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Some Financial Applications of Backward Stochastic Differential Equations with jump

– Utility, Investment, and Pricing –

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Chapter 1

Introduction

1.1 Motivation for this study

Utility maximization problems occupy an important role in Mathematical Finance. This thesis utilizes Backward Stochastic Differential Equations (BSDEs) to investigate utility maximization problems in incomplete markets whose states are driven by jump-diffusion processes. This paper focuses on two types of problems: Stochastic Differential Utility (SDU) and Utility Indifference Pricing.

Firstly, we look at a portfolio-consumption strategy optimization problem that maximizes SDU with jump. The conventional asset pricing model in financial economics is structured under the assumption that agents’ preferences have a time-additive von Neumann-Morgenstern representation. However, this model has been criticized for related two reasons: First, it has failed to perform well when applied to real market data. The equity premium puzzle of Mehra and Prescott [40] is one of the most famous counter examples. Second, the above specification of the utility confounds risk aversion and intertemporal substitutability, two aspects of preference that are conceptually different.

Motivated by these drawbacks of the conventional model, Epstein and Zin [14] studied recursive intertemporal utility functions that generalize the conventional, time-additive, expected utility in a discrete-time setting. In their following paper [15], the empirical results showed that the recursive utility improved the goodness-of-fit of the model. Duffie and Epstein [10] developed the stochastic differential utility (SDU), which was a continuous-time analogue of the recursive utility of Epstein and Zin [14]. These utility functions allow for a degree of separation between substitution and risk aversion. This means that it is possible to model the timing preference of the resolution of uncertainty, while a usual static utility is indifferent to it. In their subsequent paper [11], the state-price density process of the normalized SDU was formulated explicitly by using the utility gradient approach
of Duffie and Skiadas [13] in the case of the information generated by the Brownian motion. El Karoui, Peng, and Quenez [16] showed that the maximization problem of SDU can be solved by using the BSDE.

In chapter 2, we first extend the result of El Karoui, Peng, and Quenez [16] to the case of an incomplete market whose states are driven by a jump-diffusion process, which includes the so-called regime-shifting model. To solve the problem, we describe the first order condition and show that the condition is necessary and sufficient. The optimal wealth and utility processes are characterized as the solutions of a forward-backward system.

Next, we discuss a reaction-diffusion system in chapter 3. The regime shift phenomena, which have recently been highlighted in financial industry, are observed under real conditions with both synchronous and non-synchronous regime-shifting drift and volatility. These real conditions include structural changes of economy which provide abrupt asset price changes in the financial markets; unexpected changes of the health state of an insured person that induce some pre-specified payments to the insurance policy holder, and so forth. Recently, Becherer and Schweizer [3] have described this type of regime shift phenomena by reaction-diffusion systems which are known to appear in biological pattern formations or chemical reactions, etc. In their model, the regime shift is simply represented by the Poisson jumps of an integer-valued index.

In the chapter, we study the optimal investment/consumption strategies based upon SDU in the case that the asset prices follow the reaction-diffusion system by utilizing the result obtained in chapter 2. We shall furthermore propose a numerical procedure for solving the associated nested quasi-linear partial differential equations (PDEs). Concrete optimal portfolio and consumption are obtained.

Next, Utility Indifference Pricing is studied. The valuation of dynamic risks has been one of the fundamental issues concerning financial institutions. When there is a financial market that is complete, the price of a contingent claim can be determined uniquely. The pricing theory in the complete market is based on a strategy in which one creates a portfolio that replicates the payoff of the claim. The risk associated with the contingent claim is completely removed. The unique price of the contingent claim is provided by the initial wealth necessary to fund the replicating portfolio. However, in the incomplete market where it is impossible to construct the replicating portfolio, there are many different prices that are consistent with no-arbitrage and each corresponds to a different martingale measure. In addition to that, financial institutions’ valuation and their (partial) hedging strategy should be taken into account along with their attitudes toward and preferences for risks. These institutions can maximize their utility of wealth and can pay (or sell) a certain amount today for the right to receive (or pay) the claim.
such that they are no worse off in terms of terminal utility than they would have been without the claim. This concept was introduced by Hodges and Neuberger [22] for the valuation of European calls in the presence of transaction costs. This premium principle is known as the principle of equivalent utility among actuaries; see Gerber and Pafumi [20] for a detailed overview. The pricing and reserving of insurance contracts are important problems for actuaries. Gerber [19] extended this principle to calculate reserves, the so-called utility indifference reserve.

In this thesis, we use BSDE to calculate a utility indifference pricing of a contingent claim that contains untradable risks in the market. The BSDE approach is introduced by El Karoui and Rouge [47] to calculate the value function and the optimal portfolio. They formulated the dual value function as BSDE and the optimal portfolio under the constrained portfolio situation is obtained. More directly, Müller et al. [24] and Lim [30, 31] showed the value function itself as BSDE.

In chapter 4, we propose another type of formulation of value function in BSDE form, which has some tractable properties. The formulation is applied to the utility indifference pricing and reserving problem when a contingent claim contains untradable risks which follow Brownian motion and Poisson jump. Moreover, we present some concrete numerical examples by applying a result of the Malliavin calculus.

1.2 Abstract and focus for each chapter

We state the main result of this thesis by chapter as follows.

Chapter 1
This Chapter.

Chapter 2
This chapter studies the maximization problem of the SDU in a market whose states are driven by a jump-diffusion process in continuous-time settings. The stochastic maximum principle is obtained. We assume that the drifts and volatilities of risky assets follow the compound Poisson process with a finite jump distribution. The settings include the so-called regime-shifting model.

The first contribution of this chapter is to show the comparison theorem of BSDEs that are driven by jump-diffusion processes. To do so, we prove the existence and uniqueness of the adjoint process of linear BSDEs with jumps. The method of proof of the comparison theorem is different from one by Situ [51]. The second contribution is to derive the first order condition for optimality and a state-price density process explicitly by using the comparison theorem. The optimal wealth and utility processes are characterized as the solutions of a forward-backward sys-
tem, which is applicable to similar problems. Finally, we consider a normalization of aggregator of SDU with jump and derive new 4 types of normalized aggregator.

Chapter 3
We study the dynamic investment strategies in continuous-time settings based upon SDU. We assume that the asset prices follow interacting Itô-Poisson processes, which are known to be the so-called reaction-diffusion systems. Stochastic maximum principle for stochastic control problems described by some BSDEs that are driven by Poisson jump processes allows us to derive the optimal investment strategies as well as optimal consumption. We shall furthermore propose a numerical procedure for solving the associated nested quasi-linear partial differential equations (PDEs).

Chapter 4
This chapter deals with the utility indifference approach to the pricing and reserving problem in incomplete markets when the utility is time-additive. We propose new BSDE method to calculate a utility indifference price and reserve of a contingent claim that contains untradable risks in the market. This chapter makes the following contributions. Firstly, we derive the linear BSDE expression of the value function. Owing to the characteristics of a linear BSDE, the value function is formulated as an expectation of forward processes. The method enables a forward calculation of the value function. Secondly, we derive an expression of the utility indifference price where it is decomposed into the present value under Minimum Martingale Measure and its certainty equivalent. Thirdly, in addition to the case that the untradable risk follows Brownian motion, we also consider the case of Poisson jump. Finally, we present some concrete numerical examples by applying a result of the Malliavin calculus.

Chapter 5
Conclusion.
Chapter 2

Stochastic Differential Utility whose states are driven by jump diffusion process

2.1 Introduction

In this chapter, we consider the continuous-time portfolio-consumption problem of an agent with recursive utility who is operating in a market whose states are driven by a jump-diffusion process (compound Poisson processes with a finite jump distribution).

The conventional asset pricing model in financial economics is structured under the assumption that agents’ preferences have a time-additive von Neumann-Morgenstern representation. However, this model has been criticized since the specification confuses risk aversion with intertemporal substitutability, two aspects of preference that are conceptually different.

Duffie and Epstein [10] developed the stochastic differential utility (SDU), which is a continuous-time analogue of the recursive utility of Epstein and Zin [14]. These utility functions allow for a degree of separation between substitution and risk aversion. This means that it is possible to model the timing preference of the resolution of uncertainty, while a usual static utility is indifferent to it.

In their subsequent paper [11], the state-price density process of the normalized SDU was formulated explicitly by using the utility gradient approach of Duffie and Skiadas [13] in the case of the information generated by the Brownian motion. Recently, there was an argument on the CCAPM under SDU in an economy with a jump-diffusion process [28].

El Karoui, Peng, and Quenez [16] showed that the maximization problem of SDU can be solved by using the backward stochastic differential equation
(BSDE), which was advocated by Pardoux and Peng [45]. By using this result, they solved the optimization problem of SDU when the agent faces a nonlinear wealth process under the influence of the taxation system or uncompetitive price formation [17].

Lazrak and Zapatero [29] characterized a set of consumption processes which are optimal for a given SDU by using the forward-backward system of El Karoui et al. when the consumer’s beliefs about future asset returns are unknown. This problem is known as the inverse problem of Mas-Colell [39], and also known that the parameters of a utility function are not able to be determined uniquely only from a given demand curve. Lazrak and Zapatero presented the martingale conditions and advocated the verification method which uses a short rate.

In this chapter, we consider a small agent who operating within a market in which the stocks’ drifts and volatilities are driven by a jump-diffusion process, including the so-called regime-shifting model. First, this chapter aims to show the comparison theorem for BSDEs with jumps. To do so, we consider the adjoint process of a linear BSDE and its solution in a closed form. The method we use here to prove the comparison theorem is different from one by Situ [51]. Using this property, the backward formulation of the optimization problem is obtained. Second, this chapter aims to derive a first order condition that yields a necessary and sufficient condition of optimality. From this condition, the optimal wealth and utility and their associated deflators are obtained as the solutions of a forward-backward system.

The rest of the chapter is organized as follows: In section 2, we define the wealth and the utility processes and establish the problem formulation. In section 3, we study the comparison theorem and the backward formulation of the main problem. In section 4, we describe the first order condition and show that the condition is necessary and sufficient. In section 5, we derive a forward-backward system. In section 6, we consider a normalization of aggregator of SDU with jump and derive new 4 types. Finally, in section 7, we discuss the inverse problem when the consumer’s beliefs are driven by a jump-diffusion process.

2.2 Problem formulation

2.2.1 Backward stochastic differential equations with jumps

For the sake of analysis, certain notations need to be defined. Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), an \(\mathbb{R}^m\)-valued Brownian motion \(W\) and one random measure \(N(dt, de)\), which has a predicable measure \(v(de)dt\). \(\{\mathcal{F}_t; t \in [0, T]\}\) denotes the filtration generated by a Brownian motion \(W\) and a random measure \(N(dt, de)\), i.e., \(\mathcal{F}_t := \sigma(W_s, s \leq t) \vee \sigma(N(s, e), s \leq t) \vee \mathcal{N}\), where \(\mathcal{N}\) denotes the totality
of $\mathbb{P}$-null sets. $\tilde{N}(dt,de)$, the compensated martingale measure, is denoted such that $\tilde{N}(dt,de) = N(dt,de) - \nu(de)dt$. For $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm such that $|x| = \sqrt{\text{trace}(x^tx)}$, $\langle x, y \rangle$ denotes an inner product such that $\langle x, y \rangle = \text{trace}(x^ty)$, and $^*$ denotes a transpose.

Let $\mathbb{H}^2(\mathbb{R}^n)$ be the space of all measurable random variables $X : \Omega \rightarrow \mathbb{R}^n$ satisfying $||X|| < \infty$, where $||X|| := (E[|X|^2])^{\frac{1}{2}}$, and $\mathbb{H}^2_{\nu}(\mathbb{R}^n)$ be the space of all predictable process $\phi : \Omega \times [0,T] \rightarrow \mathbb{R}^n$ satisfying $||\phi|| < \infty$, where $||\phi|| := (E[\int_0^T |\phi(t)|^2 dt])^{\frac{1}{2}}$. Further, let $\mathbb{L}_{\nu(\cdot)}(\mathbb{R}^n)$ be the space of all measurable $\mathbb{R}^n$-valued functions $\phi$ such that $||\phi||_{\nu(\cdot)} < \infty$, where $||\phi||_{\nu(\cdot)} := (\int_\Omega |\phi(e)|^2 \nu(de))^\frac{1}{2}$. For $\beta > 0$ and $\phi \in \mathbb{H}^2_{\nu,\beta}(\mathbb{R}^n)$, $\|\phi\|_{\beta}$ denotes $E[\int_0^T e^{\beta |\phi|^2(t)dt}]$. $\mathbb{H}^2_{T,\nu,\beta}(\mathbb{R}^n)$ denotes the space $\mathbb{H}^2_{\nu}(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{\beta}$. The norm $\|\phi\|_{\nu(\cdot),\beta}$ also denotes $E[\int_0^T e^{\beta |\phi|^2(t)N(dt,de)}]$, where $\phi \in \mathbb{H}^2_{T,\nu,\beta}(\mathbb{R}^n)$. $\mathbb{H}^2_{T,\nu,\beta}(\mathbb{R}^n)$ denotes the space $\mathbb{H}^2_{T,\nu,\beta}(\mathbb{R}^n)$ endowed with the norm $\|\cdot\|_{\nu(\cdot),\beta}$.

Consider the BSDE,

$$-dY_t = f(t, Y_{t^-}, Z_t, \Lambda_t(\cdot))dt - Z_t^wdW_t - \int_{\mathbb{R} \setminus [0]} \Lambda_t(e)\tilde{N}(dt,de), \quad Y_T = \zeta, \quad (2.1)$$

which is equivalent to its integral equation

$$Y_t = \zeta + \int_t^T f(s, Y_{s^-}, Z_s, \Lambda_s(\cdot))ds - \int_t^T Z_s^wdW_s - \int_t^T \int_{\mathbb{R} \setminus [0]} \Lambda_s(e)\tilde{N}(ds,de), \quad (2.2)$$

where

$$f(t, Y_{t^-}, Z_t, \Lambda_t(\cdot)) = \int_{\mathbb{R} \setminus [0]} \zeta(t, Y_{t^-}, Z_t, \Lambda_t(e))\nu(de).$$

Here, let us introduce the standard parameters of the BSDE.

**Definition 2.2.1.** $(f, \zeta)$ are said to be the standard parameters for the BSDE, where $\zeta \in \mathbb{H}^2(\mathbb{R}^n)$, $f \in \mathbb{H}^2_{T,\nu(\cdot)}(\mathbb{R}^n)$, and $f$ is uniformly Lipschitz, i.e., there exists $L$ such that

$$|f(\omega, t, y_1, z_1, \lambda_1(\cdot)) - f(\omega, t, y_2, z_2, \lambda_2(\cdot))| \leq L (|y_1 - y_2| + |z_1 - z_2| + |\lambda_1 - \lambda_2|_{\nu(\cdot)})$$

for $\forall (y_1, z_1, \lambda_1(e)), \forall (y_2, z_2, \lambda_2(e))$. 

7
It is shown by Tang and Li [52] that there exists a unique triple \((Y, Z, \Lambda(e))\) that solves the BSDE possessing the standard parameters.

**Theorem 2.2.2.** Given the standard parameters \((f, \zeta)\), there exists a unique solution \((Y, Z, \Lambda(e)) \in \mathbb{H}_T^2(\mathbb{R}^n) \times \mathbb{H}_T^2(\mathbb{R}^{m \times n}) \times \mathbb{H}_T^2(\mathbb{R})\), which solves (2.1).

**Proof.** In this paper, we employ a different approach to the proof of this problem from Tang and Li. See the Appendix.

### 2.2.2 The utility process

Duffie and Epstein [10] introduced a stochastic differential utility in a continuous case under uncertainty. This utility permits the risk aversion to be separated from the degree of intertemporal substitutability. Let us consider a small agent who consumes \(c_t\) at time \(t \in [0, T]\). The utility at time \(t\) is the expectation of the function of the future consumption and utility. The stochastic differential utility at time \(t\) is recursively defined by

\[
Y_t := E \left[ \int_t^T f(s, c_s, Y_s, Z_s, \Lambda_s(\cdot)) ds + Y_T | \mathcal{F}_t \right].
\] (2.3)

In the remaining sections of this paper, we mainly use the utility process, which is expressed in the following differential form:

\[-dY_t = f(t, c_t, Y_t, Z_t, \Lambda_t(\cdot)) dt - Z_t^* dW_t - \int_{\mathbb{R} \setminus 0} \Lambda_t(e) \tilde{N}(dt, de) \quad Y_T = \zeta, \] (2.4)

where the driver of this BSDE is

\[f(t, c_t, Y_t, Z_t, \Lambda_t(\cdot)) = \int_{\mathbb{R} \setminus 0} g(t, c_t, Y_t, Z_t, \Lambda_t(e)) \nu(de).\]

In section 2.7, we give concrete examples of the stochastic differential utility, including the standard time-additive utility.

We make certain assumptions in order to ensure that the BSDE has a unique solution.

**Assumption 2.1.** \(f\) is supposed to be uniformly Lipschitz with respect to \(Y, Z,\) and \(\Lambda(y)\).

**Assumption 2.2.** There exists some constants \(k_1\) and \(k_2\) such that

\[|f(t, c, 0, 0, 0)| \leq k_1 + k_2 \frac{c^p}{p} \quad \text{a.s. with } 0 < p < 1 \quad \forall c \in \mathbb{R}^+.\]
These assumptions ensure that the BSDE (2.4) has a unique solution \( (Y, Z, \Lambda) \in H^2_T(R) \times H^2_T(R^{m \times 1}) \times H^2_{T,v(\cdot)}(R) \) for each consumption \( c \in H^2_{T,v(\cdot)} \) and terminal utility \( Y_T \in H^2(R) \).

Two more additional assumptions may be necessary in the later sections.

**Assumption 2.3.** \( f \) is strictly concave with respect to \( c, Y, Z, \) and \( \Lambda(e) \), and \( f \) is a strictly nondecreasing function with respect to \( c \).

By using the comparison theorem (Theorem 2.3.2), which is to be discussed in the next section, Assumption 2.3 ensures the usual properties of utility functions, that is, monotonicity with respect to the terminal value and to the consumption as well as concavity with respect to the consumption.

In general, the terminal value \( Y_T \) is a utility of the terminal wealth \( X_T \). We assume that \( Y_T = u(X_T) \) where \( u \) satisfies the assumption given below.

**Assumption 2.4.** \( u \) is a strictly concave and nondecreasing real function defined on \( R \times \Omega \), which is \( \mathcal{F}_t \times \mathcal{B}(R) \) measurable and which also satisfies
\[
|u(x)| \leq k_1 + k_2 x^p \quad \text{a.s. with } 0 < p < 1 \quad \forall x \in R^+.
\]

This assumption ensures that the variable \( u(X_T) \in H^{2/p}(R) \subset H^2(R) \) for each \( X_T \in H^2(R) \) and that the stochastic differential utility associated with this terminal value is increasing and concave with respect to terminal wealth.

### 2.2.3 The wealth process

In this subsection, we focus on the specification of the dynamics of the wealth process. Suppose that there exists \( n+1 \) assets in which the agent can invest some of his wealth in the market. One is a riskless asset, the price process \( P_0^0 \) of which is
\[
dP_0^0 = P_0^0 r_t dt, \tag{2.5}
\]
where \( r_t \) is a short rate. In addition to the bond, \( n \) risky assets (the stocks) are traded continuously. The price process of \( i \)th stock \( P_i^0 \) is
\[
\frac{dP_i^t}{P_i^0} = a_i^t(\eta_{t-})dt + \sum_{j=1}^{m} \sigma_{i,j}^t(\eta_{t-})dW_j^t \tag{2.6}
\]
\[
d\eta_t = \int_{R \setminus 0} d\eta_t(e)N(dt, de). \tag{2.7}
\]
The finite state process \( \eta \) driven by compound Poisson process may be specified as the total loss of a catastrophic event in a country or as a technological innovation.
and so on. The volatility matrix $\sigma_t(\eta) = (\sigma_t^{i,j}(\eta))$ is supposed to have full rank for $\forall \eta$.

The agent invests in these assets with a portfolio $\pi_t = (\pi_t^1, \pi_t^2, \cdots, \pi_t^n)^*$ and $\pi_t^0 = X_t - \sum_{i=1}^n \pi_t^i$. In these settings, the wealth process of the agent is

$$dX_t = (X_t - \sum_{i=1}^n \pi_t^i) \frac{dP^0_t}{P_t^0} + \sum_{i=1}^n \pi_t^i \frac{dP_t^i}{P_t^i} - c_t dt$$

$$= (X_t r_t + \pi_t^* \sigma_t(\eta_t^-) \theta_t(\eta_t^-) - c_t) dt + \pi_t^* \sigma_t(\eta_t^-) dW_t,$$

where $\theta_t$ is a risk premium vector, such that $a_t(\eta_t^-) - r_t 1 = \sigma_t(\eta_t) \theta_t(\eta_t)$.

A general setting for the wealth process can be given by

$$-dX_t = a(t, c_t, X_t, \pi_t^* \sigma_t(\eta_t^-)) dt - \pi_t^* \sigma_t(\eta_t^-) dW_t,$$

$$X_0 = x.$$  (2.9)

Let us consider similar assumptions to the utility process with respect to the driver $a$ of the wealth process.

**Assumption 2.5.** $a$ is supposed to be uniformly Lipschitz with respect to $X$, $\pi^* \sigma$.

**Assumption 2.6.** There exists a positive constant $k$ such that for $\forall c \in \mathbb{R}^+$, $|a(t, c, 0, 0)| \leq kc a.s.$

**Assumption 2.7.** $a$ is convex with respect to $c, x$, and $\pi^* \sigma$, and $a$ is nondecreasing with respect to $c$.

**Assumption 2.8.** $a(t, c, 0, 0) \geq 0 a.s.$ for $\forall c \in \mathbb{R}^+$.

Let $(X_t^{(x,c,\pi)}; 0 \leq t \leq T)$ be the wealth process associated with initial wealth $x$ and strategy $(c, \pi) \in \mathbb{H}^2_T,\nu(\mathbb{R}) \times \mathbb{H}^2_T,\nu(\mathbb{R}^n)$. One may notice that given an initial wealth and portfolio, there exists a pathwise unique wealth process that solves the forward equation (2.9) since $a$ is Lipschitz. The agent has to choose a portfolio-consumption strategy $(c, \pi)$ feasible for the initial wealth $x$, that is, $(c, \pi) \in \mathbb{H}^2_T,\nu(\mathbb{R}) \times \mathbb{H}^2_T,\nu(\mathbb{R}^n)$ with $c_t \geq 0$ and $X_t^{c,\pi} \geq 0$ $dP \otimes dt$ a.s. $\mathcal{A}(x)$ denotes the set of portfolio strategies $\pi$ feasible for the initial wealth $x$.

### 2.2.4 Problem formulation of the maximization of recursive utility

Let us consider an agent who is endowed with initial wealth $x > 0$, invests at each time $t$ with portfolio $\pi_t$, and consumes $c_t$. Recall that the wealth process of the agent is

$$-dX_t^{(x,c,\pi)} = a(t, c_t, X_t^{(x,c,\pi)}, \pi_t^* \sigma_t(\eta_t^-)) dt - \pi_t^* \sigma_t(\eta_t^-) dW_t,$$

$$X_0^{(x,c,\pi)} = x.$$  (2.10)
The agent may choose a portfolio-consumption strategy that maximizes his or her utility of consumption and terminal wealth. His or her utility process is

\[-dY^{(x,c,\pi)}_t = f(t, c_t, Y^{(x,c,\pi)}_t, Z_t, \Lambda_t(\cdot))dt - Z^*_t dW_t - \int_{\mathbb{R}\setminus{0}} \Lambda_t(e)\tilde{N}(dt, de)\]

\[Y^{(x,c,\pi)}_T = u(X_T).\] (2.11)

Under these assumptions, we define the maximization problem of the stochastic differential utility as follows:

**Problem 1.** Find the optimal consumption and portfolio \((c, \pi) \in \mathcal{A}(x)\) satisfying

\[
\sup_{(c, \pi) \in \mathcal{A}(x)} Y^{(x,c,\pi)}_0.
\] (2.12)

### 2.3 Backward formulation of the maximization problem

In this section, we first prove the comparison theorem (Theorem 2.3.2) of BSDE driven by a jump-diffusion process. By using the theorem, the backward formulation of the optimization problem is obtained.

Let us start with the following proposition, which is necessary in order to prove comparison theorem.

**Proposition 2.3.1.** (Adjoint process) Let \((\alpha, \beta, \gamma(e))\) be a bounded \(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}\)-valued predictable process for \(\forall e \in \mathbb{R}\setminus\{0\}\). Suppose that \(\varphi \in \mathbb{H}^2(\mathbb{R})\), \(\zeta \in \mathbb{H}^2(\mathbb{R})\) and \(\Lambda \in L^2_{\mathcal{V}(\cdot)}(\mathbb{R})\). Then, the linear BSDE

\[-dY_t = \left[\varphi_t + Y_t - \alpha_t + Z^*_t \beta_t + \int_{\mathbb{R}\setminus{0}} \Lambda_t(e)\gamma(e)\nu(de)\right]dt - Z^*_t dW_t\]

\[-\int_{\mathbb{R}\setminus{0}} \Lambda_t(e)\tilde{N}(dt, de),\]

\[Y_T = \zeta,\]

has a unique solution \((Y, Z, \Lambda(e))\) and \(Y_t\) is given by the closed form

\[Y_t = E \left[ \Gamma_T \zeta + \int_T^t \Gamma_s \varphi_s ds \mid \mathcal{F}_t \right],\]

where \(\Gamma_t\) is the adjoint process defined by the linear forward SDE,

\[\frac{d\Gamma_t}{\Gamma_t} = \alpha_t dt + \beta_t^* dW_t + \int_{\mathbb{R}\setminus{0}} \gamma(e)\tilde{N}(dt, de), \quad \Gamma_t = 1.\] (2.13)
Proof. Since $\alpha$, $\beta$, and $\gamma(e)$ are bounded processes, the linear driver $f(t, Y, Z, \Lambda(\cdot)) = \varphi_t + \alpha_t Y + \beta_t^* Z + \int_\mathbb{R} \gamma(e) \Lambda(e) \nu(de)$ satisfies

$$|f(t, Y_1, Z_1, \Lambda_1(\cdot)) - f(t, Y_2, Z_2, \Lambda_2(\cdot))| \leq L(|Y_1 - Y_2| + |Z_1 - Z_2| + |\Lambda_1 - \Lambda_2|_\nu),$$

for some constant $L$. By virtue of Theorem 2.2.2, there exists a unique solution $(Y, Z, \Lambda)$ of the linear BSDE.

Applying Ito’s formula to $\Gamma_t Y_t$, we find that

$$d(\Gamma_t Y_t) = -\varphi_t \Gamma_t - \Gamma_t \{Y_t - \beta_t^* + Z_t^*\} dW_t + \int_{\mathbb{R}\setminus\{0\}} \{Y_t - \Gamma_t - \gamma(e) + \Gamma_t - \Lambda_t(e) + \gamma_t(e)\} \tilde{N}(dt, de).$$

Therefore, the process $d(\Gamma_t Y_t) + \varphi_t \Gamma_t dt$ is a local martingale. By integrating and computing the conditional expectation, we obtain

$$E\left[\Gamma_T Y_T - \Gamma_t Y_t + \int_t^T \varphi_s \Gamma_s ds \bigg| \mathcal{F}_t\right] = 0.$$

This leads to the proposition.

An additional assumption is needed for the next theorem.

**Assumption 2.9.** The function $g$ is supposed to be Lipschitz with respect to $\Lambda_t(e)$ for $\forall e$, i.e., there exists $L$ such that

$$|g(t, c, y, z, \Lambda_1(e)) - g(t, c, y, z, \Lambda_2(e))| \leq L |(\Lambda_1(e) - \Lambda_2(e))|.$$

Owing to Proposition 2.3.1 and Assumption 2.9, we can obtain the following comparison theorem.

**Theorem 2.3.2.** (Comparison theorem) Suppose that the assumptions hold, let the two BSDEs be

$$-dY_t^i = f^i(t, c_t, Y_t^i, Z_t^i, \Lambda_t^i(\cdot)) dt - Z_t^i dW_t - \int_{\mathbb{R}\setminus\{0\}} \Lambda_t^i(e) \tilde{N}(dt, de),$$

$$Y_T^i = \xi^i,$$

where $i = 1, 2$; further, let us suppose that

$$\xi^1 \geq \xi^2,$$

$$f^1(t, c, Y, Z, \Lambda(\cdot)) \geq f^2(t, c, Y, Z, \Lambda(\cdot)).$$

Then, we have

$$Y_t^1 \geq Y_t^2, \quad \text{a.s.} \quad \forall t \in [0, T].$$
Proof. Let \( \delta Y_t = Y^1_t - Y^2_t, \delta Z_t = Z^1_t - Z^2_t \) and \( \delta \Lambda_t(e) = \Lambda^1_t(e) - \Lambda^2_t(e) \). The following BSDE is immediately obtained:

\[
-d\delta Y_t = (f^1(t, c_t, Y^1_{t-}, Z^1_{t-}, \Lambda^1_t(\cdot)) - f^2(t, c_t, Y^2_{t-}, Z^2_{t-}, \Lambda^2_t(\cdot))) dt - \delta Z_t dW_t
\]

\[
-\int_{\mathbb{R} \setminus \{0\}} \delta \Lambda_t(e) \tilde{N}(dt, de),
\]

\[
\delta Y_T = \xi^1 - \xi^2.
\]

Then, we obtain

\[
f^1(t, c_t, Y^1_{t-}, Z^1_{t-}, \Lambda^1_t(\cdot)) - f^2(t, c_t, Y^2_{t-}, Z^2_{t-}, \Lambda^2_t(\cdot)) = \delta_2 f_t + \Delta_Y f^1(t) \delta Y_{t-} + \Delta_Z f^1(t)^* \delta Z_t + \int_{\mathbb{R} \setminus \{0\}} \Delta g^1(t, e) \delta \Lambda_t(e) \nu(de),
\]

where

\[
\delta_2 f_t = f^1(t, c_t, Y^2_{t-}, Z^2_{t-}, \Lambda^2_t(\cdot)) - f^2(t, c_t, Y^2_{t-}, Z^2_{t-}, \Lambda^2_t(\cdot)),
\]

\[
\Delta_Y f^1(t) = \frac{f^1(t, c_t, Y^1_{t-}, Z^1_t, \Lambda^1_t(\cdot)) - f^1(t, c_t, Y^2_{t-}, Z^1_t, \Lambda^1_t(\cdot))}{Y^1_{t-} - Y^2_{t-}},
\]

\[
\Delta_Z f^{1,i}(t) = \frac{f^1(t, c_t, Y^2_{t-}, \bar{Z}^i_{t-}, \Lambda^1_t(\cdot)) - f^1(t, c_t, Y^2_{t-}, \bar{Z}^i_{t-}, \Lambda^1_t(\cdot))}{\bar{Z}^{i,1}_{t-} - \bar{Z}^{i,2}_{t-}}.
\]

\[
\Delta g^1(t, e) = \frac{g^1(t, c_t, Y^2_{t-}, Z^2_{t-}, \Lambda^1_t(e)) - g^1(t, c_t, Y^2_{t-}, Z^2_{t-}, \Lambda^2_t(e))}{\Lambda^1_t(e) - \Lambda^2_t(e)}.
\]

Here, \( \bar{Z}^i_t \) denotes the vector of \( (Z^{2,1}_{t-}, ..., Z^{2,i}_{t-}, Z^{1,i+1}_{t-}, ..., Z^{1,m}_{t-}) \).

Since the driver \( f^1 \) is uniformly Lipschitz with respect to \((Y, Z, \Lambda(e))\), \( \Delta_Y f^1(t) \) and \( \Delta_Z f^{1,i}(t) \) are bounded processes. By virtue of Assumption 2.9, \( \Delta g^1(t, e) \) is bounded for \( \forall e \). Applying Proposition 2.3.1, we can find that there is a unique adjoint process that satisfies

\[
\Gamma_t \delta Y_t = E \left[ \Gamma_T (\xi^1 - \xi^2) + \int_t^T \Gamma_s \delta_2 f_s d\mathcal{F}_s \right],
\]

and the solution \( \delta Y_t \) of the BSDE is nonnegative since the terminal value \( \xi^1 - \xi^2 \) and \( \delta_2 f_t \) are nonnegative for \( \forall s \in [t, T] \).

\( \square \)

**Remark 2.3.1.** Assumption 2.9 is a sufficient condition for the comparison theorem. If the process \( \int_{\mathbb{R} \setminus \{0\}} \Delta g^1(t, e) \delta \Lambda_t(e) \nu(de) \) is Lipschitz to \( \delta \Lambda_t(e) \) in the space of \( \mathbb{L}_\nu(\cdot) \), the comparison theorem is proved.
Let us define the maximal reward by
\[ V(x) := \sup_{(c, \pi) \in \mathcal{A}(x)} Y_0^{(x, c, \pi)}. \] (2.15)

With respect to the two optimization problems, we state the following proposition based on the comparison theorem.

**Proposition 2.3.3.** Let \((a^1, f^1, u^1), (a^2, f^2, u^2)\) be two standard parameters satisfying the above assumptions with
\[ u^1(x) \leq u^2(x), \]
\[ f^1(t, c, y, z, \lambda(\cdot)) \leq f^2(t, c, y, z, \lambda(\cdot)), \]
\[ a^1(t, c, x, \pi^* \sigma(\eta)) \geq a^2(t, c, x, \pi^* \sigma(\eta)). \]

Let \(V^1(x)\) (respectively \(V^2(x)\)) be the maximal reward associated with \((a^1, f^1, u^1)\) (respectively \((a^2, f^2, u^2)\)). Then,
\[ V^1(x) \leq V^2(x). \] (2.16)

**Proof.** By using the comparison theorem, \(-a^1 \leq -a^2\) leads to \(X^{1, x, c, \pi}_t \leq X^{2, x, c, \pi}_t\) for \(\forall (c, \pi) \in \mathcal{A}(x)\). Since \(u\) is nondecreasing, \(u(X^{1, x, c, \pi}_T) \leq u(X^{2, x, c, \pi}_T)\), it derives \(Y^{1, x, c, \pi}_0 \leq Y^{2, x, c, \pi}_0\), and the result follows. \(\square\)

The positive constraint on the wealth process \(X^{x, c, \pi}_t \geq 0\) for \(\forall t \in [0, T]\) is equivalent to the positive constraint on the terminal wealth \(X^{x, c, \pi}_T \geq 0\) since Assumption 2.8 and the comparison theorem hold. Using this property and noting \(\sigma_t\) has full rank, it is possible to consider the terminal wealth as a control variable instead of the portfolio process.

Let \(\mathcal{D}\) denote a consumption space, the subset of the predictable measurable positive process \(c_t\) which belongs to \(\mathcal{H}^2_T(\mathbb{R})\). Let \(\mathcal{L}\) denote a terminal wealth space, the set of the square-integrable \(\mathcal{F}_T\) measurable positive random variable \(\xi\).

**Definition 2.3.4.** A pair \((\xi, c) \in \mathcal{L} \times \mathcal{D}\) is a “consumption plan.” \((X_t^{(\xi, c)}, \pi_t^{(\xi, c)})\) denotes the wealth and the portfolio associated with given the consumption plan \((\xi, c)\), which is the solution of the BSDE
\[-dX_t^{(\xi, c)} = a(t, c_t, X_t^{(\xi, c)}, \pi_t^{(\xi, c)} \sigma_t(\eta_t)\)dt \- \pi_t^{(\xi, c)} \sigma_t(\eta_t)\)dW_t, \]
\[X_T^{(\xi, c)} = \xi.\]
\((Y_t^{(\xi,c)}, Z_t^{(\xi,c)}, \Lambda_t^{(\xi,c)}(e))\) denotes the utility of the consumption plan \((\xi, c)\), which is the solution of the BSDE

\[
-dY_t^{(\xi,c)} = f(t, c_t, Y_t^{(\xi,c)}, Z_t^{(\xi,c)}, \Lambda_t^{(\xi,c)}(\cdot)) dt - Z_t^{(\xi,c)*} dW_t - \int_{\mathbb{R}\setminus\{0\}} \Lambda_t^{(\xi,c)}(e) \mathcal{N}(dt, de),
\]

\[Y_T^{(\xi,c)} = u(\xi).\]

In the context of consumption plans, the initial wealth \(x\) is considered as a constraint. In the case that \(a\) is nonlinear, the set of the consumption plan such that \(X_0^{(\xi,c)} = x\) is not convex. To obtain a convex set, the milder constraint \(X_0^{(\xi,c)} \leq x\) is required.

**Definition 2.3.5.** A consumption plan \((\xi, c) \in \mathcal{L} \times \mathcal{D}\) is said to be admissible for initial wealth \(x\) if and only if \(X_0^{(\xi,c)} \leq x\). Let \(\mathcal{A}(x)\) be the set of consumption plans admissible for the initial wealth \(x\).

It follows that the optimization problem (Problem 1) can be written using the backward formulation.

**Problem 2.** Find the optimal consumption plan \((\xi, c) \in \mathcal{L} \times \mathcal{D}\) satisfying

\[
V(x) = \sup_{(\xi, c) \in \mathcal{A}(x)} Y_0^{(\xi,c)}.
\]

\[\text{(2.17)}\]

Consequently, the agent optimizes a consumption plan \((\xi, c)\) belonging to \(\mathcal{A}(x)\) so that it maximizes the stochastic differential utility function given by \(Y_0^{(\xi,c)}\). In the following sections, we consider Problem 2 instead of Problem 1.

### 2.4 Maximum principle

This section focuses on the *utility gradient approach* of Duffie and Skiadas [13]. First, we consider introducing the Lagrange multiplier \(\nu\) for the budget constraint. Owing to the assumption that \(f\) and \(u\) are concave and that \(a\) is convex, the functions defined on \(\mathcal{L} \times \mathcal{D}\) by

\[
(\xi, c) \mapsto x - X_0^{(\xi,c)}
\]

\[
(\xi, c) \mapsto Y_0^{(\xi,c)}
\]
are concave. Using the result of a convex analysis, we find that there exists a constant $\nu > 0$ such that

$$\sup_{(\xi, c) \in \mathcal{Z} \times \mathcal{D}} \left( Y_0^{(\xi, c)} + \nu \left( x - X_0^{(\xi, c)} \right) \right).$$  \tag{2.18}$$

Further, if the supreme is achieved in (2.17) by $(\xi^0, c^0)$, then it is achieved in (2.18) by $(\xi^0, c^0)$ with $X_0^{(\xi, c)} = x$. In order to maximize the Lagrangian, we derive the utility gradient approach in the presence of the jumps. The rest of this section is devoted to deriving a necessary and sufficient condition of the optimization problem of (2.18) under additional assumptions. Let us start with the introduction of the differentiability assumption.

**Assumption 2.10.** $u$ is supposed to be continuously differentiable and $u'$ is bounded. Moreover, $f$ (respectively $a$) is continuously differentiable with respect to $(c, Y, Z)$ (respectively $(c, X, \pi', \sigma)$) and $g$ is continuously differentiable with respect to $\Lambda$. $f_c$ and $a_c$ (the partial differentials of $f$ and $a$ with respect to $c$) are supposed to be bounded.

Let $(\xi^0, c^0)$ be an optimal consumption plan for (2.18), and let $(Y_0^\alpha, Z_0^\alpha, \Lambda_0^\alpha(\nu))$ and $(X_0^\alpha, \pi_0^\alpha)$ be the utility and wealth/portfolio processes associated with $(\xi^0, c^0)$. Let $(\xi, c)$ be a consumption plan such that $\xi - \xi^0, c - c^0$ are uniformly bounded. Then, $(\xi^0 + \alpha(\xi - \xi^0), c^0 + \alpha(c - c^0))$ is the consumption plan for $\forall \alpha, 0 \leq \alpha \leq 1$. Let $(Y_\alpha^\alpha, Z_\alpha^\alpha, \Lambda_\alpha^\alpha(\nu))$ and $(X_\alpha^\alpha, \pi_\alpha^\alpha)$ be the utility and wealth/portfolio processes associated with $(\xi^0 + \alpha(\xi - \xi^0), c^0 + \alpha(c - c^0))$.

**Proposition 2.4.1.** The function $\alpha \mapsto (Y_\alpha^\alpha, Z_\alpha^\alpha, \Lambda_\alpha^\alpha(\nu))$ is right-differentiable with derivatives given by $(\partial_\alpha Y_\alpha^\alpha, \partial_\alpha Z_\alpha^\alpha, \partial_\alpha \Lambda_\alpha^\alpha(\nu))$, which is the solution of the following linear BSDE:

$$-d\partial_\alpha Y_t^\alpha = \left[ f^\alpha_0(t)(c_t - c_t^0) + f^\alpha_1(t)\partial_\alpha Y_t^\alpha - f^\alpha_2(t)^* \partial_\alpha Z_t^\alpha + \int_{\mathbb{R} \setminus 0} g^\alpha_0(t, e)\partial_\alpha \Lambda_\alpha^\alpha(\nu)(de) \right] dt$$

$$- \partial_\alpha Z_t^\alpha dW_t - \int_{\mathbb{R} \setminus 0} \partial_\alpha \Lambda_\alpha^\alpha(\nu) \mathcal{N}(dt, de),$$

$$\partial_\alpha Y_T^\alpha = u'(\xi^0)(\xi - \xi^0),$$

where $f^\alpha_0(t) = \frac{\partial}{\partial c} f(t, c_t^0, Y_0^0, Z_0^0, \Lambda_0^0(\cdot))$, $f^\alpha_1(t) = \frac{\partial}{\partial Y} f(t, c_t^0, Y_0^0, Z_0^0, \Lambda_0^0(\cdot))$, $f^\alpha_2(t) = \frac{\partial}{\partial Z} f(t, c_t^0, Y_0^0, Z_0^0, \Lambda_0^0(\cdot))$, $g^\alpha_0(t, e) = \frac{\partial}{\partial \Lambda} g(t, c_t^0, Y_0^0, Z_0^0, \Lambda_0^0(\nu))$.

**Proof.** The proof is a simple extension of the one in the paper of El Karoui et al. [16]. See Appendix for details.
Applying this result to \((X_t^\alpha, \pi_t^\alpha)\), we find a pair \((\partial_\alpha X_t^0, \partial_\alpha \pi_t^0)\) to be a solution of

\[
-d\partial_\alpha X_t^0 = \left[ a_t^0(t)(c_t^0 - c_t^0) + a_t^0(t)\partial_\alpha X_t^0 + a_t^0 + \sigma_t^0(\partial_\alpha \pi_t^0)\sigma_t(\eta^-_t) \right] dt \\
- \partial_\alpha \pi_t^0 \sigma_t(\eta^-_t) dW_t,
\]

where

\[
a_t^0(t) = \frac{\partial}{\partial c} a(t, c_t^0, X_t^0, \pi_t^0, \sigma_t(\eta^-_t)), \quad a_t^0(t) = \frac{\partial}{\partial X} a(t, c_t^0, X_t^0, \pi_t^0, \sigma_t(\eta^-_t)), \quad a_t^0 + \sigma_t(t) = \frac{\partial}{\partial \pi} a(t, c_t^0, X_t^0, \pi_t^0, \sigma_t(\eta^-_t)),
\]

These results derive the state-price density process explicitly.

Before proceeding to the next theorems, an additional assumption — the so-called Inada condition — needs to be made.

**Assumption 2.11.** (Inada condition) \(u\) and \(f\) satisfy \(u'(0) = +\infty\) and \(\frac{\partial}{\partial c} f(t, 0, Y, Z, \Lambda(\cdot)) = +\infty\).

**Theorem 2.4.2.** Suppose that the assumptions hold. Let \((Y_t^0, Z_t^0, \Lambda_t^0(e))\) and \((X_t^0, \pi_t^0)\) be the utility and the wealth/portfolio process associated with \((\xi_t^0, c_t^0)\), which is an optimal consumption plan. The maximum principle can be written as

\[
\Gamma_T u'(\xi^0) = v H_T, \\
\Gamma_T f_t^0(t) = v H_t a_t^0(t), \quad 0 \leq t \leq T.
\]

**Proof.** See Appendix.

**Theorem 2.4.3.** Suppose that the assumptions hold. Let \((\xi_t^0, c_t^0)\) be a consumption plan. Let \((Y_t^0, Z_t^0, \Lambda_t^0(e))\) and \((X_t^0, \pi_t^0)\) be the utility and the wealth processes associated with \((\xi_t^0, c_t^0)\). If (2.19) and (2.20) are satisfied, \((\xi_t^0, c_t^0)\) is optimal.

**Proof.** See Appendix.

These results derive the state-price density process explicitly.
Corollary 2.4.4. (State-price density process) The state-price density process \( \phi_t \) in this economy can be expressed as

\[
\phi_t = f^0_c(t) \exp \left( \int_0^t \left\{ f^0_Y(s) - \frac{1}{2} |f^0_Z(s)|^2 - \int_{\mathbb{R}\setminus 0} g^0_t(s,e) \nu(de) \right\} ds \right)
+ \int_0^t f^0_Z(t)^* dW_s + \int_0^t \int_{\mathbb{R}\setminus 0} \log(1 + g^0_t(s,e)) N(ds,de), \quad t \in [0,T),
\]

(2.21)

\[
\phi_T = u'(\xi^0) \exp \left( \int_0^T \left\{ f^0_Y(s) - \frac{1}{2} |f^0_Z(s)|^2 - \int_{\mathbb{R}\setminus 0} g^0_t(s,e) \nu(de) \right\} ds \right)
+ \int_0^T f^0_Z(t)^* dW_s + \int_0^T \int_{\mathbb{R}\setminus 0} \log(1 + g^0_t(s,e)) N(ds,de) \right). \tag{2.22}
\]

Proof. Using the utility gradient approach, \( \partial \alpha Y_t \) has to be equal to

\[
E[\phi_T (\xi - \xi^0) + \int_t^T \phi_{cs} (cs - c^0_s) ds].
\]

On the other hand, applying Proposition 2.3.1, \( \partial \alpha Y_t = E[\Gamma_T u'(\xi^0)(\xi - \xi^0) + \int_t^T \Gamma_s - f^0_c(s)(cs - c^0_s) ds]. \)

Comparing these two equations, we obtain a desired result. \( \square \)

2.5 Forward-Backward system

This section focuses on the forward-backward system, which is derived from the maximal principle. Using Theorem 2.4.2, the optimal terminal wealth satisfies

\[
\xi^0 = u^{-1} \left( \nu H_T \Gamma_T \right).
\]

In this section, the arguments shall be made under an additional assumption for simplification.

Assumption 2.12. The driver of the wealth can be expressed as

\[
a(t, c_t, X_t, \pi^*_t, \sigma_t(\eta_{-})) = a(t, X_t, \pi^*_t, \sigma_t(\eta_{-})) + c,
\]

for \( \forall (t, c, X, \pi) \).

In this case, the optimal consumption \( c^0_t \) is

\[
c_t^0 = I(t, \nu H_t \Gamma_t^{-1}, Y_{t-}, Z_t, \Lambda_t(\cdot))
\]

where the function \( I \) is an inverse function defined as

\[
f_c(t, I(t, \nu H_t \Gamma_t^{-1}, Y_{t-}, Z_t, \Lambda_t(\cdot)), Y_{t-}, Z_t, \Lambda_t(\cdot)) = \nu H_t \Gamma_t^{-1}.
\]

Based on Theorem 2.4.2, the theorem given below holds.
Theorem 2.5.1. Let \((Y_t, Z_t, \Lambda_t(e))\) and \((X_t, \pi_t)\) be the utility and the wealth/portfolio process, and let \(\Gamma_t\) and \(H_t\) be the associated adjoint processes. These processes are the optimal utility and wealth processes and their deflators if and only if they are the unique solution of the forward-backward system.

**Backward components**

\[
\begin{align*}
-dX_t &= a(t, I(t, vH_t\Gamma_t^{-1}, Y_{t-}, Z_t, \Lambda_t(\cdot)), X_t, \pi_t^* \sigma_t(\eta_{t-}))dt - \pi_t^* \sigma_t(\eta_{t-})dW_t \\
X_T &= u^{t-1}(vH_T\Gamma_T^{-1}), \\
-dY_t &= f(t, I(t, vH_t\Gamma_t^{-1}, Y_{t-}, Z_t, \Lambda_t(\cdot)), Y_{t-}, Z_t, \Lambda_t(\cdot))dt - Z_t^* dW_t \\
&\quad - \int_{\mathbb{R}^d} \Lambda_t(e)\tilde{N}(dt, de), \\
Y_T &= u(u^{t-1}(vH_T\Gamma_T^{-1})).
\end{align*}
\]

**Forward components**

\[
\begin{align*}
\frac{d\Gamma_t}{\Gamma_t} &= f_y(t, I(t, vH_t\Gamma_t^{-1}, Y_{t-}, Z_t, \Lambda_t(\cdot)), Y_{t-}, Z_t, \Lambda_t(\cdot))dt \\
&\quad + f_z(t, I(t, vH_t\Gamma_t^{-1}, Y_{t-}, Z_t, \Lambda_t(\cdot)), Y_{t-}, Z_t, \Lambda_t(\cdot))^* dW_t \\
&\quad + \int_{\mathbb{R}^d} g_{\Lambda}(t, I(t, vH_t\Gamma_t^{-1}, Y_{t-}, Z_t, \Lambda_t(\cdot)), Y_{t-}, Z_t, \Lambda_t(e))\tilde{N}(dt, de), \\
\Gamma_0 &= 1, \\
\frac{dH_t}{H_t} &= a_{x}(t, I(t, vH_t\Gamma_t^{-1}, Y_{t-}, Z_t, \Lambda_t(\cdot)), X_t, \pi_t^* \sigma_t(\eta_{t-}))dt \\
&\quad + a_{\pi^* \sigma}(t, I(t, vH_t\Gamma_t^{-1}, Y_{t-}, Z_t, \Lambda_t(\cdot)), X_t, \pi_t^* \sigma_t(\eta_{t-}))^* dW_t \\
H_0 &= 1.
\end{align*}
\]

The following corollary can be derived immediately by applying Ito’s formula to \(A_t = \log(vH_t\Gamma_t^{-1}).\)

**Corollary 2.5.2.** Let \((Y_t, Z_t, \Lambda_t(e))\) and \((X_t, \pi_t)\) be the utility and the wealth/portfolio process. They are the optimal processes if and only if there exists a process \(A_t\) such that \((Y_t, Z_t, \Lambda_t(e)), (X_t, \pi_t),\) and \(A_t\) form the unique solution of the following forward-backward system.

**Backward components**

\[
\begin{align*}
-dX_t &= a(t, I(t, e^{A_t-}, Y_{t-}, Z_t, \Lambda_t(\cdot)), X_t, \pi_t^* \sigma_t(\eta_{t-}))dt - \pi_t^* \sigma_t dW_t \\
X_T &= u^{t-1}(e^{A_T}), \\
-dY_t &= f(t, I(t, e^{A_t-}, Y_{t-}, Z_t, \Lambda_t(\cdot)), Y_{t-}, Z_t, \Lambda_t(\cdot))dt - Z_t^* dW_t \\
&\quad - \int_{\mathbb{R}^d} \Lambda_t(e)\tilde{N}(dt, de), \\
Y_T &= u(u^{t-1}(e^{A_T})).
\end{align*}
\]
Forward components

\[ dA_t = \phi(t,A_t, X_t, Y_t, \pi_t, Z_t, A_t(\cdot)) dt + \psi(t,A_t, X_t, Y_t, \pi_t, Z_t, A_t(\cdot)) dW_t \]

\[ + \int_{\mathbb{R} \setminus 0} \chi(t,A_t, X_t, Y_t, \pi_t, Z_t, A_t(e)) N(dt,de), \]

\[ A_0 = \log \nu, \]

where

\[ \phi(t,A_t, X_t, Y_t, \pi_t, Z_t, A_t(\cdot)) = \]

\[ a_X(t, I(t, e^{A_t-}, Y_t-), Z_t, A_t(\cdot), X_t, \pi_t^* \sigma_t(\eta_t-)) \]

\[ - f_Y(t, I(t, e^{A_t-}, Y_t-), Z_t, A_t(\cdot), Y_t-, Z_t, A_t(\cdot)) \]

\[ - \frac{1}{2} [a_{Xt} \sigma_t(t, I(t, e^{A_t-}, Y_t-), Z_t, A_t(\cdot), X_t, \pi_t^* \sigma_t(\eta_t-))]^2 \]

\[ + \frac{1}{2} [f_{Zt}(t, I(t, e^{A_t-}, Y_t-), Z_t, A_t(\cdot), Y_t-, Z_t, A_t(\cdot))]^2 \]

\[ + \int_{\mathbb{R} \setminus 0} g_A(t, I(t, e^{A_t-}, Y_t-), Z_t, A_t(\cdot), Y_t-, Z_t, A_t(e)) \nu(de), \]

\[ \psi(t,A_t, X_t, Y_t, \pi_t, Z_t, A_t(\cdot)) = a_{Xt} \sigma_t(t, I(t, e^{A_t-}, Y_t-), Z_t, A_t(\cdot), X_t, \pi_t^* \sigma_t(\eta_t-)) \]

\[ - f_{Zt}(t, I(t, e^{A_t-}, Y_t-), Z_t, A_t(\cdot), Y_t-, Z_t, A_t(\cdot)), \]

and

\[ \chi(t,A_t, X_t, Y_t, \pi_t, Z_t, A_t(e)) = - \log \left(1 + g_A(t, I(t, e^{A_t-}, Y_t-), Z_t, A_t(e), Y_t-, Z_t, A_t(e))\right). \]

By using the forward-backward system, it is possible to solve the maximization problem of stochastic differential utility, where the states are driven by the jump-diffusion process.

### 2.6 Normalized SDU

In this section, we introduce aggregators that defines stochastic differential utility in a consistent manner in the case information generated by the Brownian motion and Poisson Jump.

First, based on the notation given by Duffie and Epstein [10], we would like to introduce a certainty equivalent denoted by operator \(m\), which satisfies (i) \(m\
is a function of probability measure \( p \), defined on the space of probability measures; (ii) \( m(\delta(x)) = x \), where \( \delta(x) \) denotes the Dirac measure; (iii) (Monotonicity) \( m(p') \geq m(p) \) if \( p' \) exhibits first order stochastic dominance over \( p \); and (iv) for all \( p \) in the space of probability measures, \( m(p) \leq m(\bar{\delta}()) \), where \( \bar{\delta} \) is the mean of \( p \).

One such function that appeared in their paper is an expected-utility based certainty equivalent. Here, we would like to consider the specification of the certainty equivalent. We define the expected-utility based certainty equivalent \( m \) as

\[
m(F_Y(\cdot)|\mathcal{F}_t) := h^{-1}(\mathbb{E}[h(Y)|\mathcal{F}_t]),
\]

(2.23)

where \( Y \) is a real valued integrable random variable, \( F_Y \) denotes its distribution function \( F_Y(x) = \Pr(Y \leq x) \). \( h \) denotes the von Neumann-Morgenstern index, which is continuous and strictly increasing and which satisfies a growth condition, i.e., there exists some constants \( k \) such that \( h(x) \leq k(1 + |x|) \).

With this certainty equivalent, the stochastic differential utility \( Y_t \) is defined recursively such that

\[
Y_t = \lim_{s \to 0} m(F_{Y_{t+s}}(\cdot)|\mathcal{F}_t).
\]

(2.24)

Assuming a differentiability of \( Y_t \) with respect to infinitesimal time, let us define a measurable function \( \mathcal{G} : \mathcal{D} \times \mathbb{R} \to \mathbb{R} \),

\[
\mathcal{G}(c_t, Y_t) := \lim_{s \to 0} \frac{\partial}{\partial s} m(F_{Y_{t+s}}(\cdot))
\]

With this background, it is convenient to take a certainty equivalent \( m \) and a measurable function \( \mathcal{G} \) as primitives for the stochastic differential utility. Let the pair \((\mathcal{G}, m)\) be an aggregator. In these settings, it is natural to assume that the utility process \( Y_t \) is as follows:

\[
-dY_t = f_t dt - Z_t^* dW_t - \int_{\mathbb{R} \setminus \{0\}} \Lambda_t(e) \tilde{N}(dt, de)
\]

(2.25)

where \( f \) and \( Z, \Lambda(e) \) are progressively measurable processes.

**Proposition 2.6.1.** Let \((\mathcal{G}, m)\) be an aggregator and \( m \) is the expected-utility based certainty equivalent defined by (2.23). The drift term of the utility process is

\[
f_t = \mathcal{G}(c_t, Y_{t-}) + \frac{1}{2} \frac{h''(Y_{t-})}{h'(Y_{t-})} |Z_t|^2
\]

\[
+ \int_{\mathbb{R} \setminus \{0\}} \frac{h(Y_{t-} + \Lambda_t(e)) - h(Y_{t-}) - \Lambda_t(e) h'(Y_{t-})}{h'(Y_{t-})} \nu(de).
\]

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Proof. Consider a function \( v(t, \tau, Y_t) = E_t[h(Y_\tau)] \). Since \( v \) is martingale, we arrive at

\[
\frac{\partial v}{\partial t} - \frac{\partial v}{\partial Y} f_t + \frac{1}{2} \frac{\partial^2 v}{\partial Y^2} |Z_t|^2 + \int_{\mathbb{R} \setminus 0} \left\{ v(t, \tau, Y_{t-} + \Lambda_t(e)) - v(t, \tau, Y_{t-}) - \frac{\partial v}{\partial Y} \Lambda_t(e) \right\} v(de) = 0.
\]

On the other hand,

\[
\mathcal{G}(c_t, Y_t) = \lim_{\tau \to t} \frac{\partial}{\partial t} m(Y_{\tau} | \mathcal{F}_t) = \lim_{\tau \to t} \frac{\partial}{\partial t} h^{-1}(v(t, \tau, Y_t)) = \lim_{\tau \to t} \frac{\partial v}{\partial Y} \frac{1}{h'(h^{-1}(E_t[Y_{\tau}])))}.
\]

By the definition of \( h, h'(x) \neq 0 \) for all \( x \in \mathbb{R} \) and we find

\[
\lim_{\tau \to t} \frac{\partial v}{\partial Y} = h'(Y_t) \mathcal{G}(c_t, Y_t). \tag{2.26}
\]

From \( \lim_{\tau \to t} v = h(Y_t), \lim_{\tau \to t} \partial v/\partial Y = h'(Y_t), \) and \( \lim_{\tau \to t} \partial^2 v/\partial Y^2 = h''(Y_t) \), we obtain the desired result.

\[\square\]

As we see, it is not to be said that \( f_t \) is Lipschitz with respect to \( Z \) and \( \Lambda(e) \) generally. To avoid difficulty in computation, we may consider a change of variables as any \( \varphi: \mathbb{R} \to \mathbb{R} \) that is strictly increasing and continuous \( C^2 \) function. Let us consider a new utility \( \tilde{Y} \). The two utilities \( Y \) and \( \tilde{Y} \) are ordinally equivalent if there is a change of variables \( \varphi \) such that \( \tilde{Y} = \varphi(Y) \). Applying Ito’s formula, the process of \( \tilde{Y} \) is

\[
-d\tilde{Y}_t = \left[ \varphi'(Y_{t-}) + \frac{1}{2} \frac{h''(Y_{t-})}{h'(Y_{t-})} \varphi''(Y_{t-}) \right] |Z_t|^2 + \int_{\mathbb{R} \setminus 0} \left\{ \frac{\varphi'(Y_{t-})}{h'(Y_{t-})} h(Y_{t-} + \Lambda_t(e)) - \varphi(Y_{t-} + \Lambda_t(e)) - \frac{\varphi'(Y_{t-})}{h'(Y_{t-})} \varphi'(Y_{t-}) + \varphi(Y_{t-}) \right\} v(de) dt - \varphi'(Y_{t-}) Z_t^* dW_t - \int_{\mathbb{R} \setminus 0} (\varphi(Y_{t-} + \Lambda_t(e)) - \varphi(Y_{t-})) \tilde{N}(dt, de)
\]

\[
= f(t, c_t, \tilde{Y}_{t-}, Z_t, \tilde{\Lambda}(\cdot)) dt - Z_t^* dW_t - \int_{\mathbb{R} \setminus 0} \tilde{\Lambda}_t(e) \tilde{N}(dt, de)
\]

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By taking an appropriate $\phi$, we can obtain a tractable utility process of which the driver $\tilde{f}$ is a function of only $c_t$ and $\tilde{Y}_t$. Following the notation of Duffie and Epstein [10], hereinafter we refer to it as the normalized stochastic differential utility. The next theorem is obtained.

**Theorem 2.6.2.** Suppose that the assumptions hold. There exist normalized ordinally equivalent SDU $\tilde{Y}$ by considering the change of variables as $\phi \equiv kh$ (where $k$ is constant) for any SDU $Y$ generated by the expected-utility based aggregator $(\mathcal{G}, m)$.

**Proof.** In order to eliminate the terms with respect to $Z$ and $\Lambda(e)$,

$$\frac{h''(x)}{h'(x)}\phi'(x) = \phi''(x), \quad \frac{\phi'(x)}{h'(x)}h(y) = \phi(y) \quad \text{for } x, y,$$

must be satisfied. The second equation requests that $\phi(x)/h(x)$ and $\phi'(y)/h'(y)$ are constant (hereinafter referred to as $k$) for $\forall x, y$. The change of variables $\phi \equiv kh$ also satisfies the first equation.

**Remark 2.6.1.** The normalized ordinally equivalent utility in the case of the information generated by the Brownian motion is obtained by taking $\frac{\phi''}{\phi'} = \frac{h''}{h'}$. (See Duffie and Epstein [10] and Duffie and Lions [12] for details.)

**Example 2.6.3. (Standard Additive Utility)**

The standard additive expected-utility function with the utility process $Y_t = E_t\left[ \int_t^T u(c_s)e^{-\beta(s-t)}ds + u_2(X_T) \right]$ corresponds to the aggregator $(\mathcal{G}, m)$, where

$$\mathcal{G}(c, y) = u(c) - \beta y \quad \text{and} \quad m(F_Y(\cdot)) = E(Y).$$

**Example 2.6.4. (Kreps-Porteus Utility [27])**

Let $0 \neq \rho < 1$, $0 \leq \beta$, $0 \neq \alpha < 1$, and define

$$\mathcal{G}(c, y) = \frac{c^\rho}{\rho} - \beta y \quad \text{and} \quad m(F_Y(\cdot)) = \left[ E(Y^\alpha) \right]^{\frac{1}{\alpha}}.$$

Let us consider its ordinally equivalent $Y_t = \phi(Y_t) = (Y_t)^\alpha$. The driver of the BSDE of the ordinally equivalent utility process is

$$\tilde{f} = \alpha \left( \frac{c^\rho}{\rho} (Y_t)^{\frac{a-1}{\alpha}} - \beta Y_t \right).$$

**Example 2.6.5. (Schroder-Skiadas Log Utility [48])**

Let $0 \leq \beta$, $\alpha \leq \beta$, and define

$$\mathcal{G}(c, y) = \log c - \beta y \quad \text{and} \quad m(F_Y(\cdot)) = \frac{1}{\alpha} \log \left( 1 + \alpha E \left[ \frac{e^{\alpha Y} - 1}{\alpha} \right] \right).$$
Let us consider its ordinally equivalent $\bar{Y}_t = \varphi(Y_t) = \frac{e^{\alpha Y_t} - 1}{\alpha}$. The driver of the BSDE of the ordinally equivalent utility process is

$$\bar{f} = (1 + \alpha \bar{Y}_t) \left( \log c - \frac{\beta}{\alpha} \log(1 + \alpha \bar{Y}_t) \right).$$

As we see, the aggregator is characterized by aggregator $G$ and index $h$. Let us say that $h(x) = \alpha x$ is Linear index, $h(x) = x^\alpha$ is Kreps-Porteus index, and $h(x) = e^{\alpha x} - 1$ is Schroder-Skiadas index.

We can define new 4 types of SDU as follows.

**Example 2.6.6. (Exponential aggregator with Kreps-Porteus index)**
Let $\rho > 0$, $0 \leq \beta$, $0 \neq \alpha < 1$, and define

$$G(c, y) = -\exp(-\rho c) - \beta y \quad \text{and} \quad m(F_Y(\cdot)) = [E(Y^\alpha)]^{1/\alpha}.$$

Let us consider its ordinally equivalent $\bar{Y}_t = (Y_t)^\alpha$. The driver of the BSDE of the ordinally equivalent utility process is

$$\bar{f} = \alpha \left( -\exp(-\rho c)(\bar{Y}_t)^{\frac{\alpha - 1}{\alpha}} - \beta \bar{Y}_t \right).$$

**Example 2.6.7. (Log aggregator with Kreps-Porteus index)**
Let $0 \leq \beta$, $0 \neq \alpha < 1$, and define

$$G(c, y) = \log(c) - \beta y \quad \text{and} \quad m(F_Y(\cdot)) = [E(Y^\alpha)]^{1/\alpha}.$$

Let us consider its ordinally equivalent $\bar{Y}_t = (Y_t)^\alpha$. The driver of the BSDE of the ordinally equivalent utility process is

$$\bar{f} = \alpha \left( \log(c)(\bar{Y}_t)^{\frac{\alpha - 1}{\alpha}} - \beta \bar{Y}_t \right).$$

**Example 2.6.8. (Power aggregator with Schroder-Skiadas index)**
Let $0 \neq \rho < 1$, $0 \leq \beta$, $\alpha \leq \beta$, and define

$$G(c, y) = \frac{\rho^\alpha}{\rho} - \beta y \quad \text{and} \quad m(F_Y(\cdot)) = \frac{1}{\alpha} \log \left( 1 + \alpha E \left[ \frac{e^{\alpha Y} - 1}{\alpha} \right] \right).$$

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Let us consider its ordinally equivalent $\bar{Y}_t = \frac{e^{\alpha Y_t} - 1}{\alpha}$. The driver of the BSDE of the ordinally equivalent utility process is

$$\bar{f} = (1 + \alpha \bar{Y}_t) \left( \frac{c^p}{\rho} - \frac{\beta}{\alpha} \log(1 + \alpha \bar{Y}_t) \right).$$

Example 2.6.9. (Exponential aggregator with Schroder-Skiadas index)

Let $0 \neq \rho < 1$, $0 \leq \beta$, $\alpha \leq \beta$, and define

$$\mathcal{G}(c, y) = -\exp(-\rho c) - \beta y \quad \text{and} \quad m(F_Y(\cdot)) = \frac{1}{\alpha} \log \left( 1 + \alpha E \left[ \frac{e^{\alpha Y} - 1}{\alpha} \right] \right).$$

Let us consider its ordinally equivalent $\bar{Y}_t = \frac{e^{\alpha Y_t} - 1}{\alpha}$. The driver of the BSDE of the ordinally equivalent utility process is

$$\bar{f} = (1 + \alpha \bar{Y}_t) \left( -\exp(-\rho c) - \frac{\beta}{\alpha} \log(1 + \alpha \bar{Y}_t) \right).$$

2.7 An application to inverse problem

Lazrak and Zapatero [29] formulated the inverse problem of recursive preference as FBSDEs. They argued the testability of the model when the beliefs of the agent are unknown. In this section, we consider the inverse problem when the representative agent has a SDU preference and invests in a financial market; the agent’s intertemporal contingent consumption is the outcome of trading risky financial assets. We assume that the risky assets only follow Brownian motion, however, the agent believes that the probability of jump can not be ignored. We shall define some additional notations.

Density generator of beliefs. We define $Y$, the set of possible "beliefs" as the set of progressively measurable processes and predictable mappings $(\gamma, h(e)) : (\Omega \times [0, T], \Omega \times [0, T] \times \mathbb{R} \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R})$ with the integrability condition,

$$E \left[ \exp \left\{ -\frac{1}{2} \int_0^t |\gamma_s|^2 ds - \int_0^t \gamma_s dW_s + \int_0^t \int_{\mathbb{R} \setminus \{0\}} \log(1 + h_s(e)) N(ds, de) \right. \right.$$ 

$$\left. - \int_0^t \int_{\mathbb{R} \setminus \{0\}} h_s(e) \nu(de) ds \right\} = 1, \quad \forall t \in [0, T].$$

Let $P^\Xi$ be a probability measure on $(\Omega, \mathcal{F})$ equivalent to $P$, with a Radon Nikodym derivative of the form $\frac{dP^\Xi}{dP} = \Xi^{\gamma, h}_t$, where $\Xi^{\gamma, h}_t$ is the martingale,

$$\Xi^{\gamma, h}_t = \exp \left\{ -\frac{1}{2} \int_0^t |\gamma_s|^2 ds - \int_0^t \gamma_s dW_s \right\} \cdot \exp \left\{ \int_0^t \int_{\mathbb{R} \setminus \{0\}} \log(1 + h_s(e)) N(ds, de) - \int_0^t \int_{\mathbb{R} \setminus \{0\}} h_s(e) \nu(de) ds \right\}.$$
The state price deflator is defined without jump term,

\[ H_t = \exp \left\{ - \int_0^t r_s ds \right\} \exp \left\{ - \frac{1}{2} \int_0^t |\theta_s|^2 ds - \int_0^t \theta_s^* dW_s \right\}. \]

Therefore, the utility process \( Y_t = E_T^x \left[ \int_0^T f(c_s, Y_s) ds \right] \) is formulated in a differential form as follows,

\[-dY_t = \left( f(c_t, Y_t) - \gamma_t Z_t + \int_{\mathbb{R}\setminus\{0\}} \Lambda_t(e) h_t(e) \nu(de) \right) dt \]
\[-Z_t^* dW_t - \int_{\mathbb{R}\setminus\{0\}} \Lambda_t(e) \tilde{N}(dt, de), \quad Y_T = 0. \]

We formulate the consumption optimization problem of consumer when the intertemporal aggregator is given by \( f \) and beliefs are given by the density generator \((\gamma, h) \in \Upsilon\), as

\[ \sup_{c \in C} Y_0, \quad \text{s.t.} \quad E \left[ \int_0^T H_t c_t \right] \leq w_0, \]

where \( C \) is a consumption set and \( w_0 \) is an initial wealth.

Applying Corollary 2.5.2, there exists a process \( A_t \) which solve a forward-backward system,

\[ dA_t = - \left( r_t + f_Y(I(e^{A_t}, X_t), Y_t) + \frac{1}{2}(|\theta_t|^2 - |\gamma_t|^2) + \int_{\mathbb{R}\setminus\{0\}} h_t(e) \nu(de) \right) dt \]
\[-(\theta_t - \gamma_t) dW_t + \int_{\mathbb{R}\setminus\{0\}} \log \frac{1}{1 + h_t(e)} N(dt, de), \quad A_0 = \log(v), \]

where \( I \) is an inverse function of \( f \) such that \( e^{A_t} = f_c(I(e^{A_t}, Y_t), Y_t) \), and \( v \) is a Lagrange multiplier satisfying budget constraint \( E \left[ \int_0^T H(s) I(e^{A_s}, Y_s) ds \right] \leq w_0 \).

We make an assumption that the optimal consumption follows the given process,

\[ dc_t = \kappa(\bar{c} - c_t) dt + c_t^\varepsilon \rho_t^* dW_t, \quad 0 < \varepsilon \leq 1. \]

Under the optimal consumption process, next theorem holds.

**Theorem 2.7.1.** A consumption process \( c \in C \) that satisfies the dynamics (2.29) is optimal for fixed intertemporal aggregator \( f \) when the beliefs density generator \( \gamma \) and \( h(y) \) as follows,

\[ \gamma_t = \theta_t + f_{cc}(c_t, Y_t) c_t^\varepsilon \rho_t + \frac{f_{cY}(c_t, Y_t)}{f_c(c_t, Y_t)} Z_t, \]
\[ h_t(e) = \frac{f_c(c_t, Y_t)}{f_c(c_t, Y_t + \Lambda_t(e))} - 1. \]
Moreover, the following equation with respect to a short rate holds,

\[
0 = r_t + f_Y(c_t, Y_t) - \frac{f_{cc}(c_t, Y_t)}{f_c(c_t, Y_t)} \left(c_t^E \rho_t^* \gamma_t - \kappa(c_t - \bar{c}_t)\right) \\
+ c_t^2 \left| \rho_t \right|^2 \left(\frac{1}{2} \frac{f_{ccc}(c_t, Y_t)}{f_c(c_t, Y_t)} - \frac{f_{cc}(c_t, Y_t)}{f_c(c_t, Y_t)} \right)^2 + \frac{1}{2} \frac{f_{ccY}(c_t, Y_t)}{f_c(c_t, Y_t)} |Z_t|^2 \\
- \frac{f_Y(c_t, Y_t)}{f_c(c_t, Y_t)} \left(f(c_t, Y_t) + \frac{f_{cc}(c_t, Y_t)}{f_c(c_t, Y_t)} c_t^E \rho_t^* Z_t\right) + \frac{f_{ccY}(c_t, Y_t)}{f_c(c_t, Y_t)} c_t^E \rho_t^* Z_t \\
- \int_{\mathbb{R} \setminus 0} \frac{f_{Y}(c_t, Y_t)}{f_c(c_t, Y_t + \Lambda_t(e))} \Lambda_t(e)v(de) \\
- \int_{\mathbb{R} \setminus 0} \left(\frac{f_{Y}(c_t, Y_t)}{f_c(c_t, Y_t + \Lambda_t(e))} - 1\right)v(de).
\]  

(2.30)

**Proof.** By the definition of \( A_t \) and the first order condition of optimality,

\[ f_c(c_t, Y_t) = \nu \frac{H_t}{\Gamma_t}, \]

is hold when \( c_t \) is optimal, where the process \( \Gamma_t \) is formulated in a differential form,

\[
\frac{d \Gamma_t}{\Gamma_t} = f_Y(c_t, Y_t) dt - \gamma^e dW_t + \int_{\mathbb{R} \setminus 0} h_t(e) \tilde{N}(dt, de).
\]

By decomposing \( \Gamma_t \) into the term of beliefs density generator \((\gamma, h)\) and others, we find,

\[
\frac{H_t e^{-\int_0^t f_Y(c_s, Y_s) ds}}{f_c(c_t, Y_t)} = \frac{1}{\nu} \mathbb{E}_{t}^{\gamma, h}.
\]

By applying Itô formula to the right and left hand side of this equality and comparing drift term, diffusion term, and jump term, we find desired results.  \( \square \)

**Remark 2.7.1.** The identity of a short rate \( r_t \) (2.30) contains the utility \( Y_t \). This fact makes it clear that the short rate is driven by jump-diffusion process via the fluctuation of the utility level in an assumed economy. Conversely, this model is consistent with the case that the short rate is driven by jump-diffusion process which is not interrelated with the fluctuation of risky assets.

### 2.8 Summary and conclusions

In this chapter, we studied the maximization problem of the stochastic differential utility in a market whose states are driven by jump-diffusion processes. To fulfill
this purpose, we have described the comparison theorem for BSDEs with jumps. To extend the result of El Karoui et al. [16], we have studied differentiability of the solutions of BSDEs with jumps. The first order condition that yields the necessary and sufficient condition for optimality is also derived by using the property of differentiability. Moreover, it is established that the optimal wealth and utility and their associated deflators are the unique solutions of a forward-backward system. These results can be easily extended to not only multidimensional jumps but also cases wherein the drift and diffusion coefficient follow the non-synchronous jump-diffusion process. Furthermore, we consider a normalization of aggregator of SDU with jump and derive new 4 types of normalized aggregator. Finally, the inverse problem is discussed where the agent’s beliefs contain jump diffusion. The beliefs density generator and the identity of short rate are obtained under given optimal consumption process.

2.9 Appendix

This appendix contains some proofs that were omitted from the main body of the chapter.

2.9.1 Proof of Theorem 2.2.2

First, we state the a priori estimates of the spread between the solutions of two BSDEs with jumps, from which we prove the existence and uniqueness of the solution.

**Lemma 2.9.1.** (A priori estimates) Let \(((f^i, \zeta^i); i = 1, 2)\) be two standard parameters of the BSDEs and \(((Y^i, Z^i, \Lambda^i(e)); i = 1, 2)\) be the solutions. Put \(\delta Y_t = Y^1_t - Y^2_t\) and \(\delta Z_t = Z^1_t - Z^2_t\), \(\delta \Lambda_t(e) = \Lambda^1_t(e) - \Lambda^2_t(e)\), \(\delta_2 f^i = f^1(t, Y^2_t, Z^2_t, \Lambda^2_t(\cdot)) - f^2(t, Y^2_t, Z^2_t, \Lambda^2_t(\cdot))\). For any \((\kappa, \lambda, \mu)\) such that \(\kappa^2 > L\), \(\lambda^2 > L\), \(\beta \geq L(2 + \kappa^2 + \lambda^2) + \mu^2\), it follows that

\[
\|\delta Y\|^2_\beta \leq T \left\{ E[e^{\beta T} |\delta Y_T|^2] + \frac{1}{\mu^2} \|\delta_2 f\|^2_\beta \right\},
\]

\[
\|\delta Z\|^2_\beta \leq \frac{\kappa^2}{\kappa^2 - L} \left\{ E[e^{\beta T} |\delta Y_T|^2] + \frac{1}{\mu^2} \|\delta_2 f\|^2_\beta \right\},
\]

\[
\|\delta \Lambda\|^2_\beta \leq \frac{\lambda^2}{\lambda^2 - L} \left\{ E[e^{\beta T} |\delta Y_T|^2] + \frac{1}{\mu^2} \|\delta_2 f\|^2_\beta \right\}.
\]
Proof of Lemma 2.9.1. Let us consider two solutions \((Y^1, Z^1, \Lambda^1)\) and \((Y^2, Z^2, \Lambda^2)\) associated with \((f^1, \xi^1)\) and \((f^2, \xi^2)\), respectively. Applying Ito’s formula to \(e^{\beta t}|\delta Y_t|^2\), it follows that

\[
\begin{align*}
d(e^{\beta t}|\delta Y_t|^2) &= \beta e^{\beta t}|\delta Y_t|^2 dt + e^{\beta t}|\delta Z_t|^2 dt + 2e^{\beta t} \langle \delta Y_t, \delta Z_t \rangle dW_t \\leq & -2e^{\beta t} \langle \delta Y_t, f^1(t, Y_{t-}^1, Z_t^1, \Lambda_t^1(\cdot)) - f^2(t, Y_{t-}^2, Z_t^2, \Lambda_t^2(\cdot)) \rangle dt \\
&+ 2 \int_{\mathbb{R}^2} e^{\beta t} \delta Y_t \delta \Lambda_t(e) \tilde{N}(dt, de) + \int_{\mathbb{R}^2} e^{\beta t} |\delta \Lambda_t(e)|^2 N(dt, de).
\end{align*}
\]

Since \(\sup_{t \leq T} |\delta Y_t|\) belongs to \(L^2_T(\mathbb{R})\), \(\int_t^T e^{\beta s} \langle \delta Y_s, \delta Z_s \rangle dW_t\) and \(\int_{\mathbb{R}} e^{\beta t} \delta Y_t \delta \Lambda_t(e) \tilde{N}(dt, de)\) are \(\mathbb{P}\)-integrable with zero expectation, by integrating and computing the expectation, we can derive the following:

\[
\begin{align*}
E[e^{\beta t}|\delta Y_t|^2] + \beta E[\int_t^T e^{\beta s}|\delta Y_s|^2 ds] + E[\int_t^T e^{\beta s}|\delta Z_s|^2 ds] \\
+ E[\int_t^T \int_{\mathbb{R}^2} e^{\beta s}|\delta \Lambda_s(e)|^2 N(ds, de)] \\
= E[e^{\beta T}|\delta Y_T|^2] \\
+ 2E[\int_t^T e^{\beta s} \langle \delta Y_s, f^1(s, Y_s^1, Z_s^1, \Lambda_s^1(\cdot)) - f^2(s, Y_s^2, Z_s^2, \Lambda_s^2(\cdot)) \rangle ds].
\end{align*}
\]

By virtue of the assumption of the standard parameter, we obtain

\[
\begin{align*}
&\langle f^1(s, Y_s^1, Z_s^1, \Lambda_s^1(\cdot)) - f^2(s, Y_s^2, Z_s^2, \Lambda_s^2(\cdot)) \rangle \\
\leq & |f^1(s, Y_s^1, Z_s^1, \Lambda_s^1(\cdot)) - f^1(s, Y_s^2, Z_s^2, \Lambda_s^2(\cdot))| + |\delta f_s| \\
\leq & L(|Y_s^1 - Y_s^2| + |Z_s^1 - Z_s^2|) + |\Lambda_s^1 - \Lambda_s^2| + |\delta f_s|.
\end{align*}
\]

The inequality \(2y(L(z + l) + t) \leq L \left( \frac{z^2}{\kappa^2} + \frac{\mu^2}{\lambda^2} \right) + \frac{L(\lambda^2 + \mu^2)}{\kappa^2} \) \((\kappa, \lambda, \mu \neq 0)\) implies

\[
\begin{align*}
2E[\int_t^T e^{\beta s} \langle \delta Y_s, f^1(s, Y_s^1, Z_s^1, \Lambda_s^1(\cdot)) - f^2(s, Y_s^2, Z_s^2, \Lambda_s^2(\cdot)) \rangle ds] \\
\leq LE[\int_t^T e^{\beta s} \left( \frac{|\delta Z_s|^2}{\kappa^2} + \frac{|\delta \Lambda_s|^2}{\lambda^2} \right) ds] + E[\int_t^T e^{\beta s} |\delta f_s|^2 ds] \\
+ (\mu^2 + L(2 + \lambda^2 + \mu^2))E[\int_t^T e^{\beta s} |\delta Y_s|^2 ds].
\end{align*}
\]

By choosing \(\beta \geq L(2 + \kappa^2 + \lambda^2) + \mu^2, \kappa^2 > L\) and \(\lambda^2 > L\), we obtain the following from these inequalities:

\[
E[e^{\beta t}|\delta Y_t|^2] \leq E[e^{\beta T}|\delta Y_T|^2] + E[\int_t^T e^{\beta s} |\delta f_s|^2 ds].
\]
The control of the norm of the process $\delta Y_t$ is obtained by integration. The control of the norm of the process $\delta Z_t$ and $\delta \Lambda_t(e)$ can then be obtained.

The proof of the existence and uniqueness of the solution is obtained from Lemma 2.9.1.

**Proof of Theorem 2.2.2.** Consider the mapping $(y,z,t(e))$ onto the solution $(Y,Z,\Lambda(e))$ of the BSDE with generator $f(t,y,z,\dot{t}())$, i.e.,

$$Y_t = \zeta + \int_t^T f(s,y_s,z_s,t_s(\cdot))ds - \int_t^T Z_s^r dW_s - \int_t^T \int_{\mathbb{R}\setminus\{0\}} \Lambda_t(e)N(dt,de).$$

We use a fixed-point theorem for the mapping $(y,z,t(e)) \in \mathbb{H}^2_T(\mathbb{R}^n) \times \mathbb{H}^2_T(\mathbb{R}^{m \times n}) \times \mathbb{H}^2_T,\nu(\cdot)(\mathbb{R})$ into $(Y,Z,\Lambda(e)) \in \mathbb{H}^2_T(\mathbb{R}^n) \times \mathbb{H}^2_T(\mathbb{R}^{m \times n}) \times \mathbb{H}^2_T,\nu(\cdot)(\mathbb{R})$ here.

Let $(y^1,z^1,t^1(e))$, $(y^2,z^2,t^2(e))$ be two elements of this space, and let $(Y^1,Z^1,\Lambda^1(e))$, $(Y^2,Z^2,\Lambda^2(e))$ be the associated solutions. By using a priori estimates, we obtain

$$\|\delta Y\|_\beta^2 \leq \frac{T}{\mu^2} E \left[ \int_t^T e^{\beta s} |f(s,y_s^1,z_s^1,t_s^1(\cdot)) - f(s,y_s^2,z_s^2,t_s^2(\cdot))|^2 ds \right]$$

$$\|\delta Z\|_\beta^2 \leq \frac{\kappa}{\kappa^2 - L \mu^2} E \left[ \int_t^T e^{\beta s} |f(s,y_s^1,z_s^1,t_s^1(\cdot)) - f(s,y_s^2,z_s^2,t_s^2(\cdot))|^2 ds \right]$$

$$\|\delta \Lambda\|_\beta^2 \leq \frac{\lambda}{\lambda^2 - L \mu^2} E \left[ \int_t^T e^{\beta s} |f(s,y_s^1,z_s^1,t_s^1(\cdot)) - f(s,y_s^2,z_s^2,t_s^2(\cdot))|^2 ds \right].$$

Since $f$ is Lipschitz, we obtain

$$\|\delta Y\|_\beta^2 + \|\delta Z\|_\beta^2 + \|\delta \Lambda\|_\beta^2 \leq \left( T + \frac{\kappa}{\kappa^2 - L} + \frac{\lambda}{\lambda^2 - L} \right) \frac{2L^2}{\mu^2} \left( \|\delta y\|_\beta^2 + \|\delta z\|_\beta^2 + \|\delta t\|_\beta^2 \right).$$

Considering $\mu$ to be sufficiently large, this mapping is a contraction from $\mathbb{H}^2_T(\mathbb{R}^n) \times \mathbb{H}^2_T(\mathbb{R}^{m \times n}) \times \mathbb{H}^2_T,\nu(\cdot)(\mathbb{R})$ onto itself, and we can see that there exists a fixed point, which is the unique solution of the BSDE.

**2.9.2 Proof of Proposition 2.4.1**

*Proof.* For notational convenience, let us assume that the dimensions of the Brownian motion are equal to one. Put $\Delta_{\alpha}c_t = \frac{e^\alpha - e^{\alpha t}}{\alpha}$, $\Delta_{\alpha}Y_t = \frac{y_t - y_t^0}{\alpha}$, $\Delta_{\alpha}Z_t = \frac{Z_t^0 - Z_t}{\alpha}$, and
\[ \Delta_{\alpha} \Lambda_{\alpha}(e) = \frac{\Lambda^\alpha(e) - \Lambda^0(e)}{\alpha}. \]

Similar to the approach adopted for the proof of Theorem 2.3.2, it is considered as a linear equation,

\[
-d \Delta_{\alpha} Y_t = \psi(\alpha, t, \Delta_{\alpha} c_t, \Delta_{\alpha} Y_{t-}, \Delta_{\alpha} Z_t, \Delta_{\alpha} \Lambda_t(\cdot))dt - \Delta_{\alpha} Z^\alpha_t dW_t
\]

\[
\Delta_{\alpha} Y_T = \frac{\sigma_\alpha - \sigma_0}{\alpha},
\]

where \( \psi(\alpha, t, c, y, z, l) = A^\alpha(t)c + B^\alpha(t)y + C^\alpha(t)z + \int_e D^\alpha(t, e)v(de)l \)

and

\[
A^\alpha(t) = \begin{cases} 
\frac{f(a, t, c^0_t, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot)) - f(a, t, c^0_t, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot))}{c^0_t - c_0^0} & \text{if } c^\alpha_t \neq c_0^0, \\
\partial_c f(a, t, c_t^0, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot)) & \text{otherwise;}
\end{cases}
\]

\[
B^\alpha(t) = \begin{cases} 
\frac{f(a, t, c^0_t, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot)) - f(a, t, c^0_t, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot))}{Y^0_t - Y^0_t} & \text{if } Y^\alpha_t \neq Y^0_t, \\
\partial_Y f(a, t, c_t^0, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot)) & \text{otherwise;}
\end{cases}
\]

\[
C^\alpha(t) = \begin{cases} 
\frac{f(a, t, c^0_t, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot)) - f(a, t, c^0_t, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot))}{Z^\alpha_t - Z^0_t} & \text{if } Z^\alpha_t \neq Z^0_t, \\
\partial_Z f(a, t, c_t^0, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot)) & \text{otherwise;}
\end{cases}
\]

\[
D^\alpha(t, e) = \begin{cases} 
\frac{g(a, t, c^0_t, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot)) - g(a, t, c^0_t, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot))}{\Lambda^\alpha(\cdot) - \Lambda^0(\cdot)} & \text{if } \Lambda^\alpha_t(e) \neq \Lambda^0_t(e), \\
\partial_{\Lambda^\alpha} g(a, t, c^0_t, Y^0_t, Z^0_t, \Lambda^\alpha(\cdot)) & \text{otherwise.}
\end{cases}
\]

Put

\[
\psi(0, t, c, y, z, l) = \partial_c f(0, t, c^0_t, Y^0_t, Z^0_t, \Lambda^0(\cdot))c + \partial_Y f(0, t, c^0_t, Y^0_t, Z^0_t, \Lambda^0(\cdot))y + \partial_Z f(0, t, c^0_t, Y^0_t, Z^0_t, \Lambda^0(\cdot))z + \int_e \partial_{\Lambda^\alpha} g(0, t, c^0_t, Y^0_t, Z^0_t, \Lambda^0(\cdot))v(de)l.
\]

We have to prove that \((\Delta_{\alpha} Y, \Delta_{\alpha} Z, \Delta_{\alpha} \Lambda)\) converges to \((\partial_{\alpha} Y^0, \partial_{\alpha} Z^0, \partial_{\alpha} \Lambda^0)\) when \(\alpha \to 0\) in \(H^2_T,\beta(\mathbb{R}^n) \otimes H^2_{T,\beta}(\mathbb{R}^{m\times n}) \otimes H^2_{T,\nu(\cdot),\beta}(\mathbb{R})\). To use the convergence property of a priori estimates (Lemma 2.9.1), we show that \(\psi(\alpha, t, \Delta_{\alpha} c_t^0, \Delta_{\alpha} Y^0_t, \Delta_{\alpha} Z^0_t, \Delta_{\alpha} \Lambda^\alpha_t(\cdot))\) converges to \(\psi(0, t, \Delta_{\alpha} c_t^0, \Delta_{\alpha} Y^0_t, \Delta_{\alpha} Z^0_t, \Delta_{\alpha} \Lambda^0_t(\cdot))\) when \(\alpha \to 0\) in \(H^2_T,\beta(\mathbb{R}^n) \otimes H^2_{T,\beta}(\mathbb{R}^{m\times n}) \otimes H^2_{T,\nu(\cdot),\beta}(\mathbb{R})\). Note that

\[
D^\alpha(t, e) = \int_0^1 \partial_{\Lambda^\alpha} \left( \alpha, t, c^0_t, Y^0_t, Z^0_t, \Lambda^\alpha_t(e) + \lambda (\Lambda^\alpha_t(e) - \Lambda^0_t(e)) \right) d\lambda.
\]

Since \(g\) is Lipschitz to \(\Lambda\), based on Assumption 2.9 and Fubini’s theorem, it fol-
By using a priori estimates (Lemma 2.9.1), the solution converges to 0 as
\[ \lim_{\alpha \to 0} (\partial_\alpha g(\alpha, t, e, Y_0, Z_0, \Lambda_\alpha(t))) \partial_\alpha \Lambda_\alpha(t) \| \mathbf{v}(\cdot) \| = 0 \]

The above integral is split into two terms corresponding to the set \( \{ |\Lambda_\alpha - \Lambda_0|_{\mathbf{v}(\cdot)} \leq \eta \} \) and its complement because \( \partial_\alpha g \) is continuous and bounded by a constant \( K \). It follows that for each \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that

\[
|\partial_\alpha \Lambda_\alpha(t)|^2_{\mathbf{v}(\cdot)} \leq \varepsilon^2 |\partial_\alpha \Lambda_\alpha(t)|^2_{\mathbf{v}(\cdot)} + K^2 E \left[ \int_0^T 1_{\{ |\Lambda_\alpha - \Lambda_0|_{\mathbf{v}(\cdot)} > \eta \}} |\partial_\alpha \Lambda_\alpha(t)|^2_{\mathbf{v}(\cdot)} dt \right].
\]

On splitting the last term of the right-hand side of the inequality into two terms on
the set \( \{ |\partial_\alpha \Lambda_\alpha|_{\mathbf{v}(\cdot)} \leq M \} \) and its complement and applying Markov inequality to
\( \Lambda_\alpha - \Lambda_0 \), we obtain

\[
E \left[ \int_0^T 1_{\{ |\Lambda_\alpha - \Lambda_0|_{\mathbf{v}(\cdot)} > \eta \}} |\partial_\alpha \Lambda_\alpha(t)|^2_{\mathbf{v}(\cdot)} dt \right] \leq \frac{M^2}{\eta^2} |\Lambda_\alpha - \Lambda_0|^2_{\mathbf{v}(\cdot)} + E \left[ \int_0^T 1_{\{ |\partial_\alpha \Lambda_\alpha|_{\mathbf{v}(\cdot)} > M \}} |\partial_\alpha \Lambda_\alpha(t)|^2_{\mathbf{v}(\cdot)} dt \right].
\]

Based on the Lebesgue theorem, since \( \partial_\alpha \Lambda_\alpha \) is a square integrable, the last term converges to 0 as \( M \to \infty \). Choosing a sufficiently large \( M \), it follows that

\[
\lim_{\alpha \to 0} \| (D^\alpha(\cdot, \cdot) - \partial_\alpha g(\alpha, t, e, Y_0, Z_0, \Lambda_\alpha(\cdot))) \partial_\alpha \Lambda_\alpha(\cdot) \|_{\mathbf{v}(\cdot)} = 0
\]

By employing the same process, we can see that \( \lim_{\alpha \to 0} \| (\partial_\alpha g(\alpha, t, e, Y_0, Z_0, \Lambda_\alpha(\cdot))) \|_{\mathbf{v}(\cdot)} = 0 \). Therefore, we obtain that \( \lim_{\alpha \to 0} \| (D^\alpha(\cdot, \cdot) - \partial_\alpha g(0, t, e, Y_0, Z_0, \Lambda(\cdot))) \partial_\alpha \Lambda(\cdot) \|_{\mathbf{v}(\cdot)} = 0 \). The proof of the convergences of other terms can be found in the paper [16] of El Karoui et al. As a result, we can show that \( \psi(\alpha, t, \partial_\alpha c^0, \partial_\alpha Y_0, \partial_\alpha Z^0, \partial_\alpha \Lambda(\cdot)) \) converges to \( \psi(0, t, \partial_\alpha c^0, \partial_\alpha Y_0, \partial_\alpha Z^0, \partial_\alpha \Lambda(\cdot)) \) when \( \alpha \to 0 \) in \( H^{2}_{T, \beta}(\mathbb{R}^n) \otimes H^{m \times n}_{T, \beta}(\mathbb{R}) \).

By using a priori estimates (Lemma 2.9.1), the solution \((\Delta_\alpha Y, \Delta_\alpha Z, \Delta_\alpha \Lambda)\) converges to \((\partial_\alpha Y_0, \partial_\alpha Z^0, \partial_\alpha \Lambda(\cdot))\) when \( \alpha \to 0 \) in \( H^{2}_{T, \beta}(\mathbb{R}^n) \otimes H^{m \times n}_{T, \beta}(\mathbb{R}) \).
2.9.3 Proof of Theorem 2.4.2

Proof. Since \((\xi_0, c_0)\) is an optimal consumption plan,

\[
Y_0 - uX_0 \leq Y_0 - uX_0.
\]

Dividing by \(\alpha\) and letting to 0, we have

\[
\partial_\alpha Y_0 - u\partial_\alpha X_0 \leq 0.
\]

Using the property of the adjoint process, we have

\[
\partial_\alpha Y_0 - u\partial_\alpha X_0 = E \left[ (\Gamma_T u'(\xi_0) - uH_T)(\xi - \xi_0) \right] + E \left[ \int_0^T (\Gamma_t - f_0^0(t) - uH_ta_0^0(t)) (c_t - c_t^0) dt \right].
\]

Therefore, we find two inequalities such that

\[
E \left[ (\Gamma_T u'(\xi_0) - uH_T)(\xi - \xi_0) \right] \leq 0 \quad (2.31)
\]

\[
E \left[ \int_0^T (\Gamma_t - f_0^0(t) - uH_ta_0^0(t)) (c_t - c_t^0) dt \right] \leq 0. \quad (2.32)
\]

Considering \(A = \{\Gamma_T u'(\xi_0) - uH_T > 0\}, \xi = \xi_0 + 1_A\) contradicts this inequality. Further, considering \(B = \{\Gamma_T u'(\xi_0) - uH_T < 0, \xi_0 \geq \epsilon > 0\}, \xi = \xi_0 - \epsilon 1_A\) also contradicts this inequality; considering \(C = \{\Gamma_T u'(\xi_0) - uH_T < 0, \xi_0 = 0\}\) contradicts the Inada condition. Therefore, we derive the desired result. The same arguments can be adopted for the consumption process.

\[
\square
\]

2.9.4 Proof of Theorem 2.4.3

Proof. Let \((\xi, c)\) be a consumption plan and let \((Y, Z, \Lambda), (X, \pi)\) be the associated solutions. We denote the variations of each variable as follows:

\[
\Delta X = X_t - X_0^t,
\]

\[
\Delta \pi_t = \pi_t - \pi_0^t,
\]

\[
\Delta Y_t = Y_t - Y_0^t,
\]

\[
\Delta Z_t = Z_t - Z_0^t,
\]

\[
\Delta \Lambda_t(e) = \Lambda_t(e) - \Lambda_0^t(e).
\]

Then, the trio \((\Delta Y, \Delta Z, \Delta \Lambda(e))\) is a solution of the following BSDE:

\[
-d\Delta Y_t = f_1(t, c_t, \Delta Y_t, \Delta Z_t, \Delta \Lambda_t(\cdot)) dt - \Delta Z_t^* dW_t
\]

\[
- \int_{\mathbb{R}} \Delta \Lambda_t(e) \tilde{N}(dt, de),
\]

\[
\Delta Y_T = u(\xi) - u(\xi_0),
\]

33
where \( f_1(t, c, y, z, \lambda(\cdot)) = f(t, c_0^0 + c, Y_t^0 + y, Z_t^0 + z, \Lambda_t^0(\cdot) + \lambda(\cdot)) - f(t, c_0^0, Y_t^0, Z_t^0, \Lambda_t^0(\cdot)) \). Similarly, from Proposition 2.4.1, we derive the trio \((\partial_\alpha Y_t^0, \partial_\alpha Z_t^0, \partial_\alpha \Lambda_t^0(\cdot))\) that is a solution of the following BSDE:

\[
-d\partial_\alpha Y_t^0 = f_2(t, c, \partial_\alpha Y_t^0, \partial_\alpha Z_t^0, \partial_\alpha \Lambda_t^0(\cdot))dt - \partial_\alpha Z_t^0 dW_t - \int_{\mathbb{R}\setminus 0} \partial_\alpha \Lambda_t^0(e) \mathcal{N}(dt, de),
\]

\[
\partial_\alpha Y_t^0 = u'(\xi^0)(\xi - \xi^0),
\]

where \( f_2(t, c, y, z, \lambda(\cdot)) = f_c^0(t)(c_t - c_t^0) + f_y^0(t)y + f_z^0(t)z + \int_{\mathbb{R}\setminus 0} g_\lambda^0(t, e) \lambda(e) \mathcal{N}(de) \). According to the assumption of concavity, we arrive at

\[
u(\xi) - u(\xi^0) \leq u'(\xi^0)(\xi - \xi^0),
\]

\[
f_1(t, c, y, z, \lambda(\cdot)) \leq f_2(t, c, y, z, \lambda(\cdot))
\]

Since the comparison theorem holds, we obtain \( \Delta Y_t \leq \partial_\alpha Y_t^0 \). By adopting the same arguments, we have \( \Delta X_t \geq \partial_\alpha X_t^0 \). It follows that

\[
\Delta Y_t - \nu \Delta X_t \leq \partial_\alpha Y_t^0 - \nu \partial_\alpha X_t^0
\]

The conditions (2.19) and (2.20) yield \( \partial_\alpha Y_t^0 - \nu \partial_\alpha X_t^0 = 0 \), thus we find that

\[
\Delta Y_t - \nu \Delta X_t \leq 0, \quad \forall (\xi, c) \in \mathcal{L} \times \mathcal{D}.
\]

Therefore, the consumption plan \((\xi^0, c^0)\) is optimal.
Chapter 3

Dynamic Investment Strategies to Reaction-Diffusion Systems Based upon Stochastic Differential Utilities

3.1 Introduction

In this chapter, our study focuses on optimal consumption and investment issues for regime-shift models, which have been recently highlighted in financial industry, with either synchronous regime-shifting drift and volatility or non-synchronous ones. The regime shift phenomena are observed in reality, for instance, as structural changes of economy which provide abrupt asset price changes in the financial markets, unexpected changes of the health state of an insured person that induce some pre-specified payments to the insurance policy holder, and so forth.

Recently, Becherer and Schweizer [3] have described this sort of regime shift phenomena by reaction-diffusion systems which are known to appear in biological pattern formations or chemical reactions, etc. In their model, the regime shift is simply represented by the Poisson jumps of an integer-valued index.

In this chapter, we first address the optimal consumption and investment issues in such reaction-diffusion systems by working with a more wider class of utility theories, namely the so-called stochastic differential utilities (SDU). The SDU is a continuous-time analog of the recursive utility of Epstein and Zin [14], and nesting the usual time-additive static utility model. The SDU has a recursive structure through the “intertemporal aggregator” dependent of an investor’s preference for consumption and future utility levels. As a result, it has an advantage over usual static utility theories in a sense that the investor with a usual static utility is indifferent to the timing of resolution of uncertainty, while the investor with a SDU allows for flexibility to model the preference for early or late resolution of uncer-
tainty. We shall explore the financial implications of employing SDUs instead of the usual static utility functions.

Next, we shall derive the necessary conditions for optimality in several SDU-based maximization problems of the reaction-diffusion systems and furthermore prove those sufficiency with the help of a powerful theorem of BSDE theory, that is, the comparison theorem [45, 16] and the stochastic gradient approach of El Karoui, Peng, and Quenez [17]. We also elucidate that the optimal terminal wealth and consumption are characterized by the FBSDEs that are driven by Poisson jump processes. Especially, in a Markovian setting, we can compute the solution of a forward-backward system by using the jump version of well-known four-step scheme (see for the detail, Ma, Protter, and Yong [35], Ma, Protter, San Martín, and Torres [36], Ma and Yong [37], Yong and Zhou [53]).

In our several numerical examples, we shall attempt to solve the nested PDEs with Poisson jump terms, employing the practical fixed-point algorithm, as stated in [41], so that we can get the optimal portfolio, consumption and utility processes.

The remainder of the chapter is organized as follows. In Section 2, we provide the setup of the SDU-based continuous-time consumption and investment problems. Employing the stochastic maximum principle, we derive the optimal consumption and investment policies which are characterized by some FBSDE system. In Section 3, we propose a viable numerical procedure of computing nested quasi-linear PDE system, conducting numerical analyses in such a framework. Section 4 is devoted to the summary and concluding remarks.

### 3.2 Maximum Principle in the Model with Stochastic Differential Utility

#### 3.2.1 Optimal investment in regime-shifting assets

First of all, we shall start with the general settings necessary to our reaction-diffusion systems. Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), an \(n_B\)-dimensional Brownian motion \(W := (W_1, \cdots, W_{n_B})\) and an \(n_P\)-dimensional independent Poisson random measure \(N(dt, de) := (N_1(dt, de), \cdots, N_{n_P}(dt, de))\), each of which has a predictable measure \(v_i(de)dt\), \((i = 1, \cdots, n_P)\}. \{\mathcal{F}_t; t \in [0,T]\}\} denotes the filtration generated by a Brownian motion \(W\) and a random measure \(N(dt, de)\). \(\tilde{N}(dt, de)\), the compensated martingale measure is denoted such that \(\tilde{N}(dt, de) = N(dt, de) - v(de)dt\). For \(x \in \mathbb{R}^n\), \(|x|\) denotes its Euclidean norm such that \(|x| = \sqrt{\text{trace}(x^*x)}\), \(\langle x, y \rangle\) denotes an inner product such that \(\langle x, y \rangle = \text{trace}(x^*y)\), and \(^*\) denotes a transpose.

Let \(\mathbb{H}^2(\mathbb{R}^n)\) be the space of all measurable random variables \(X : \Omega \rightarrow \mathbb{R}^n\) sat-
isfying $||X|| < \infty$, where $||X|| := (E[|X|^2])^{1/2}$, and $\mathbb{H}_T^2(\mathbb{R}^n)$ be the space of all predictable process $\varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ satisfying $||\varphi|| < \infty$, where $||\varphi|| := (E[\int_0^T |\varphi_t|^2 dt])^{1/2}$. Further, let $L_{v(\cdot)}(\mathbb{R}^n)$ be the space of all measurable $\mathbb{R}^n$-valued functions $\varphi(e)$ such that $|\varphi|_{v(\cdot)} < \infty$, where $|\varphi|_{v(\cdot)} := \int_{\mathbb{R}} |\varphi(e)|v(de)$, $E := \mathbb{R} \setminus 0$, and $\mathbb{H}_{T,v(\cdot)}^2(\mathbb{R}^n)$ be the space of all predictable function processes $\varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^n$ satisfying $||\varphi||_{v(\cdot)} < \infty$, where $||\varphi||_{v(\cdot)} := (E[\int_0^T \int_{\mathbb{R}} |\varphi_t(e)|^2 N(dt, de)])^{1/2}$. For $\beta > 0$ and $\varphi \in \mathbb{H}_T^2(\mathbb{R}^n)$, $||\varphi||_\beta^2$ denotes $E[\int_0^T e^{\beta t} |\varphi_t|^2 dt]$. $\mathbb{H}_{T,\beta}^2(\mathbb{R}^n)$ denotes the space $\mathbb{H}_T^2(\mathbb{R}^n)$ endowed with the norm $|| \cdot ||_\beta$. The norm $||\varphi||_{v(\cdot), \beta}^2$ also denotes $E[\int_0^T e^{\beta t} \int_{\mathbb{R}} |\varphi_t(e)|^2 N(dt, de)]$, where $\varphi \in \mathbb{H}_{T,v(\cdot)}^2(\mathbb{R}^n)$. $\mathbb{H}_{T,v(\cdot), \beta}^2(\mathbb{R}^n)$ denotes the space $\mathbb{H}_{T,v(\cdot)}^2(\mathbb{R}^n)$ endowed with the norm $|| \cdot ||_{v(\cdot), \beta}$.

The first model, say model (I), that we are dealing with in this chapter is of the following form:

**Model (I)** The stock price evolves according to the following regime-shifting dynamics:

$$
\frac{dS_t}{S_t} = \mu(\eta_t-)dt + \sigma(\eta_t-)dW_t
$$

$$
d\eta_t = \sum_{k,j=1}^{m} (j-k)\mathbf{1}_{\{k\}}(\eta_t-)d\tilde{N}_t^{kj},
$$

(3.1)

where $W_t$ denotes one-dimensional Brownian motion ($n_B = 1$), and we suppose a concrete form for the general setting of Poisson random measures, namely, we take as $\Lambda_t^{kj} (k,j = 1, \cdots, m)$ an $m^2$-dimensional independent Poisson process $(nP = m^2, \nu_{kj}(de) = \delta_1(de))$ with a constant intensity $\lambda^{kj}$. The finite-state $\eta$ driven by Poisson processes may be specified, for instance, as a credit rating in modelling the credit risk, or as a health state in pricing the insurance contracts, etc. The remarkable feature of model (I) is that the drift and diffusion terms change synchronously when the regime-shift occurs.

Now, let us introduce the stochastic differential utility (SDU) defined by

$$
Y_t = E_t \left[ \int_t^T f(s, c_s, Y_{s-}, Z_s, \Lambda_s(\cdot))ds + U(X_T) \right],
$$

(3.2)

or equivalently

$$
dY_t = -f(t, c_t, Y_{t-}, Z_t, \Lambda_t(\cdot))dt + Z_tdW_t + \sum_{k,j=1}^{m} \Lambda_t^{kj}d\tilde{N}_t^{kj}, \quad Y_T = U(X_T),
$$

(3.3)

1This model was recently studied in Becherer and Schweizer [3], and Bielecki, Jeanblanc, and Rutkowski [5].
with

\[ f(t, c_t, Y_t, Z_t, \lambda_t(\cdot)) := \sum_{k,j=1}^{m} \lambda_t^{kj} g(c_t, Y_t, Z_t, \Lambda_t^{kj}), \]

where \( \tilde{N}_t^{kj} := N_t^{kj} - \int_0^t \lambda_t^{kj} ds \) denotes a compensated Poisson process. Let \( X_t \) denotes the wealth process. Suppose that the investor invests his money \( \pi_t \) in the risky asset defined in (3.1) and the remaining amount \( X_t - \pi_t \) in the money market account (riskless asset) denoted by \( \beta(t) := e^{\int_0^t r(u) du} \) (\( r_t \) stands for the interest rate).

Then, the wealth process becomes

\[
dX_t = \pi_t dS_t + (X_t - \pi_t) \frac{d\beta_t}{\beta_t} - c_t dt = -b(t, c_t, X_t, \pi_t, \eta_{t-}) dt + \pi_t \sigma(\eta_{t-}) dW_t, \quad X_0 = x, \tag{3.4}
\]

where \( c_t \) denotes the consumption process, \( x \) stands for an initial wealth, and

\[ b(t, c_t, X_t, \pi_t, \eta_{t-}) := -r_t X_t - \pi_t [\mu(\eta_{t-}) - r_t] + c_t. \]

For the convenience of formulation, by using the comparison theorem, we suppose that the wealth process (3.4) is regarded as the BSDE with a given \( \mathcal{F}_T \) measurable terminal wealth \( \xi \), not but as a FSDE, where the terminal wealth \( \xi \) is thought of as a control variable, as described in El Karoui, Peng, and Quenez [17] as backward formulation.

We make several assumptions according to El Karoui, Peng and Quenez [17] in the sequel to ensure the existence and uniqueness of adapted solutions of BSDEs with jump term [2, 50, 52].

**Assumption 3.1.** The intertemporal aggregator \( f \) satisfies the assumptions of standard drivers. More precisely, \( f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{nB} \times \mathbb{L}^2_{\mathbb{V}(\cdot)}(\mathbb{R}^{nB}) \rightarrow \mathbb{R} \) is progressively measurable with respect to all its variables and it is supposed to be uniformly Lipschitz with respect to \( y, z, \lambda(\cdot); \) that is, there exists a constant \( L > 0 \) such that

\[
|f(\omega, t, c, y_1, z_1, \lambda_1(\cdot)) - f(\omega, t, c, y_2, z_2, \lambda_2(\cdot))| \\
\leq L \left( |y_1 - y_2| + |z_1 - z_2| + |\lambda_1 - \lambda_2|_{\mathbb{V}(\cdot)} \right),
\]

for \( \forall(\omega, t, c, y_1, z_1, \lambda_1(\cdot)), \forall(\omega, t, c, y_2, z_2, \lambda_2(\cdot)). \) Moreover \( g \) is Lipschitz with respect to \( \lambda(\cdot) \) for \( \forall \epsilon; \) there exists a \( L_0 > 0 \) such that \( |g(\omega, t, c, y, z, \lambda_1(\epsilon)) - g(\omega, t, c, y, z, \lambda_2(\epsilon))| \leq L_0 |\lambda_1(\epsilon) - \lambda_2(\epsilon)|. \)

The drift of wealth, \( b, \) also satisfies the assumptions of a standard driver. In particular, it is Lipschitz with respect to \( x, \pi, \) uniformly with respect to \( (\omega, t, c). \)
Assumption 3.2. There exists some constants $k_1, k_2$ such that,

$$|f(t, c, 0, 0, 0)| \leq k_1 + k_2 \frac{c^p}{p} \text{ a.s. with } 0 < p < 1 \quad \forall c \in \mathbb{R}^+.$$ 

These assumptions ensure that, for each $c \in H^2_{T, v(\cdot)}$, and each terminal reward $Y_T \in \mathbb{H}^2$, BSDE (3.3) has a unique solution $(Y, Z, \Lambda) \in H^2_{T} (\mathbb{R}^1) \times H^2_{T} (\mathbb{R}^n) \times H^2_{T, v(\cdot)} (\mathbb{R}^np)$ and the comparison theorem of BSDEs that are driven by jump-diffusion processes hold. We also make the following natural assumptions, which are also crucial in the proof of sufficient conditions for optimality, as shown later.

Assumption 3.3. $f$ is strictly concave with respect to $c, Y, Z$ and $\Lambda(\epsilon)$ and $f$ is a strictly nondecreasing function with respect to $c$.

By the comparison theorem of BSDE theory, assumption 3.3 ensures the usual properties of utility functions, that is, monotonicity with respect to the terminal value and to the consumption, and concavity with respect to the consumption.

The terminal value $Y_T$ of the SDU is joined to the static utility of terminal wealth $X_T$ as $Y(\omega) = U(X_T(\omega), \omega)$. The assumption to be made for the static utility function is the following usual one.

Assumption 3.4. $U$ is strictly concave and strictly nondecreasing real function defined on $\mathbb{R} \times \Omega$ which is $\mathcal{F}_T \times \mathcal{B}(\mathbb{R})$ measurable, and which also satisfies,

$$|U(x)| \leq k_1 + k_2 \frac{x^p}{p} \text{ a.s. with } 0 < p < 1 \quad \forall x \in \mathbb{R}^+.$$ 

This assumption 3.4 ensures that the variable $U(X_T) \in \mathbb{H}^{2/p} \subset \mathbb{H}^2$ for each $X_T \in \mathbb{H}^2$, and that the recursive utility associated with this terminal value is increasing and concave with respect to terminal wealth.

We suppose similar assumptions to the utility process on the driver $b$ of the wealth process.

Assumption 3.5. $b$ is supposed to be uniformly Lipschitz with respect to $X$ and $\pi' \sigma$.

Assumption 3.6. There exists a positive constant $k$ such that for $\forall c \in \mathbb{R}^+$ and $\forall \eta \in \{1, \cdots, m\}$, $|b(t, c, 0, 0, \eta)| \leq kc$ a.s.

Assumption 3.7. $b$ is convex with respect to $c, x$ and $\pi' \sigma$, and $b$ is nondecreasing with respect to $c$.

Assumption 3.8. $b(t, c, 0, 0, \eta) \geq 0$ a.s. for $\forall c \in \mathbb{R}^+$ and $\forall \eta \in \{1, \cdots, m\}$. 

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Let \((X^t_c, \pi; 0 \leq t \leq T)\) be the wealth process associated with initial wealth \(x\) and strategy \((c, \pi) \in H^2_{T, \nu}(\mathbb{R}) \times H^2_{T, \nu}(\mathbb{R})\). One may notice that given an initial wealth and portfolio, there exists a pathwise unique wealth process which solves forward equation (3.4) since \(b\) is Lipschitz.

It should be noted that the wealth processes have the diffusion coefficient given exogenously, but those have the controlled drift due to the optimally chosen consumptions. According to Duffie and Skiadas [13], we take a consumption space \(\mathcal{C}\) to be the subset of predictable measurable positive processes \(c_t\) which belong to \(H^2\) (i.e., such that \(\mathbb{E}\left[\int_0^T c_t^2 dt\right] < +\infty\)), and a terminal value space \(\mathcal{W}\) to be the set of square integrable \(\mathcal{F}_T\)-measurable positive random variable \(\xi(\mathcal{W} = (H^2)^+)\). A consumption plan \((\xi, c) \in \mathcal{W} \times \mathcal{C}\) is called feasible for the initial wealth \(x\) if and only if \(X^\xi, c(0) \leq x\). We will denote by \(\mathcal{A}(x)\) the set of consumption plans feasible for the initial wealth \(x\).

Now, we proceed to the maximization problem:

\[
\sup_{\xi, c \in \mathcal{A}(x)} Y_t \quad \text{ (3.5)}
\]

s.t. \(X_0 \leq x\). The SDU together with the wealth process formulated in backward form are written as

\[
\begin{align*}
\frac{dY_t}{dt} &= -f(t, c_t, Y_t, Z_t, \Lambda_t) + Z_t dW_t + \sum_{j,k=1}^{m} \Lambda^{kj}_t dN^{kj}_t, \\
Y_T &= U(\xi), \\
\frac{dX_t}{dt} &= -b(\pi_t, c_t, X_t, \eta_{t-}) + \pi_t \sigma(\eta_{t-}) dW_t, \\
X_T &= \xi.
\end{align*}
\]

As usual in the optimization procedure, we define the Lagrangian as

\[
L(0) = Y_0 - \lambda (X_0 - x).
\]

In order to maximize this Lagrangian, we employ the stochastic gradient approach, described by El Karoui, Peng, and Quenez [17]. Let \(\Delta c_t := c_t - c_t^0\) and
\[ \Delta \xi := \xi - \xi^0. \] The stochastic gradients\(^2\) of \((Y,X)\) are given by
\[ d\nabla Y_t = - \left( f_x(t) \Delta c_t + f_y(t) \nabla Y_t + f_z(t) \nabla Z_t + \int_E g_{\lambda}(t,e) \nabla \Lambda_t(e) \nabla (de) \right) dt + \nabla Z_t dW_t + \int_E \nabla \Lambda_t(e) \tilde{N}(dt,de), \] (3.8)
\[ \nabla Y_T = \partial U(\xi^0) \nabla X_T, \]
d\[ d\nabla X_t = - (b_x(t) \Delta c_t + b_y(t) \nabla X_t + b_\pi(t) \Delta \pi_t) dt + \Delta \pi_t \sigma(\eta_t) dW_t, \] (3.9)
\[ \nabla X_T = \Delta \xi, \]
where \(f_x, f_y, f_z, b_x, b_y, b_\pi\) etc. denote partial derivatives with respect to each subscript variable, \(\partial U\) also denotes the partial derivative with respect to the wealth, and we abbreviate \(X^{\pi,x}\) to \(X\). Thanks to the assumption 3.3, the driver of (3.8) is uniformly Lipschitz with respect to the above FSDE of \(G\)
\[ \Delta \xi := \xi - \xi^0. \] The stochastic gradients\(^2\) of \((Y,X)\) are given by
\[ d\nabla Y_t = - \left( f_x(t) \Delta c_t + f_y(t) \nabla Y_t + f_z(t) \nabla Z_t + \int_E g_{\lambda}(t,e) \nabla \Lambda_t(e) \nabla (de) \right) dt + \nabla Z_t dW_t + \int_E \nabla \Lambda_t(e) \tilde{N}(dt,de), \] (3.8)
\[ \nabla Y_T = \partial U(\xi^0) \nabla X_T, \]
d\[ d\nabla X_t = - (b_x(t) \Delta c_t + b_y(t) \nabla X_t + b_\pi(t) \Delta \pi_t) dt + \Delta \pi_t \sigma(\eta_t) dW_t, \] (3.9)
\[ \nabla X_T = \Delta \xi, \]
where \(f_x, f_y, f_z, b_x, b_y, b_\pi\) etc. denote partial derivatives with respect to each subscript variable, \(\partial U\) also denotes the partial derivative with respect to the wealth, and we abbreviate \(X^{\pi,x}\) to \(X\). Thanks to the assumption 3.3, the driver of (3.8) is uniformly Lipschitz with respect to \(\nabla Y_t, \nabla Z_t, \nabla \Lambda_t(\cdot)\), so the BSDE has unique solution. These two BSDEs can be solved by using the adjoint processes introduced in Proposition 2.3.1:
\[ \frac{dG^X(t)}{G^X(t)} = b_x(t) dt + b_\pi(t) dW_t, \quad G^X(0) = 1, \]
\[ \frac{dG^Y(t)}{G^Y(t)} = f_x(t) dt + f_\pi(t) dW_t + \int_E g_{\lambda}(t,e) \tilde{N}(dt,de), \quad G^Y(0) = 1, \]
Since \(b_x(t,x,\pi,c,\eta) = r(t), b_\pi(t,x,\pi,c,\eta) = -\sigma^{-1}(\eta)(\mu(\eta) - r(t)) = -\Psi(t)\), the above FSDE of \(G^X\) has a solution,
\[ G^X(T) = \exp \left( - \int_0^T r(u) du - \int_0^T \Psi(u) dW(u) - \frac{1}{2} \int_0^T \Psi(u)^2 du \right) = H(T). \]
The linear SDEs of (3.8) and (3.9) are easily solved in terms of \(G^X\) and \(G^Y\), which have the following stochastic representations,
\[ \nabla X_t = \mathbb{E}_t \left[ G^X(T) \nabla X_T + \int_t^T G^X(s) b_c(s) \Delta c_s ds \right], \]
\[ \nabla Y_t = \mathbb{E}_t \left[ G^Y(T) \nabla Y_T + \int_t^T G^Y(s) f_c(s) \Delta c_s ds \right]. \]
Thus we have the stochastic gradient of Lagrangian,
\[ \nabla L(t) = \mathbb{E}_t \left[ \left\{ G^Y(T) \partial U(\xi^0) - \lambda G^X(T) \right\} \right] \Delta \xi + \mathbb{E}_t \left[ \int_t^T \left\{ G^Y(s) f_c(s) - \lambda G^X(s) b_c(s) \right\} \Delta c_s ds \right]. \]
\(^2\)As studied in [13], the stochastic gradient \(\nabla Y(t,\tilde{c})\) is the Gateaux derivative that is defined by
\[ \nabla Y(t,\tilde{c}) := \lim_{\alpha \to 0} (Y(t,\tilde{c} + \alpha(c - \tilde{c})) - Y(t,\tilde{c}))/\alpha \] provided the limit exists and finite.
Defining
\[ \Gamma^X(T) := G^Y(T) \partial U(\xi^0) - \lambda G^X(T), \]
\[ \Gamma^c(t) := G^Y(t) f_c(t) - \lambda G^X(t) b_c(t). \]

\( \nabla L(0) \) is written as
\[ \nabla L(0) = \mathbb{E} \left[ \Gamma^X(T) \Delta \xi + \int_0^T \Gamma^c(s) \Delta c_s ds \right]. \]

Before giving the maximum principle, an additional assumption — the so-called Inada condition — needs to be made.

**Assumption 3.9.** \( U \) and \( f \) satisfies \( \partial U(0) = +\infty \) and \( f_c(t,0,Y_t,Z_t,A_t(\cdot)) = +\infty. \)

Thanks to the theorem 2.4.2 and 2.4.3, it is said that \((\xi^0,c^0) \in \mathcal{A}(x)\) is the optimal solution for the problem (3.5) if and only if the following maximum conditions hold:
\[ \Gamma^X_0(T) = 0, \quad \Gamma^c_0(t) = 0. \quad (3.10) \]

Let \( I(\cdot) := (\partial U)^{-1}(\cdot) \). From equation (3.10), we find the optimal terminal wealth and intermediate consumption.

**Proposition 3.2.1.** The optimal terminal wealth \( \xi^0 \) is of the form
\[ \xi^0 = I(\lambda H_Y(T)), \quad H_Y(t) := \frac{G_t^X}{G_t^Y}, \quad (3.11) \]
and the optimal consumption \( c^0 \) is
\[ c^0(t) = (f_c)^{-1}(t,\lambda H_Y(t)b_c(t),Y_t,Z_t,A_t(\cdot)). \quad (3.12) \]

**Proof.** The results follow immediately from equation (3.10). \( \square \)

Now, in order to construct a Markovian FBSDE system, it is convenient to introduce a variable defined by
\[ A_t = \log \left( \frac{\lambda G_t^X}{G_t^Y} \right). \quad (3.13) \]

Then
\[ dA_t = \phi(t,\Theta_t)dt + \psi(t,\Theta_t)dW_t + \int_{\mathcal{E}} \chi(t,e,\Theta_t)N(dt,de), \quad (3.14) \]
where
\[
\phi(t, \Theta_t) = b_x - f_y - \frac{1}{2} b_z^2 + \frac{1}{2} f_z^2 + \int_E g_\Lambda(\Lambda(e)) v(\text{d}e),
\]
\[
\psi(t, \Theta_t) = b_z - f_z,
\]
\[
\chi(t, v, \Theta_t) = -\log (1 + g_\Lambda(\Lambda(e))).
\]

where we set the argument \( \Theta_t = (t, X_t, Y_t, \pi_t, Z_t, \Lambda(\cdot)) \). Let us guess the form of SDU and wealth processes for model (I).

\[
Y_t = \hat{u}(t, A_t, \eta_{t-}) = \sum_{i=1}^{m} \mathbf{1}_{\{t\}}(\eta_{t-}) u(t, A_t, i),
\]
\[
X_t = \hat{v}(t, A_t, \eta_{t-}) = \sum_{i=1}^{m} \mathbf{1}_{\{t\}}(\eta_{t-}) v(t, A_t, i).
\]

(3.15)

Let by \( L_t^{a, \eta} \) denote the infinitesimal generator which is decomposed as a sum of the continuous and jump parts involving the time-evolution of \( A \) and \( \eta \):

\[
L_t^{a, \eta} u(t, a, k) = \mathcal{A}_t^{a, \eta} u(t, a, k) + \mathcal{B}_t^{a, \eta} u(t, a, k),
\]

where
\[
\mathcal{A}_t^{a, \eta} u(t, a, k) := \phi(t, a, k) \partial_a u + \frac{1}{2} \psi(t, a) \partial_{aa} u,
\]
\[
\mathcal{B}_t^{a, \eta} u(t, a, k) := \sum_{j=1, j \neq k}^{m} \{u(t, a + \chi_{kj, j}) - u(t, a, k)\} \lambda^{kj}(t, a).
\]

With this, the SDEs of \( Y_t \) and \( X_t \) become

\[
dY_t = \{\partial_t \hat{u} + \mathcal{A}_t^{a, \eta} \hat{u} + \mathcal{B}_t^{a, \eta} \hat{u}\} \text{d}t
\]
\[
+ \psi \partial_u \hat{u} \text{d}W_t + \sum_{k=1}^{m} \mathbf{1}_{\{k\}}(\eta_{t-}) \sum_{j=1, j \neq k}^{m} \{u(t, A_t + \chi_{kj, j}) - u(t, A_t, k)\} \lambda^{kj},
\]
\[
dX_t = \{\partial_t \hat{v} + \mathcal{A}_t^{a, \eta} \hat{v} + \mathcal{B}_t^{a, \eta} \hat{v}\} \text{d}t
\]
\[
+ \psi \partial_v \hat{v} \text{d}W_t + \sum_{k=1}^{m} \mathbf{1}_{\{k\}}(\eta_{t-}) \sum_{j=1, j \neq k}^{m} \{v(t, A_t + \chi_{kj, j}) - v(t, A_t, k)\} \lambda^{kj}.
\]

More explicitly, the PDE system we have to deal with is of the form;

\[
\partial_t u(t, a, k) + \mathcal{A}_t^{a, \eta} u(t, a, k) + \mathcal{B}_t^{a, \eta} u(t, a, k) + f(c_t, u(t, a, k)) = 0,
\]
\[
u(T, a, k) = U (v(T, a, k)),
\]

(3.16)
and
\[ \partial_t v(t, a, k) + \mathcal{A}_t^a \eta v(t, a, k) + \mathcal{B}_t^a \eta v(t, a, k) + b(\pi_t, c_t, v(t, a, k)) = 0, \]
\[ v(T, a, k) = I(e^a), \] (3.17)
where we used the expression of the optimal terminal wealth (3.11) and the notation of the forward process \( A_t \) of (3.13). Since the PDE system satisfies conditions (A2) of Theorem 3.5 in [2], they have unique viscosity solutions. With those solutions, it readily follows that the diffusion and jump coefficients of (3.6) and (3.7) are represented by
\[ Z_t = \psi(t, A_t, \eta_t) \frac{\partial \hat{u}}{\partial a}(t, A_t, \eta_t), \]
\[ \Lambda_{k,j} = \{u(t, A_t + \chi_{k,j}, j) - u(t, A_t, k)\} \lambda^{k,j}, \]
\[ \pi'_\sigma(\eta_t) = \psi(t, A_t, \eta_t) \frac{\partial \hat{v}}{\partial a}(t, A_t, \eta_t). \]
The final expression gives the optimal investment strategy via solving numerically the nested quasi-linear PDEs of (3.16) and (3.17). In the subsequent section, we will provide a viable numerical procedure.

### 3.2.2 Non-synchronous regime shift model

Next, we shall generalize the model (I) in the following way.

**Model (II)** Another modelling is that the drift and diffusion coefficient follow the non-synchronous stochastic processes:
\[ \frac{dS_t}{S_t} = \mu(\eta_{t-})dt + \sigma(\zeta_{t-})dW_t, \]
\[ d\eta_t = \sum_{k,j=1}^{m_\mu} (j-k)1_{(k)}(\eta_{t-})dN_{1t}^{kj}, \]
\[ d\zeta_t = \sum_{l,j=1}^{m_\sigma} (j-l)1_{(l)}(\zeta_{t-})dN_{2t}^{lj}, \] (3.18)
where \( W_t \) denotes one-dimensional Brownian motion \((n_B = 1)\), and \( N_{1t}^{kj} (k, j = 1, \ldots, m_\mu) \) and \( N_{2t}^{kj} (k, j = 1, \ldots, m_\sigma) \) denote \( m_\mu^2 \) and \( m_\sigma^2 \)-dimensional independent Poisson processes \((n_P = m_\mu^2 + m_\sigma^2, \nu_{1,kj}(de) = \nu_{2,kj}(de) = \delta_1(de))\), each of which has constant intensity \( \lambda_{1}^{kj} \) and \( \lambda_{2}^{kj} \), respectively. The essential point of model (II) is that these finite states \( \eta \) and \( \zeta \) driven by two types of independent Poisson processes allow us to model non-synchronous changes of drift and
diffusion coefficient, for instance, due to the unexpected drastic changes of the economic structure. Then the SDU and wealth processes are given by

\[ dY_t = -f(c_t, Y_t, Z_t, \Lambda(\cdot))dt + Z_t dW_t + \sum_{j,k=1}^{m_u} \Lambda_{1t}^{kj} dN_{1t}^{kj} + \sum_{j,k=1}^{m_g} \Lambda_{2t}^{kj} dN_{2t}^{kj}, \]
\[ dX_t = -b(\pi_t, c_t, X_t, \eta_{t-})dt + \pi_t \sigma(\zeta_{t-})dW_t, \]

where we denoted \( \Lambda = (\Lambda_1, \Lambda_2) \) for simplicity.

Let us define the integral operator associated with two-dimensional Poisson processes\(^3\):

\[ \mathcal{A}_t^{a, \eta, \zeta} u(t, a, k, l) := \phi(t, a, k, l) \partial_a u + \frac{1}{2} \psi(t, a) \partial_{aa} u, \]
\[ \mathcal{B}_t^{a, \eta, \zeta} u(t, a, k, l) := \sum_{j=1, j \neq k}^{m_u} \{ u(t, a + \chi_{k\lambda j}, j, l) - u(t, a, k, l) \} \lambda_1^{kj} + \sum_{j=1, j \neq l}^{m_g} \{ u(t, a + \chi_{klj}, k, j) - u(t, a, k, l) \} \lambda_2^{lj}. \]

For the model (II), we shall guess the form of SDU and wealth processes as

\[ Y_t = \hat{u}(t, A_t, \eta_{t-}, \zeta_{t-}) = \sum_{i=1}^{m_u} \sum_{j=1}^{m_g} 1_{\{i\}} (\eta_{t-}) 1_{\{j\}} (\zeta_{t-}) u(t, A_t, i, j), \]
\[ X_t = \hat{v}(t, A_t, \eta_{t-}, \zeta_{t-}) = \sum_{i=1}^{m_u} \sum_{j=1}^{m_g} 1_{\{i\}} (\eta_{t-}) 1_{\{j\}} (\zeta_{t-}) v(t, A_t, i, j), \]

(3.19)

where we let \( u(t, a, k, l), v(t, a, k, l) \) be the solutions of the following PDEs associated with the relevant reaction-diffusion system;

\[ \partial_t u(t, a, k, l) + \mathcal{A}_t^{a, \eta, \zeta} u(t, a, k, l) + \mathcal{B}_t^{a, \eta, \zeta} u(t, a, k, l) + f(c_t, u(t, a, k, l)) = 0, \]
\[ u(T, a, k, l) = U(v(T, a, k, l)), \]

(3.20)

and

\[ \partial_t v(t, a, k, l) + \mathcal{A}_t^{a, \eta, \zeta} v(t, a, k, l) + \mathcal{B}_t^{a, \eta, \zeta} v(t, a, k, l) + b(\pi_t, c_t, v(t, a, k, l)) = 0, \]
\[ v(T, a, k, l) = I(e^a). \]

(3.21)

\(^3\)We shall neglect the simultaneous jumps of \( N_1 \) and \( N_2 \), because its intensity is approximately \( \lambda_1 \times \lambda_2 \), which seems to be considerably small.
With those solutions, we find that

\[ Z_t = \psi(t, A_t, \eta_t, \zeta_t) \frac{\partial \hat{u}}{\partial a}(t, A_t, \eta_t, \zeta_t), \]

\[ \Lambda_{1, kl; jl} = u(t, A_t + \chi_{kl; jl}, j, l) - u(t, A_t, k, l), \]

\[ \Lambda_{2, kl; km} = u(t, A_t + \chi_{kl; km}, k, m) - u(t, A_t, k, l), \]

\[ \pi' \sigma(\zeta_t) = \psi(t, A_t, \eta_t, \zeta_t) \frac{\partial \hat{v}}{\partial a}(t, A_t, \eta_t, \zeta_t). \]

The final equation yields the optimal investment strategy via solving numerically the nested quasi-linear PDEs of (3.20) and (3.21). In the subsequent section, we will provide a viable numerical procedure.

### 3.3 Numerical computation

In this section, we provide several numerical results of our models described earlier. For simplicity we assume the intertemporal aggregator of the simple form, \( f(c_s, Y_s) \), for whole numerical computation.

#### 3.3.1 Nested PDE system of synchronous regime shift model

We shall consider three regimes case \((m = 3)\) of model (I), which may be interpreted as the risky asset dynamics (3.1) changing among three states such as credit ratings or business cycles. Then the PDE (3.16) becomes

\[
\frac{\partial u(t, a, 1)}{\partial t} + \mathcal{S}^{a, \eta} u(t, a, 1) + \lambda^{12} (u(t, a + \chi_{12}, 2) - u(t, a, 1)) \\
+ \lambda^{13} (u(t, a + \chi_{13}, 3) - u(t, a, 1)) + f(c^0(t), u(t, a, 1)) = 0,
\]

\[
\frac{\partial u(t, a, 2)}{\partial t} + \mathcal{S}^{a, \eta} u(t, a, 2) + \lambda^{21} (u(t, a + \chi_{21}, 1) - u(t, a, 2)) \\
+ \lambda^{23} (u(t, a + \chi_{23}, 3) - u(t, a, 2)) + f(c^0(t), u(t, a, 2)) = 0,
\]

\[
\frac{\partial u(t, a, 3)}{\partial t} + \mathcal{S}^{a, \eta} u(t, a, 3) + \lambda^{31} (u(t, a + \chi_{31}, 1) - u(t, a, 3)) \\
+ \lambda^{32} (u(t, a + \chi_{32}, 2) - u(t, a, 3)) + f(c^0(t), u(t, a, 3)) = 0
\]

(3.22)
with the boundary conditions, \( u(T,a,k) = U(I(e^a)) \) for \( k = 1, 2, 3 \), and the PDE (3.17) becomes

\[
\frac{\partial v(t,a,1)}{\partial t} + \mathcal{A}^{a,n} v(t,a,1) + \lambda^{12} (v(t,a + \chi_{12}, 2) - v(t,a,1)) \\
\quad + \lambda^{13} (v(t,a + \chi_{13}, 3) - v(t,a,1)) + b(\pi^0(t), c^0(t), v(t,a,1)) = 0,
\]

\[
\frac{\partial v(t,a,2)}{\partial t} + \mathcal{A}^{a,n} v(t,a,2) + \lambda^{21} (v(t,a + \chi_{21}, 1) - v(t,a,2)) \\
\quad + \lambda^{23} (v(t,a + \chi_{23}, 3) - v(t,a,2)) + b(\pi^0(t), c^0(t), v(t,a,2)) = 0,
\]

\[
\frac{\partial v(t,a,3)}{\partial t} + \mathcal{A}^{a,n} v(t,a,3) + \lambda^{31} (v(t,a + \chi_{31}, 1) - v(t,a,3)) \\
\quad + \lambda^{32} (v(t,a + \chi_{32}, 2) - v(t,a,3)) + b(\pi^0(t), c^0(t), v(t,a,3)) = 0
\]

(3.23)

with the boundary conditions, \( v(T,a,k) = I(e^a) \) for \( k = 1, 2, 3 \), where we put \( c^0_{ik} = (f_c)^{-1}(e^a, u(t,a,k)) \) and \( \pi^0_{ik} = \sigma^{-1} \psi(t,a,k) \frac{\partial v}{\partial a}(t,a,k) \) for \( k = 1, 2, 3 \).

**Fixed Point Algorithm**

We propose the iterative algorithm to compute the PDE system (3.22) and (3.23).

1. Set the terminal boundary condition to \( u(T,a,k) \) for all \( k \).

2. Repeat the following iterative phase until the stopping criterion is met.

   a. Given \( u(t,a,2), u(t,a,3) \), compute \( u(t,a,1) \) by using the discretized approximation of relevant PDE, and put the value to \( \hat{u}(t,a,1) \).

   b. Given \( u(t,a,1), u(t,a,3) \), compute \( u(t,a,2) \) by using the discretized approximation of relevant PDE, and put the value to \( \hat{u}(t,a,2) \).

   c. Given \( u(t,a,1), u(t,a,2) \), compute \( u(t,a,3) \) by using the discretized approximation of relevant PDE, and put the value to \( \hat{u}(t,a,3) \).

   d. If \( \sum_{k=1}^{3} |u^{(i)}(t,a,k) - u^{(i-1)}(t,a,k)| < \varepsilon \), then go to the end, else return to the iterative phase.

3. end.

If the third state is absorbing one, that is, default state in credit risky security, dead state in insurance contract, etc., then we may take \( \lambda^{31} = \lambda^{32} = 0 \) in (3.22).

For the numerical analysis, we have the following candidates of the intertemporal aggregator:
(i) a logarithmic type function (Schroder and Skiadas (1999) [48])

\[
f(c, y) = (1 + \alpha y) \left[ \log c - \frac{\beta}{\alpha} \log (1 + \alpha y) \right],
\]

with \( \beta \geq \min(0, \alpha) \).

(ii) a power type function (continuous time version of the Kreps-Porteus recursive utility)

\[
f(c, y) = (1 + a) \left[ \frac{c^v}{y |y|^{\frac{\nu}{1+a}}} - \beta y \right],
\]

with \( \nu < \min(1, 1/(1 + a)), a \in (-1, 1), \beta > 0, \nu \neq 0 \).

In whole numerical analysis, we used the above power type intertemporal aggregator (3.25). In order to solve numerically the quasi-linear PDEs such as (3.22) and (3.23) under certain terminal conditions we employed the fixed point algorithm stated above. See Nakamura [41] for its detail and the other numerical applications.

Figure 3.1 depicts the optimal SDUs and consumptions of three regimes, which are obtained by solving numerically the quasi-linear PDEs (3.22) with the fixed point algorithm stated earlier being applied. Employing this numerical scheme for PDEs (3.23), we can also obtain the optimal wealth and trading strategy of each state. When the regime represented by \( \eta \) shifts from one state to another in simulating the forward process \( A \) numerically, we can observe that each path of \( X \) and \( Y \) jumps between the corresponding surfaces of Fig.3.1.

### 3.3.2 Nested PDE system of non-synchronous regime shift model

Next, we shall consider two regimes case \( m\mu = m\sigma = 2 \) for non-synchronous regime shift model (II). These four states may be thought of as a combination of two categories, e.g., good or bad state of the firm’s operating activity times stable or volatile state reflecting a good or bad choice of risky projects with uncertain
Figure 3.1: 3D plot of the solution \((Y, C)\) of three regimes \(\eta = 1, 2, 3\) (corresponding respectively to the first, second and third row), where we take the SDU with the Kreps-Porteus type intertemporal aggregator (3.25), the static utility for bequest, \(U(x) = x^{1-\gamma}/(1-\gamma)\), and set the parameters as \(1/(1+a) = 1.1\), \(\nu = 0.5\min(1, 1/(1+a))\), \(\beta = 0.05\), \(\gamma = 2\), \(\lambda = 1\), \(T = 0.25\), \(r = 0.01\), \(\mu = 0.1\), \(\sigma = 0.9\), intensities of the Poisson processes, \(\lambda_{12} = \lambda_{21} = 6\), \(\lambda_{13} = \lambda_{31} = 4.5\), \(\lambda_{23} = \lambda_{32} = 9\), division numbers of time and \(A\) are 100 and 50, respectively. The x-axis indicates \(A(t)\), and the y-axis does the time \(t\) (year). Let \(h\) denote the small interval of the division of \(A(t)\) region. We take the jump sizes as \(\chi_{12} = \chi_{21} = 20h\), \(\chi_{13} = \chi_{31} = 10h\), \(\chi_{23} = \chi_{32} = 30h\). The fixed point algorithm is used for the numerical computation of the quasi-linear PDEs (3.22). The first column of the graph shows \(Y = u(t, A)\) of each state \(\eta = 1, 2, 3\), and the second column shows the corresponding consumptions, \(c\) (3.12).
future revenue. Then the PDE (3.20) becomes

\[
\begin{align*}
\frac{\partial u(t,a,1,1)}{\partial t} + \omega^a \eta \xi u(t,a,1,1) + \lambda_1^{12}(u(t,a + \chi_{11;21}, 2, 1) - u(t,a,1,1)) \\
+ \lambda_2^{12}(u(t,a + \chi_{11;12}, 1, 2) - u(t,a,1,1)) + f(c^0(t), u(t,a,1,1)) = 0, \\
\frac{\partial u(t,a,1,2)}{\partial t} + \omega^a \eta \xi u(t,a,1,2) + \lambda_1^{12}(u(t,a + \chi_{12;22}, 2, 2) - u(t,a,1,2)) \\
+ \lambda_2^{12}(u(t,a + \chi_{12;11}, 1, 1) - u(t,a,1,2)) + f(c^0(t), u(t,a,1,2)) = 0,
\end{align*}
\]

(3.26)

with the boundary conditions, \( u(T,a,k,l) = U(I(e^a)) \) for \( k,l = 1,2 \), and the PDE (3.21) becomes

\[
\begin{align*}
\frac{\partial v(t,a,1,1)}{\partial t} + \omega^a \eta \xi v(t,a,1,1) + \lambda_1^{12}(v(t,a + \chi_{11;21}, 2, 1) - v(t,a,1,1)) \\
+ \lambda_2^{12}(v(t,a + \chi_{11;12}, 1, 2) - v(t,a,1,1)) + b(\pi^0(t), c^0(t), v(t,a,1,1)) = 0, \\
\frac{\partial v(t,a,1,2)}{\partial t} + \omega^a \eta \xi v(t,a,1,2) + \lambda_1^{12}(v(t,a + \chi_{12;22}, 2, 2) - v(t,a,1,2)) \\
+ \lambda_2^{12}(v(t,a + \chi_{12;11}, 1, 1) - v(t,a,1,2)) + b(\pi^0(t), c^0(t), v(t,a,1,2)) = 0,
\end{align*}
\]

(3.27)

with the boundary conditions, \( v(T,a,k,l) = I(e^a) \) for \( k,l = 1,2 \), where we put \( c^0_{kl} = (f_c)^{-1}(e^a, u(t,a,k,l)) \) and \( \pi^0_{kl} = \sigma^{-1}\psi(t,a,k,l) \frac{\partial \psi}{\partial a}(t,a,k,l) \) for \( k,l = 1,2 \). According to the numerical procedure shown in the previous subsection, we can compute the nested PDEs of (3.26) and (3.27). As the result is similar to the previous computational example, we will omit it.

### 3.4 Summary and Concluding Remarks

In this chapter, we have studied two SDU-based optimization problems for reaction-diffusion systems; each asset price dynamics is described by either synchronous
or non-synchronous asset price process. We have solved these optimization problems, employing the stochastic maximum principle. Consequently, the optimal solutions such as wealth process and associated investment strategy as well as a pair of SDU and its diffusion coefficient are characterized by a FBSDE system.

In the numerical analysis, we have computed the nested PDEs with Poisson jump terms by employing the fixed point algorithm. This procedure is a straightforward extension of the four-step scheme [35] of FBSDE system based upon diffusion processes in such a way to incorporate the Poisson jump terms. As mentioned earlier, there has been a renewed interest in dynamic investment models with regime-shift in the past few years. Our studies show that the reaction-diffusion system sheds light on the study of regime-shifting phenomena and provides much potential scope of modeling the state dynamics.

Finally, we shall make a few remarks about the remaining issues. First, we have mainly investigated the numerical properties based on the Kreps-Porteus type intertemporal aggregator. We are furthermore interested in exploring what optimal solutions the other choices of intertemporal aggregators provide. Second, we have solved the optimization problems with constrained terminal conditions. What happens if we are going to constrain the wealth process or the trading strategies at intermediate time. This may be solved by an obstacle problem to which the reflected BSDE theory is applied, as shown in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [18]. Cvitanic, Karatzas, and Soner [9] have also studied the constrained strategy problem. Their ideas may have applicability to our SDU-based maximization problems. Fourth, as mentioned in Becherer and Schweizer [3], the reaction-diffusion system can describe the defaultable claims, the credit quality of which changes between discrete states. Recently Shouda [49] studied the indifference pricing of defaultable bonds in the framework with an unpredictable recovery rate in which two regimes are regarded as non-default and default states. It is quite interesting to explore what investment strategy in defaultable assets is optimal in our SDU-based multi-regime model corresponding to the credit-rating transition. Those subjects stated above are of course open problems. We leave those for future research.
Chapter 4

A BSDE Approach to Utility Indifference Pricing and Reserving

4.1 Introduction

The valuation of dynamic risks has been one of the fundamental issues concerning financial institutions. When there is a financial market that is complete, the price of a contingent claim can be determined uniquely. The pricing theory in a complete market is based on a strategy in which one creates a portfolio that replicates the payoff of the claim. The risk associated with the contingent claim is completely removed. The unique price of the contingent claim is provided by the initial wealth necessary to fund the replicating portfolio. However, in an incomplete market where it is not always possible to construct the replicating portfolio, there are many different prices that are consistent with no-arbitrage and each corresponds to a different martingale measure.

This chapter studies a utility indifference approach to the pricing and reserving problem in incomplete markets where there are untradable risks when the utility is time-additive. The pricing and reserving of a contingent claim that contains untradable risks is a main problem of financial institutions who offer products that have an insurance function. For example, a weather derivative has various types of payoffs that depend on the temperature or rainfall, snowfall, typhoon, etc., but the weather conditions cannot be replicated in the market. A variable annuity pays a death benefit, which is a function of stock price; in addition, some products guarantee a maturity benefit for survivors at the beginning of the annuity. Indeed, the exact time of the death of the insured cannot be replicated. However, financial institutions’ valuation and their (partial) hedging strategy should be taken into account along with their attitudes toward and preferences for the risks. These institutions can maximize their utility of wealth and can pay (or sell) a certain amount
today for the right to receive (or pay) the claim such that they are no worse off in terms of terminal utility than they would have been without the claim. This concept was introduced by Hodges and Neuberger [22] for the valuation of European calls in the presence of transaction costs. This premium principle is known as the principle of equivalent utility among actuaries; see Gerber and Pafumi [20] for detail. The pricing and reserving of insurance contracts are important problems for actuaries. Gerber [19] extended this principle to calculate reserves, the so-called utility indifference reserve. Recently, Young [54] applied the principle of equivalent utility to price and reserve equity-indexed life insurance. Most studies have focused on how to solve the partial differential equation (PDE). Young also solved the Hamilton-Jacobi-Bellman (HJB) equation by using the explicit finite difference method. However, the finite difference method encounters problems in the case of multidimensional risks. This chapter applies the principle of equivalent utility to the price and reserve of a financial product that contains untradable risks in the market, studies the utility maximization problem of a small investor (who does not influence market prices), and deals with the calculation method of the value function.

The Malliavin calculus developed by Malliavin [38] was originally designed to study the smoothness of the densities of the solutions to stochastic differential equations. After Karatzas and Ocone [25] demonstrated the usefulness of the representation theorem in terms of the Malliavin derivative in finance, there was a massive interest in the Malliavin calculus, as a result of which the theory has been generalized and many applications have been found. In the case of the Brownian motion, the chain rule is proved through the definition of the derivative as a weak derivative on the canonical space (see Nualart [42]). Nualart and Vives [43] proved that the Malliavin derivative coincides with a difference operator on the canonical space in the case of the Poisson process. The Clark-Ocone formula, which has been frequently used to obtain hedging portfolios for derivatives, holds for the Lévy process (see Løkka [33]).

We use the backward stochastic differential equation (BSDE) to calculate a utility indifference selling price (buying price) and reserve (asset valuation) of the writer (or buyer) of a contingent claim that contains untradable risks in the market. We put a literature review in section 2.4 in order to summarize the earlier studies of the BSDE approach.

The first contribution of this chapter is to derive the linear BSDE expression of the value function. Owing to the characteristics of the adjoint process of a linear BSDE, the value function could be computed as an expectation of forward processes. The second contribution is to derive the expression of utility indifference price where it is decomposed into the present value under Minimal Martingale Measure and its certainty equivalent. These formulations are derived from the requirement of the dynamic programming principle. The third contribution is to
apply the same methodology not only to the case that the untradable risk follows a Brownian motion but also to a jump-diffusion process.

The remaining chapter is organized as follows. The problem formulation is established and a utility indifference price and reserve is introduced in section 2. In section 3, we study and obtain the utility indifference pricing and reserving where the untradable risk follows a Brownian motion. In section 4, we study the price and the reserve of a contingent claim, which is not traded in the market, containing both market tradable risks and a non-financial jump risk. In section 5, we consider examples that illustrate the characteristics of utility indifference prices for the reinsurance contract.

4.2 Problem formulation

4.2.1 Notation

First, we will begin with the general setting necessary for analysis. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a complete filtered probability space such that \(\mathcal{F}_0\) is augmented by all the \(P\)-null sets of \(\mathcal{F}\). Let

\[ \bar{W}_t := (W_1(t)^*, W_2(t)^*)^* := (W_{1:1}(t), \cdots, W_{1:m}(t), W_{2:1}(t), \cdots, W_{2:l}(t))^* \quad (4.1) \]

for \(m \geq 1\) and \(l \geq 0\) be an \((m+l)\)-dimensional standard Brownian motion. And let \(N(dt, de)\) be a random measure which has a predictable measure \(\nu(de)dt\). They are defined on this space such that \(\{\mathcal{F}_t\}_{t \geq 0}\) is generated by \(\bar{W}\) and \(N\). In this paper, we use \(W_2\) and \(N\) to model the untradable risk, with \(l = 0\) and \(\nu(de) = 0\) corresponding to the case of the complete market. \(\bar{N}(dt, de)\) is the compensated martingale measure such that \(\bar{N}(dt, de) = N(dt, de) - \nu(de)dt\). For \(x \in \mathbb{R}^n\), \(|x|\) denotes its Euclidean norm such that \(|x| = \sqrt{\text{trace}(x^*x)}\), and \(\langle x, y \rangle\) denotes an inner product such that \(\langle x, y \rangle = \text{trace}(x^*y)\) and * denotes a transpose.

Let \(L^2_{\mathcal{F}}(\mathbb{R}^n)\) be the space of all measurable functions \(\varphi: [0, T] \rightarrow \mathbb{R}^n\) satisfying \(|\varphi|_{\mathcal{F}} < \infty\), where \(|\varphi|_{\mathcal{F}} := (\int_0^T |\varphi(t)|^2 dt)^{1/2}\). Let \(H^2(\mathbb{R}^n)\) be the space of \(\mathcal{F}_t\) measurable random variables \(X: \Omega \rightarrow \mathbb{R}^n\) satisfying \(|X| < \infty\), where \(|X| := (E[|X|^2])^{1/2}\), and \(H^2_{\mathcal{F}}(\mathbb{R}^n)\) be the space of all predictable process \(\varphi: \Omega \times [0, T] \rightarrow \mathbb{R}^n\) satisfying \(|\varphi| < \infty\), where \(|\varphi| := (E[\int_0^T |\varphi|^2 dt])^{1/2}\). Let \(S^2(\mathbb{R}^n)\) be the space of all progressively measurable processes \(\varphi: \Omega \times [0, T] \rightarrow \mathbb{R}^n\), where \(E(\sup_{0 \leq t \leq T} |\varphi|^2) < \infty\).

Denote also that \(L^2_{\mathcal{F}_t}(\mathbb{R}^n)\) be the space of \(\mathcal{F}_t\) measurable \(\mathbb{R}^n\)-valued functions \(\varphi(e)\) such that \(|\varphi|_{\mathcal{F}} < \infty\), where \(|\varphi|_{\mathcal{F}} := (\int_0^T |\varphi(e)|^2 \nu(de))^{1/2}\), and \(E := \mathbb{R} \setminus 0\) and \(H^2_{\mathcal{F}_t}(\mathbb{R}^n)\) be the space of all predictable function process \(\varphi: \Omega \times [0, T] \times \mathbb{R} \setminus 0 \rightarrow \mathbb{R}^n\) satisfying \(|\varphi| < \infty\) where \(|\varphi| := (E[\int_0^T \int_E |\varphi(e)|^2 N(dt, de)])^{1/2}\).
4.2.2 The financial market and the wealth process

We consider a market in which trading takes place continuously. There exist \( n + 1 \) assets in which the agent can invest some of his wealth in the market. One is a riskless asset in which price process \( P^0_t \) is

\[
dP^0_t = P^0_t r_t dt,
\]

where \( r_t \) is the short rate. The others are \( n \) risky assets (the stocks). The price process of the \( i \)th stock \( P^i_t \) is

\[
dP^i_t = a^i_t dt + \sum_{j=1}^{m} \sigma^i_{t,j} dW_{1:j}(t).
\]

We assume that the \( n \times m \) volatility matrix \( \sigma_t = (\sigma^i_{t,j}) \) has full rank and that \( n \geq m \).

The agent invests in these assets with portfolio \( \pi^*_t = (\pi^1_t, \pi^2_t, \ldots, \pi^n_t) \) and \( \pi^0_t = X_t - \sum_{i=1}^{n} \pi^i_t \). In these settings, the wealth process of the agent is

\[
dX_t = (X_t - \sum_{i=1}^{n} \pi^i_t) \frac{dP^0_t}{P^0_t} + \sum_{i=1}^{n} \pi^i_t \frac{dP^i_t}{P^i_t} dt + \pi^*_t \theta_t dW_t,
\]

where \( \theta_t \) is the risk premium vector such that \( a^i_t - r_t 1 = \sigma_t^i \theta_t \). Let \( (X^{x,\pi}_t, 0 \leq t \leq T) \) be the wealth process associated with initial wealth \( x \) and strategy \( \pi \). The agent has to choose a portfolio strategy \( \pi \) feasible for the initial wealth \( x \), that is, \( \pi \in \mathcal{H}^T_{\mathbb{R}^n} \), with \( X^{x,\pi}_t \geq 0 \) \( dP \otimes dt \) a.s. By \( \mathcal{A} \), we denote the set of portfolio strategies \( \pi \) that are feasible for initial wealth \( x \). In addition, we shall assume that the market parameters, \( \theta_t \in L^2_T(\mathbb{R}^m), \sigma_t \in L^2_T(\mathbb{R}^{n \times m}), \) and \( \sigma_t \theta_t \in L^2_T(\mathbb{R}^m) \) are deterministic functions of \( t \).

4.2.3 Utility indifference pricing and reserving

Let us consider a contingent claim \( \xi \) that is assumed to be \( \mathcal{F}_T \) measurable and bounded. A utility indifference price \( p \) is the price at which the investor’s utility is indifferent between receiving nothing and not having the liability \( \xi \) and receiving \( p \) and having the liability \( \xi \) at time \( T \). Let \( U \) be a utility function of the agent. Let us consider the agent who has \( k \) units of liability \( \xi \), whose value is assumed to be \( \mathcal{F}_T \) measurable and bounded, but necessarily positive. A natural way to define the agent’s preference is to model his attitude toward risk using utility functions. The maximization of the expected utility is as follows:

\[
V(x,k,t) = \sup_{\pi \in \mathcal{A}} E_t \left[ U(X^{x,\pi}_T - k\xi) | X_t = x \right].
\]
Under these assumptions, we define the utility indifference pricing problem as follows.

**Problem 3.** Find the utility indifference price \( p(x, k) \) and an admissible portfolio \( \pi \in \mathcal{A} \) that satisfies

\[
V(x + p(x, k), k, 0) = V(x, 0, 0).
\]  
(4.6)

**Remark 4.2.1.** Defined above is the so-called utility indifference selling (or asking) price. A utility indifference buying (or bidding) price \( p^b(x, k) \) is defined as \( p^b(x, k) := -p(x, -k) \). With this in mind, for \( \forall k \in \mathbb{R} \), we let \( p(x, k) \) denote the solution to (4.6).

For a special case, if the market is complete or if the claim \( \xi \) is perfectly replicable, the utility indifference price \( p(x, k) \) is equivalent to the complete market price of \( k \) units. This fact is shown by many authors, for example, we can find the proof in the work of Henderson and Hobson [21].

After writing the contingent claim, the writer should set aside the reserve for the payoff at maturity \( T \). The reserve at time \( t \in [0, T] \) is the amount \( v(x, k, t) \) such that the writer is indifferent between continuing to underwrite the contingent claim and paying \( v \) to another writer to assume its risk. We define a utility indifference reserving problem as follows.

**Problem 4.** Find the utility indifference reserve \( v(x, k, t) \) and an admissible portfolio \( \pi \in \mathcal{A} \) at time \( t \in [0, T] \) that satisfies

\[
V(x, k, t) = V(x - v(x, k, t), 0, t).
\]  
(4.7)

**Remark 4.2.2.** At the beginning of this contract (time 0), the utility indifference price is equivalent to the utility indifference reserve, which is shown by

\[
V(x + p(x, k), k, 0) = V(x + p(x, k) - v(x, k, 0), 0, 0) = V(x, 0, 0).
\]  
(4.8)

**Remark 4.2.3.** When the agent is a buyer of the contingent claim \( \xi \), a utility indifference asset valuation \( v^b(x, k, t) \) is defined as \( v^b(x, k, t) := -v(x, k, t) \), which is similar to the utility indifference buying price \( p^b(x, k) \). At the beginning of the contract, the utility indifference asset valuation is equivalent to the utility indifference buying price.

4.2.4 Literature review

We use the backward stochastic differential equation (BSDE) to calculate a utility indifference selling price (buying price) and reserve (asset valuation) of the
writer (or buyer) of a contingent claim that contains untradable risks in the market. The BSDE approach was used by El Karoui and Rouge [47] to calculate the value function and the optimal portfolio. They formulated the dual value function as BSDE and led the optimal portfolio under the constrained portfolio situation. More directly, Müller et al. [24] showed the value function itself as BSDE. They considered the process satisfying BSDE as
\[ U(X_t - y_t) \Rightarrow U(X_T) \]
\[ y_t \Rightarrow 0, \]
where \( U \) denotes the utility function of terminal wealth and \( X_t \) represents the wealth process, and they get an adjusting process \( y_t \) that makes the process \( U(X_t - y_t) \) be a value function. Further, Jeanblanc et al. [4] presented a similar formulation to study the indifference pricing of defaultable claims. Another approach was proposed by Lim [30], who considered the following process,
\[ y_t U(X_t) \Rightarrow U(X_T) \]
\[ y_t \Rightarrow 1, \]
and formulated the adjusting process \( y_t \) in the case of mean-variance hedging. His subsequent paper [31] applied the abovementioned method to mean-variance hedging in an incomplete market where the underlying assets were driven by jump-diffusion processes. However, the existence of unique solution of adjusting process \( y_t \) is proved only in the case of quadratic utility. It is not applied to the other utility functions.

In contrast, our paper considers the process of
\[ U(F(t, X_t)) + y_t \Rightarrow U(X_T) \]
\[ y_t \Rightarrow 0 \]
\[ F(T, x) = x \quad \forall x \in \mathbb{R}, \]
instead of the approaches used in referenced papers. This formulation has some tractable properties.

4.3 Case of Brownian motion

This section focuses on a case that untradable risk in the market follows Brownian motion. the approach of completion of squares of the value function that appeared in many papers; for example, El Karoui and Rouge [47] computed the value function and the optimal strategy for exponential utility by means of BSDE under a constrained portfolio environment. Further, assuming an additional auxiliary process, Müller et al. [24] succeeded in constructing the value function directly.
addition, Lim applied these concepts to the quadratic hedging problem in \cite{30}. In this section, we consider the utility indifference price of the contingent claim $\xi$ that has an untradable component $W_2(t)$ under a deterministic interest rate circumstance. To compute the utility indifference price $p(x,k)$, let us begin with the arguments concerning the value function $V(X_t,k,t)$.

First, let us define a fictitious price $h_t$ by

$$h_t := E^Q \left[ e^{-\int_t^T r_s ds} \xi \big| \mathcal{F}_t \right],$$

where $Q$ is the risk-neutral measure that is priced in the market and considers an expectation under real measure for untradable risk $W_2$. In other words, the fictitious price $h_t$ is said to be the price of the claim $\xi$ under the minimum martingale measure which does not have a market price of risk of $W_2$. If the contingent claim becomes tradable in the market, its price needs an additional market price of risk for the new risk. Let $\Gamma_t$ be a discounted Girsanov density process defined by the forward SDE

$$d\Gamma_t = -r_t dt + \theta_t^* dW_1(t) + \theta_t^* dW_2(t), \quad \Gamma_0 = 1.$$ 

Then we have that the fictitious price $h_t = E[\Gamma_T \xi | \mathcal{F}_t]$. With the help of the adjoint process of the linear BSDE, the fictitious price $h_t$ is a unique solution of BSDE of

$$dh_t = \{r_t h_t + \lambda_{1t}^* \theta_t \} dt + \lambda_{1t}^* dW_1(t) + \lambda_{2t}^* dW_2(t) \quad h_T = \xi. \quad (4.9)$$

By considering the BSDE (4.9) of the fictitious price, it is possible to argue about the problem by taking advantage of the property of the BSDE.

Second, let us assume that the process $y_t$ is an adjusting process that makes the process $U(P(t,T)^{-1}(X_t - kh_t)) + y_t$ equivalent to the value function of this problem $V(x,k,t)$ where $P(t,T)$ be the price of a discount bond paying 1 at maturity $T$. In this section, we assume that the interest rate is deterministic $r_t$ and $P(t,T) = e^{-\int_t^T r_s ds}$. Let the observable accumulated net capital be denoted by $C(t,T,r_t,X_t,h_t) := P(t,T)^{-1}S_t := P(t,T)^{-1}(X_t - kh_t)$. For notational convenience, we use $C(t,X_t)$ or $C_t$ as an abbreviation of $C(t,T,r_t,X_t,h_t)$, and $\tilde{C}(t,X_t)$ or $\tilde{C}_t$ as $C(t,T,r_t,X_t,0,h_t)$.

Let us consider the following BSDE,

$$dy_t = g(t,X_t)dt + \varepsilon_{1t}^* dW_1(t) + \varepsilon_{2t}^* dW_2(t), \quad y_T = 0, \quad (4.10)$$

with a process $g(t,X_t)$ that depends on the utility functions. Before proceeding to the proposition, we have to make assumptions on the utility function.

**Assumption 4.1.** $U(x)$ is a twice continuously-differentiable, strictly concave, and strictly nondecreasing real function on $\mathbb{R}$ that is $\mathcal{B}(\mathbb{R})$ measurable. $|U''(x)|^2 < \infty$ and $|U''(x)|^2 < \infty$ for $-\infty \leq x \leq \infty$. 

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Assumption 4.2. Suppose \( \hat{U}(x) := U''(x) \left( \frac{U'(x)}{U''(x)} \right)^2 \kappa_1 - \kappa_2 \) for \( 0 \leq \kappa_1 \leq \infty \) and \( 0 \leq \kappa_2 \leq \infty \), \( |\hat{U}(x)|^2 < \infty \) for \( -\infty \leq x \leq \infty \), \( \hat{U} \) is a differentiable, and \( \lim_{x \to \infty} \hat{U}'(x) = 0 \).

The next proposition is found under the assumptions.

Proposition 4.3.1. The following FBSDE has a unique solution.

\[
\begin{align*}
    dX_t^0 &= (X_t^0 r_t + \pi_t^0 \sigma_t \theta_t) \, dt + \sigma_t^0 \sigma_t dW_1(t), \quad X_0^0 = x, \\
    dy_t^0 &= g(t, X_t^0) dt + z_t^0 dW_1(t) + z_t^0 dW_2(t), \quad y_0^0 = 0,
\end{align*}
\]

where

\[
\pi_t^0 = \theta_t^{-1} \left( k\lambda_{1t} - P(t, T) \frac{U'(C(t, X_t^0))}{U''(C(t, X_t^0))} \right)
\]

and the function \( g(t, x) \) is given by

\[
g(t, x) := \frac{1}{2} \frac{U''(C(t, x))}{U''(C(t, x))} |\theta_t|^2 - \frac{1}{2} U''(C(t, x)) P(t, T)^{-2} k^2 |\lambda_{2t}|^2.
\]

Proof. By using the comparison theorem and noting \( \sigma_t \) has full rank, we suppose that the wealth process is regarded as the BSDE with a given \( \mathcal{F}_T \) measurable terminal wealth \( \zeta \). Now we consider,

\[
\begin{align*}
    dX_t^0 &= (X_t^0 r_t + \pi_t^0 \sigma_t \theta_t) \, dt + \sigma_t^0 \sigma_t dW_1(t), \quad X_0^0 = \zeta, \\
    dy_t^0 &= g(t, X_t^0) dt + z_t^0 dW_1(t) + z_t^0 dW_2(t), \quad y_0^0 = 0.
\end{align*}
\]

As \( g(t, x) = -1/2 \cdot \hat{U}(P(t, T)^{-1}(x - kh_t)) \) where \( \kappa_1 = |\theta_t|^2 \) and \( \kappa_2 = P(t, T)^{-2} k^2 |\lambda_{2t}|^2 \), thanks to assumption 4.2 and \( h_t \) is bounded, \( g(t, x) \) is Lipschitz with respect to \( x \). In addition, noting that the \( \lambda_{2t} \) is a solution of BSDE (4.9), we find \( g(t, x) \in L^2_t (\mathbb{R}) \) since \( \sigma_t \theta_t \in L^2_t (\mathbb{R}^m) \), \( \lambda_{2t} \in L^2_t (\mathbb{R}) \), and \( X_t \in H^2_t (\mathbb{R}) \). Therefore the BSDEs have unique solutions. \( \square \)

Lemma 4.3.2. The process

\[
U(C(t, X_t)) + y_t
\]

is a supermartingale, where \( (X, y) \) is a solution to the FBSDE:

\[
\begin{align*}
    dX_t &= (X_t r_t + \pi_t^* \sigma_t \theta_t) \, dt + \pi_t^* \sigma_t dW_1(t), \quad X_0 = x, \\
    dy_t &= g(t, X_t) dt + z_t^* dW_1(t) + z_t^* dW_2(t), \quad y_T = 0.
\end{align*}
\]

for \( \forall \pi \in \mathcal{A} \subset H^2_t (\mathbb{R}^n) \).

Especially, the process

\[
U(C(t, X_t^0)) + y_t^0
\]

is a martingale.
Proof. Applying Itô’s formula to the process $U(C_t) + y_t$, we find
\[
d(U(C_t, X_t)) + y_t = \left[ \frac{1}{2} U''(C(t, X_t))P(t, T)^{-2} \left| \sigma_t^* \pi_t - \left(k\lambda_{1t} - P(t, T) \frac{U'(C(t, X_t))}{U''(C(t, X_t))} \theta_t \right) \right|^2 \right] dt
+ \left( U'(C(t, X_t))P(t, T)^{-1}(\pi_t^* \sigma_t - k\lambda_{1t}^*) + z_{1t}^* \right) dW_1(t)
+ \left( -U'(C(t, X_t))P(t, T)^{-1}k\lambda_{2t}^* + z_{2t}^* \right) dW_2(t).
\]

Let us denote
\[
D_1(t, X_t) := \frac{1}{2} U''(C(t, X_t))P(t, T)^{-2} \left| \sigma_t^* \pi_t - \left(k\lambda_{1t} - P(t, T) \frac{U'(C(t, X_t))}{U''(C(t, X_t))} \theta_t \right) \right|^2,
\]
\[
D_2(t, X_t, z_{1t}) := U'(C(t, X_t))P(t, T)^{-1}(\pi_t^* \sigma_t - k\lambda_{1t}^*) + z_{1t}^*,
\]
\[
D_3(t, X_t, z_{2t}) := -U'(C(t, X_t))P(t, T)^{-1}k\lambda_{2t}^* + z_{2t}^*.
\]

The process $d(U(C_t, X_t)) + y_t$ is integrable in $[0, T]$ because the processes $D_1(t, X_t) \in \mathbb{H}^2_T(\mathbb{R})$, $D_2(t, X_t, z_{1t}) \in \mathbb{H}^2_T(\mathbb{R}^m)$, and $D_3(t, X_t, z_{2t}) \in \mathbb{H}^2_T(\mathbb{R}^l)$. Since $U''$ is non-positive, $U(C_t, X_t) + y_t$ is a supermartingale.

Furthermore, applying also Itô’s formula to the process $U(C(t, X_t^0)) + y_t^0$, we find
\[
d(U(C(t, X_t^0)) + y_t^0)
+ \left( U'(C(t, X_t^0))P(t, T)^{-1}(\pi_t^0 \sigma_t - k\lambda_{1t}^*) + z_{1t}^0 \right) dW_1(t)
+ \left( -U'(C(t, X_t^0))P(t, T)^{-1}k\lambda_{2t}^* + z_{2t}^0 \right) dW_2(t).
\]

Since $D_2(t, X_t^0, z_{1t}^0) \in \mathbb{H}^2_T(\mathbb{R}^m)$ and $D_3(t, X_t^0, z_{2t}^0) \in \mathbb{H}^2_T(\mathbb{R}^l)$, $U(C(t, X_t^0)) + y_t^0$ is a martingale.

In order to prove the optimality of $\pi_t^0$, we make a concrete assumption on the utility function instead of assumption 4.1 and 4.2.

**Assumption 4.3.** $U(x)$ is an exponential utility function, i.e. $U(x) = e^{-\rho x}$.

**Remark 4.3.1.** The exponential utility ($U(x) = e^{-\rho x}$ and $\dot{U}(x) = -e^{-\rho x}(\kappa_1 - \rho^2 \kappa_2)$) satisfies the assumption 4.1 and 4.2.

Next proposition is found under the assumption.

**Proposition 4.3.3.** Suppose that the assumption 4.3 hold. The optimal portfolio of the problem (4.5) is solved uniquely,
\[
\pi_t^0 = \sigma_t^{-1} \left( k\lambda_{1t} - P(t, T) \frac{U'(C(t, X_t^0, \pi_t^0))}{U''(C(t, X_t^0, \pi_t^0))} \theta_t \right).
\] (4.11)
Further, the maximization of the expected utility is

\[ V(x, k, 0) = U(C(0, X_0^0)) + y_0^0. \] (4.12)

**Proof.** Consider the value function \( J(t, S_t) \) where \( S_t = X_t - kh_t \). The Hamilton-Jacobi-Bellman equation of \( J \) is,

\[
\max_x \left[ J_t + J_S(S_t r_t + \pi^*_t \sigma_t - k \lambda^*_t \theta_t) + J_{SS} |\pi^*_t \sigma_t - k \lambda^*_t|^2 \right] = 0.
\]

Assuming that the value function \( J(t, S_t) = A(t) \cdot (e^{-\rho P(t, T)^{-1} S_t}) \), the HJB equation is solved where \( A(t) = E[\exp(-\int_t^T |\Theta_s|^2 ds)] \) with \( \Theta_t = \frac{1}{2}(\theta_t^2 - \rho^2 P(t, T)^{-2} \cdot k^2 |\lambda_{2t}|^2) \).

Because the optimal portfolio of the HJB equation \( \sigma^{-1} \left( k \lambda^*_t + \frac{1}{\rho} P(t, T) \theta_t \right) \) is equivalent to \( \pi^*_t \) and martingale property proved in lemma 4.3.2, \( U(C(t, X_0^0)) + y_0^0 \) is a value function.

**Remark 4.3.2.** If we only consider utility maximization problem without underwriting the untradable risk, i.e. \( k = 0 \), proposition 4.3.3 also holds in case of power utility \( (U(x) = \frac{x^p}{p}) \).

Using the proposition, the following theorem is found immediately.

**Theorem 4.3.4.** Suppose that the assumptions hold. The utility indifference price \( p(x, k) \) and the utility indifference reserve \( v(x, k, t) \) of \( \mathcal{F}_T \) measurable contingent claim \( \xi \) are

\[
p(x, k) = kh_0 + P(0, T)(U)^{-1}(U(P(0, T)^{-1}x) + \alpha_0) - x \quad (4.13)
\]
\[
v(X_t, k, t) = kh_t - P(t, T)(U)^{-1}(U(P(t, T)^{-1}(X_t - kh_t)) - \alpha_t) + (X_t - kh_t) \quad (4.14)
\]

where

\[
\alpha_t = \frac{1}{2} E_t \left[ \int_t^T \left\{ \frac{U''(C_s)^2}{U''(C_s)} |\theta_s|^2 - \frac{U''(C_s)^2}{U''(C_s)} P(s, T)^{-2} k^2 |\lambda_{2s}|^2 - \frac{U''(C_s)^2}{U''(C_s)} |\theta_s|^2 \right\} ds \right].
\]

**Proof.** The value functions can be written as

\[
V(x + p(x, k), k, 0) = U(P(0, T)^{-1}(x + p(x, k) - kh_0)) + y_0
\]
\[
V(x, 0, 0) = U(P(0, T)^{-1}x) + y_0.
\]

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Substituting the utility indifference equation $V(x + p(x,k), k, 0) = V(x, 0, 0)$ and solving $p(x,k)$, we find
\[ p(x,k) = kh_0 + P(0,T)(U)^{-1}(U(P(0,T)^{-1}x) + \tilde{y}_0 - y_0) - x, \quad (4.15) \]
where $\tilde{y}_0$ denotes the solution of BSDE,
\[ d\tilde{y}_t = \frac{1}{2} \frac{U'(C_t)}{U''(C_t)} |\theta_t|^2 dt + \tilde{z}_1^w dW_1(t) + \tilde{z}_2^w dW_2(t), \quad \tilde{y}_T = 0. \]

Hereinafter, $\tilde{y}$ and $\tilde{z}$ denote the solution in the case of not having the liability $\xi$. With the same argument, we have
\[ V(X_t, k, t) = V(X_t - v(X_t, k, t), 0, t) \iff v(X_t, k, t) = kh_t - P(t, T)(U)^{-1}(U(P(t, T)^{-1}(X_t - kh_t)) - \tilde{y}_t + y_t \]
\[ + (X_t - kh_t), \quad (4.16) \]
and thus, the utility indifference reserve is obtained.

Since the driver of the BSDE of $\tilde{y}_t$ (respectively $y_t$) depends on neither $\tilde{y}_t$ (respectively $y_t$), $\tilde{z}_{1t}$ (respectively $z_{1t}$) nor $\tilde{z}_{2t}$ (respectively $z_{2t}$), the adjoint process of the BSDE exists uniquely. The proof of existence and uniqueness of the adjoint process for the linear BSDE is found in the paper of El Karoui et al. [16]. Let us denote $\Gamma_t$ (respectively $\tilde{\Gamma}_t$) as an adjoint process of $y_t$. We find $\Gamma_t \equiv 1$ ($\tilde{\Gamma}_t \equiv 1$), and the solutions of the BSDEs can be written as
\[ y_0 = E \left[ y_T - \int_0^T \left\{ \frac{1}{2} \frac{U'(C_t)}{U''(C_t)} |\theta_t|^2 - \frac{1}{2} \frac{U''(C_t)}{U''(C_t)} P(t, T)^{-1} k^2 |\lambda_{zt}|^2 \right\} dt \right] \]
\[ \tilde{y}_0 = E \left[ \tilde{y}_T - \int_0^T \frac{1}{2} \frac{U'(C_t)}{U''(C_t)} |\theta_t|^2 dt \right]. \]
Substituting the solutions $y_0$ and $\tilde{y}_0$ into (4.15) and (4.16), we obtain the desired result. \hfill \Box

**Remark 4.3.3.** The first term of the utility indifference price does not include the price of the untradable risk. The remaining part $P(0,T)(U)^{-1}(U(P(0,T)^{-1}x) + \alpha) - x$ corresponds to the *indifference price of the risk* of $W_{2t}$, and $(U)^{-1}(U(P(0,T)^{-1}x) + \alpha)$ can be regarded as the *certainty equivalent* of the contingent claim.

**Remark 4.3.4.** Since the solution $y_t$ depends on the optimal wealth $X_t$, it also depends on initial wealth $x + p(x,k)$ in the case of having the liability $\xi$ and receiving $p(x,k)$. Therefore, this pricing requires a convergent calculation like Newton-Raphson method.
Let us consider the price process that is consistent with the utility indifference price and reserve. The next corollary is derived.

**Corollary 4.3.5.** Suppose that the price process $\tilde{h}_t$ is equivalent to the utility indifference price and reserve for $\forall t \in [0, T]$. Then, $\tilde{h}_t$ satisfies the BSDE of

$$
\begin{align*}
    d\tilde{h}_t &= \left\{ r_t \tilde{h}_t + \lambda^*_{1t} \theta_t - \frac{1}{2k} \left\{ \frac{U'(C_t)}{U''(C_t)} - \frac{U''(\tilde{C}_t)}{U''(C_t)U''(\tilde{C}_t)} \right\} P(t, T)|\theta_t|^2 \\
    &\quad + \frac{1}{2} \frac{U''(C_t)}{U'(C_t)} P(t, T)^{-1} k |\lambda_{2t}|^2 \right\} dt + \lambda^*_{1t} dW_1(t) + \lambda^*_{2t} dW_2(t), \\
    \tilde{h}_T &= \xi.
\end{align*}
$$

(4.17)

**Proof.** In order to obtain the existence of solutions of the BSDE (4.17), we apply Theorem 2.3 of Kobylanski [26].

Consider that $\theta_{2t}$ is the price of the risk of $W_{2t}$, and that price process $\tilde{h}_t$ is formulated as

$$
\begin{align*}
    d\tilde{h}_t &= \left\{ r_t \tilde{h}_t + \lambda^*_{1t} \theta_t + \lambda^*_{2t} \theta_{2t} \right\} dt + \lambda^*_{1t} dW_1(t) + \lambda^*_{2t} dW_2(t), \\
    \tilde{h}_T &= \xi.
\end{align*}
$$

When $\alpha_t = 0$ for $\forall t \in [0, T]$, the price $\tilde{h}_t$ is equivalent to the utility indifference price $p(x, k)$ at time 0 and to the utility indifference reserve $v(X_t, k, t)$ for $\forall t \in [0, T]$. The condition of the price of the risk $\theta_{2t}$, which makes $\alpha_t$ equivalent to 0, is obtained by using the procedure followed for the previous theorem.

$\square$

### 4.4 Case of Poisson jump

In this section, we consider the utility indifference price of the reinsurance contract $\xi$ that has an untradable component $N(t, e)$ under a deterministic interest rate circumstance. Let us assume a fictitious price $h_t$, which is defined as

$$
h_t := E^Q \left[ e^{-\int^T_t r_s ds} \xi | \mathcal{F}_t \right],
$$

where $Q$ is the risk-neutral measure that is priced in the market and considers an expectation under real measure for untradable jump risk $N(t, e)$. With the help of the adjoint process of the linear BSDE, the fictitious price $h_t$ process can be expressed as BSDE form,

$$
\begin{align*}
    dh_t &= \{ r_t h_t + \lambda^*_{1t} \theta_t \} dt + \lambda^*_{1t} dW_1(t) + \int_{\mathbb{R}\setminus{0}} \Lambda(t, e) \tilde{N}(dt, de), \\
    h_T &= \xi.
\end{align*}
$$

(4.18)
Let us consider the following BSDE of adjusting process,

\[ dy_t = g(t, X_t)dt + z^0_t dW_1(t) + \int_{\mathbb{R}\setminus \{0\}} \eta(t, e)\tilde{N}(dt, de), \quad y_T = 0, \quad (4.19) \]

with a process \( g(t, X_t) \) that depends on the utility functions. The next proposition is obtained.

**Proposition 4.4.1.** Suppose that the assumption 4.1 and 4.2 hold. The following FBSDE has a unique solution.

\[
\begin{align*}
dX_t^0 &= (x_0^r_t + \pi_t^0 \sigma_t \theta_t) dt + \pi_t^0 \sigma_t dW_1(t), \quad X_0^0 = x, \\
dy_t^0 &= g(t, X_t^0)dt + z_t^0 dW_1(t) + \int_{\mathbb{R}\setminus \{0\}} \eta^0(t, e)\tilde{N}(dt, de), \quad y_T^0 = 0,
\end{align*}
\]

where

\[
\pi_t^0 = \sigma_t^{-1} \left( k\lambda_t - P(t, T) \frac{U'(C(t, X_t^0))}{U''(C(t, X_t^0))} \theta_t \right).
\]

and the function \( g(t, X_t) \) is given as

\[
g(t, X_t) = \frac{1}{2} \frac{U''(C(t, X_t^0))}{U'''(C(t, X_t^0))} \theta_t^2 \\
- \int_{\mathbb{R}\setminus \{0\}} \left[ U(C(t, X_t-)) - kP(t, T)^{-1}\Lambda(t, e) - U(C(t, X_t-)) \right] \nu(de) \\
+ U'(C(t, X_t-))kP(t, T)^{-1}\Lambda(t, e) \nu(de).
\]

**Proof.** By using the comparison theorem, we suppose that the wealth process is regarded as the BSDE with a given \( \mathcal{F}_T \) measurable terminal wealth \( \zeta \). Now we consider,

\[
\begin{align*}
dx_t^0 &= (x_0^r_t + \pi_t^0 \sigma_t \theta_t) dt + \pi_t^0 \sigma_t dW_1(t), \quad X_0^0 = \zeta, \\
dy_t^0 &= g(t, X_t^0)dt + z_t^0 dW_1(t) + \int_{\mathbb{R}\setminus \{0\}} \eta^0(t, e)\tilde{N}(dt, de), \quad y_T^0 = 0.
\end{align*}
\]

The function \( g \) is decomposed into \( \hat{U} \) and integral of jump size \( e \), i.e., \( g(t, x) = -1/2 \cdot \hat{U}(P(t, T)^{-1}(x - kh_t)) - \int k_3(t, x, e)\nu(de) \) where \( \kappa_1 = |\theta_t|^2 \), \( \kappa_2 = 0 \), and \( \kappa_3(t, x, e) = U(C(t, x) - kP(t, T)^{-1}\Lambda(t, e)) - U(C(t, x)) + U'(C(t, x))kP(t, T)^{-1}\Lambda(t, e). \)

Thanks to assumption 4.2 and \( h_t \) is bounded, \( g(t, x) \) is Lipschitz with respect to \( x \).

In addition, noting that the \( \Lambda(t, e) \) is a solution of BSDE (4.18), \( g(t, x) \in \mathbb{H}^2_T(\mathbb{R}) \) since \( \theta_t \in \mathbb{H}^2_T(\mathbb{R}^m) \), \( \Lambda(t, e) \in \mathbb{H}^2_T,\nu(\cdot)(\mathbb{R}^l) \), and \( X_t \in \mathbb{H}^2_T(\mathbb{R}) \). Therefore the BSDEs have unique solutions. \[\square\]
Lemma 4.4.2. The process

\[ U(C(t, X_t)) + y_t \]

is a supermartingale, where \((X, y)\) is a solution to the FBSDE:

\[
\begin{align*}
    dX_t &= (X_t r_t + \pi^*_t \sigma_t) dt + \pi^*_t \sigma_t dW_1(t), \quad X_0 = x, \\
    dy_t &= g(t, X_t) dt + z^*_t dW_1(t) + \int_{\mathbb{R} \setminus \{0\}} \eta(t, e) \tilde{N}(dt, de), \quad y_T = 0.
\end{align*}
\]

for all \( \pi \in \mathcal{A} \subset \mathbb{H}^2_T(\mathbb{R}^n) \).

Especially, the process \( U(C(t, X^0_t)) + y^0_t \) is martingale.

Proof. Applying Ito’s formula to the process \( U(C_t) + y_t \), we find

\[
\begin{align*}
    d(U(C(t, X_t)) + y_t) &= \left[ \frac{1}{2} U''(C(t, X_t)) P(t, T)^{-2} \left| \sigma^*_t \pi_t - \left( k\lambda_{1t} - P(t, T) \frac{U'(C(t, X_t))}{U''(C(t, X_t))} \theta_t \right) \right|^2 \right] dt \\
    &\quad + \left( U'(C(t, X_t)) P(t, T)^{-1} (\pi^*_t \sigma_t - k\lambda^*_t + z^*_t) \right) dW_1(t) \\
    &\quad + \int_{\mathbb{R} \setminus \{0\}} \left[ U(C(t, X_{t-}) - kP(t, T)^{-1} \Lambda(t, e)) - U(C(t, X_{t-})) \right] \tilde{N}(dt, de).
\end{align*}
\]

Let us denote

\[
\begin{align*}
    D_1(t, X_t) &:= \frac{1}{2} U''(C(t, X_t)) P(t, T)^{-2} \left| \sigma^*_t \pi_t - \left( k\lambda_{1t} - P(t, T) \frac{U'(C(t, X_t))}{U''(C(t, X_t))} \theta_t \right) \right|^2, \\
    D_2(t, X_t, z_t) &:= U'(C(t, X_t)) P(t, T)^{-1} (\pi^*_t \sigma_t - k\lambda^*_t + z_t), \\
    D_3(t, X_{t-}, \Lambda(t, e)) &:= U(C(t, X_{t-}) - kP(t, T)^{-1} \Lambda(t, e)) - U(C(t, X_{t-})).
\end{align*}
\]

The process \( d(U(C(t, X_t)) + y_t) \) is integrable in \([0, T]\) because the processes \( D_1(t, X_t) \in \mathbb{H}^2_T(\mathbb{R}) \), \( D_2(t, X_t, z_t) \in \mathbb{H}^2_T(\mathbb{R}^n) \), and \( D_3(t, X_{t-}, \Lambda(t, e)) \in \mathbb{H}^2_T,\mathcal{V}(\mathbb{R}^l) \). Since \( U'' \) is non-positive, \( U(C(t, X_t)) + y_t \) is a supermartingale.

Furthermore, applying also Ito’s formula to the process \( U(C(t, X^0_t)) + y^0_t \), we find

\[
\begin{align*}
    d(U(C(t, X^0_t)) + y^0_t) &= \left[ \frac{1}{2} U''(C(t, X^0_t)) P(t, T)^{-2} \left| \sigma^*_t \pi_t - \left( k\lambda^*_t - P(t, T) \frac{U'(C(t, X^0_t))}{U''(C(t, X^0_t))} \theta_t \right) \right|^2 \right] dt \\
    &\quad + \left( U'(C(t, X^0_t)) P(t, T)^{-1} (\pi^*_t \sigma_t - k\lambda^*_t + z^*_t) \right) dW_1(t) \\
    &\quad + \int_{\mathbb{R} \setminus \{0\}} \left[ U(C(t, X^0_{t-}) - kP(t, T)^{-1} \Lambda(t, e)) - U(C(t, X^0_{t-})) \right] \tilde{N}(dt, de),
\end{align*}
\]

Since \( D_2(t, X^0_t, z^*_t) \in \mathbb{H}^2_T(\mathbb{R}^m) \), and \( D_3(t, X^0_{t-}, \Lambda(t, e)) \in \mathbb{H}^2_T,\mathcal{V}(\mathbb{R}^l) \), \( U(C(t, X^0_t)) + y^0_t \) is a martingale. \qed
**Proposition 4.4.3.** Suppose that the assumption 4.3 holds. The optimal portfolio of this problem is solved uniquely,

\[
\pi_t^0 = \sigma_t^{-1} \left( k\lambda_t - P(t,T) \frac{U'(C(t,X_t^0))}{U''(C(t,X_t^0))} \theta_t \right).
\]  

(4.21)

Further, the maximization of the expected utility is

\[
V(x,k,0) = U(C_0) + y_0.
\]  

(4.22)

**Proof.** Consider the value function \( J(t,S_t) \) where \( S_t = x_t - kh_t \). The Hamilton-Jacobi-Bellman equation of \( J \) is,

\[
\max_{\pi} \left[ J_t + J_S(S_t,\pi_t + \pi_t^* \sigma_t \theta_t - k\lambda_{1\pi}^* \theta_t) + J_{SS} \left| \pi_t^* \sigma_t - k\lambda_{1\pi}^* \right|^2 + J_S \int_{\mathbb{R}} \Lambda(t,e) V(de) \right.
\]

\[
+ \int_{\mathbb{R}} \left\{ J(t,S_t - k\Lambda(t,e)) - J(t,S_t) \right\} V(de) = 0.
\]

Assuming that the value function \( J(t,S_t) = A(t) \cdot (e^{-\rho P(t,T)^{-1}}) \), the HJB equation is solved where \( A(t) = E[\exp(-\int_t^T |\Theta_s|^2 ds)] \) with

\[
\Theta_t = \frac{1}{2} |\theta_t|^2 + \rho P(t,T) k \int_{\mathbb{R}} \Lambda(t,e) V(de) + \int_{\mathbb{R}} [1 - e^{2P(t,T)k\Lambda(t,e)}] V(de).
\]

Because the optimal portfolio of the HJB equation \( \sigma^{-1} \left( k\lambda_{1\pi}^* + \frac{1}{p} P(t,T) \theta_t \right) \) is equivalent to \( \pi_t^0 \) and martingale property proved in lemma 4.4.2, \( U(C(t,X_t^0)) + \pi_t^0 \) is a value function.

Using the proposition, the following theorem is found immediately.

**Theorem 4.4.4.** Suppose that the assumptions hold. The utility indifference price \( p(x,k) \) and the utility indifference reserve \( v(x,k,t) \) of \( \mathcal{F}_T \) measurable reinsurance contract \( \xi \) are

\[
p(x,k) = kh_0 + P(0,T)(U)^{-1} \left( U(P(0,T)^{-1}x) + \alpha_0 \right) - x
\]  

(4.23)

\[
v(x_t,k,t) = kh_t - P(t,T)(U)^{-1} \left( U(P(t,T)^{-1}(x_t - kh_t)) - \alpha_t \right) - (X_t - kh_t)
\]  

(4.24)

where

\[
\alpha_t = \frac{1}{2} E_t \left[ \int_t^T \left\{ \frac{U'(C_s)^2}{U''(C_s)} |\theta_s|^2 - \int_{\mathbb{R}} \left\{ U(C_s - kP(s,T)^{-1} \Lambda(s,e)) - U(C_s)
\right.ight.
\]

\[
iU'(C_s)kP(s,T)^{-1} \Lambda(s,e)\right\} V(de) \right. - \left. \frac{U'(C_s)^2}{U''(C_s)} |\theta_s|^2 \right\} ds \right].
\]  

(4.25)
Proof. See Theorem 4.3.4. □

Let us consider the price process that is consistent with the utility indifference price and reserve, next corollary is derived.

Corollary 4.4.5. Suppose that the price process $\bar{h}_t$ is equivalent to the utility indifference price and reserve for $\forall t \in [0, T]$. Then, $\bar{h}_t$ satisfies the BSDE of

$$d\bar{h}_t = \left\{ r\bar{h}_t + \lambda^* \theta_t - \frac{1}{2k} \left\{ \frac{U'(C_t)}{U''(C_t)} - \frac{U''(C_t)}{U'(C_t)} \right\} P(t, T)|\theta_t|^2 \right.\
+ \left. \int_{\mathbb{R}\setminus 0} \left[ \frac{P(t, T)}{U'(C_t)} (U(C_t - kP(t, T)^{-1} \Lambda(t, e)) - U(C_t)) + k\Lambda(t, e) \right] \nu(de) \right\} dt \right.$$

$$+ \lambda^* dW_1(t) + \int_{\mathbb{R}\setminus 0} \Lambda(t, e) \tilde{N}(dt, de), \quad \bar{h}_T = \xi.$$  (4.26)

Proof. See Corollary 4.3.5 □

4.5 Numerical computation

In this section, we provide several numerical exmples of the utility indifference price and reserve of the contingent claim $\xi$ which has an untradable jump component $N(t, e)$. We consider the forward contract and European call option of an insurance claim which is not traded in the market. (The contingent claims correspond to the quota-share reinsurance which is one of the proportional reinsurance, and the aggregate excess of loss reinsurance which is one of the non-proportional reinsurance, respectively.)

Let us assume that the untradable asset $P_t^u$, which represents insurance claim underwrited directly by an insurer, follows

$$\frac{dP_t^u}{P_t^u} = a^u dt + \sigma^u dW_1(t) + \int_{\mathbb{R}\setminus 0} \{\exp(e) - 1\} \tilde{N}(dt, de).$$  (4.27)

Note that the untradable asset has a market-coupled risks $W_1$ through a $1 \times m$ volatility vector $\sigma_t^u$. To compute a differential form of the contingent claim, we apply the Malliavin calculus to our problem. Let us consider an $\mathcal{F}_T$-measurable random variable $F(\omega)$ such that

$$F(\omega) = \xi = \begin{cases} KP_T^- & \text{Quota-share reinsurance} \\ (P_T^u - K)^+ & \text{Aggregate excess of loss reinsurance} \end{cases}$$
where $\kappa$ is a proportion and $K$ is a retention (which is interpreted as the strike price of the European call option). By applying Clark-Ocone formula, we find

$$F = E^Q[F] + \int_0^T E^Q[D_tF|\mathcal{F}_t]dW_1^Q(t) + \int_0^T \int_{\mathbb{R}\setminus\{0\}} E^Q[D_{t,e}F|\mathcal{F}_t]\tilde{N}^Q(dt,de),$$

(4.28)

where $D_t$, $D_{t,e}$ are the derivatives in the Wiener and Poisson random measure direction respectively. Note that the Poisson random measure $\tilde{N}$ under minimal martingale measure is equal to $N$ because only Brownian motion $W_1$ is priced in the market. Comparing with (4.18), we find followings and it is possible to perform the general calculation.

$$\lambda_t = E^Q[D_tF|\mathcal{F}_t],$$

$$\Lambda(t,e) = E^Q[D_{t,e}F|\mathcal{F}_t].$$

In the case of the quota-share reinsurance,

$$D_tP_t^\mu = \sigma_t^\mu P_t^\mu,$$

$$D_{t,e}P_t^\mu = \{\exp(e) - 1\}P_t^\mu,$$

are obtained.

In the case of the aggregate excess of loss reinsurance, we use the standard procedure of approximating the function $(x - K)^+$ by a sequence of $C^1$ functions (see Øksendal [44] and Løkka [32] for details) to compute $D_t[(P_T^\mu - K)^+]$ and $D_{t,e}[(P_T^\mu - K)^+]$. We obtain,

$$D_t[(P_T^\mu - K)^+] = \sigma_t^\mu P_t^\mu 1_{P_T^\mu > K},$$

$$D_{t,e}[(P_T^\mu - K)^+] = (P_T^\mu + \{\exp(e) - 1\}P_T^\mu - K)^+ - (P_T^\mu - K)^+.$$

By using these results, we calculate following numerical examples.

**Example 4.5.1. (Exponential Utility)**

Assuming the exponential utility ($U(x) = -e^{-\rho x}$), the adjusting value $y_0$ is a solution of

$$dy_t = U(P(t,T)^{-1}(X_t - kh_t))$$

$$+ \left( \frac{1}{2} |\theta_t|^2 - \int_{\mathbb{R}\setminus\{0\}} [e^{\rho kP(t,T)^{-1}\Lambda(t,e)} - 1 - \rho kP(t,T)^{-1}\Lambda(t,e)]\nu(de) \right) dt$$

$$+ \xi_t^*dW_1(t) + \int_{\mathbb{R}\setminus\{0\}} \eta(t,e)\tilde{N}(dt,de), \quad y_T = 0,$$
and \( \tilde{y}_0 \) is a solution of
\[
d\tilde{y}_t = \frac{1}{2} U(P(t,T)^{-1}\tilde{X}_t) |\theta_t|^2 dt + \tilde{z}_t^* dW_1(t), \quad \tilde{y}_T = 0.
\]
The utility indifference price is then given by
\[
p(x,k) = kh_0 - \frac{1}{\rho} P(0,T) \log \left( e^{-\rho P(0,T)^{-1}x + \tilde{y}_0 - y_0} - x \right)
\]
and the utility indifference reserve is given by
\[
v(x,k,t) = kh_t + (X_t - kh_t) + \frac{1}{\rho} P(t,T) \log \left( e^{-\rho P(t,T)^{-1}(X_t - kh_t)} - \tilde{y}_t + y_t \right).
\]
The optimal portfolio is
\[
\pi_t = \sigma_t^{* -1} \left( k\lambda_t + P(t,T) \frac{1}{\rho} \theta_t \right).
\]

In the followings, we calculate the utility indifference prices of the quota-share reinsurance (figure 4.1 in case the jump size distribution follows normal distribution and figure 4.2 in case of exponential distribution) and the aggregate excess loss contract (figure 4.3 in case the jump size distribution follows normal distribution and figure 4.4 in case of exponential distribution).

### 4.6 Conclusion and Remarks

In this chapter, we studied the utility indifference pricing and reserving problem when a contingent claim contains untradable risks which follow Brownian motion and Poisson jump. We use the BSDE method to calculate the utility indifference prices and reserves of the contingent claim. As a result, we obtained the utility indifference pricing formula using the certainty equivalent. Owing to the characteristics of the adjoint process of a linear BSDE, the value function is formulated as an expectation of forward processes. Finally, the numerical examples is provided by applying the result of the Malliavin calculus, it enables for us to evaluate many types of contingent claims whose jump size distributions follow not only the normal distribution or the exponential distribution which are illustrated in the example. Finally, we shall make a few remarks about the remaining issues. First, the results obtained here are premised on the assumption that the agent has an exponential utility. If \( y_0^0 \geq y_0 \) is proved, our results could be extended to general utilities. Comparing the results with its numerical solution of optimization problem is of interest. Second, we have obtained value function without any constraint
Figure 4.1: (Quota-share reinsurance, Normal distribution) The utility indifference price using the Monte Carlo simulation method for a quota-share reinsurance of the untradable asset with exponential utility. The parameter set is $(r, \sigma, \theta, \rho, X_0, k) = (0.03, 0.2, 0.1, 0.05, 1000, 1)$ and the jump size follows log normal distribution $N(0.01, 0.04)$ with intensity 0.075. The upper left graph shows the optimal wealth without the liability; the upper right graph, the fictitious price of the reinsurance contract; and the lower left graph, the optimal wealth with the liability. The utility indifference price in the lower right graph is $p(x, k) = 113.0002$ (the fictitious price $h_0 = 100$).
Figure 4.2: (Quota-share reinsurance, Exponential distribution) The utility indifference price using the Monte Carlo simulation method for a quota-share reinsurance of the untradable asset with exponential utility. The parameter set is $(r, \sigma, \theta, \rho, X_0, k) = (0.03, 0.2, 0.1, 0.05, 1000, 1)$ and the jump size follows exponential distribution $P(0.10)$ with intensity 0.075. The upper left graph shows the optimal wealth without the liability; the upper right graph, the fictitious price of the reinsurance contract; and the lower left graph, the optimal wealth with the liability. The utility indifference price in the lower right graph is $p(x, k) = 109.7134$ (the fictitious price $h_0 = 100$).
Figure 4.3: (Aggregate excess of loss reinsurance, Normal distribution) The utility indifference price using the Monte Carlo simulation method for an aggregate excess of loss reinsurance of the untradable asset with exponential utility, in case of unconstrained portfolio. The parameter set is \((r, \sigma, \theta, \rho, X_0, k) = (0.03, 0.2, 0.1, 0.05, 1000, 1)\) and the jump size follows log normal distribution \(N(0.01, 0.04)\) with intensity 0.075. The upper left graph shows the optimal wealth without the liability; the upper right graph, the fictitious price of the reinsurance contract; and the lower left graph, the optimal wealth with the liability. The utility indifference price in the lower right graph is \(p(x,k) = 14.8048\) (the fictitious price \(h_0 = 5.5489\)).
Figure 4.4: *(Aggregate excess of loss reinsurance, Exponential distribution)*

The utility indifference price using the Monte Carlo simulation method for an aggregate excess of loss reinsurance of the untradable asset with exponential utility, in case of unconstrained portfolio. The parameter set is \((r, \sigma, \theta, \rho, X_0, k) = (0.03, 0.2, 0.1, 0.05, 1000, 1)\) and the jump size follows exponential distribution \(P(0.10)\) with intensity 0.075. The upper left graph shows the optimal wealth without the liability; the upper right graph, the fictitious price of the reinsurance contract; and the lower left graph, the optimal wealth with the liability. The utility indifference price in the lower right graph is \(p(x, k) = 11.2689\) (the fictitious price \(h_0 = 5.2103\)).
on wealth. It is interesting to explore what happens if we are going to constrain the wealth process or the trading strategies at intermediate time. Third, we do not consider possibility the agent will bankrupt due to the fluctuation of untradable risk. In reality, the financial institutions who underwrite untradable risks are always exposed to bankruptcy risk because there are no hedging instruments. It is quite interesting to explore how the utility indifference price change and what kinds of suggestions can be derived, when the possibility of bankruptcy of the agent is considered. We leave those subjects stated above for future research.
Chapter 5

Conclusion

Reaching the end of this thesis, the conclusion of our study and suggestions for future work are detailed.

In this thesis, we have investigated utility maximization problems in incomplete markets whose states are driven by jump-diffusion processes by applying Backward Stochastic Differential Equations. We have focused on the maximization problem of Stochastic Differential Utility and the Utility Indifference Pricing.

In chapter 2, we have provided the theoretical framework which is necessary for applications of SDU with jump. We have studied the maximization problem of the SDU in a market whose states are driven by a jump-diffusion process. To extend the result of El Karoui et al. [16], we have studied differentiability of the solutions of BSDEs with jumps. The first order condition that yields the necessary and sufficient condition for optimality is derived by using the property of the differentiability. Moreover, it is established that the optimal wealth and utility and their associated deflators are the unique solutions of a forward-backward system.

In chapter 3, we have studied two SDU-based optimization problems for reaction-diffusion systems; each asset price dynamics is described by either synchronous or non-synchronous asset price process. We have solved these optimization problems, employing the stochastic maximum principle by taking same approach as in chapter 2. In the numerical analysis, we have computed the nested PDEs with Poisson jump terms by employing the fixed point algorithm. This procedure is a straightforward extension of the four-step scheme [35] of FBSDE system based upon diffusion processes in such a way to incorporate the Poisson jump terms. Our studies show that the reaction-diffusion system sheds light on the study of regime-shifting phenomena and provides much potential scope of modeling the state dynamics.

In chapter 4, we have studied the utility indifference pricing and reserving problem when the contingent claim contains untradable risks which follow Brownian motion and Poisson jump. We have proposed new BSDE method to calculate...
the utility indifference prices and reserves. We have obtained the utility indifference pricing expression using the present value under Minimum Martingale Measure and the certainty equivalent. Furthermore, the numerical examples is provided by applying the result of the Malliavin calculus, it enables for us to evaluate many types of contingent claims whose jump size distributions follow not only the normal distribution or the exponential distribution which are illustrated in the example.

Finally, we shall make a few remarks about the remaining issues. In chapter 2, we propose new 4 types of normalized aggregator of SDU with jump. Examining the characteristics of the 4 aggregators are interesting research themes. Moreover, the inverse problem is discussed where the agent’s beliefs contain jump-diffusion. The beliefs density generator and the identity of short rate are obtained under given optimal consumption process. It is interesting to explore which types of aggregator and beliefs are most suitable for actually observed consumption data.

In chapter 3, we have mainly investigated the numerical properties based on the Kreps-Porteus type intertemporal aggregator. We are furthermore interested in exploring what optimal solutions the other choices of intertemporal aggregators provide. In addition, we have solved the optimization problems with constrained terminal conditions. What happens if we are going to constrain the wealth process or the trading strategies at intermediate time. This may be solved by an obstacle problem to which the reflected BSDE theory is applied, as shown in El Karoui, Kapoudjian, Pardoux, Peng, and Quenez [18]. Cvitanic, Karatzas and Soner [9] have also studied the constrained strategy problem. Their ideas may have applicability to our SDU-based maximization problems. Furthermore, as mentioned in Becherer and Schweizer [3], the reaction-diffusion system can describe the defaultable claims, the credit quality of which changes between discrete states. It is quite interesting to explore what investment strategy in defaultable assets is optimal in our SDU-based multi-regime model corresponding to the credit-rating transition.

In chapter 4, we have obtained the results premised on the assumption that the agent has an exponential utility. Comparing with its numerical solution of optimization problem in cases of other utilities is of interest. Moreover, we have obtained value function without any constrain on wealth. It is interesting to explore what happens if we are going to constrain the wealth process or the trading strategies at intermediate time. In addition, we have not considered any possibility that the agent would bankrupt due to the fluctuation of untradable risk. It is quite interesting to explore how the utility indifference price change and what kinds of suggestions can be derived, when the possibility of bankruptcy of the agent is considered. Those subjects stated above are of course open problems. We leave those for future research.


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