beta \leq 1$.
By Propositions 3.1 and 3.2, we have $\rho_{\alpha\beta}(n; \mathcal{E}, f) > \frac{1}{2} - \varepsilon$ and $\rho_{1-\beta,1-\alpha}(n; \mathcal{E}, f) > \frac{1}{2} - \varepsilon$. (1) and (2) imply that for all $n > n_0$, $\rho_{\alpha\beta}(n; \mathcal{E}, f) < \frac{1}{2} + \varepsilon$. Now, choose δ which is greater than both the δ in Propositions 3.1 and 3.2, and also greater than $2\|K\|^2\sqrt{n}$, where $x \in K$ implies $x \leq \|K\|$. (Recall that K is compact.) It follows that for this δ, the Theorem holds. (Note that, since $p \in \Delta$, $\text{dev}(f) < \delta \,\big/\, \sqrt{n}$ for $n < n_0$.)

Case 2: Let $0 \leq \alpha < \frac{1}{2} < \beta \leq 1$.
By definition,

$$\rho_{\alpha\beta}(n; \mathcal{E}, f) = \frac{s_{\alpha,\frac{1}{2}}(n; \mathcal{E}, f) - s_{\frac{1}{2},\frac{1}{2}}(n; \mathcal{E}, f) + s_{\frac{1}{2},\beta}(n; \mathcal{E}, f)}{s_{\alpha,\frac{1}{2}}(n) - s_{\frac{1}{2},\frac{1}{2}}(n) + s_{\frac{1}{2},\beta}(n)}\,.$$

Note that for any positive numbers a_i, b_i, $i = 1, \ldots, k$,

$$\min_{1 \le i \le k} \frac{a_i}{b_i} \le \frac{\sum_{i=1}^{k} a_i}{\sum_{i=1}^{k} b_i} \le \max_{1 \le i \le k} \frac{a_i}{b_i} \, . \tag{3}$$

Applying (2) and (3) together with Case 1, the result follows. $\square$

4 Appendix I

Since Proposition 3.3 is used in the proof of the two preceding propositions above, we begin by proving it. For this purpose, we need the following two Lemmata:

Lemma 4.1 *For all* $0 \le \alpha < \beta \le 1$,

$$\frac{s_{0\alpha}(n)}{s_{0\beta}(n)} \le \frac{\binom{n}{[\alpha n]}}{\binom{n}{[\beta n]}} \, ,$$

where $[r]$ *denotes the greatest integer not exceeding* r.

Proof By (3),

$$\frac{s_{0\alpha}(n)}{s_{0\beta}(n)} = \frac{\sum_{i=0}^{[\alpha n]} \binom{n}{i}}{\sum_{j=0}^{[\beta n]} \binom{n}{j}} \le \frac{\sum_{i=0}^{[\alpha n]} \binom{n}{i}}{\sum_{j=[\beta n]-[\alpha n]}^{[\beta n]} \binom{n}{j}} \le \max_{0 \le k \le [\alpha n]} \frac{\binom{n}{[\alpha n]-k}}{\binom{n}{[\beta n]-k}} = \frac{\binom{n}{[\alpha n]}}{\binom{n}{[\beta n]}}.$$

$\square$

Lemma 4.2 *For all* $0 \le \alpha < \beta \le \frac{1}{2}$, *and for all* n *with* $[\beta n] > [\alpha n]$,

$$\frac{\binom{n}{[\alpha n]}}{\binom{n}{[\beta n]}} < \left(\frac{\beta}{1-\alpha}\right)^{(\beta-\alpha)n-1} .$$

Proof Note that

$$\frac{\binom{n}{[\alpha n]}}{\binom{n}{[\beta n]}} = \frac{[\beta n]! \, (n - [\beta n])!}{(n - [\alpha n])!} = \prod_{k=1}^{[\beta n]-[\alpha n]} c_k,$$

where $c_k = ([\alpha n]+k)/(n-[\beta n]+k)$. Since for all k,

$$c_k \leq \frac{[\beta n]}{n-[\alpha n]} \leq \frac{\beta}{1-\alpha}\ , \text{ and } \frac{\beta}{1-\alpha} < 1,$$

we have

$$\frac{\binom{n}{[\alpha n]}}{\binom{n}{[\beta n]}} \leq \left(\frac{\beta}{1-\alpha}\right)^{[\beta n]-[\alpha n]} < \left(\frac{\beta}{1-\alpha}\right)^{(\beta-\alpha)n-1} .$$

□

Proof of Proposition 3.3

Let $0 \leq \alpha < \beta \leq \frac{1}{2}$. Then, by Lemmata 4.1 and 4.2, for all n with $[\beta n] > [\alpha n]$, we have

$$\frac{s_{0\alpha}(n)}{s_{0\beta}(n)} < \left(\frac{\beta}{1-\alpha}\right)^{(\beta-\alpha)n-1} .$$

Recalling that $\beta/(1-\alpha) < 1$, we get

$$\lim_{n\to\infty} \frac{s_{0\alpha}(n)}{s_{0\beta}(n)} = 0\,. \tag{4}$$

Let $\frac{1}{2} \leq \alpha < \beta \leq 1$. Note that for all $0 \leq t \leq 1$, $s_{0,t}(n) = s_{1-t,1}(n)$. Thus,

$$\lim_{n\to\infty} \frac{s_{1-\alpha,1}(n)}{s_{1-\beta,1}(n)} = \lim_{n\to\infty} \frac{s_{0\alpha}(n)}{s_{0\beta}(n)} = 0.$$

The last equality follows from (4). □

In order to prove Proposition 3.1, we first show the following:

Proposition 4.3 *Let E and K be as given in the Theorem. Then, for any $0 < \varepsilon < \frac{1}{2}$, and any $0 < \alpha < \beta \leq \frac{1}{2}$, there exists a constant $\delta > 0$ such that for any finite set N with n agents, for any economy $\mathcal{E} : N \to E$ and any Pareto optimal allocation $f : N \to K$, $f \notin \mathcal{WPO}_\delta(\mathcal{E})$, the following holds:*

$$\rho_{\alpha\beta}(n; \mathcal{E}, f) > \frac{1}{2} - \varepsilon\,.$$

Proof Let $N, \mathcal{E} : N \to K, f : N \to K, 0 < \varepsilon < \frac{1}{2}$, and $0 < \alpha < \beta \leq \frac{1}{2}$ be given. As in [14], p.212, the constants a, G, r and κ are determined. (See Lemmata 5.1 - 5.3 in Appendix II). Define $\eta, \lambda, t_0, J, \delta$ (in this order) such that:

$$\frac{1}{2} - \varepsilon < \eta < \frac{1}{2},\ \lambda = \frac{\eta - \left(\frac{1}{2} - \varepsilon\right)}{4},$$

$$1-\Phi(t_0)=\frac{\frac{1}{2}+\eta}{2},\ \text{where}\ \ \Phi(t)\ \text{is the}\ (0,1)\ \text{normal distribution},$$

$$J\geq\sqrt{\max\left[\frac{\ln G-\ln\lambda}{r\alpha a^2},\frac{2\kappa t_0}{\alpha\sqrt{a}(\frac{1}{2}-\eta)}\right]},\ \delta=\frac{\alpha J^2}{t_0}\,.$$

Put $\gamma^2\equiv\text{dev}(f)^2$, $\pi=\pi(f)$, $f_i=f(i)$, $g_i=e_i-f_i$ (note that g_i is the *excess supply* of agent$i$) and $M=\{m\in\mathbb{Z}_+|\,\alpha n\leq m\leq\beta n\}$ where $\mathbb{Z}_+$ is the set of nonnegative integers. For every $m\in M$, define:

$$\begin{aligned}
\gamma_m^2&=m\left(1-\frac{m}{n}\right)\gamma^2\,,\ N_m=\{S\subset N|\,s=m\}\\
C_m^1&=\left\{S\subset N_m\;\middle|\;\left\|\sum\nolimits_{i\in S}g_i\right\|\leq\frac{aJm}{\sqrt{n}}\right\}\\
C_m^2&=\left\{S\subset N_m\;\middle|\;\frac{\sum_{i\in S}\pi\cdot g_i}{\left\|\sum_{i\in S}g_i\right\|}\geq\frac{J}{\sqrt{n}}\right\}\\
C_m^3&=\left\{S\subset N_m\;\middle|\;\sum\nolimits_{i\in S}\pi\cdot g_i\geq t_0\delta\sqrt{\frac{m}{2n}}\right\}\\
C_m^4&=\left\{S\subset N_m\;\middle|\;\sum\nolimits_{i\in S}\pi\cdot g_i\geq\gamma_m t_0\right\}\\
c_m^j&=\frac{\sharp C_m^j}{\binom{n}{m}},\quad j=1,2,3,4\\
c&=\frac{\sharp\left\{\cup_{m\in M}\left(C_m^1\cap C_m^2\right)\right\}}{\sum_{m\in M}\binom{n}{m}},
\end{aligned}$$

where $\sharp Q$ stands for the cardinality of the set Q.

Suppose $\gamma>\dfrac{\delta}{\sqrt{n}}$. Then, as was proved in [14],

$$S\in C_m^1\cap C_m^2\implies S\in\mathcal{B}(n;\mathcal{E},f). \tag{5}$$

(Note that for such a coalition $\sum_{i\in S}\pi\cdot g_i>0$.) (3) implies that

$$c\geq\min_{n\in M}\frac{\sharp\left\{C_m^1\cap C_m^2\right\}}{\binom{n}{m}}\geq\min_{m\in M}\left\{c_m^1+c_m^2-1\right\}\,.$$

By using (5), we obtain

$$\rho_{\alpha\beta}(n;\mathcal{E},f)\geq c\geq\min_{m\in M}\left\{c_m^1+c_m^2-1\right\}. \tag{6}$$

To conclude the proof of the Proposition, it suffices to show that the last term is greater than $\frac{1}{2}-\varepsilon$. Now, as in [14], p.214, Lemma 5.2 immediately yields that

$$c_m^1\geq 1-\lambda\ \ \text{for all}\ \ m\in M\,. \tag{7}$$

In order to evaluate c_m^2 observe that $\gamma_m\sqrt{2n} \leq \delta\sqrt{m}$. Since $m \leq n/2$, we have $C_m^4 \subset C_m^3$ and $C_m^1 \cap C_m^3 \subset C_m^2$. Therefore,

$$c_m^2 \geq c_m^1 + c_m^3 - 1 \geq c_m^1 + c_m^4 - 1\,. \tag{8}$$

Now, from Lemma 5.3 (with $d_i = \pi \cdot g_i$) it follows that for any $m \in M$,

$$c_m^4 \geq \frac{-\kappa}{\gamma\sqrt{m}} + 1 - \Phi(t_0) \geq \frac{-\kappa\sqrt{n}}{\delta\sqrt{m}} + 1 - \Phi(t_0) \geq \frac{-\kappa t_0}{aJ^2\sqrt{\alpha}} + 1 - \Phi(t_0)\,.$$

By the choice of our constants, therefore, for all $m \in M$,

$$c_m^4 \geq -\frac{\frac{1}{2}-\eta}{2} + \frac{\frac{1}{2}+\eta}{2} = \eta. \tag{9}$$

Therefore, using (7) – (9), we finally get

$$\rho_{\alpha\beta}(n;\mathcal{E},f) \geq c_{m^*}^1 + c_{m^*}^2 - 1 \geq 2c_{m^*}^1 + c_{m^*}^4 - 2 = \eta - 2\lambda = \frac{\eta + \left(\frac{1}{2}-\varepsilon\right)}{2} > \frac{1}{2} - \varepsilon\,,$$

where m^* is the value of m at which the minimum in (6) is attained. □

Proof of Proposition 3.1

We have to show that the assertion of Proposition 4.3 holds also for $\alpha = 0$. Let $\theta = \beta/2 < \beta$. By Proposition 4.3, there exists $\delta' > 0$ such that $\mathrm{dev}(f) > \dfrac{\delta'}{\sqrt{n}}$ implies $\rho_{\theta\beta}(n;\mathcal{E},f) \geq q$ for some $q > \frac{1}{2} - \varepsilon$. Now,

$$\rho_{\theta\beta}(n;\mathcal{E},f) \geq \frac{s_{\theta\beta}(n;\mathcal{E},f)}{s_{0\theta}(n) + s_{\theta\beta}(n)} \geq \frac{q \cdot s_{\theta\beta}(n)}{s_{0\theta}(n) + s_{\theta\beta}(n)} = \frac{q}{1 + \frac{s_{0\theta}(n)}{s_{\theta\beta}(n)}}.$$

By Proposition 3.3, $\dfrac{s_{0\theta}(n)}{s_{\theta\beta}(n)} \to 0$ when $n \to \infty$. Therefore, there exists n_1 such that for all $n > n_1$, $\rho_{0\beta}(n;\mathcal{E},f) \geq \frac{1}{2} - \varepsilon$. Since $f_i \in K$, $\|f_i\| \leq \|K\|$ for all $i \in N$, and therefore, $\mathrm{dev}(f) < 2\|K\|^2$. (Recall that $\pi \in \Delta$). Choosing δ such that $\delta > \max\{2\|K\|\sqrt{n}\,,\ \delta'\}$, completes the proof. □

Proposition 4.4 *Let E and K be as given in the Theorem. Then, for any $0 < \varepsilon < \frac{1}{2}$, and any $0 < \alpha < \beta \leq \frac{1}{2}$, there exists a constant $\delta > 0$ such that for any finite set N with n agents, for any economy $\mathcal{E} : N \to E$ and any Pareto optimal allocation $f : N \to K$, $f \notin \mathcal{WPO}_\delta(\mathcal{E})$, the following holds:*

$$\rho_{1-\beta,1-\alpha}(n;\mathcal{E},f) > \frac{1}{2} - \varepsilon\,.$$

Proof First note that the proof of Proposition 4.3 cannot be directly applied here, since there we used the fact that $m \leq \frac{1}{2}n$. We therefore alter the definitions of the sets C_m^j as follows:

$$
\begin{aligned}
C_m^1 &= \left\{ S \subset N_m \;\middle|\; \left\|\sum\nolimits_{i\in S} g_i\right\| \leq \frac{aJm}{\sqrt{n}} \right\} \\
C_m^2 &= \left\{ S \subset N_m \;\middle|\; \frac{\sum_{i\in S} \pi \cdot g_i}{\left\|\sum_{i\in S} g_i\right\|} \leq -\frac{J}{\sqrt{n}} \right\} \\
C_m^3 &= \left\{ S \subset N_m \;\middle|\; \sum\nolimits_{i\in S} \pi \cdot g_i \leq -t_0 \delta \sqrt{\frac{m}{2n}} \right\} \\
C_m^4 &= \left\{ S \subset N_m \;\middle|\; \sum\nolimits_{i\in S} \pi \cdot g_i \leq -\gamma_m t_0 \right\} .
\end{aligned}
$$

All the constants and notations are the same as in the proof of Proposition 4.3. Now, instead of looking at a coalition S of relative size between $1-\beta$ and $1-\alpha$, we consider its complement $\overline{S}$ whose relative size is, therefore, between α and β, with $\beta \leq \frac{1}{2}$. The fact that $\sum_{i\in N} g_i = 0$ implies the following equalities:

$$
\left\|\sum\nolimits_{i\in S} g_i\right\| = \left\|\sum\nolimits_{i\in \overline{S}} g_i\right\| \quad \text{and} \quad \sum\nolimits_{i\in S} \pi \cdot g_i = -\sum\nolimits_{i\in \overline{S}} \pi \cdot g_i
$$

Using these relations, the proof of Proposition 4.3 yields that for all $m \in M$

(i) $S \in C_m^1 \cap C_m^2$ implies $\overline{S}$ blocks f

(ii) $C_m^4 \subset C_m^3$ and $C_m^1 \cap C_m^3 \subset C_m^2$

(iii) $c_m^1 \geq 1 - \lambda$

(iv) $c_m^4 \geq F_m(-t_0) \geq -\dfrac{\kappa}{\gamma\sqrt{m}} + \Phi(-t_0) = -\dfrac{\kappa}{\gamma\sqrt{m}} + 1 - \Phi(t_0) \geq \eta$

Following the proof of Proposition 4.3, we get

$$
\rho_{1-\beta,1-\alpha}(n; \mathcal{E}, f) \geq c_{m^*}^1 + c_{m^*}^2 - 1 \geq \frac{\eta + \left(\frac{1}{2} - \varepsilon\right)}{2} > \frac{1}{2} - \varepsilon \, .
$$

□

Proof of Proposition 3.2

Again, the only case to be considered is $\alpha = 0$. The proof here is similar to the one given for Proposition 3.1, but now using the second part of Proposition 3.3. □

5 Appendix II

We reproduce here the three Lemmata which were used in [14], and from which the constants a, G, r and κ are determined. Their original references are indicated and we point out the particular choice of the sets in our case.

Lemma 5.1 *Let $E \subset \mathcal{P} \times X$ be a compact set. There is $a > 0$ such that for all $(\succsim, e) \in E$, $x \in X$, and $y \in \mathbb{R}^\ell$, if $\|x\| < \|K\|$ and $p \cdot y \neq 0$, then*

$$x + \frac{\alpha p \cdot y}{\|y\|^2} y \succ x \,,$$

where $p = p(x)$.

See Mas-Colell [14], p.211.

Lemma 5.2 *Let $\Gamma \subset \mathbb{R}^\ell$ be nonempty and compact. There are constants $G > 0$ and $r > 0$ such that if $y_i \in \Gamma, i \in N, \#N = n$, and $\sum_{i \in N} y_i = 0$, then for every $\tau > 0$ and $m \leq n$,*

$$\sharp \left\{ S \subset N_m \;\middle|\; \left\| \sum_{i \in S} y_i \right\| \geq \tau m \right\} < \binom{n}{m} G e^{-r\tau^2 m} \,.$$

See Hoeffding [12], $Th.1$, Section 6. The set Γ used in our paper is:

$$\Gamma = \{(x - e) | \, x \in K \text{ and there exists } \succsim \in \mathcal{P} \text{ such that } (\succsim, e) \in E\}$$

(and of course, $y_i = f_i - e_i, i \in N$).

Lemma 5.3 *Let $\Gamma \subset \mathbb{R}$ be nonempty and compact. Then there is a constant $\kappa > 0$ such that if $d_i \in \Gamma, i \in N, \sum_{i \in I} d_i = 0$, and $m \leq n/2$, then, letting*

$$\gamma^2 = \frac{1}{n} \sum_{i \in I} d_i^2$$

$$\gamma_m^2 = m \left(1 - \frac{m}{n}\right) \gamma^2$$

$$F_m(t) = \frac{1}{\binom{n}{m}} \sharp \left\{ S \subset N_m \;\middle|\; \sum_{i \in N} d_i < \gamma_m t \right\}$$

and $\Phi(t)$ be the normal $(0,1)$ distribution, we have

$$\sup_{t \in \mathbb{R}} |F_m(t) - \Phi(t)| \leq \frac{\kappa}{\gamma \sqrt{m}} \,.$$

See Bikelis [3]. The set Γ used for the application to our context is

$$\Gamma = \{p(x - e) | \, p \in \Delta, x \in K, \text{ and there exists } \succsim \in \mathcal{P} \text{ such that } (\succsim, e) \in E\}$$

(and, of course, $d_i = \pi \cdot f_i - \pi \cdot e_i, i \in N$).

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