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A MODEL OF EQUITY PRICES WITH HETEROGENEOUS BELIEFS*

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Abstract

This paper analyzes the effect of interaction among heterogeneous investors on equity prices. We classify investors into three groups according to their information sets and beliefs: informed investors, trend followers, and contrarians. Then, the equity price is derived through the market clearing condition. Our model explains many anomalous phenomena in the equity markets, including excess volatility, the momentum effect, and the mean-reverting effect. Further, the empirical analysis shows that the difference in returns behavior between small- and large-cap equities in the U.S. market can be explained by differences in the composition of investors.

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JEL Classification: G11, G12

I. *Introduction*

The standard asset pricing model, which assumes a representative investor with rational beliefs, fails to explain many dynamic properties of equity prices.[1] For example, Shiller (1981) points out that equity prices are too volatile compared with their fundamentals. If investors in the equity market have rational expectations about future fundamentals, equity prices should be less volatile than their fundamentals. Further, many empirical findings suggest that equity returns are predictable. Lo and Mackinlay (1988) and Jegadeesh and Titman (1993) find that equity returns exhibit a momentum effect in the short term, and Debondt and Thaler (1985), among others, find evidence of a mean-reverting effect in the long term. These findings challenge the rational expectation model.[2]

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[1] Campbell (2000) presents a comprehensive survey of existing asset pricing models and anomalous phenomena in asset markets.

[2] Equity prices do not necessarily follow a random walk, even if investors have rational expectations [see, e.g., Leroy (1973) and Lucas (1978)]. However, if one could not relate the apparently abnormal behavior of equity prices to meaningful economic factors, the reliability of the rational expectations model would be undermined.

To address these anomalies in equity markets, the current paper proposes an alternative model by allowing heterogeneous investors. We assume information asymmetry among investors. Informed investors can observe the fundamental value of equity. Although uninformed investors cannot directly observe the fundamental value, they infer it from the realized values of equity prices.

Wang (1993) analyzes the effect of asymmetric information among investors on equity prices within the rational expectations framework. He shows that the existence of uninformed investors increases equity volatility and negative autocorrelations in equity returns. However, his model fails to explain the momentum effect in equity returns.

Therefore, we further classify uninformed investors into two different groups according to their beliefs. It is shown that, if uninformed investors believe that the equity price is less volatile than its fundamental value, they will act as trend followers who increase their equity demands when realized past equity returns are relatively high. By contrast, if uninformed investors believe that the equity price is more volatile than its fundamental value, they will act as contrarians that increase their equity demands when past equity returns are relatively low.

The existence of contrarians makes the equity price respond sluggishly to fluctuations in fundamental values, which can generate the momentum effect. The existence of trend followers causes equity prices to respond excessively to changes in their fundamental value, which can generate excess volatility and the mean-reverting effect. Interaction among these investors can simultaneously generate excess volatility, the momentum effect, and the mean-reverting effect. Further, our empirical analysis shows that differences in the behavior of returns between small- and large-cap equities in the U.S. market can be explained by differences in the composition of investors.

Our model is closely related to asset pricing models in the field of behavioral finance. Having assumed heterogeneous investors with different information sets and different beliefs, Hong and Stein (1999) attribute both the momentum effect and the mean-reverting effect to interaction among different investors.[3] In this sense, we adopt a similar approach to that of Hong and Stein (1999). However, they assume that news about fundamentals spreads only gradually among informed investors, which is an assumption required in their model to prevent the equity price from converging immediately to its fundamental value. In the current paper, it is difficult to ascribe the momentum effect, which can persist for months, to information lags experienced by informed investors. Indeed, there are no information lags in the current model. Nevertheless, there remains the mispricing of equity, unless the informed investors are risk neutral.

The reminder of this paper is organized as follows. In Section II, we describe the structure of the model. Then, the equity price is derived through the market clearing condition. It is then shown that excess volatility, the momentum effect, and the mean-reverting effect of equity returns can be explained by the current model. In Section III, the parameter values of equity price process are estimated. It is shown that the current model can explain differences in the behavior of returns between large- and small-cap equities. Finally, Section VI presents concluding remarks.

[3] Meanwhile, Barberis, Shleifer, and Vishny (1998), and Daniel, Hirshleifer, and Subrabmanyam (1998) postulate an irrational representative investor, and ascribe both phenomena to the investor's cognitive decision making.

II. *The Model*

This section describes the basic setting of the economy assumed in this paper. Then, the equity price is derived from the market clearing condition.

1. The Economy

We assume a continuous time economy with two tradable assets in the financial market. One of these assets is a riskless asset, and the other is a risky asset (equity). The riskless interest rate is normalized to zero through time. The equity price is stochastic in every period, and the logarithm of the dividend-cum-equity price at time t is denoted by $p(t)$. In the current paper, the equity price is determined through both its fundamental value and investors' net demands for the equity. The fundamental value of equity might be represented by the expectation of the sum of discounted future dividends. The logarithm of the fundamental value at time t, denoted by $f(t)$, is assumed to be governed by the following stochastic process:

$$df(t)=\theta dt+\kappa dw^{f}(t), \tag{1}$$

where $w^{f}(t)$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}(t)\}, \mathbb{P})$.

2. Investors' Beliefs

There are three types of investors, each with their own subjective beliefs. Each investor believes that the fundamental value of equity evolves as follows:

$$df(t)=\theta_i dt+\kappa_i dw_i^{f}(t),\ i=0,\ 1,\ 2, \tag{2}$$

where index $i=0, 1, 2$ emphasizes that each value is meaningful only in the subjective beliefs of type-i investors. Each $w_i^{f}(t)$ is a standard Brownian motion under the subjective filtered probability space of type-i investors, $(\Omega, \mathscr{F}_i, \{\mathscr{F}_i(t)\}, \mathbb{P}_i)$. While all investors correctly believe that the fundamental value of equity follows a random walk, as in equation (1), their subjective drift rates θ_i and volatilities κ_i may differ from the corresponding true values θ and κ, respectively.

Each investor also has his/her own subjective beliefs about expected instantaneous equity returns as follows:

$$\mu_i(t)=\theta_i+\gamma_i\ \{f(t)-p(t)\},\ \gamma_i>0,\ i=0,\ 1,\ 2. \tag{3}$$

From equation (3), each investor believes that the current expected equity return consists of two components. The first component is the expected growth rate in the fundamental value θ_i, and the second component is the correction to the current pricing error. Each investor expects the equity return to be relatively high (low) when the current equity price falls below (exceeds) its fundamental value. The parameter γ_i represents the speed with which type-i investors believe that pricing errors are eradicated.

While the investors have different beliefs about expected equity returns, all investors know the correct value of the instantaneous volatility of equity returns, denoted by σ, which is constant over time. This is because investors can infer accurately the volatility of equity returns from the quadratic variation of the observed equity returns process.[4] And, the equity returns process under type-i investors' beliefs is as follows:

$$dp(t)=\mu_i(t)dt+\sigma dw_i^p(t),\ i=0,\ 1,\ 2, \tag{4}$$

where $w_i^p(t)$ is a standard Brownian motion under $(\Omega,\ \mathscr{F}_i,\ \{\mathscr{F}_i(t)\},\ \mathbb{P}_i)$, and the correlation coefficient between $w_i^p(t)$ and $w_i^f(t)$ under $(\Omega,\ \mathscr{F}_i,\ \{\mathscr{F}_i(t)\},\ \mathbb{P}_i)$ is denoted by ρ_i^{pf}.

As shown in the following subsection, the equilibrium equity returns process has the same form as equation (4). In particular, it has a time-varying drift term with a constant volatility term, as does equation (4). In the current model, investors are not fully rational in the sense that they have incorrect beliefs about expected equity returns. However, it would be difficult for investors to understand the true structure of the time-varying expected equity returns process. Therefore, it is, arguably, reasonable to assume that investors rely on their own beliefs about expected returns.

Besides the heterogeneity in their beliefs, we assume that investors also differ in their information sets. Only informed investors can observe the fundamental value of equity. Hereafter, informed investors are represented by the index $i=0$. Hence, the information set of informed investors at time t, $\mathscr{F}_0(t)$, contains, at least, the realized values of both equity prices and the fundamental value up to time t. Uninformed investors, of which there are two types (indexed by $i=1,\ 2$), cannot observe the current fundamental value directly. That is, the information sets of these investors, $\mathscr{F}_1(t)$ and $\mathscr{F}_2(t)$, contain the realized equity prices up to time t, but not the realized fundamental value. To make their investment decisions, these investors must infer the current fundamental value from the realized equity returns up to the current period.

From standard filtering theory, given the observable process $p(t)$ in equation (4) and the unobservable process $f(t)$ in equation (2), it can be shown that the least squares estimators of the fundamental value under the uninformed investors' beliefs, denoted by $\hat{f}_1(t)$ and $\hat{f}_2(t)$, respectively, evolve as follows:[5]

$$d\hat{f}_i(t)=\theta_i dt+\frac{\sigma\kappa_i\rho_i^{pf}+s_i(t)\gamma_i}{\sigma^2}\left\{dp(t)-\left[\theta_i+\gamma_i\left\{\hat{f}_i(t)-p(t)\right\}\right]dt\right\},\ i=1,\ 2, \tag{5}$$

where:

$$s_i(t)=E_i\left[\left\{\hat{f}_i(t)-f(t)\right\}^2 | \mathscr{F}_i(t)\right], \tag{6}$$

and $E_i[\cdot]$ denotes the expectation under $\mathbb{P}_i$. Hence, $s_i(t)$ is the mean squared error of the estimator $\hat{f}_i(t)$ under $(\Omega,\ \mathscr{F}_i,\ \{\mathscr{F}_i(t)\},\ \mathbb{P}_i)$, and this evolves deterministically as follows:

$$ds_i(t)=\left[\kappa_i^2-\left\{\frac{\sigma\kappa_i\rho_i^{pf}+s_i(t)\gamma_i}{\sigma^2}\right\}^2\right]dt. \tag{7}$$

Equation (7) implies that when $s_i(0)$ is set at the steady state level $\sigma\kappa_i(1-\rho_i^{pf})/\gamma_i$, then

[4] For details, see Williams (1977).

[5] See, e.g., Liptser and Shiryaev (2001).

$s_i(t)$ is constant over time. In this case, the least squares estimator $\hat{f}_i(t)$, for $i=1, 2$, can be expressed simply as follows:

$$\hat{f}_i(t)=p(t)+\frac{\delta_i}{\gamma_i}x_i(t),\ i=1, 2, \tag{8}$$

where:

$$x_i(t)=\int_0^t \alpha_i e^{-\alpha_i(t-u)}\ \{dp(u)-\theta_i du\}, \tag{9}$$

and:

$$\alpha_i=\frac{\gamma_i\kappa_i}{\sigma}, \tag{10}$$

$$\delta_i=1-\frac{\sigma}{\kappa_i}. \tag{11}$$

By substituting $\hat{f}_i(t)$ into $f(t)$ in equation (3), the expected equity return for type-i investors ($i=1, 2$) becomes:

$$\hat{\mu}_i(t)=\theta_i+\delta_i x_i(t),\ i=1, 2. \tag{12}$$

From equation (9), $x_i(t)$ can be interpreted as the weighted average of abnormal equity returns realized until time t, and α_i determines the weights on past returns. If α_i is large, type-i investors put relatively heavy weights on recent equity returns. If α_i is small, type-i investors estimate the current fundamental value based on equity returns from the distant past. From equation (10), α_i is increasing in both γ_i and κ_i. When γ_i is large, type-i investors believe that the current difference between $p(t)$ and $f(t)$ will vanish quickly (recall equation (3)), and they therefore believe that equity returns from the distant past contain little information about the current fundamental value. When κ_i is large, type-i investors believe that the fundamental value is rather volatile, so again they believe that equity returns from the distant past are of little use for inferring the current fundamental value.

The above setup allows us to classify uninformed investors into two different groups: trend followers and contrarians. Equation (11) shows that:

$$\delta_i \gtreqless 0 \text{ as } \kappa_i \gtreqless \sigma. \tag{13}$$

This relation and equation (8) indicate that uninformed investors characterized by $\kappa_i>\sigma$ expect the current equity price to be lower than its fundamental value when past equity returns $x_i(t)$ is relatively high. These investors believe that the equity price is less volatile than its fundamental value, and that the equity price underreacts to a change in the fundamental value. Good performance by past equity returns suggests that fundamental values grew rather rapidly in the past, which in turn suggests, to these investors, that the current equity price is undervalued relative to its fundamental value. Therefore, from equation (12), the expected equity return for uninformed investors with $\kappa_i>\sigma$ is high when past equity returns $x_i(t)$ are relatively high, which leads these investors to increase their equity demands. Because of this behavior, such investors are referred to as trend followers.

Uninformed investors characterized by $\kappa_i<\sigma$ believe that the equity price is more volatile than its fundamental value, and believe that the equity price tends to overreact to a change in the fundamental value. Hence, from equation (12), the expected equity return for these investors is low when past equity returns $x_i(t)$ are relatively high, which leads them to reduce their equity demands. Because of this behavior, uninformed investors characterized by $\kappa_i < \sigma$ are referred to as contrarians. It is henceforth assumed that $\kappa_1 > \sigma$ and $\kappa_2 < \sigma$, i.e., type-1 investors are trend followers and type-2 investors are contrarians.

3. The Equity Price

Next, consider the optimal portfolio of each investor. In this paper, we assume that all investors have constant absolute risk aversion (CARA) utility. Under this assumption, investors' wealth levels do not affect their optimal portfolios, which are instead completely determined by the relationship between each asset's expected return and risk. Because the riskless interest rate is normalized to zero, the equity demand function for each investor at time t, $D_i(t)$, is as follows:

$$D_0(t) = \phi_0 \frac{\mu_0(t)}{\sigma} + C_0, \tag{14}$$

$$D_i(t) = \phi_i \frac{\hat{\mu}_i(t)}{\sigma} + C_i,\ i = 1, 2, \tag{15}$$

where C_i is constant and ϕ_i is an inverse measure of the risk aversion of type-i investors.

We can derive an equilibrium equity price from the market clearing condition for the equity market. To obtain an explicit solution for the equity price, it is further assumed that the number of type-i investors is constant through time; we denote this number by N_i. If the supply of equity $S(t)$ is fixed through time, then $S(t) = S$ for each t, and the market clearing condition is:

$$\sum_{i=0}^{2} N_i D_i(t) = S, \textit{ for each } t. \tag{16}$$

From the market clearing condition in equation (16), the equity price is obtained as follows:[6]

$$p(t) = f(t) + \lambda_1 x_1(t) - \lambda_2 x_2(t) + const, \tag{17}$$

where:

$$\lambda_1 = \frac{\delta_1 \phi_1 N_1}{\gamma_0 \phi_0 N_0}, \tag{18}$$

$$\lambda_2 = -\frac{\delta_2 \phi_2 N_2}{\gamma_0 \phi_0 N_0}, \tag{19}$$

are constant coefficients with positive values, each of which expresses the effective number of type-i uninformed investors.

Consider the basic property of the equity price in equation (17). If the number of

[6] The derivation of equations (17) and (20) is explained in the Appendix located at the end of this paper.

uninformed investors is zero ($N_1=N_2=0$), or if the informed investors are risk neutral ($\phi_0=\infty$), then $\lambda_1=\lambda_2=0$, and the equity price reflects only its fundamental value. This is the product of an economy in which all investors can effectively observe the fundamental value of equity. Were this not to occur, because uninformed investors infer the current fundamental value from past equity returns, the current equity price would also be influenced by past equity returns through $x_i(t)$.

From equation (17), the equity returns process can be derived. For analytical simplicity, it is further assumed that $\theta_i=\theta$ for $i=0, 1, 2$. That is, the subjective drift rate of the fundamental value for each investor coincides with its true value. Then, by applying Ito's formula to equation (17), the equity returns process can be written as follows:

$$dp(t)=\theta dt+\frac{\{\alpha_2\lambda_2x_2(t)-\alpha_1\lambda_1x_1(t)\}dt+\kappa dw^f(t)}{1-\alpha_1\lambda_1+\alpha_2\lambda_2}. \tag{20}$$

From equation (20), it can be shown that the unconditional expected equity return is identical to the drift rate of the fundamental value:

$$E\,[dp(t)]=\theta dt. \tag{21}$$

Although the equity price can deviate from its fundamental value in each period, the average equity return coincides with the growth rate of the fundamental value as long as each investor correctly computes the drift rate of the fundamental value θ.

The volatility of equity returns is:

$$\sigma=\frac{\kappa}{1-\alpha_1\lambda_1+\alpha_2\lambda_2}\gtreqless\kappa \text{ as } \alpha_1\lambda_1\gtreqless\alpha_2\lambda_2. \tag{22}$$

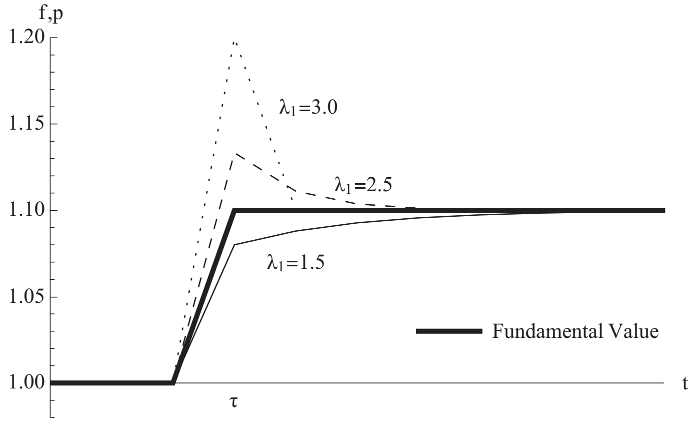
The volatility of equity returns depends on λ_i. Trend followers increase their equity demands when the equity price rises, which further pushes up the equity price. Hence, the existence of trend followers amplifies the volatility of equity returns. By contrast, contrarians reduce their equity demands when the equity price rises. Hence, the existence of contrarians lowers the volatility of equity returns.

The volatility of equity returns also depends on α_i. When α_1 is relatively large, trend followers respond immediately to a change in the equity price, and this increases equity price volatility. Contrarians, by contrast, when α_2 is relatively large, aggressively counter a change in the equity price, which moderates equity price volatility. Hence, the current model can explain the excess volatility of equity prices, particularly when there are many trend followers in the equity market and/or when trend followers have a short-term outlook.

Next, consider the effect of past equity returns on the current equity return. From equation (20), when the effective number of contrarians λ_2 is large relative to the number of trend followers λ_1, past equity returns positively affect the current equity return. As noted above, the existence of contrarians moderates the fluctuation in equity returns. Therefore, when there are many contrarians in the equity market, the equity price tends to underreact to a change in the fundamental value. This underreaction of the equity price generates the momentum effect.

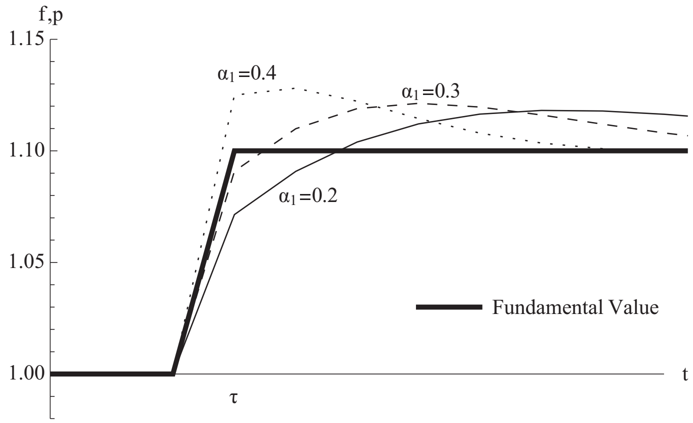
Equation (20) shows that α_i determines the duration of the momentum effect. The smaller α_1 is, the longer the momentum effect lasts. When α_1 is relatively small, trend followers respond gradually to past equity returns. Hence, the performance of past equity returns has a

FIG. 1. THE EFFECT OF λ_1 ON EQUITY PRICES



Notes: The other parameters $(\lambda_2, \alpha_1, \alpha_2)$ are fixed at (2.0,0.5,0.5).

FIG. 2. THE EFFECT OF α_1 ON EQUITY PRICES



Notes: The other parameters $(\lambda_1, \lambda_2, \alpha_2)$ are fixed at (3.0,2.0,0.5).

persistent effect on subsequent equity prices, and equity returns tend to exhibit momentum in that period. The longer the momentum effect lasts, the larger the deviation between the equity price and its fundamental value becomes. Then, informed investors begin to take advantage of this opportunity, and the equity price reverts to its fundamental value. Therefore, the current model can simultaneously explain the momentum effect of equity returns in the short run, and the mean-reverting effect in the long run.

To convey the intuition behind the effect of uninformed investors on the equity price, Figure 1 and Figure 2 illustrate the response of the equity price to a shock to the fundamental value. In these figures, the fundamental value increases by ten percent at time τ, then remains at that level.

Figure 1 shows the effect of λ_1 on the path of equity prices. As noted above, when λ_1 is relatively small, the equity price at time τ underreacts to the shock to the fundamental value, and then converges gradually to the fundamental value. When λ_1 is relatively large, the equity price overreacts to the shock to the fundamental value at time τ, then it returns to its fundamental value. Hence, the effective number of trend followers λ_1 relative to the number of contrarians λ_2 strongly affects the volatility of equity prices.

Similarly, Figure 2 shows the effect of α_1 on the path of equity prices. When α_1 is relatively small, the equity price continues to rise long after period τ. When α_1 is relatively large, the trend in equity prices disappears quickly, and the equity price reverts to its fundamental value. This result indicates that the value of α_1 relative to α_2 influences the persistence of the momentum effect.

III. *Empirical Analysis*

In this section, the parameter values of the equity returns process in equation (20) are estimated. The results indicate that differences in the behavior of returns among equity portfolios sorted by market values can be explained by the current model.

1. Data

The data series used in the estimation are monthly equity returns in the U.S. markets, which are obtained from the Center for Research in Security Prices (CRSP). It is well known that the equity returns of large firms and of small firms behave quite differently.[7] Hence, the equities listed on the NYSE, AMEX, and NASDAQ are sorted into five groups (CAP1 to CAP5) according to their market values. The CAP1 portfolio comprises equities whose market values belong to the bottom quintile in the markets. The CAP2 portfolio represents the second quintile in the market value, and the CAP5 portfolio represents the top quintile. The portfolios are rearranged at the beginning of every month according to the market values at the end of the preceding month. The sample period is from January 1979 to December 2004.

2. Basic Statistics

Table 1 presents the basic statistics of each portfolio in the sample period. The estimates $\hat{\mu}$ and $\hat{\sigma}^2$ denote the sample mean and variance of monthly returns in each portfolio, respectively, and $\widehat{VR}_n$ denotes the variance ratio of the nth successive monthly returns in each portfolio. As has been found in many empirical studies, for our sample period, Table 1 shows that the portfolios of small-cap equities earn higher average returns than the portfolios of large-cap equities. In particular, the average return of the CAP1 portfolio (0.025) is quite high, and it is about twice the average return in the CAP5 portfolio (0.012). The volatility of equity returns is high in the small-cap portfolios relative to that in large-cap portfolios.

To reveal the autocorrelations in the returns of each portfolio, columns three to seven of

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[7] For example, Jegadeesh and Titman (2001) report that momentum effects and mean-reverting effects differ in magnitude and persistence between portfolios of large-cap equities and those of small-cap equities.

TABLE 1. BASIC STATISTICS OF PORTFOLIO RETURNS

	$\hat{\mu}$	$\hat{\sigma}^2$	$\widehat{VR}_3$	$\widehat{VR}_6$	$\widehat{VR}_{12}$	$\widehat{VR}_{24}$	$\widehat{VR}_{36}$
CAP1	0.0251	0.0057	1.242*	1.183	0.996	0.720	0.678
CAP2	0.0134	0.0037	1.321*	1.253	1.135	0.751	0.778
CAP3	0.0131	0.0037	1.221*	1.059	0.914	0.510	0.497
CAP4	0.0132	0.0035	1.150	0.946	0.799	0.420	0.386
CAP5	0.0123	0.0024	1.055	0.895	0.816	0.541	0.512

Note: The asterisks indicate that the corresponding variance ratios are different from unity at the five percent significance level under the null hypothesis that any successive returns are uncorrelated, allowing for heteroscedasticity.

the table present the variance ratios for various months. The variance ratios of the nth successive monthly returns indicate whether there are autocorrelations among these returns. A variance ratio above (below) unity suggests positive (negative) autocorrelations among nth successive monthly returns. The variance ratios of three-month returns exceed unity in all portfolios, and these ratios are significantly different from unity in the CAP1 to CAP3 portfolios. This suggests that equity returns are positively autocorrelated in the short term, and that there is a clear momentum effect in small-cap portfolios. The fact that the variance ratios decrease as the number of successive returns increases suggest a mean-reverting effect in equity returns in the long term. Although the evidence for this mean-reverting effect is statistically weak, Table 1 shows that the variance ratios decrease more rapidly in large-cap portfolios. Therefore, the mean-reverting effect in equity returns is more apparent in large-cap portfolios.

In summary, the mean and the variance of equity returns are higher in small-cap portfolios than in large-cap portfolios. The momentum effect is stronger and more persistent in small-cap portfolios, and the mean-reverting effect is more apparent in large-cap portfolios.

3. Estimation

In this subsection, maximum likelihood is used to estimate the parameter values of the equity returns process for each portfolio. The parameters to be estimated are $\Phi^j=(\theta^j, (\kappa^j)^2, \lambda_1^j, \lambda_2^j, \alpha_1^j, \alpha_2^j)$, where the superscript $j=1, 2, \cdots, 5$ represents each portfolio. To estimate the parameter values, the equity returns process in equation (20) is approximated by the following AR(12) process:[8]

$$R^j(t)=\theta^j+\frac{\alpha_2^j\lambda_2^j X_2^j(t)-\alpha_1^j\lambda_1^j X_1^j(t)+\kappa^j\varepsilon^j(t)}{1-\alpha_1^j\lambda_1^j+\alpha_2^j\lambda_2^j},\ j=1, 2, \cdots, 5 \tag{23}$$

where:

$$X_i^j(t)=\sum_{s=1}^{12}\alpha_i^j(1-\alpha_i^j)^{s-1}R^j(t-s),\ i=1, 2, \tag{24}$$

$$\varepsilon^j(t) \overset{i.i.d}{\sim} N(0, 1), \tag{25}$$

[8] As a referee points out, the discretized process represented by equation (23) incorporating monthly data would be a rather rough approximation of the continuous process in equation (20), and it may be interpreted as a marginal case.

Table 2. Estimated Parameter Values

	$\hat{\theta}$	$\hat{\kappa}^2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
CAP1	0.0249	0.0052	1.456	0.741	0.394	0.743
	(0.0046)	(0.0018)	(0.539)	(0.372)	(0.255)	(0.070)
CAP2	0.0133	0.0039	2.070	1.677	0.592	0.772
	(0.0037)	(0.0011)	(0.256)	(0.247)	(0.204)	(0.128)
CAP3	0.0130	0.0033	2.317	1.804	0.623	0.784
	(0.0035)	(0.0009)	(0.223)	(0.218)	(0.177)	(0.128)
CAP4	0.0130	0.0027	2.048	1.417	0.584	0.775
	(0.0033)	(0.0008)	(0.235)	(0.225)	(0.166)	(0.114)
CAP5	0.0122	0.0018	1.598	1.024	0.553	0.738
	(0.0028)	(0.0005)	(0.263)	(0.250)	(0.228)	(0.158)

Note: The numbers in parentheses are their asymptotic estimated standard errors.

and $R^j(t)$ denotes the equity return of portfolio j at time t.

By using the approximated returns process in equation (23), we can construct the likelihood function for the sample returns of each portfolio. The likelihood function for the portfolio returns is as follows:

$$L^j = \sum_{t=1}^{T} \log h\ (R^j(t)\,|\,\mathcal{R}^j(t-1),\ \Phi^j),\ j=1, 2, \cdots, 5, \tag{26}$$

where:

$$h(R^j(t)\,|\,\mathcal{R}^j(t-1),\ \Phi^j) = \frac{1}{\sqrt{2\pi(\sigma^j)^2}} \exp\left\{-\frac{(R^j(t)-\mu^j(t))^2}{2(\sigma^j)^2}\right\}, \tag{27}$$

$$\mu^j(t) = \theta^j + \frac{\alpha_2^j \lambda_2^j X_2^j(t) - \alpha_1^j \lambda_1^j X_1^j(t)}{1-\alpha_1^j \lambda_1^j + \alpha_2^j \lambda_2^j}, \tag{28}$$

$$(\sigma^j)^2 = \left(\frac{\kappa^j}{1-\alpha_1^j \lambda_1^j + \alpha_2^j \lambda_2^j}\right)^2, \tag{29}$$

and $\mathcal{R}^j(t-1) = (R(1), R(2), \cdots, R(t-1))$ denotes the realized returns of portfolio j up to time $t-1$. Then, a search is conducted for a combination of parameter values Φ^j that maximizes equation (26) for each j.

Table 2 presents the estimation results. First, the estimated values of θ^j are almost identical to the average returns of the corresponding portfolios presented in Table 1. This is a direct consequence of equation (21), which implies that the average equity return coincide unconditionally with the growth rate of the fundamental values.

Second, the estimated values of $(\kappa^j)^2$ are somewhat smaller than the volatilities of the corresponding portfolio returns, except for the CAP2 portfolio. From the estimated values of λ_1^j and λ_2^j, the effective number of trend followers exceed the number of contrarians in all equity markets. This market structure generates excess volatility in equity returns in our model. However, the level of excess volatility is below that reported by Shiller (1981).

Table 2 shows that the values of α_1^j fall short of those of α_2^j in all equity markets, which generates the momentum effect in our model. In particular, the differences between α_2^j and α_1^j

are large for the CAP1 portfolios, which makes the momentum effect more persistent in these portfolios.

To summarize the results, there are more trend followers than contrarians in the U.S. equity markets. This exacerbates volatility in equity returns. Further, uninformed investors, particularly trend followers, behave differently in the equity markets of small and large firms. Trend followers operating in the market for the small firms believe that trends in equity prices persist for long time. Hence, current equity returns have a persistent effect on the equity demands of trend followers. This persistence generates strong and lasting momentum effects in the equities of small firms. By contrast, trend followers in the market for equity of large firms believe that trends in equity prices hardly persist at all. Hence, their response to past equity returns is transient, and this subsequently generates a mean-reverting effect.

IV. *Concluding Remarks*

In the current paper, we developed an equity pricing model in which investors are heterogeneous in both their information sets and their beliefs. In particular, we showed that an uninformed investor who mistakenly believes that the equity price is less (more) volatile than is its fundamental value behaves like a trend follower (contrarian). Then, the equity price, which reflects the beliefs of uninformed investors, can deviate from its fundamental value. In particular, the current model can simultaneously explain excess volatility, the momentum effect, and the mean-reverting effect in equity returns. Further, our empirical results indicate that the model can explain differences in the behavior of the returns of large-cap and small-cap equities.

However, we have made strong assumptions to derive the equity price. For example, all investors are irrational in the sense that they do not utilize all of the information accessible to them, and in the sense that they stick to their own incorrect beliefs having observed a large sample of equity prices. Unless obstacles prevent investors from learning about the processes that determine economic variables, investors should correct their errors having observed enough samples.

Further, investors who continue to misunderstand the processes determining important economic variables would leave the market in the long run. In this sense, our model lacks dynamics describing market structure. Therefore, the next step is to introduce these dynamics into the model and determine their effect on equity prices. This is a task for our future research.

Appendix

This appendix explains how equations (17) and (20) are obtained. First, we derive the equity price in equation (17). Because informed investors can observe the fundamental value of equity, their expected equity return at time t is $\mu_0(t)$ in equation (3) with $i=0$. Substituting this into equation (14) yields the following expression for the equity demand of informed investors:

$$D_0(t)=\phi_0\left(\frac{\theta_0+\gamma_0\{f(t)-p(t)\}}{\sigma}\right)+C_0. \tag{A.1}$$

The expected equity return for type-i uninformed investors is expressed by equation (12). By substituting this into equation (15) yields the following expression for the equity demand of uninformed investors:

$$D_i(t)=\phi_i\left(\frac{\theta_i+\delta_i x_i(t)}{\sigma}\right)+C_i, \text{ for } i=1, 2. \tag{A.2}$$

By substituting equations (A.1) and (A.2) into the market clearing condition in equation (16), and rearranging terms, the equity price in equation (17) is obtained.

Second, we derive the equity returns process in equation (20). From equation (17), the equity return at time t is:

$$dp(t)=df(t)+\lambda_1 dx_1(t)-\lambda_2 dx_2(t). \tag{A.3}$$

Under $\mathbb{P}$, the fundamental value follows the stochastic process represented by equation (1). By applying Ito's formula to equation (9), the fluctuation of $x_i(t)$ for $i=1$, 2 can be expressed as follows:

$$dx_i(t)=\alpha_i\{dp(t)-\theta_i dt-x_i(t)dt\},\ i=1, 2. \tag{A.4}$$

Substituting equations (1) and (A.4) into equation (A.3), and using the assumption that $\theta_i=\theta$ for all i, yields the equity returns process represented by equation (20).

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