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Trend Inflation and Exchange Rate Dynamics
A New Keynesian Approach

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Trend Inflation and Exchange Rate Dynamics
A New Keynesian Approach

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Abstract
This study examines the exchange rate implications of trend inflation within a two-country New
Keynesian (NK) model. An NK Phillips curve generalized by trend inflation makes the two-country
inflation differential smoother, more persistent, and less sensitive to the real exchange rate. The
equilibrium of the model, which is supported under Taylor rule-type monetary policies with a
persistent trend inflation shock and strong interest rate smoothing, mimics a series of empirical reg-
ularities in real and nominal exchange rates that are hard to be reconciled jointly within a standard
NK model. Trend inflation helps explain the empirical puzzles of exchange rate dynamics.

Key Words: Real and Nominal Exchange Rates, Trend Inflation, New Keynesian Models
JEL Classification Numbers: E31, E52, F31, F41

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1. Introduction

The empirical reality of exchange rates keeps confounding researchers of international finance. Real exchange rates are persistent as well as volatile. Near random-walk nominal exchange rates have no significant dependence on current or past information on macroeconomic fundamentals. Real and nominal exchange rates move together closely. Nominal currency regimes affect real exchange rate dynamics significantly. These stylized facts jointly refute the theoretical challenges of canonical open-economy models with price stickiness. The elucidation of the complicated life of exchange rates in the literature on New Keynesian (NK) open-economy models is still far from satisfactory.\textsuperscript{1}

In this paper, I scrutinize the crucial roles of trend inflation in exchange rate dynamics.\textsuperscript{2} My investigation is based on an otherwise standard symmetric two-country NK model equipped with the Calvo-type time-dependent pricing behaviors of monopolistic competitive final good firms conducting pricing-to-market strategies in terms of local currencies, Taylor rule-type monetary policies with high interest rate smoothing, incomplete international financial markets, and permanent labor productivity shocks cointegrated across the two countries. I then extend the standard NK model by allowing for trend inflation that stochastically fluctuates around a positive long-run mean in the two countries. To my best knowledge, this study is the first attempt to embed trend inflation into a two-country NK model explicitly and extract its implications on exchange rate dynamics. Importantly, such a small change in the canonical two-country NK model regarding the low-frequency property of inflation has significant impacts on exchange rate dynamics.

As in Ireland (2007) and Cogley et al. (2010), the source of the fluctuations in trend inflation in this model is a time-varying inflation target that the central bank in each country sets following the Taylor rule. As specified by Cogley et al. (2010), the inflation target follows a persistent stochastic process with a positive long-run mean. On the one hand, a positive long-run mean of

\textsuperscript{1}Burstein and Gopinath (2014) and Engel (2014) provide excellent surveys of the recent empirical findings related to international real and nominal relative prices.

\textsuperscript{2}Ascari and Sbordone (2014) offer an excellent summary of recent research on trend inflation, focusing on the inflation dynamics in the United States.
trend inflation is undebatable for post-war data in the world economy. On the other hand, there are three rationales behind stochastic variations in the inflation target. First, as discussed by Cogley et al. (2010), the central bank only imperfectly knows the true economic structure and its learning process generates fluctuations in the inflation target. Second, private economic agents might suffer from an imperfect information problem about the inflation target. In this case, stochastic variations in the inflation target are likely to reflect the inferential errors of private agents about the long-run inflation target set by the central bank.

Third, a persistent shock to the inflation target is observationally equivalent to a persistent monetary policy disturbance in the Taylor rule with high interest rate smoothing in this paper. Coibion and Gorodnichenko (2012) provide a series of empirical evidence that the large persistence observed in the actual data on the short-run nominal interest rate in the United States stems largely from the policy inertia caused by interest rate smoothing under the Taylor rule, but not from the exogenous persistence of the monetary policy disturbance. Importantly, their empirical result suggests that a misspecification problem could lead to a high estimate of the serial correlation of the monetary policy disturbance unless the time-varying inflation target is correctly specified in the Taylor rule. Therefore, a persistent shock to the inflation target might be identified as a persistent monetary policy disturbance in the Taylor rule, which has been frequently emphasized in the literature on monetary policy.

Allowing for persistent trend inflation with a positive long-run mean within a two-country NK model produces fundamentally richer equilibrium dynamics of real and nominal exchange rates. Trend inflation changes the equilibrium dynamics of the two-country differential in the inflation rate essentially. A positive long-run mean of trend inflation substantially alters the shape of the log-linearized New Keynesian Phillips curve (NKPC) in each country, as examined by Cogley and Sbordone (2008) and Ascari and Sbordone (2014). In this NKPC generalized by trend inflation (hereafter, generalized NKPC, or GNKPC), the current inflation rate follows a second-order expectational difference equation depending on the rational expectations of the one- and two-period future inflation rates. This dependence of the current inflation rate on the higher-order expecta-
tions in each country results in a persistent and smooth movement of the two-country inflation differential. Moreover, the GNKPCs in the two countries imply a looser relationship between the cross-country inflation differential and real exchange rate than that in the conventional model: the current inflation differential becomes insensitive to a current development in the real exchange rate.

Importantly, these properties of the inflation differential in the GNKPC are passed into the cross-country differential in the inflation gap, i.e., the gap between the current inflation rate and inflation target. The interest rate inertia created by the monetary policy framework plays a crucial role here. With a high degree of interest rate smoothing, the resulting persistent and smooth dynamics of the inflation gap differential are amplified into large and persistent movements in the nominal interest rate differential. As a result, a positive persistent shock to the inflation target has a cumulative effect on the nominal interest rate differential, which increases sluggishly with a strong hump shape. The real interest rate differential shares this sluggish and persistent response of the nominal interest rate differential; therefore, the real interest rate differential becomes persistent and insensitive to the real exchange rate as well.

The persistent real interest rate differential, which is insensitive to the real exchange rate, is the key driver of the persistent and volatile movements in the real exchange rate in the model. The real uncovered interest rate parity (RUIP) condition as the no-arbitrage condition in incomplete financial markets determines the real exchange rate as the negative of the expected present values of future real interest differentials with the unit discount factor. The real interest rate acts as a persistent and exogenous economic fundamental for the real exchange rate. As claimed by Engel and West (2005), the Beveridge and Nelson random-walk trend component of a near nonstationary economic fundamental, i.e., the real interest rate differential, dominates the stochastic property of the real exchange rate.\(^3\) The real exchange rate then follows a near random walk with large volatility, as suggested by Engel and West’s (2005) proposition. This main mechanism is preserved and even strengthened with a small sensitivity of the policy rate to the output gap in the Taylor rule.

\(^3\)The random-walk property of the nominal exchange rate within two-country general equilibrium models with explicit real money demand functions is investigated by Nason and Rogers (2008) and Kano (2016).
Under a plausible calibration, the proposed trend inflation (TI) model indeed generates a highly persistent real exchange rate: the sum of the AR(5) roots of the real exchange rate is 0.993. As emphasized in the recent literature on real exchange rate dynamics including Cheung and Lai (2000), Steinsson (2008), Iversen and Söderström (2014), and Burstein and Gopinath (2014), the impulse response function (IRF) of the real exchange rate to a reduced-form shock is hump-shaped with a peak around a year and the corresponding half-life of the real exchange rate is longer than four years. A decomposition of the IRF into structural shocks uncovers that the trend inflation shock, even with a smaller standard deviation than those of the productivity shock and conventional monetary policy shock, dominates the amplification and propagation mechanism of the real exchange rate by generating a persistent hump-shaped IRF. As shown by Eichenbaum and Evans (1995) and Bouakez and Normandin (2010), the TI model successfully yields a significant delayed overshooting of the exchange rate to a monetary easing shock identified as a surprise change in the long-run inflation target.

As emphasized by Steinsson (2008), the hump-shaped real exchange rate to a trend inflation shock stems from the theoretical outcome that the IRF of the real interest rate differential flips its sign in several periods after the impact. More specifically, the IRF of the real interest rate differential to a positive trend inflation shock stays negative for 10 quarters after the impact and then turns positive afterward toward the long-run equilibrium. The RUIP condition then implies that the real exchange rate appreciates at the impact, keeps appreciating for some periods, and then starts depreciating. This conditional covariance between the real exchange rate and real interest rate differential explains two important empirical facts. First, the PPP puzzle (Rogoff 1996) can be reconciled with the strong hump-shaped IRF of the real exchange rate to a shock to the inflation rate target. Second, an inflation surprise generated by a positive shock to the inflation target corresponds to a simultaneous currency appreciation: bad news about inflation is good news for exchange rates (Clarida and Waldman 2008).

The TI model also mimics the large volatility of the real exchange rate as in the actual data. The simulated volatility ratio of the real currency return to the nominal one is close to one.
The trend inflation specification in fact implies an almost perfect correlation between the real and nominal currency returns. In other words, Mussa’s (1986) well-known observation of the one-to-one correspondence between the real and nominal exchange rates is replicated in equilibrium. Moreover, when the TI model is allowed for a managed exchange rate regime, the volatility of the real exchange rate is dampened sharply, whereas the volatility of the inflation differential is almost unchanged. Hence, the TI model presented in this paper also approaches Mussa’s observations plausibly. The results of the calibration exercises strongly support trend inflation, i.e., a time-varying inflation rate target with a long-run mean, as a relevant hypothesis for better understanding exchange rate dynamics.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 reports the results of the calibration exercises. Section 4 concludes.

## 2. The model

To extract the theoretical implications of trend inflation for exchange rate dynamics as clearly as possible, I investigate a plain vanilla version of a symmetric two-country NK model. The model is equipped with the Calvo-type sticky local currency pricing strategies of monopolistic competitive firms and Taylor rule-type monetary policies under incomplete international financial markets. It is well known that this canonical model generates neither the persistence and volatility of real exchange rates nor the one-to-one comovement between real and nominal exchange rates, as observed in the actual data. Below, I show that persistent trend inflation with a positive steady state greatly improves the empirical fit of the canonical model to the reality of exchange rates.

### 2.1. Household sectors

There are home and foreign countries in this model. Throughout this paper, any variable with an asterisk corresponds to a foreign variable, while a variable without an asterisk denotes the home counterpart. The two countries are endowed with representative households whose objectives
are the following lifetime utility functions:

\[ \sum_{j=0}^{\infty} \beta^j E_t \left\{ \ln C_{t+j} - \frac{(N_{t+j})^{1+\eta}}{1+\eta} \right\} \quad \text{and} \quad \sum_{j=0}^{\infty} \beta^j E_t \left\{ \ln C^*_{t+j} - \frac{(N^*_{t+j})^{1+\eta}}{1+\eta} \right\}, \]

for the home and foreign countries, respectively, where \( C_t \) and \( N_t \) represent the home country’s consumption basket and hours worked and \( C^*_t \) and \( N^*_t \) their foreign country’s counterparts. For simplicity, I assume that the two countries share the identical lifetime utility specified by the same subjective discount factor \( \beta \in (0, 1) \) and the Frisch elasticity of labor supply \( \eta > 1 \).

The representative households in the two countries consume both home \((h)\) and foreign \((f)\) final goods. Home consumption basket \( C_t \) consists of home and foreign product aggregators \( C_{h,t} \) and \( C_{f,t} \), while foreign consumption basket \( C^*_t \) consists of home and foreign product aggregators \( C^*_{h,t} \) and \( C^*_{f,t} \):

\[
C_t = \left[ (C_{h,t})^{\frac{\zeta}{\xi}} + (C_{f,t})^{\frac{\zeta}{\xi}} \right]^{\frac{\xi}{\zeta-1}} \quad \text{and} \quad C^*_t = \left[ (C^*_{h,t})^{\frac{\zeta}{\xi}} + (C^*_{f,t})^{\frac{\zeta}{\xi}} \right]^{\frac{\xi}{\zeta-1}},
\]

where \( \zeta > 0 \) is the elasticity of substitution between the home and foreign product aggregators.

The static cost minimization problems of home and foreign households derive the demand functions for \( C_{j,t} \), \( C^*_{j,t} \), \( C^*_{h,t} \), and \( C^*_{f,t} \) for \( j = \{h, f\} \),

\[
C_{j,t} = \left( \frac{P_{j,t}}{P_t} \right)^{-\zeta} C_t, \quad \text{and} \quad C^*_{j,t} = \left( \frac{P^*_{j,t}}{P^*_t} \right)^{-\zeta} C^*_t,
\]

where \( P_t \) and \( P^*_t \) are the aggregate consumer price indices (CPIs) of the home and foreign countries:

\[
P_t = \left[ (P_{h,t})^{1-\zeta} + (P_{f,t})^{1-\zeta} \right]^{\frac{1}{1-\zeta}} \quad \text{and} \quad P^*_t = \left[ (P^*_{h,t})^{1-\zeta} + (P^*_{f,t})^{1-\zeta} \right]^{\frac{1}{1-\zeta}}.
\]

\[\text{Because of the stochastic trends due to the permanent labor productivity shocks in the two countries, the model assumes the log-utility functions over consumption to guarantee the existence of a balanced growth path in the two-country equilibrium.}\]

\[\text{In this paper, I assume no home bias over each country’s consumption preference. Steinsson (2008) and Iversen and Söderström (2014), for example, exploit home bias as an important propagation for real exchange rates.}\]
Here, $P_{h,t}$, $P_{f,t}$, $P_{h,t}^*$, and $P_{f,t}^*$ are, respectively, the aggregate price over the home goods in the home country, that over the foreign goods in the home country, that over the home goods in the foreign country, and that over the foreign goods in the foreign country, as defined below.

The home and foreign aggregators of the home and foreign final goods, $C_{h,t}$, $C_{f,t}$, $C_{h,t}^*$, and $C_{f,t}^*$, consist of a continuum of the home and foreign final goods, each of which is produced by a monopolistically competitive firm residing either in the home or in the foreign countries. Each final good is placed within the unit interval $[0, 1]$. The home and foreign final good aggregators in the home country are of the following Dixit–Stiglitz type: for $j = \{h, f\}$,

$$C_{j,t}(z) = \left( \int_0^1 C_{j,t}(z)^{1-\xi} \, dz \right)^{\frac{\xi}{1-\xi}}$$

and

$$C_{j,t}^*(z) = \left( \int_0^1 C_{j,t}(z)^{1-\xi} \, dz \right)^{\frac{\xi}{1-\xi}},$$

where $C_{j,t}(z)$ and $C_{j,t}(z)^*$ are home and foreign demand for the particular home and foreign final goods indexed by $z \in [0, 1]$, respectively. Here, $\xi > 0$ represents the elasticity of demand for each final good with respect to its own price.\(^6\)

The static cost minimization problems of the representative households in the two countries derive the two countries' demand functions for the home and foreign final goods. Given the home and foreign prices of the home and foreign goods indexed by $z$, $P_{h,t}(z)$, $P_{f,t}(z)$, $P_{h,t}^*(z)$, and $P_{f,t}^*(z)$, they are

$$C_{j,t}(z) = \left( \frac{P_{j,t}(z)}{P_{j,t}} \right)^{-\xi} C_{j,t} \quad \text{and} \quad C_{j,t}^*(z) = \left( \frac{P_{j,t}^*(z)}{P_{j,t}^*} \right)^{-\xi} C_{j,t}^*,$$

for $j = \{h, f\}$. Price aggregators $P_{h,t}$, $P_{f,t}$, $P_{h,t}^*$, and $P_{f,t}^*$ are then given as

$$P_{j,t} = \left( \int_0^1 P_{j,t}(z)^{1-\xi} \, dz \right)^{\frac{1}{\xi}}$$

and

$$P_{j,t}^* = \left( \int_0^1 P_{j,t}^*(z)^{1-\xi} \, dz \right)^{\frac{1}{\xi}},$$

for $j = \{h, f\}$.

The home representative household needs to satisfy the budget constraint to maximize lifetime

---

\(^6\)For simplicity, I assume the same price elasticity of demand for an individual final product, $\xi$, as that of demand for an aggregate consumption basket.
utility:

\[ B_{h,t} + S_{t}B_{f,t} + P_{t}C_{t} \leq (1 + i_{h,t-1})B_{h,t-1} + S_{t}(1 + i_{f,t-1})B_{f,t-1} + W_{t}N_{t} + \Lambda_{t}, \]

where \( B_{h,t} \), \( B_{f,t} \), \( i_{h,t} \), \( i_{f,t} \), \( W_{t} \), and \( S_{t} \) denote the home country’s holdings of the home country’s nominal bonds, the home country’s holdings of the foreign country’s nominal bonds, the home country’s nominal interest rate for the home country’s bonds, the home country’s nominal interest rate for the foreign country’s bonds, the home country’s nominal wage, and the static profit from the home country’s monopolistically competitive firms, and the level of the bilateral nominal exchange rate of the home currency per the foreign country, respectively. Similarly, the representative household in the foreign country maximizes its lifetime utility subject to the following budget constraint:

\[ \frac{B_{h,t}^{*}}{S_{t}^{*}} + B_{f,t}^{*} + P_{t}^{*}C_{t}^{*} \leq (1 + i_{h,t-1}^{*})\frac{B_{h,t-1}^{*}}{S_{t}^{*}} + (1 + i_{f,t-1}^{*})B_{f,t-1}^{*} + W_{t}^{*}N_{t}^{*} + \Lambda_{t}^{*}, \]

Where \( B_{h,t}^{*} \), \( B_{f,t}^{*} \), \( i_{h,t}^{*} \), \( i_{f,t}^{*} \), \( W_{t}^{*} \), and \( \Lambda_{t}^{*} \) denote the foreign country’s holdings of the home country’s nominal bonds, the foreign country’s holdings of the foreign country’s nominal bonds, the foreign country’s nominal interest rate for the home country’s bonds, the foreign country’s nominal interest rate for the foreign country’s bonds, the foreign country’s nominal wage, and the static profit from the foreign country’s monopolistically competitive final good firms.

The first-order necessary conditions (FONCs) for the home and foreign households’ lifetime utility maximization problems consist of the home and foreign Euler equations:

\[ \frac{1}{P_{t}C_{t}} = \beta(1 + i_{h,t})E_{t} \frac{1}{P_{t+1}C_{t+1}} \quad \text{and} \quad \frac{1}{P_{t}^{*}C_{t}^{*}} = \beta(1 + i_{f,t}^{*})E_{t} \frac{1}{P_{t+1}^{*}C_{t+1}^{*}}; \]

the home and foreign utility-based uncovered interest parity conditions (UIPs):

\[ E_{t}\frac{1}{P_{t+1}^{*}C_{t+1}^{*}} \left\{ (1 + i_{h,t}) - (1 + i_{f,t})\frac{S_{t+1}^{*}}{S_{t}} \right\} = 0 \quad \text{and} \quad E_{t}\frac{1}{P_{t+1}^{*}C_{t+1}^{*}} \left\{ (1 + i_{f,t}^{*}) - (1 + i_{h,t}^{*})\frac{S_{t}}{S_{t+1}} \right\} = 0; \]
and the home and foreign intratemporal optimality conditions for hours worked

\[ N_t^\eta = \frac{W_t}{P_t C_t} \quad \text{and} \quad N_t^{\eta*} = \frac{W_t^*}{P_t^* C_t^*} \]

Suitable transversality conditions for international bond holdings should be satisfied.

2.2. Final good sectors

Facing the corresponding demand functions, a final good firm with index \( z \in [0, 1] \) acts as an identical monopolistically competitive price setter. When setting its current optimal price, each firm follows a Calvo-type time-dependent pricing strategy: in each period, a firm cannot reset its optimal price with a probability \( \mu \in (0, 1] \). Moreover, to generate endogenous fluctuations of the real exchange rate in this model, I assume that all the firms adopt a local currency pricing strategy as in Betts and Devereux (1996, 2000): each firm sets its optimal price differently between the two countries in terms of the local currencies.

Under such a local currency pricing–Calvo pricing strategy, the objective function of a home firm is

\[
\max_{P_{h,t},P_{h,t}^*} E_t \sum_{i=0}^{\infty} \mu^i \Gamma_{t+i} \left\{ \left( \frac{P_{h,t}}{P_{h,t+i}} \right) - m_{c_{t+i}} \right\} \left( \frac{P_{h,t}}{P_{h,t+i}} \right)^{-\zeta} C_{h,t+i}
\]

\[
+ E_t \sum_{i=0}^{\infty} \mu^i \Gamma_{t+i} \left\{ \left( \frac{S_{t+i} P_{h,t}^*}{P_{h,t+i}} \right) - m_{c_{t+i}} \right\} \left( \frac{P_{h,t}^*}{P_{h,t+i}} \right)^{-\zeta} C_{h,t+i},
\]

where \( P_{h,t}, P_{h,t}^*, m_{c_t}, \) and \( \Gamma_t \) are the optimal price of the home good in the home country, the optimal price of the home good in the foreign country, the real marginal cost of the home firm, and the home country’s real stochastic discount factor \( \Gamma_{t+i} \equiv \beta^i (C_t / C_{t+i}) \), respectively. Similarly, the
objective function of a foreign firm is

$$
\max_{\mathbf{P}_{f,t}, \mathbf{P}^*_f} \ E_t \sum_{i=0}^{\infty} \mu_i \Gamma^*_t \left\{ \left( \frac{\mathbf{P}_{f,t}}{S_{i+1}^t \mathbf{P}^*_{f,t+i}} \right) - mc^*_t \right\} \left( \frac{\mathbf{P}^*_f}{\mathbf{P}_{f,t+i}} \right)^{-\zeta} C_{f,t+i}
$$

$$
+ E_t \sum_{i=0}^{\infty} \mu_i \Gamma^*_t \left\{ \left( \frac{\mathbf{P}^*_f}{\mathbf{P}^*_{f,t+i}} \right) - mc^*_{t+i} \right\} \left( \frac{\mathbf{P}^*_f}{\mathbf{P}^*_{f,t+i}} \right)^{-\zeta} C_{f,t+i},
$$

where $\mathbf{P}_{f,t}$, $\mathbf{P}^*_f$, $mc^*_t$, and $\Gamma^*_t$ are the optimal price of the foreign good in the home country, the optimal price of the foreign good in the foreign country, the real marginal cost of the foreign firm, and the foreign country’s real stochastic discount factor $\Gamma^*_{t+i} = \beta^i (C^*_t/C^*_{t+i})$, respectively.

The FONCs for the optimal prices the home firm sets for the home and foreign countries are

$$
\mathbf{P}^*_h \ E_t \sum_{i=0}^{\infty} \mu_i \Gamma^*_t \left( \frac{1}{\mathbf{P}^*_{h,t+i}} \right) C^{1-\zeta}_{h,t+i} = \frac{\zeta}{\zeta - 1} E_t \sum_{i=0}^{\infty} \mu_i \Gamma^*_t mc^*_{t+i} \left( \frac{1}{\mathbf{P}^*_{h,t+i}} \right)^{-\zeta} C_{h,t+i},
$$

$$
\mathbf{P}^*_f \ E_t \sum_{i=0}^{\infty} \mu_i \Gamma^*_t \left( \frac{1}{\mathbf{P}^*_{f,t+i}} \right) C^{1-\zeta}_{f,t+i} = \frac{\zeta}{\zeta - 1} E_t \sum_{i=0}^{\infty} \mu_i \Gamma^*_t mc^*_{t+i} \left( \frac{1}{\mathbf{P}^*_{f,t+i}} \right)^{-\zeta} C_{f,t+i},
$$

respectively. Similarly, the FONCs for the optimal prices the foreign firm sets for the home and foreign countries are

$$
\mathbf{P}_{f,t} \ E_t \sum_{i=0}^{\infty} \mu_i \Gamma^*_t \left( \frac{\mathbf{P}^*_{f,t+i}}{S_{i+1}^t \mathbf{P}^*_{f,t+i}} \right) \left( \frac{1}{\mathbf{P}^*_{f,t+i}} \right)^{-\zeta} C_{f,t+i} = \frac{\zeta}{\zeta - 1} E_t \sum_{i=0}^{\infty} \mu_i \Gamma^*_t mc^*_{t+i} \left( \frac{1}{\mathbf{P}^*_{f,t+i}} \right)^{-\zeta} C_{f,t+i},
$$

$$
\mathbf{P}^*_h \ E_t \sum_{i=0}^{\infty} \mu_i \Gamma^*_t \left( \frac{1}{\mathbf{P}^*_{h,t+i}} \right) C^{1-\zeta}_{h,t+i} = \frac{\zeta}{\zeta - 1} E_t \sum_{i=0}^{\infty} \mu_i \Gamma^*_t mc^*_{t+i} \left( \frac{1}{\mathbf{P}^*_{h,t+i}} \right)^{-\zeta} C_{h,t+i},
$$

Given the optimal prices, price aggregators $P_{h,t}$, $P_{f,t}$, $P^*_h$, and $P^*_f$ follow the laws of motion:

$$
P^1_{j,t} = (1 - \mu) P^1_{j,t-1} + \mu P^1_{j,t-1}, \quad \text{and} \quad P^1_{j,t} = (1 - \mu) P^1_{j,t-1} + \mu P^1_{j,t-1},
$$

where $j = \{h, f\}$.

In this paper, I assume that each final good firm produces its product by using only the labor input hired from the domestic competitive labor market. The production functions of the
home and foreign goods are \( Y_t(z) = A_t L_t(z) \) and \( Y^*_t(z) = A^*_t L^*_t(z) \), where \( A_t \) and \( A^*_t \) are the labor productivities in the home and foreign countries. In this case, the real marginal costs that the home and foreign firms face are

\[
mc_t = \frac{W_t}{A_t P_{h,t}} \quad \text{and} \quad mc^*_t = \frac{W^*_t}{A^*_t P^*_{f,t}}.
\]

The home and foreign static profits are

\[
\Lambda_t \equiv \int_0^1 \left\{ P_{h,t}(z)C_{h,t}(z) + S_t P^*_{h,t}(z)C^*_{h,t}(z) + W_t N_t(z) \right\} dz
\]

\[
= P_{h,t} \left( \frac{P_{h,t}}{P_t} \right)^{-\zeta} C_t + S_t P^*_{h,t} \left( \frac{P^*_{h,t}}{P^*_t} \right)^{-\zeta} C^*_t - W_t N_t,
\]

\[
S_t \Lambda^*_t \equiv \int_0^1 \left\{ P_{f,t}(z)C_{f,t}(z) + S_t P^*_{f,t}(z)C^*_{f,t}(z) - S_t W^*_t N_t(z)^* \right\} dz
\]

\[
= P_{f,t} \left( \frac{P_{f,t}}{P_t} \right)^{\zeta} C_t + S_t P^*_{f,t} \left( \frac{P^*_{f,t}}{P^*_t} \right)^{\zeta} C^*_t - S_t W^*_t N^*_t,
\]

respectively.

2.3. Monetary policies with trend inflation

The monetary policies in the two countries are characterized by Taylor rules. The central banks of the two countries set their short-term domestic nominal interest rate as \((1 + i_{h,t})\) and \((1 + i^*_{f,t})\) depending on the past interest rate levels \((1 + i_{h,t-1})\) and \((1 + i^*_{f,t-1})\), current inflation rates \(\gamma_{t} \equiv P_t/P_{t-1}\) and \(\gamma^*_t \equiv P^*_t/P^*_{t-1}\), and detrended output levels \(y_t \equiv Y_t/A_t\) and \(y^*_t \equiv Y^*_t/A^*_t\). In this paper, I further allow for exogenous slow-moving trend inflation \(\gamma^*_{t} \equiv P^*_{t-1}/P^*_{t-1}\) and \(\gamma^*_{t} \equiv P^*_{t-1}/P^*_{t-1}\) in the two countries, where \(P_{t-1}\) and \(P^*_{t-1}\) are the exogenous permanent components of the aggregate price levels of the two countries generated by trend inflation. As specified by Ireland (2007) and Cogley et al. (2010), the central banks of the two countries target the time-varying trend
inflation levels:

\[(1 + i_{h,t}) = (1 + i) \left( (1 + i_{h,t-1})^{\rho_i} \left[ \left( \frac{\gamma_{\tau,t}}{\gamma_{\tau,t}} \right)^{\alpha_y} \right] \right)^{1-\rho_i} \exp(\epsilon_{i,t}),\]

\[(1 + i_{f,t}^*) = (1 + i^*) \left( (1 + i_{f,t-1}^*)^{\rho_i} \left[ \left( \frac{\gamma_{\tau,t}^*}{\gamma_{\tau,t}^*} \right)^{\alpha_y} \right] \right)^{1-\rho_i} \exp(\epsilon_{i,t}^*),\]

where \(i\) and \(i^*\) are the deterministic steady-state values of the home and foreign nominal interest rates and \(\rho_i \in (0, 1)\) captures the degree of interest rate smoothing. In this paper, I allow the trend inflation rates \(\gamma_{\tau,t}\) and \(\gamma_{\tau,t}^*\) to be stochastic with exogenous AR(1) processes in the logarithmic term:

\[\ln \gamma_{\tau,t} = (1 - \rho_t) \ln \gamma_t + \rho_t \ln \gamma_{\tau,t-1} + \epsilon_{\tau,t}\]

and

\[\ln \gamma_{\tau,t}^* = (1 - \rho_t) \ln \gamma_t^* + \rho_t \ln \gamma_{\tau,t-1}^* + \epsilon_{\tau,t}^*,\]

where \(\gamma_t\) is the common long-run mean of the trend inflation rate, \(\rho_t \in [0, 1]\) is the AR(1) root of the trend inflation rate, and \(\epsilon_{\tau,t}\) and \(\epsilon_{\tau,t}^*\) are the i.i.d. trend inflation shocks.\(^7\) As in a closed-economy NK model by Ascari and Sbordone (2014), the time-invariant common unconditional mean \(\gamma_t\) makes the log-linearization exercise of this study tractable when characterizing the deterministic steady state.\(^8\) Taking a difference between the two countries’ trend inflation equations above yields the following stochastic process of the trend inflation differential:

\[\ln \gamma_{\tau,t}^d = \rho_t \ln \gamma_{\tau,t-1}^d + \epsilon_{\tau,t}^d,\]

where \(\epsilon_{\tau,t}^d = \epsilon_{\tau,t} - \epsilon_{\tau,t}^*\).

I assume that the monetary policy disturbances in the home and foreign countries, \(\epsilon_{i,t}\) and \(\epsilon_{i,t}^*\), are serially uncorrelated i.i.d. shocks. Coibion and Gorodnichenko (2012) provide a series of empirical evidence that the large persistence observed in the actual data on the short-run nominal interest rate in the United States stems largely from the policy inertia caused by interest rate short-term fluctuations.\(^7\) Cogley et al. (2010) discuss as a primary reason for the stochastic variations in the central banks’ long-run inflation targets that the central banks only imperfectly know the true economic structure and their learning process generates endogenous updating of the inflation targets.\(^8\) Ascari and Sbordone (2014) do not allow for stochastic variations in trend inflation.
smoothing following the Taylor rule, but not from the exogenous persistence of the monetary policy disturbance, as claimed by Rudebusch (2002, 2006). In particular, their empirical result suggests that the time-varying inflation target that the conventional Taylor rule misses could lead to a high estimate of the serial correlation of the monetary policy disturbance. Indeed, the persistent trend inflation shock $\gamma_{\tau,t}$ is indistinguishable from a persistent monetary policy disturbance if $\epsilon_{t,t}$ is set to $-(1 - \rho_i) a \ln \gamma_{\tau,t}$. Following the above notion, I allow for both interest rate smoothing and the time-varying inflation target in the Taylor rules in this study.

2.4. Market clearing and productivity shocks

To guarantee a stationary distribution of the net foreign asset positions of the two countries within incomplete international financial markets, I allow for a debt-elastic risk premium in the interest rates faced only by the home country: for $j = \{h, f\}$,

$$i_{j,t} = i^*_{j,t}[1 + \psi\{\exp(-b_{j,t} + \bar{d}) - 1\}], \quad \bar{d} \leq 0, \quad \psi > 0,$$

where $b_{j,t}$ is the transitory component of the home country’s holdings of country $j$’s bonds, which is precisely defined below. The home country needs to pay a risk premium over the interest rate level the foreign household faces when the transitory component of the home country’s net borrowing positions $b_{j,t} < 0$ is beyond its threshold level $\bar{d}$. The risk premium is given as an externality: the household does not take into account the effect of the debt position on the risk premium when maximizing its lifetime utility function. On the contrary, I do not attach a risk premium to the foreign country’s interest rates.$^9$

The market-clearing conditions of the two countries’ bond markets are

$$B_{h,t} + B^*_h = 0 \quad \text{and} \quad B_{f,t} + B^*_f = 0.$$

$^9$Since the elasticity of the risk premium toward the debt position, $\psi$, is set to a small number, this asymmetric treatment of the debt-elastic risk premium between the home and foreign countries does not affect the equilibrium outcome significantly.
In other words, along an equilibrium path, the world net supply of nominal bonds is zero on a period-by-period basis. The market-clearing conditions of the home and foreign final goods are

\[ A_t N_t = \int_0^1 \{ C_{h,t}(z) + C^*_{h,t}(z) \} dz = \Omega_{h,t} C_{h,t} + \Omega^*_{h,t} C^*_{h,t}, \]

\[ A^*_t N^*_t = \int_0^1 \{ C_{f,t}(z) + C^*_{f,t}(z) \} dz = \Omega_{f,t} C_{f,t} + \Omega^*_{f,t} C^*_{f,t}, \]

where the variables

\[ \Omega_{j,t} \equiv \int_0^1 \left( \frac{P_{j,t}(z)}{P_t} \right)^{-\zeta} dz, \quad \text{and} \quad \Omega^*_{j,t} \equiv \int_0^1 \left( \frac{P^*_{j,t}(z)}{P^*_t} \right)^{-\zeta} dz, \quad \text{for} \ j = h, f \]

capture the degrees of price dispersions in the four final good markets. As discussed by Ascari and Sbordone (2014), the price dispersion variables are greater than one under price stickiness, but are one if all the prices are identical within each final good market in each country under flexible price adjustments. The market-clearing conditions then imply that variables \( \Omega_{j,t} \) and \( \Omega^*_{j,t} \) represent the resource costs of price dispersion: given the output level, the higher the price dispersion variable, the lower the amount allocated to consumption. It is shown that the price dispersion variables follow the transitions

\[ \Omega_{j,t} = (1 - \mu) \left( \frac{P_{j,t}}{P_t} \right)^{-\zeta} + \mu \left( \frac{P_{j,t}}{P_{j,t-1}} \right)^{\zeta} \Omega_{j,t-1}, \]

\[ \Omega^*_{j,t} = (1 - \mu) \left( \frac{P^*_{j,t}}{P^*_t} \right)^{-\zeta} + \mu \left( \frac{P^*_{j,t}}{P^*_{j,t-1}} \right)^{\zeta} \Omega^*_{j,t-1}, \]

for \( j = h, f \).

I assume that the logarithms of labor productivities, \( \ln A_t \) and \( \ln A^*_t \), are of I(1). To guarantee a balanced growth path of this two-country model, I assume that the labor productivity differential \( \ln a_t \equiv \ln A_t - \ln A^*_t \) is of I(0). As investigated by Mandelman et al. (2011), Rabanal et al. (2011), and Ireland (2013), the I(1) labor productivities and stationary productivity differential

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10 The full derivation of the transition equations of the price dispersion variables is found in Ascari and Sbordone (2014).
jointly imply that the labor productivity of the home country must be cointegrated with that of the foreign country. Following Kano (2016), I assume that the home and foreign growth rates of labor productivity, \( \gamma_{A,t} \equiv \ln A_t - \ln A_{t-1} \) and \( \gamma_{A,t}^* \equiv \ln A_t^* - \ln A_{t-1}^* \), are generated by error correction processes:

\[
\gamma_{A,t} = \ln \gamma_A - \frac{\lambda}{2} \ln a_{t-1} + \epsilon_{A,t}, \quad \text{and} \quad \gamma_{A,t}^* = \ln \gamma_A + \frac{\lambda}{2} \ln a_{t-1} + \epsilon_{A,t}^*;
\]

where \( \gamma_A > 1 \) is the common drift term and \( \lambda \in [0, 1) \) is the speed of error correction. The error correction mechanism implies that the cross-country labor productivity differential is of I(0) because

\[
\ln a_t = (1 - \lambda) \ln a_{t-1} + \epsilon_{A,t}^d, \tag{2}
\]

where \( \epsilon_{A,t}^d \equiv \epsilon_{A,t} - \epsilon_{A,t}^* \). Importantly, if the adjustment speed \( \lambda \) is sufficiently close to zero, the cross-country labor productivity differential can be realized near I(1).

Because the model contains nonstationary components, \( A_t, A_t^*, P_{r,t}, \) and \( P_{r,t}^* \), I stochastically detrend the FONCs by these stochastic trend components to characterize the unique deterministic steady state, as shown in the accompanying appendix in detail.\(^{11}\) In doing so, I define the stochastically detrended versions of home consumption by \( c_t \equiv C_t / A_t \); foreign consumption \( c_t^* \equiv C_t^* / A_t^* \); the home price of home goods \( p_{h,t} \equiv P_{h,t} A_t / P_{r,t} \); the home price of foreign goods \( p_{f,t} \equiv P_{f,t} A_t / P_{r,t} \); the foreign price of home goods \( p_{h,t}^* \equiv P_{h,t}^* A_t^* / P_{r,t}^* \); the foreign price of foreign goods \( p_{f,t}^* \equiv P_{f,t}^* A_t^* / P_{r,t}^* \); the optimal home price of home goods \( p_{h,t}^* \equiv P_{h,t} A_t / P_{r,t} \); the optimal home price of foreign goods \( p_{f,t}^* \equiv P_{f,t} A_t / P_{r,t} \); the optimal foreign price of home goods \( p_{h,t}^{*,*} \equiv P_{h,t} A_t^* / P_{r,t}^* \); the optimal foreign price of foreign goods \( p_{f,t}^{*,*} \equiv P_{f,t} A_t^* / P_{r,t}^* \); the home CPI \( p_t \equiv P_t A_t / P_{r,t} \); the foreign CPI \( p_t^* \equiv P_t A_t^* / P_{r,t}^* \); the home holdings of the home bond \( b_{h,t} \equiv B_{h,t} / P_{r,t} \); the home holdings of the foreign bond \( b_{f,t} \equiv B_{f,t} / P_{r,t} \); the foreign holdings of the home bond \( b_{h,t}^* \equiv B_{h,t}^* / P_{r,t}^* \); the foreign holdings of the foreign bond \( b_{f,t}^* \equiv B_{f,t}^* / P_{r,t}^* \); the home nominal wage \( w_t \equiv W_t / P_{r,t} \); the foreign nominal wage \( w_t^* \equiv W_t^* / P_{r,t}^* \); and the nominal exchange rate \( s_t \equiv S_t P_{r,t}^* / P_{r,t} \). The real exchange rate is given by \( q_t \equiv S_t P_{r,t}^* / P_t = s_t a_t p_t^* / p_t \). To derive the corresponding linear rational expectations

\(^{11}\)The appendix is available upon request.
(LRE) models, I take a log-linear approximation of the stochastically detrended FONCs around the unique deterministic steady state. For a variable $x_t$, let $\hat{x}_t$ denote the percentage deviation from its deterministic steady-state value $x$, i.e., $\hat{x}_t \equiv (x_t - x)/x$, and $\tilde{x}_t$ the deviation from the deterministic steady-state value, i.e., $\tilde{x}_t \equiv x_t - x$.

For simplicity, I assume two symmetric countries with $\tilde{d} = 0$ throughout this paper. In this case, the deviation from the law of one price is also symmetric across the home and foreign goods:

$$\hat{q}_t = \hat{s}_t + \hat{a}_t + \hat{p}^*_h, t - \hat{p}_h, t, \quad \text{and} \quad \hat{q}_t = \hat{s}_t + \hat{a}_t + \hat{p}^*_f, t - \hat{p}_f, t.$$

Hence, $\hat{p}_h, t - \hat{p}^*_h, t = \hat{p}_f, t - \hat{p}^*_f, t$, i.e., the cross-country price differential is identical for the home and foreign goods. Moreover, from the home and foreign CPIs, this condition implies $\hat{p}_t - \hat{p}^*_t = \hat{p}_h, t - \hat{p}^*_h, t = \hat{p}_f, t - \hat{p}^*_f, t$, i.e., the cross-country CPI differential is also identical to the price differentials of the individual goods across the two countries.

Let $\hat{\pi}_{h,t} \equiv \hat{p}_h, t - \hat{p}_h, t-1$ denote the rate of change in the detrended home good price in the home country and $\hat{\pi}^*_h, t \equiv \hat{p}^*_h, t - \hat{p}^*_h, t-1$ denote the rate of change in the detrended home good price in the foreign country. In addition, let $\hat{\pi}_t \equiv \hat{p}_t - \hat{p}_{t-1}$ denote the rate of change in the detrended home CPI. The log-linearized home good price inflation in the home country is given by $\Delta \ln P_{h,t} \equiv \hat{\pi}_{h,t} + \hat{\gamma}_{r,t} - \hat{\gamma}_{A,t}$. The log-linearized home and foreign CPI inflations are given by $\Delta \ln P_t \equiv \hat{\pi}_t + \hat{\gamma}_{r,t} - \hat{\gamma}_{A,t}$ and $\Delta \ln P^*_t \equiv \hat{\pi}^*_t + \hat{\gamma}^*_r, t - \hat{\gamma}^*_A, t$, respectively.

2.5. A GNKPC with trend inflation

The accompanying appendix shows the whole derivation of the GNKPC for the home good price inflation in the home country, $\Delta \ln P_{h,t}$. With the definitions of the parameters, $\varphi_0 \equiv \frac{1-\beta \gamma}{\beta - \gamma}$, $\varphi_1 \equiv \beta \mu \gamma \xi^{-1} \in (0, 1)$, and $\varphi_2 \equiv \beta \mu \gamma \xi \in (0, 1)$, where $\gamma \equiv \gamma_r / \gamma_A$ is the productivity-adjusted
steady-state trend inflation, the corresponding GNKPC is

\[
\Delta \ln P_{h,t} = \beta \gamma E_t \Delta \ln P_{h,t+1} + \varphi_0 (1 - \varphi_2) \hat{m}c_t + \frac{\varphi_0 (\varphi_1 - \varphi_2)}{\varphi_1} \sum_{i=1}^{\infty} \varphi_i^1 E_t \Delta \ln P_{h,t+i} \]

\[
- \frac{\varphi_0 (\varphi_1 - \varphi_2)}{\varphi_1} \zeta \sum_{i=1}^{\infty} \varphi_i^1 E_t \Delta \ln P_{t+i}.
\]

Therefore, as claimed by Cogley and Sbordone (2008), GNKPC (3) depends on the higher-order leads of the home good price inflation as well as the home CPI inflation. Moreover, coefficients \( \varphi_0, \varphi_1, \text{and} \varphi_2 \) are given as nonlinear functions of primitives \( \beta, \mu, \text{and} \bar{\gamma} \).

Higher steady-state trend inflation \( \bar{\gamma} > 1 \) implies a smaller weight on the current real marginal cost \( \hat{m}c_t \) and larger weights on the expected future home good price inflation \( E_t \Delta \ln P_{h,t+i} \) and the expected future home CPI inflation \( E_t \Delta \ln P_{t+i} \). This is because long-run trend inflation affects the expected future profits of monopolistic competitive producers through its effect on the relative prices \( P_{h,t+i}/P_{t+i} \) in the demand functions toward an infinite future. The higher long-run trend inflation, the larger are the weights that producers place on expected future profits when they decide the current optimal price. Producers place a smaller weight on a current variable such as the real marginal cost.

The appendix also displays the full derivation of the GNKPC for the home good price inflation in the foreign country, \( \Delta \ln P^*_h \). Under the assumption of two symmetric countries, the relation \( \hat{\pi}_t - \hat{\pi}^*_t = \hat{\pi}_{h,t} - \hat{\pi}^*_{h,t} = \hat{\pi}_{f,t} - \hat{\pi}^*_{f,t} \) holds. Define by \( x_t^d \equiv x_t - x^*_t \) the cross-country differential for any home and foreign variable \( x_t \) and \( x^*_t \). Subtracting the GNKPC of \( \Delta \ln P^*_h \) from that of \( \Delta \ln P_h \)

\[12\]GNKPC (3) turns out to be the conventional NKPC under the restrictive parameterization of \( \bar{\gamma} = 1 \). Because in this case \( \varphi_1 = \varphi_2 = \beta \mu \) and \( \varphi_0 = (1 - \mu)/\mu \), the first, fourth, and fifth terms on the RHS of eq. (3) vanish and the conventional NKPC of the home good price inflation \( \Delta \ln P_h \) emerges:

\[
\Delta \ln P_{h,t} = \beta E_t \Delta \ln P_{h,t+1} + \kappa_\mu \hat{m}c_t,
\]

where \( \kappa_\mu \equiv \frac{(1 - \mu)(1 - \beta \mu)}{\mu} \in (0, 1) \) and \( \Delta \equiv 1 - L \) is the first difference operator. Hence, GNKPC (3) nests the standard NKPC under the restrictive parameterization of \( \bar{\gamma} = 1 \). The standard open-economy NK model investigated in the literature on the PPP puzzle such as Chari et al. (2002) and Benigno (2004) allows for no stochastic trend of either productivity growth or inflation, i.e., \( \gamma_A = \gamma_T = 1 \). Therefore, the GNKPC in this model implies richer and more persistent dynamics of domestic inflation than that in conventional models.
then provides the following GNKPC for the inflation differential between the two countries, $\Delta \ln P^d_t$:

$$
\Delta \ln P^d_t = \beta_0 E_t \Delta \ln P^d_{t+1} + \phi_0(1 - \varphi_2) \hat{a}_t + \frac{\phi_0(1 - \varphi_1)}{\varphi_1} \phi_1 \Delta \hat{q}_t + \frac{\phi_0(1 - \varphi_2)}{\varphi_1} \phi_1 \sum_{i=1}^{\infty} E_t \Delta \hat{a}_{t+i} + \frac{\phi_0(1 - \varphi_1)}{\varphi_1} \phi_1 \sum_{i=1}^{\infty} E_t \Delta \hat{q}_{t+i}.
$$

(4)

GNKPC (4) implies that, with positive long-run trend inflation, the current inflation differential $\Delta \ln P^d_t$ becomes less sensitive to the current productivity differential $\hat{a}_t$ and real exchange rate $\hat{q}_t$ but more dependent on the present discounted values of the expected future variables such as the inflation differentials $E_t \Delta \ln P^d_{t+1}$, consumption growth differential $E_t \Delta \hat{c}^d_{t+1}$, productivity differentials $E_t \hat{a}_{t+i}$, and real exchange rates $E_t \hat{q}_{t+i}$. Therefore, only persistent slow-moving components of the expected future variables matter for the current inflation differential. This is the main reason why the inflation differential becomes more persistent, less volatile, and less sensitive to the current real exchange rate in GNKPC (4).

2.6. Terms of trade (TOT) dynamics

Another important relative price in this model is the TOT. Let $\Psi_t$ and $\Psi^*_t$ denote the TOT in the home and foreign countries, respectively. In other words, these variables represent the relative price of the home good to the foreign good in the home country, $\Psi_t \equiv P_{h,t}/P_{f,t} = p_{h,t}/p_{f,t}$, and that of the foreign good to the home good in the foreign country, $\Psi^*_t \equiv P^*_{f,t}/P^*_{h,t} = p^*_{f,t}/p^*_{h,t}$. Recall that under the symmetric equilibrium, $P_{f,t}/P_{h,t} = P^*_{f,t}/P^*_{h,t}$ and, hence, $\Psi_t = 1/\Psi^*_t$. Below, I focus only on the dynamics of the home TOT $\Psi_t$.

\[^{13}\text{Under the condition } \bar{\gamma} = 1, \text{ GNKPC (4) is degenerated to the standard NKPC: } \Delta \ln P^d_t = \beta E_t \Delta \ln P^d_{t+1} + \kappa_\mu \hat{q}_t, \text{ where } \kappa_\mu \equiv \frac{(1-\mu)(1-\beta_1)}{\mu} \in (0,1). \text{ In this restrictive case, the current inflation differential depends on the one-period-ahead expected inflation differential and real exchange rate. This is the same representation of the inflation differential dynamics as in Benigno (2004). The current inflation differential becomes sensitive to the current change in the real exchange rate.} \]
The log-linear approximation of the home TOT is \( \hat{\Psi}_t = \hat{p}_{h,t} - \hat{p}_{f,t} \). Hence, the growth rate of the home TOT is simply the inflation differential between the home good price and foreign good price in the home country: \( \Delta \hat{\Psi}_t = \Delta \ln P_{h,t} - \Delta \ln P_{f,t} \). The appendix derives the GNKPC for the foreign good price inflation in the home country \( \Delta \ln P_{f,t} \). Subtracting the GNKPC of \( \Delta \ln P_{f,t} \) from that for \( \Delta \ln P_{h,t} \) and rearranging the result provides the TOT dynamics:

\[
\Delta \hat{\Psi}_t = \beta E_t \Delta \hat{\Psi}_{t+1} + \varphi_0 (1 - \varphi_2) (\hat{m}_c^d - \hat{a}_t) - \frac{\varphi_0 \varphi_2 (1 - \varphi_1)}{\varphi_1} (\hat{q}_t - \hat{a}_t) + \frac{\varphi_0 (\varphi_1 - \varphi_2)}{\varphi_1} \sum_{i=1}^{\infty} \varphi_i E_t \Delta \hat{\Psi}_{t+i} \\
+ \frac{\varphi_0 (\varphi_1 - \varphi_2)}{\varphi_1} \sum_{i=1}^{\infty} \varphi_i E_t \hat{c}_d^{d,i} - \frac{\varphi_0 (\varphi_1 - \varphi_2) (1 - \varphi_1)}{\varphi_1} \sum_{i=1}^{\infty} \varphi_i E_t (\hat{q}_t + i - \hat{a}_t + i).
\]

(5)

As in GNKPC (4) for the inflation differential, the current TOT growth rate \( \Delta \ln \Psi_t^d \) becomes less sensitive to the current productivity differential \( \hat{a}_t \), real marginal cost differential \( \hat{m}_c^d \), and real exchange rate \( \hat{q}_t \) but more dependent on the present discounted values of the expected future variables. Hence, the current TOT growth rate becomes more persistent and less volatile in the TOT dynamics.\(^{14}\)

2.7. Symmetric two-country equilibrium

In summary, the symmetric equilibrium of this two-country model is characterized by the following 10 equations. Jointly with the home risk premium, the log-linear approximation of the home UIP is

\[
E_t \hat{s}_{t+1} - \hat{s}_t = (1 + \hat{i}_t)^d + \psi (1 - \kappa) \hat{b}_t - E_t \hat{c}_t^{d,d,1},
\]

(6)

where \((1 + \hat{i}_t)^d \equiv (1 + \hat{i}_{h,t}) - (1 + \hat{i}_{f,t})\) is the two-country interest rate differential. Taking the difference between the two countries’ Euler equations and using the home UIP characterize the forward-looking dynamics of the consumption differential \( \hat{c}_t^{d,1} + \hat{a}_t \) as a function of the real exchange

\(^{14}\)In the case with \( \bar{\gamma} = 1 \), the TOT follows

\[
\Delta \hat{\Psi}_t = \beta E_t \Delta \hat{\Psi}_{t+1} + \kappa \hat{\mu} (\hat{m}_c^d - \hat{q}_t).
\]

Hence, in this simplified version of the TOT equation, the growth rate of the home TOT depends on the expected future growth rate of the home TOT and the real marginal cost differential in terms of the home currency unit.
rate $\hat{q}_t$ and net foreign asset position $\tilde{b}_t$:

$$\hat{c}_t - \hat{q}_t + \hat{a}_t - \psi(1 - \kappa)\tilde{b}_t = E_t(\hat{c}_{t+1} - \hat{q}_{t+1} + \hat{a}_{t+1}).$$ \hfill (7)$$

The home and foreign good market-clearing conditions determine the labor supply differential $\hat{N}^d_t$ as a static function of the TOT $\hat{\Psi}_t$, productivity differential $\hat{a}_t$, and price dispersion differential $\hat{\omega}^d_t \equiv \hat{\omega}_{h,t} - \hat{\omega}_{f,t} + \hat{\omega}^*_h - \hat{\omega}^*_f = \ln \Omega_{h,t}/\Omega_{f,t} + \ln \Omega^*_{h,t}/\Omega^*_{f,t}$:

$$\hat{N}^d_t + 2\zeta\hat{\Psi}_t + 2\hat{a}_t = \hat{\omega}^d_t.$$ \hfill (8)$$

The transition of the price dispersion differential $\hat{\omega}^d$ is given as

$$\hat{\omega}^d_t = 2\zeta\mu\gamma\hat{\xi}^{-1}\Theta\Delta\hat{\Psi}_t + \mu\gamma\hat{\xi}\hat{\omega}^d_{t-1},$$ \hfill (9)$$

where $\Theta \equiv \gamma - \frac{1 - \mu\gamma\xi}{1 - \mu\gamma\xi}$.\hfill 15

The home and foreign budget constraints characterize the equilibrium transition of the net foreign asset position $\tilde{b}_t$:

$$\tilde{b}_t = \beta^{-1}\tilde{b}_{t-1} + pc^* (1 - \zeta)\hat{\Psi}_t - pc^* (\hat{c}_t^d + \hat{a}_t - \hat{q}_t),$$ \hfill (10)$$

where $pc^* = pc/4$. The home and foreign real marginal costs and home and foreign intratemporal optimality conditions jointly determine the real marginal cost differential as a static function of the TOT $\hat{\Psi}_t$, consumption differential $\hat{c}_t^d$, and labor supply differential $\hat{N}^d_t$:

$$\tilde{m}\hat{c}_t^d = -\hat{\Psi}_t + \hat{c}_t^d + \eta\hat{N}^d_t.$$ \hfill (11)$$

\hfill 15\text{Price dispersion differential }\hat{\omega}^d_t \text{ becomes irrelevant at a first order under no trend inflation. To see this property, notice that, under }\gamma = 1, \text{ the transition of price dispersion differential is } \hat{\omega}^d_t = \mu\hat{\omega}^{d-1}_t. \text{ In this case, the unique solution is } \hat{\omega}^d_t = 0.
The home and foreign Taylor rules give the interest rate differential \((1 + \hat{i})^d\):

\[
(1 + \hat{i})^d = \rho_i(1 + \hat{i}_{t-1})^d + (1 - \rho_i) \left[ a_x(\Delta \ln P_t^d - \hat{\gamma}_{\tau,t}^d) + a_y N_t^d \right] + \epsilon_{t,i}^d. \tag{12}
\]

Using the definition of the real exchange rate characterizes the relation among the real and nominal currency returns and inflation differential:

\[
\Delta \hat{q}_{t+1} = \Delta \hat{s}_{t+1} - \Delta \ln P_{t+1}^d + \hat{\gamma}_{\tau,t+1}^d. \tag{13}
\]

Rearranging the GNKPC of the inflation differential in eq. (4) provides the following second-order expectational difference equation of the inflation differential:

\[
\beta \gamma \varphi_1 E_t \Delta \ln P_{t+2}^d - [\varphi_1 + \beta \gamma + \varphi_0(\varphi_1 - \varphi_2)(1 - \zeta)] E_t \Delta \ln P_{t+1}^d + \Delta \ln P_t^d
= \varphi_0(1 - \varphi_1)\hat{q}_t - \varphi_0(1 - \varphi_1)\varphi_2 E_t \hat{q}_{t+1} + \varphi_0(\varphi_1 - \varphi_2)\hat{a}_t - \varphi_0(\varphi_1 - \varphi_2) E_t \hat{a}_{t+1}
- \varphi_0(\varphi_1 - \varphi_2) E_t \Delta \hat{e}_{t+1}. \tag{14}
\]

Similarly, rearranging the TOT dynamics (5) implies the following second-order expectational difference equation of the TOT growth:

\[
\beta \varphi_1 E_t \Delta \hat{\psi}_{t+2} - [\varphi_1 + \beta \gamma + \varphi_0(\varphi_1 - \varphi_2)] E_t \Delta \hat{\psi}_{t+1} + \Delta \hat{\psi}_t
= \varphi_0(1 - \varphi_2) E_t (\hat{m}_{\hat{e}_{t+1}} - \hat{a}_t)
- \varphi_0 \varphi_1 (1 - \varphi_2) E_t (\hat{m}_{\hat{c}_{t+1}} - \hat{a}_{t+1}) - \varphi_0 (1 - \varphi_1) (\hat{q}_t - \hat{a}_t) + \varphi_0 \varphi_2 (1 - \varphi_1) E_t (\hat{q}_{t+1} - \hat{a}_{t+1})
+ \varphi_0 (\varphi_1 - \varphi_2) E_t \Delta \hat{e}_{t+1}. \tag{15}
\]

The LRE model in this paper consists of 10 stochastic difference equations (6)–(15) and determines the following 10 endogenous variables \(\hat{c}_t, \Delta \ln P_t^d, \hat{N}_t^d, \hat{\omega}_t^d, \hat{m}_{\hat{e}_t}^d, \hat{s}_t, \hat{q}_t, \hat{\psi}_t, (1 + \hat{i})^d, \) and \(\hat{b}_t\) given the stochastic processes of three exogenous variables, namely \(\hat{\gamma}_{\tau,t}^d, \hat{a}_t,\) and \(\epsilon_{t,i}^d,\) including eqs. (1) and (2).
3. Assessing the model by calibration

3.1. Empirical moments

In this section, I assess the performance of the proposed model in terms of the empirical moments of the real and nominal exchange rates. The first row of Table 1 summarizes the empirical moments that this study targets. The first empirical moment is the sum of the autoregressive coefficients of the AR(5) process of the real exchange rate ln$q_t$, denoted by $\alpha$. Steinsson (2008) conducts $\alpha$ from the augmented Dickey–Fuller regression with the fifth lags. He reports that the estimate of $\alpha$ is 0.954 with a 90% confidence interval between 0.879 and 1.000 for the U.S. trade-weighted real exchange rate.\footnote{The corresponding half-life, calculated through the conventional formula $\log(0.5)/\log(\alpha)$, is 3.649. This number is consistent with the conventional consensus about the real exchange rate’s half-life of three to five years (Rogoff 1996).}

Steinsson (2008) and Burstein and Gopinath (2014) report a half-life measure, which is denoted by HL. The estimated ADF equation provides the IRFs of the real exchange rate with a unit reduced-form shock. The second empirical moment that this study targets, HL, is simply calculated as the maximum period before the IRF reaches 0.500, i.e., $\text{HL} = T$ such that $\text{IRF}(T-1) > 0.500$ and $\text{IRF}(T) \leq 0.500$. Burstein and Gopinath (2014) report HL for eight advanced countries: Canada, France, Germany, Italy, Japan, Switzerland, the United Kingdom, and the United States. The cross-country average of HL is 4.425 years with a minimum of 1.600 years for Switzerland and a maximum of 6.000 years for the United States.

The third empirical moment is the correlation coefficient between the real and nominal currency returns. Real and nominal exchange rates comove closely. Indeed, Burstein and Gopinath (2014) show the correlation coefficients between the real and nominal currency returns over these eight advanced countries. The average of the correlation coefficient, denoted by Corr, is 0.932 with a minimum of 0.820 for Italy and a maximum of 0.990 for Japan. The tight comovement between the real and nominal currency returns implies that the exchange rate dynamics are disconnected from the inflation differential—at least in the short run. Indeed, Burstein and Gopinath (2014)
provide the ratios of the standard deviations of the real currency return to the nominal currency return, \( \frac{\text{Std}(\Delta \ln q_t)}{\text{Std}(\Delta \ln S_t)} \), for the eight advanced economies. The cross-country average is 0.956 with a minimum of 0.870 for France and a maximum of 1.040 for the United Kingdom. A standard deviation ratio close to one implies that the exchange rate is much more volatile than the inflation differential. In Figure 1, the dotted blue line is the JPY/USD nominal currency return, the solid red line is the corresponding real currency return, and the solid black line is the inflation differential for Japan and the United States between Q2:1985 and Q4:2014. This figure confirms that the real and nominal currency returns comove almost one-to-one and are much more volatile than the inflation differential.

3.2. Calibration

The LRE model derived in Section 2 has no analytical closed-form solution. To capture the population properties, I conduct Monte Carlo simulations of the calibrated LRE model to generate synthetic time series samples of the real and nominal exchange rates and inflation differential with 1,000 quarterly periods. I then calculate the theoretical moments by using the synthetic samples to understand the model’s implications for real and nominal exchange rate dynamics.

I follow the conventional calibrations of the structural parameters of the subjective discount factor and Frisch labor supply elasticity by \( \beta = 0.990 \), \( \eta = 2.000 \), respectively. The Calvo probability of no price resetting is calibrated to \( \mu = 0.800 \). The sensitivity parameters of the Taylor rule toward the inflation gap and output gap are set to \( a_\pi = 3.500 \) and \( a_y = 0.010 \), respectively. The interest rate smoothing parameter is calibrated to \( \rho_i = 0.900 \). The calibrated values of \( a_\pi \) and \( \rho_i \) are slightly larger than the conventional values under zero trend inflation in the literature. This is particularly because Coibion and Gorodnichenko (2011) show in a closed-economy NK model that high values of \( a_\pi \) and \( \rho_i \) are required for equilibrium determinacy with positive trend inflation.

A key parameter of the model is the long-run unconditional mean of the productivity-adjusted trend inflation rate \( \bar{\gamma} \). The transition of price dispersion differential (9), GNKPC (14), and TOT dynamics (15) depend on \( \bar{\gamma} \) through the parameters \( \varphi_0, \varphi_1, \) and \( \varphi_2 \). As the benchmark calibration,
I set $\gamma = 1.0080$, which is equivalent to 3.26% at an annual rate. This calibrated value of the steady-state trend inflation $\gamma$ is slightly larger than the 3% that Coibion and Gorodnichenko (2011) calibrate with their U.S. sample in the post-Volcker disinflation period.

The calibration of the price elasticity of demand $\zeta$ depends on long-run trend inflation $\gamma$ crucially. To see this, recall that the price elasticity of demand is conventionally calibrated by fitting the model to a 11–12% markup rate at the steady state. Under the GNKPC, the steady-state markup rate is given by the inverse of the real marginal cost given by

$$mc = \left(\frac{\zeta - 1}{\zeta}\right) \left(\frac{1 - \beta \mu \gamma}{1 - \beta \mu \gamma^{-1}}\right) \left(\frac{1 - \mu \gamma^{-1}}{1 - \mu}\right)^{-1}. \tag{16}$$

Given $\gamma = 1$, the steady-state markup rate returns to the conventional one, i.e., $\zeta/(\zeta - 1)$. Under the benchmark calibration of $\beta = 0.990$, $\mu = 0.800$, and $\hat{\gamma} = 1.0080$ reported above, I set $\zeta = 22$, which implies a steady-state markup rate of 11.80%.

Finally, the three exogenous impulses are calibrated as follows. The error correction speed of the cointegrated labor productivity process, $\lambda$, is set to 0.010 to reflect a slow cross-country productivity diffusion. The AR root of the trend inflation process is set to 0.990 to capture the slow mean-reverting property of the permanent component of the inflation differential as in the data. Figure 2 plots the quarterly sample of the inflation differential between Japan and the United States and the corresponding trend component extracted by the Hodrick–Prescott filter. The figure shows substantial swings in the trend component of the two-country inflation differential over the sample period. The trend component indeed moves slowly around zero.\textsuperscript{17} The standard deviations of the productivity differential shock, trend inflation differential shock, and monetary policy differential shock are calibrated to $\sigma_A = 0.010$, $\sigma_\tau = 0.001$, and $\sigma_i = 0.002$, respectively. I calibrate the standard deviation of the trend inflation differential shock to a sufficiently small number relative to those of the other two shocks. This calibration approximately replicates the fact that the central banks change their inflation targets persistently but infrequently. The three shocks follow i.i.d.

\textsuperscript{17} The figure plots the demeaned time series of both the inflation differential and the Hodrick–Prescott trend component because the model with two symmetric countries has no implication of the unconditional mean.
normal distributions with zero means. Table 2 summarizes the benchmark calibration.

The above calibration guarantees the determinacy of the equilibrium of the LRE model. I solve the unique equilibrium of the LRE model and derive the state-space representation by using the QZ algorithm by Sims for the Monte Carlo simulations.

3.3. Results of the TI model

The second row of Table 2, which is labeled “Trend Inflation,” reports the theoretical moments simulated with the model with the benchmark calibration above. The TI model fits all the targeted empirical moments outstandingly well. This model implies a large AR root of the real exchange rate: the sum of the AR(5) coefficients, \( \alpha \), is simulated to 0.993. In fact, this number is consistent with its empirical counterpart of 0.954 and within the 90% coverage of the cross-country sample. The TI model hence can generate a high persistence of the real exchange rate as observed in the actual data. Furthermore, it yields a large half-life measure of 4.75 years, which mimics its empirical counterpart of 4.425 years closely. This long half-life stems from the fact that the TI model generates a strong hump-shaped impulse response of the real exchange rate. Figure 3 plots the simulated IRF of the real exchange rate calculated in the same way as in Steinsson (2008). The IRF peaks after a year and monotonically but slowly declines toward zero over time. This hump-shaped pattern of the real exchange rate response is emphasized in the literature on the real exchange rate including Cheung and Lai (2000), Steinsson (2008), Iversen and Söderström (2014), Burstein and Gopinath (2014), and Carvalho and Nechio (2014). The TI model, therefore, is endowed with a strong propagation mechanism of the real exchange rate.

A natural question then is which structural shock generates such high persistence of the real exchange rate with a hump-shaped impulse response. To answer this question, I calculate the IRFs of the real exchange rate to the three structural shocks from the state-space representation of the TI model. The left, center, and right windows in Figure 4 plot the IRFs of the real exchange rate to one standard deviation shocks to the productivity differential, trend inflation differential, and monetary policy shock differential, respectively. Confirm that the IRF to the trend inflation
differential shock dominates those to the other two structural shocks in absolute size. This means that although the calibrated size of the trend inflation differential shock is smaller than those of the other two shocks, the model amplifies and propagates the trend inflation differential shock greatly toward the real exchange rate. Further, only the IRF to the trend inflation differential shock is strongly hump-shaped. It is, therefore, the trend inflation differential shock that generates the hump-shaped IRF of the real exchange rate, as shown in Figure 3.

To capture intuitively the economic mechanism underlying the successful outcomes of the TI model in terms of real exchange rate dynamics, it is useful to solve the RUIP condition forward:

\[ \hat{q}_t = - \sum_{j=0}^{\infty} E_t (1 + \hat{r}_{t+j})^d + \lim_{j \to \infty} E_t \hat{q}_{t+j}, \]  \hspace{1cm} (17)

where \( (1 + \hat{r}_t)^d \equiv (1 + \hat{i}_t)^d - E_t \Delta \ln P_{t+1}^d \) is the real interest rate differential. The last limiting term on the RHS converges toward zero. Hence, the real exchange rate response depends negatively on the expected present values of the future real interest rate differentials with the unit discount factor.\(^\text{18}\) To understand the high persistence of the real exchange rate simulated by the TI model, suppose for exposition that the real interest rate differential follows an AR(1) process exogenous to the real exchange rate: \( (1 + \hat{r}_t)^d = \rho_r (1 + \hat{r}_{t-1})^d + u_{r,t} \), where \(\rho_r \in (0, 1)\) and \(u_{r,t}\) is an i.i.d. shock.

The present value representation of the equilibrium real exchange rate, eq. (17), implies that if the real interest rate differential \( (1 + \hat{r}_t)^d \) approaches a unit root process, the real exchange rate approximately follows a random walk:

\[ \lim_{\rho_r \to 1} \Delta q_t = \lim_{\rho_r \to 1} \left[ -\frac{1}{1 - \rho_r} u_{r,t} + (1 + \hat{r}_{t-1})^d \right] = \lim_{\rho_r \to 1} \frac{1}{1 - \rho_r} u_{r,t}, \]

\(^{18}\)The present value representation (17) is a restricted case of the fundamental asset pricing equation in Engel and West (2005):

\[ z_t = (1 - b) \sum_{j=0}^{\infty} b^j E_t(\alpha_1^t \mathbf{x}_{1,t+j}) + b \sum_{j=0}^{\infty} b^j E_t(\alpha_2^t \mathbf{x}_{2,t+j}), \]

where \( z_t \) is the asset price, \( b \) is the discount factor, \( \mathbf{x}_{1,t} \) is a vector of the observable economic fundamentals, \( \mathbf{x}_{2,t} \) is a vector of the unobservable economic fundamentals, \( \alpha_1 \) and \( \alpha_2 \) are the corresponding coefficient vectors. Eq. (17) is obtained by imposing \( \alpha_1 = 0 \) and \( b \to 1 \) on the fundamental asset pricing equation. As proved by Engel and West (2005), if an element of the economic fundamentals, which is the real interest rate differential in this case, is I(1), the real exchange rate converges to a random walk with an infinite variance of the real currency return.
where the second equality stems from the fact that \( \lim_{\rho_r \to 1} \frac{1}{1-\rho_r} u_{r,t} \) dominates \( (1 + \hat{r}_{t-1})^d \) in terms of volatility at the limit of \( \rho_r \to 1 \).\(^{19}\) Therefore, the TI model implies that a persistent real interest rate differential is sufficient for yielding the near random-walk behavior of the real exchange rate accompanied by large volatility. Indeed, the model simulates the persistent synthetic data of the real interest rate differential: a particular simulated dataset with 200 sample periods estimates the sum of the autoregression coefficients of the AR(5) process for \( (1 + \hat{r}_t)^d \) to be 0.886. The size of the persistence of the real interest rate differential is likely to be sufficient for a near random-walk real exchange rate with large volatility.\(^{20}\) The TI model presented in this paper thus explains the persistent and volatile equilibrium dynamics of the real exchange rate as in the actual data.

A question that should be asked then is why the TI model can generate the high persistence of the real interest rate differential independent of real exchange rate movements. Windows (a) and (b) in Figure 5 display the IRFs of selected variables to the one standard deviation shock to the trend inflation differential in the TI model (I discuss Windows (c) and (d) in Figure 5 in section 3.4). Window (a) plots the IRFs of the real interest rate differential (solid black line), expected inflation rate differential (circle-bar green line), nominal interest rate differential (dashed blue line), and inflation gap differential (dot-dashed red line). Here, by following Cogley et al. (2010), the inflation gap differential is constructed as the difference between the inflation rate differential and trend inflation differential, \( \Delta \ln P_t^{d_t} - \gamma_d^{d_t} \). Window (b) depicts the IRF of the real exchange rate (solid blue line).

Observe the persistent response of the nominal interest rate differential. In response to a positive trend inflation differential shock, the nominal interest rate differential rises in the impact period and keeps increasing for 18 consecutive quarters. The nominal interest rate differential then starts declining toward the long-run value but at a slow rate: even after 40 quarters, the corresponding IRF stays at 85% of the peak value. This highly persistent response of the nominal interest rate differential stems from the slow-moving dynamics of the inflation gap differential, which

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\(^{19}\)This proof borrows the argument found in the appendix of Engel and West (2005).

\(^{20}\)The Monte Carlo exercise conducted by Engel and West (2005) suggests that given the unit discount factor and AR root of an economic fundamental close to 0.900, it would be extremely difficult to reject a random walk of the corresponding asset price in the standard present value model.
jumps in the impact period and declines monotonically at a very slow rate. After 40 quarters, the IRF of the inflation gap differential remains 43% of the impact value (and 23% of the impact value even after 100 quarters). This high persistence of the inflation gap differential is an inherent outcome of GNKPC (14).

Furthermore, the inflation gap differential is insensitive to the real exchange rate differential. To see this, Euler equation (7) rewrites GNKPC (14) as

$$
\beta \gamma_1 E_t \Delta \ln P_{t+2}^d - [\varphi_1 + \beta \gamma + \varphi_0(\varphi_1 - \varphi_2)(1 - \zeta)] E_t \Delta \ln P_{t+1}^d + \Delta \ln P_t^d \\
= \varphi_0(1 - \varphi_2) \hat{q}_t - \varphi_0 \varphi_1(1 - \varphi_2) E_t \hat{q}_{t+1} + \varphi_0(\varphi_1 - \varphi_2) \psi(1 - \kappa) \tilde{b}_t.
$$

The TI model shows that the coefficients of $\hat{q}_t$ and $E_t \hat{q}_{t+1}$ take small numbers in absolute value.\textsuperscript{21}

The former insensitivity of the inflation differential to the current real exchange rate is a straightforward result of positive LR trend inflation: the firm places more weight on future variables than current variables in its optimal price setting because of the implied higher discount factor. The latter insensitivity of the inflation differential to the expected future real exchange rate stems from the following reason. On the one hand, the future real exchange rate affects the future real profit that the home firm earns in the foreign market. On the other hand, the future consumption differential affects the future real profit of the home firm through the foreign demand function and home stochastic discount factor. Euler equation (7) implies \textit{ex ante} risk sharing: the expected change in the consumption differential matches that in the real exchange rate. The effects of the future real exchange rate and future consumption differential then cancel out in the home firm’s optimal price setting. Therefore, the inflation differential as well as its linear transformation, the inflation gap differential, become insensitive to real exchange rate developments.

Monetary policy then propagates the inflation gap differential into the nominal interest rate differential: the high degree of interest rate smoothing in the Taylor rules then cumulatively and

\textsuperscript{21}Under the benchmark calibration, the calibrated coefficients of $\hat{q}_t$ and $E_t \hat{q}_{t+1}$ are 0.003 and -0.003, while those of $\Delta \ln P_t^d$, $E_t \Delta \ln P_{t+1}^d$, and $E_t \Delta \ln P_{t+2}^d$ are 1.000, -1.944, and 0.935, respectively. Indeed, if LR trend inflation is not allowed, the calibrated coefficients of $\hat{q}_t$ and $E_t \hat{q}_{t+1}$ sharply increase in absolute value to 0.052 and 0.041, respectively.

28
persistently feeds these positive responses of the inflation gap differential into the nominal interest rate differential. In sum, the GNKPC and interest rate smoothing jointly imply the high persistence of the nominal interest rate differential that is exogenous to real exchange rate developments. As shown by the circle-bar green line, the response of the expected inflation rate differential is also highly persistent. The persistence of the nominal interest rate differential thus results in that of the real interest rate differential: indeed, the IRF of the latter is almost a mirror image of that of the former.

The IRF of the real interest rate differential in Window (a) further reveals an outstandingly important property of the TI model. In their complete market sticky price models, Steinsson (2008) and Iversen and Söderström (2014) derive the almost same equilibrium real exchange rate equation as in eq. (17) and discuss that to generate a hump-shaped response of the real exchange rate, the response of the real interest rate differential needs to flip its sign within several quarters of the impact period. Window (a) in Figure 5 confirms that this is the case in the TI model. The IRF of the nominal interest rate differential is persistent with a hump shape. While the IRF of the expected inflation rate differential is greater in absolute value than that of the nominal interest rate differential up to 2.5 years, the latter gradually overcomes the former over time. In particular, as claimed by Steinsson (2008) and Iversen and Söderström (2014), the IRF of the real interest rate differential is negative up to 2.5 years but turns positive subsequently for long periods. Therefore, as shown in Window (b) in Figure 5, eq. (17) implies that the real exchange rate appreciates in the impact period because the expected present value of the future real interest rate differential is positive. The RUPIP condition then implies that the real exchange rate keeps appreciating when the response of the real interest rate differential stays negative; when the response of the real interest rate differential becomes zero, the real exchange rate stops appreciating; and when the response of the real interest rate differential turns positive, the real exchange rate starts depreciating. As a result, the model generates the hump-shaped IRF of the real exchange rate to a positive shock to the trend inflation differential.

The TI model implies that monetary easing by a positive trend inflation shock in the home
country results in a sudden rise in the inflation rate differential and home real exchange rate appreciation in the impact period. This conditional covariance between the inflation rate differential and real exchange rate is consistent with the well-known empirical finding by Clarida and Waldman (2008): bad news about inflation is good news for the real exchange rate.

3.4. Into the Mussa puzzle

The TI model also replicates the two influential observations established by Mussa (1986) on the joint behavior of real and nominal exchange rates within fixed and flexible exchange rate regimes before and after the breakdown of the Bretton Woods (BW) international monetary system. First, in the post-BW period, the real exchange rates of advanced economies almost perfectly comove with nominal exchange rates. Second, the short-run volatilities of real exchange rates are much smaller in the BW period than those in the post-BW period. Because it is hard for neoclassical real models with flexible prices to reconcile, these observations are known as the Mussa puzzle. In a small-open-economy NK model, Monacelli (2004) shows that the price stickiness accompanied by incomplete pass though resolves the Mussa puzzle plausibly. By inheriting the theoretical properties of Monacelli’s (2004) model, the TI model presented in this paper also passes this stringent reality check.\footnote{Monacelli (2004), however, does not touch upon the persistence puzzle emphasized by the recent literature on the real exchange rate.}

The TI model indeed implies a high positive correlation coefficient between the real and nominal currency returns. The simulated correlation coefficient of 0.961 is also close to its empirical counterpart of 0.932. The benchmark model successfully replicates the high volatility of the real exchange rate relative to the inflation differential as in the actual data: the STD ratio of the real to nominal currency returns is 0.976. This simulated value almost perfectly matches its empirical counterpart of 0.956. Figure 6 plots the simulated synthetic time series of the real currency return (solid red line), nominal currency return (dotted blue line), and inflation differential (solid black line). The real and nominal currency returns comove almost perfectly. This result means that by construction the fluctuation in the inflation differential is flat with much smaller volatility than the
volatilities of the real and nominal currency returns. Indeed, the simulated relative volatilities of the real and nominal currency returns to the inflation differential match their empirical counterparts.

To simulate the synthetic time series in the BW period, I assume a managed exchange rate regime following Benigno (2004) in which the monetary authority in the foreign country adopts the Taylor rule including the nominal currency return:

$$(1 + i^*_t) = (1 + i^{**})^{1 - \rho}(1 + i^*_{t-1})^{\rho_i} \left[ \left( \frac{\gamma^*_{\pi_t} + \gamma^*_\tau_t}{\gamma^*_{\tau_t}} \right)^{\alpha_i} (y^*_t)^{\alpha_y} \right]^{1 - \rho_i} \left( \frac{S_t}{S_{t-1}} \right)^{-\frac{\gamma^*_{\pi_t}}{\gamma_{\tau_t}}} \exp(\epsilon^*_t).$$

The parameter $\varphi \in [0,1]$ captures the degree of the managed exchange rate regime: the larger $\varphi$, the more stringent the managed regime is. If $\varphi = 0.000$, the model returns to the benchmark specification with the flexible exchange rate. If $\varphi \to 1$, the model is subject to a strict fixed nominal exchange rate regime with $S_t = 1$ for all $t$. If $\varphi \in (0,1)$, the model is characterized by a managed exchange rate regime. To calibrate the parameter $\varphi$, I follow Monacelli (2004) and set $\varphi = 0.760$.

As a robustness check, I also simulate the model under a smaller calibration of $\varphi = 0.200$.

Table 3 reports the simulated standard deviations of the real currency return and inflation differential in the TI model and managed exchange rate regime models. The first column corresponds to the TI model with $\varphi = 0.000$. Under the flexible exchange rate regime, the real currency return is much more volatile than the inflation differential: the standard deviation of the real currency return is 4.34%, while that of the inflation differential is 1.21%. The second column of the table corresponds to the managed exchange rate model with a smaller degree of the sensitivity of the foreign country’s Taylor rule to the nominal currency return $\varphi = 0.200$. In this calibration, the volatility of the real currency return is greatly weakened to 0.80%. The standard deviation of the real currency return of the TI model is about 5.5 times larger than that under the managed exchange rate model. At the same time, the volatility of the inflation differential changes little between the two exchange rate regimes: the standard deviation under the benchmark model is 1.21%, while that under the managed exchange rate regime is 0.63%. The alternative calibration for a more stringent managed exchange rate regime with $\varphi = 0.760$ yields the same inference that the volatility of the
real currency return jumps sharply from the managed regime to the flexible regime, while there is no sharp change in the volatility of the inflation differential. Figure 7 graphically verifies these successful properties of the model toward the Mussa puzzle. The figure plots the simulated time series of the real currency return (dashed blue line) and simulated inflation rate differential (solid black line) under the calibration of \( \varphi = 0.200 \). The two time series in the first 200 sample periods are simulated from the managed exchange rate model, while those in the second 200 sample periods from the benchmark model. On the one hand, the volatility of the real currency return rises sharply from the managed regime to the flexible regime. On the other hand, the volatility of the inflation differential stays at almost the same degree visually.\(^{23}\)

In sum, the model presented in this paper replicates the observed structural change in the dynamics of the real exchange rate before and after the BW period plausibly. As Monacelli (2004) infers, a regime change in the monetary policy rule should be the primary reason behind the Mussa puzzle.

### 3.4. Contributions of long-run trend inflation

The most crucial parameter of the TI model to generate successful outcomes on for the exchange rate dynamics is the long-run trend inflation, \( \bar{\gamma} \). Recall that under \( \bar{\gamma} = 1 \), price dispersion transition (9), the GNKPC (14), and the TOT dynamics (15) are degenerated to the corresponding standard specifications in the conventional two-country NK model as in Benigno (2004). To clarify the contribution that of the long-run trend inflation \( \bar{\gamma} \) makes to our the understanding of the exchange rate dynamics, I simulate the restricted model by imposing the restriction of no long-run trend inflation, \( \bar{\gamma} = 1 \) on the benchmark model.\(^{24}\)

\(^{23}\)This successful outcome of the model toward the Mussa puzzle is explained intuitively. For simplicity, consider the strict fixed exchange rate regime with \( \varphi = 1 \). In this case, \( S_t = 1 \) for all \( t \) and the real currency return is perfectly determined by the inflation differential, \( \hat{q}_t = \hat{q}_{t-1} - \Delta \ln P^d_t \). Substituting this equilibrium relationship into the GNKPC of the inflation differential (14) provides the equilibrium dynamics of the real exchange rate. The equilibrium dynamics of the real exchange rate under the fixed exchange rate regime is thus completely governed by the relative price dynamics characterized by the GNKPC rather than the asset price dynamics represented by the RUIP condition under the flexible exchange rate regime as in the benchmark model. This difference is the main reason why the volatility of the real exchange rate is smaller under the fixed exchange rate regime than that under the flexible one. The same intuition is applicable to the managed exchange rate model depending on the size of \( \varphi \).

\(^{24}\)I also recalibrate the price elasticity of demand \( \zeta \) to 10, fitting the model to the empirical markup rate of 11.11%,
The third row of Table 2, which is labeled “No LR Trend Inflation,” reports the simulated moments of the restricted model. First, the sum of the AR(5) coefficient, $\alpha$, decreases sharply to 0.781 in the restricted model from 0.993 in the TI model. Furthermore, the simulated half-life measure of 0.750 years uncovers the low persistence of the real exchange rate implied by the restricted model. This weak persistence of the real exchange rate in the restricted model results from the weak persistence of the real interest rate differential. Indeed, the restricted model generates a synthetic sample of the real interest rate differential with smaller AR roots: a particular simulated dataset with 200 sample periods estimates the sum of the autoregression coefficients of the AR(5) process for $(1 + \hat{r}_t)^d$ to be 0.712.

Windows (c) and (d) in Figure 5 display the IRFs of selected variables to a one standard deviation shock to the trend inflation differential in the restricted model. As in Windows (a) and (b), Window (c) plots the IRFs of the real interest rate differential (solid black line), expected inflation rate differential (circle-bar green line), nominal interest rate differential (dashed blue line), and inflation gap differential (dot-dashed red line). Window (d) depicts the IRF of the real exchange rate as the solid blue line. Observe in Window (d) no hump-shaped IRF of the real exchange rate in the restricted model. Furthermore, the magnitude of the IRF is much smaller than that in the benchmark model plotted in Window (b). Therefore, without LR trend inflation, the model fails to generate the persistence and magnitude of the real exchange rate to match the empirical observation. Window (c) reveals that this failure of the restricted model with respect to the IRF of the real exchange rate stems from the monotonically increasing IRF of the real interest rate differential to zero with no sign flipping. The no sign flipping of the IRF of the real interest rate differential results from a sharp increase in the expected inflation rate differential and a quick convergence of the nominal interest rate to the expected inflation rate differential due to the weak persistence of the inflation gap differential.

The restricted model with no LR trend inflation performs worse even in the other moments. The simulated correlation coefficient between the real and nominal currency returns, Corr, is 0.728,
which is far below the minimum value of 0.820 from Burstein and Gopinath’s (2014) estimates. The simulated STD ratio of the real and nominal currency returns sharply falls to 0.443 from 0.976 in the TI model. This low STD ratio implies that in the restricted model, the volatility of the inflation differential is counterfactually large. To confirm this weak property of the restricted model, Figure 8 plots the simulated synthetic time series of the real currency return (solid red line), nominal currency return (dotted blue line), and inflation differential (solid black line). The inflation differential has a large swing with high volatility. Therefore, long-run trend inflation improves the empirical fit of the benchmark model by damping down the inflation differential process.

LR trend inflation indeed strengthens the “endogenous” persistence of the real exchange rate. Following the simulated inference by Benigno (2004), Engel (2014, 2018) analytically proves that in a canonical two-country NK model, the degree of endogenous persistence is bounded above by the degrees of interest rate smoothing $\rho_i$ and price stickiness $\mu$. To identify the endogenous persistence in the model presented in this paper, I assume trend inflation differential $\gamma_{d,t}$ and productivity differential $a_t$ to be i.i.d. by setting $\rho_r = 0$ and $\lambda = 1$. Following Engel (2014, 2018), I further cut the sensitivity of the Taylor rule to the output gap with $a_y = 0$. In this case, endogenous persistence is identified approximately as the stable root of the stochastic process of the nominal interest rate differential, $(1 + \hat{i}_t)^d$.

On the one hand, the restricted model with no LR trend inflation yields an endogenous persistence of 0.701. The restricted model is therefore subject to the Benigno–Engel upper bounds for the endogenous persistence of the real exchange rate. The TI model with positive LR trend inflation, on the other hand, produces an endogenous persistence of 0.842. Hence, the benchmark model contains much stronger endogenous persistence than does the restricted model. The endogenous persistence in trend inflation is greater than the degree of price stickiness $\mu = 0.800$ but smaller than that of interest rate smoothing $\rho_i = 0.900$. This finding means that with a high degree of interest rate smoothing, the endogenous persistence of the real exchange rate can be beyond the

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25 There are three state variables in this model: $(1 + \hat{i}_{t-1})^d$, $\hat{b}_t$, and $\hat{\omega}_{d,t-1}$. Because the degree of the endogenous risk premium, $\psi$, is set to 0.001, endogenous persistence through $\hat{b}_t$ is negligible and so is endogenous persistence through $\hat{\omega}_{d,t-1}$ when $a_y = 0$. 

34
degree of price stickiness under positive LR trend inflation.

3.5. Roles of the price elasticity of demand

The most controversial calibration in the TI model might be about the price elasticity of demand $\zeta$. With positive long-run trend inflation $\bar{\gamma} > 1$, the price elasticity of demand $\zeta$ affects price dispersion transition (9), GNKPC (14), and TOT dynamics (15). Contrary to the standard NK model with no long-run trend inflation, the price elasticity parameter plays a crucial role in the exchange rate dynamics in the benchmark model.

In the TI model, I set the parameter to 22 because this large value implies an empirically plausible markup rate of 11.8% through the markup equation (16). Careful investigation, however, reveals that the markup equation (16) is a quadratic U-shaped equation, as shown in Figure 9. Hence, a conventional value of $\zeta$ of, say, around 10 also implies a markup rate of 11.11%. This means that under positive LR trend inflation, the conventional way to calibrate $\zeta$ by matching the theoretical markup rate to its empirical counterpart with the actual data fails to pin down the value of $\zeta$. Therefore, what is a plausible value of $\zeta$ within this two-country NK model with trend inflation?

The fourth row of Table 2, which is labeled “Lower Price Elasticity,” displays the implications on the exchange rate moments of the specification of the model with $\zeta = 10$. The simulated sum of the AR(5) coefficients $\alpha$ of 0.931 is largely consistent with its empirical counterpart. The implied half-life measure of 1.25 years, however, is counterfactually low. This finding means that the model is absent from a strong propagation mechanism of the real exchange rate and cannot generate a hump-shaped impulse response as found in its empirical counterpart. The correlation between the real and nominal currency returns is simulated to 0.723 and the STD ratio of the real and nominal currency returns to 0.481. These numbers are far below their empirical counterparts. A lower price elasticity clearly worsens the fit of the model to the data properties of the exchange rates.

\[\text{For example, by using post-war U.S. data, Cogley and Sbordone (2014) estimate a single-equation GNKPC and report that the median estimate of } \zeta \text{ is 9.8 with a 90\% confidence interval between 7.4 and 12.1. Ascari and Sbordone (2014) set } \zeta \text{ to 10 in their discussion of their closed-economy NK model. To my best knowledge, there has been no estimate of } \zeta \text{ within an open-economy NK model with trend inflation.}\]
Therefore, the benchmark model with a larger value of $\zeta$ around 22 performs much better in terms of the exchange rate moments than the specification of the model with a conventional value of price elasticity around 10. However, developing a better estimation of the price elasticity of demand is outside the scope of this study. I leave this as an important future task of future research.

4. Conclusion

In this paper, I argue that allowing for trend inflation in an otherwise standard two-country NK model fundamentally changes the exchange rate dynamics. Positive long-run trend inflation implies a more persistent but less volatile inflation differential with the GNKPCs of the two countries. At the same time, the GNKPCs weaken the linkage between the inflation differential and real exchange rate. The RUIP condition jointly with the Taylor rules then generates a nearly permanent real exchange rate with large volatility. Combined with the less volatile inflation differential, the real and nominal currency returns comove almost perfectly in an equilibrium of the model.

The result of this paper recasts the crucial role that monetary policy frameworks play in the exchange rate dynamics. In particular, the model identifies negligible stochastic variations in trend inflation around the long-term inflation rate target as the primary driver of volatile and persistent exchange rate fluctuations; under positive trend inflation, revisions in perceptions of market participants about the long-term policy goal of the central bank could be the primary fundamental for the real and nominal exchange rates. This result stands in stark contrast to that of an influential calibration study by Steinsson (2008), which advocates real shocks to resolve the PPP puzzle. More future researches are needed to dig deeper about structural interpretation of time-varying trend inflation.

The model presented in this paper is still missing many of the theoretical features used to approach exchange rate anomalies emphasized by the recent literature on international relative prices. For example, the benchmark specification has neither home bias nor a distributional margin. As a result, the benchmark model implies an almost one-to-one correspondence of the real exchange rate to the consumption differential—even in incomplete international financial markets. Including
these features into the benchmark model is thus an important future task to address Backus and Smith’s (1993) anomaly.

References


Table 1: Data and Simulated Moments of the Exchange Rates

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>HL</th>
<th>Corr</th>
<th>Std($\Delta \ln q_t$) / Std($\Delta \ln S_t$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empirical</td>
<td>0.954</td>
<td>4.425</td>
<td>0.932</td>
<td>0.956</td>
</tr>
<tr>
<td>90% CI or ${min, max}$</td>
<td>[0.879 1.000]</td>
<td>${1.600 6.000}$</td>
<td>${0.820 0.990}$</td>
<td>${0.870 1.040}$</td>
</tr>
<tr>
<td>Trend Inflation</td>
<td>0.993</td>
<td>4.750</td>
<td>0.961</td>
<td>0.976</td>
</tr>
<tr>
<td>No LR Trend Inflation $(\bar{\gamma} = 1)$</td>
<td>0.781</td>
<td>0.750</td>
<td>0.728</td>
<td>0.443</td>
</tr>
<tr>
<td>Lower Price Elasticity $(\zeta = 10)$</td>
<td>0.933</td>
<td>1.250</td>
<td>0.723</td>
<td>0.481</td>
</tr>
</tbody>
</table>

**Note 1:** $\alpha$ is the sum of the AR coefficients in the AR(5) process of $\ln q_t$.
**Note 2:** Half-life measure (HL) is the maximum period at which the IRF of the real exchange rate is greater than 0.5. HL is measured in years.
**Note 3:** Corr represents the correlation coefficient between $\Delta \ln q_t$ and $\Delta \ln S_t$.
**Note 4:** $\text{Std}(\Delta \ln q_t) / \text{Std}(\Delta \ln S_t)$ represents the ratio of the standard deviations of $\Delta \ln q_t$ to $\Delta \ln S_t$.
**Note 5:** In the “Empirical” row, $\alpha$ comes from Steinsson’s (2008) estimate using the U.S. trade-weighted real exchange rate. The other statistics stem from Burstein and Gopinath (2014). Each of them exhibits the average of the point estimates over the eight advanced countries.
Table 2: Benchmark Calibration of the TI Model

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$ Subjective Discount Factor</td>
<td>0.990</td>
</tr>
<tr>
<td>$\bar{\gamma}$ Mean Adjusted Trend Inflation Rate</td>
<td>1.0081</td>
</tr>
<tr>
<td>$\eta$ Labor Supply Elasticity</td>
<td>2.000</td>
</tr>
<tr>
<td>$\zeta$ Price Elasticity of Final Goods Demand</td>
<td>22.000</td>
</tr>
<tr>
<td>$\mu$ Calvo probability of No Price Resetting</td>
<td>0.800</td>
</tr>
<tr>
<td>$a_{\pi}$ Taylor Rule Parameter on Inflation Gap</td>
<td>3.500</td>
</tr>
<tr>
<td>$a_y$ Taylor Rule Parameter on Output Gap</td>
<td>0.010</td>
</tr>
<tr>
<td>$\rho_i$ Interest Rate Smoothing Parameter</td>
<td>0.900</td>
</tr>
<tr>
<td>$\lambda$ Error Correction Speed of Productivity Shock</td>
<td>0.010</td>
</tr>
<tr>
<td>$\rho_r$ Trend Inflation Differential AR(1) Coef.</td>
<td>0.990</td>
</tr>
<tr>
<td>$\sigma_A$ Productivity Differential Shock Std.</td>
<td>0.010</td>
</tr>
<tr>
<td>$\sigma_r$ Trend Inflation Differential Shock Std.</td>
<td>0.001</td>
</tr>
<tr>
<td>$\sigma_i$ Monetary Policy Shock Std.</td>
<td>0.002</td>
</tr>
</tbody>
</table>

Note 1. Price elasticity of demand $\zeta$ is calibrated to match the steady-state markup rate to 11.69%.
Note 2. Calibrated long-run adjusted trend inflation rate $\bar{\gamma}$ corresponds to 3.42% at an annual rate.

Table 3: Simulated Volatilities Between the Flexible and Managed Exchange Rate Regimes

<table>
<thead>
<tr>
<th></th>
<th>Flexible $\varphi = 0.000$</th>
<th>Managed $\varphi = 0.200$</th>
<th>Managed $\varphi = 0.760$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Std ($\Delta q_t$) %</td>
<td>4.341</td>
<td>0.800</td>
<td>0.079</td>
</tr>
<tr>
<td>Std($\Delta \ln P_t^d$) %</td>
<td>1.211</td>
<td>0.634</td>
<td>0.107</td>
</tr>
</tbody>
</table>
Figure 1: Real Currency Return, Nominal Currency Return, and Inflation Differential: Japan and the United States
Figure 2: Inflation Differential and Trend Component (Demeaned): Japan and the United States
Figure 3: IRF of the Real Exchange Rate: the TI Model
Figure 4: Impulse Responses of the Real Exchange Rate to Structural Shocks

(a) Productivity Shock

(b) Trend Inflation Shock

(c) Monetary Policy Shock
Figure 5: Impulse Responses of the Nominal Interest Rate, Inflation Rate, and Real Interest Rate to Trend Inflation Shocks
Figure 6: Simulated Synthetic Data from the TI Model
Figure 7: Inflation Differential and Real Currency Return in the Managed and Flexible Regimes
Figure 8: Simulated Synthetic Data from the No LR TI Model; \( \gamma = 1 \)
Figure 9: Markup Rates over the Price Elasticity of Demand
Appendix: Trend Inflation and Exchange Rates Dynamics: A New Keynesian Approach

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Appendix: log-linear approximation of FONCs

Stochastically detrended FONCs

The stochastically detrended versions of the FONCs for the home country consist of the budget constraint

$$p_t c_t + b_{h,t} + s_t b_{f,t} = \left(1 + i_{h,t-1}\right) b_{h,t-1} + \left(1 + i_{f,t-1}\right) s_t b_{f,t-1} + p_{h,t} \left(\frac{p_{h,t}}{p_t}\right)^{-\zeta} a_t + s_t p_{h,t} \left(\frac{p_{h,t}}{p_t}\right)^{-\zeta} c_t^*; \quad (A.1)$$

the home intratemporal optimality condition

$$N_t^\eta = \frac{w_t}{p_t c_t}; \quad (A.2)$$

the home Euler equation

$$\frac{1}{p_t c_t} = \beta(1 + i_{h,t}) E_t \left(\frac{1}{\gamma_{h,t+1} p_{h,t+1} c_{h,t+1}}\right); \quad (A.3)$$

the home UIP condition

$$s_t(1 + i_{h,t}) E_t \left(\frac{1}{p_{t+1} c_{t+1} \gamma_{h,t+1}}\right) = (1 + i_{f,t}) E_t \left(\frac{s_{t+1}}{p_{t+1} c_{t+1} \gamma_{h,t+1}}\right); \quad (A.4)$$

the debt elastic risk premium that the home country faces

$$i_{h,t} = i_{h,t}^* [1 + \psi\{\exp(-b_{h,t} + \bar{d}) - 1\}], \quad \text{and} \quad i_{f,t} = i_{f,t}^* [1 + \psi\{\exp(-b_{f,t} + \bar{d}) - 1\}]; \quad (A.5)$$

the market clearing condition for the home country

$$N_t = \Omega_{h,t} c_{h,t} + \Omega_{h,t}^* c_{h,t}^* a_t^{-1}; \quad (A.6)$$

the transition equations of price dispersions $\Omega_{h,t}$ and $\Omega_{h,t}^*$

$$\Omega_{h,t} = (1 - \mu) \left(\frac{p_{h,t}}{p_{h,t-1}}\right)^{-\zeta} + \mu \left(\frac{p_{h,t}}{p_{h,t-1}}\right)^{\zeta} \left(\frac{\gamma_{h,t}}{\gamma_{A,t}}\right)^{\zeta} \Omega_{h,t-1},$$

and

$$\Omega_{h,t}^* = (1 - \mu) \left(\frac{p_{h,t}}{p_{h,t-1}}\right)^{-\zeta} + \mu \left(\frac{p_{h,t}}{p_{h,t-1}}\right)^{\zeta} \left(\frac{\gamma_{h,t}^*}{\gamma_{A,t}}\right)^{\zeta} \Omega_{h,t-1}^*; \quad (A.7)$$

1
the optimal price setting rules of the home firm for the home and foreign markets

\[
\frac{p_{h,t}}{p_{f,t}} E_t \sum_{i=0}^{\infty} (\beta \mu)^i \frac{c_{h,t+i}}{c_{t+i}} = \frac{\zeta}{\zeta - 1} E_t \sum_{i=0}^{\infty} (\beta \mu)^i \frac{c_{h,t+i}}{c_{t+i}} mc_{t+i}, \quad \text{and}
\]

\[
\frac{p_{h,t}}{p_{f,t}} E_t \sum_{i=0}^{\infty} (\beta \mu)^i \frac{c_{h,t+i}}{c_{t+i}} = \frac{\zeta}{\zeta - 1} E_t \sum_{i=0}^{\infty} (\beta \mu)^i \frac{c_{h,t+i}}{c_{t+i}} mc_{t+i}, \quad \text{and}
\]

where cumulative inflations \( \phi_t, \phi_{h,t}, \) and \( \phi_{h,t}^* \) are defined as

\[
\phi_t \equiv \Pi_{s=1}^i \left( \frac{P_{t+i}}{P_t} \right) = \left( \frac{P_{t+i}}{P_t} \right) \Pi_{s=1}^i \left( \frac{\gamma_{t+s}}{\gamma_{A,t+s}} \right), \quad \phi_{h,t} \equiv \Pi_{s=1}^i \left( \frac{P_{h,t+i}}{P_{h,t}} \right) = \left( \frac{P_{h,t+i}}{P_{h,t}} \right) \Pi_{s=1}^i \left( \frac{\gamma_{t+s}}{\gamma_{A,t+s}} \right),
\]

and \( \phi_{h,t+i}^* \equiv \Pi_{s=1}^i \left( \frac{P_{h,t+i}^*}{P_{h,t}^*} \right) = \left( \frac{P_{h,t+i}^*}{P_{h,t}^*} \right) \Pi_{s=1}^i \left( \frac{\gamma_{t+s}}{\gamma_{A,t+s}} \right); \quad \text{(A.10)}
\]

the home and foreign demand functions for the home good

\[
c_{h,t} = \left( \frac{p_{h,t}}{p_t} \right)^{-\zeta} c_t, \quad \text{and} \quad c_{h,t}^* = \left( \frac{p_{h,t}^*}{p_t^*} \right)^{-\zeta} c_t^*; \quad \text{(A.11)}
\]

the home CPI

\[
p_t^{-1-\zeta} = p_{h,t}^{-1-\zeta} + p_{f,t}^{-1-\zeta}; \quad \text{(A.12)}
\]

the laws of motion of the home final good prices at the home and foreign countries

\[
p_{h,t}^{-1-\zeta} = (1 - \mu)p_{h,t}^{-1-\zeta} + \mu p_{h,t-1}^{-1-\zeta} \left( \frac{\gamma_{A,t}}{\gamma_{t}} \right)^{1-\zeta}, \quad \text{and} \quad p_{h,t}^{-1-\zeta} = (1 - \mu)p_{h,t}^{-1-\zeta} + \mu p_{h,t-1}^{-1-\zeta} \left( \frac{\gamma_{A,t}}{\gamma_{t}} \right)^{1-\zeta}; \quad \text{(A.13)}
\]

the real marginal cost of the home firm

\[
mc_t = \frac{w_t}{p_{h,t}}; \quad \text{(A.14)}
\]

and the Taylor rule the home central bank follows

\[
(1 + i_{h,t}) = (1 + i)^{1-\psi} (1 + i_{h,t-1})^\psi \left( \frac{p_t}{p_{t-1}} \right)^{-1-\zeta} y_t^a y_t^{\sigma} \exp(\epsilon_{t,t}). \quad \text{(A.15)}
\]

Similarly, the stochastically de-trended versions of the FONCs for the foreign country consist
of the budget constraint
\[ \frac{q_t p_t c_t^*}{a_t} - b_{h,t} - s_t b_{f,t} = \frac{(1 + i_{h,t-1}) b_{h,t-1}}{\gamma_{t,1}} - \frac{(1 + i_{f,t-1}) s_t b_{f,t-1}}{\gamma_{t,1}} + p_{f,t} \left( \frac{p_{f,t}}{p_t} \right)^{-\zeta} c_t + s_t p_{f,t}^* \left( \frac{p_{f,t}^*}{p_t^*} \right)^{-\zeta} c_t^*; \]

(A.16)

the foreign intratemporal optimality condition
\[ N_t^{*\eta} = \frac{w_t^* s_t a_t}{q_t p_t t^*}; \]

(A.17)

the foreign Euler equation
\[ \frac{a_t s_t}{q_t p_t t^*} E_t \left( \frac{a_{t+1} s_{t+1}}{q_{t+1} p_{t+1} c_{t+1}^* \gamma_{t+1}} \right) = \beta (1 + i_{f,t}) E_t \left( \frac{a_{t+1} s_{t+1}}{q_{t+1} p_{t+1} c_{t+1}^* \gamma_{t+1}} \right), \]

(A.18)

the foreign UIP condition
\[ s_t (1 + i_{h,t}) E_t \left( \frac{a_{t+1}}{q_{t+1} p_{t+1} c_{t+1}^* \gamma_{t+1}} \right) = (1 + i_{f,t}^*) E_t \left( \frac{a_{t+1} s_{t+1}}{q_{t+1} p_{t+1} c_{t+1}^* \gamma_{t+1}} \right), \]

(A.19)

the market clearing condition for the foreign country
\[ N_t^* = \Omega_{f,t} a_t c_{f,t} + \Omega_{f,t}^* c_{f,t}^*; \]

(A.20)

the transition equations of price dispersions \( \Omega_{f,t} \) and \( \Omega_{f,t}^* \)
\[ \Omega_{f,t} = (1 - \mu) \left( \frac{P_{f,t}}{P_{f,t}^*} \right)^{-\zeta} + \mu \left( \frac{p_{f,t}}{p_{f,t}^*} \right)^{\zeta} \left( \frac{\gamma_{t,1}}{\gamma_{t,1}} \right)^{\zeta} \Omega_{f,t-1}, \]

\[ \text{and} \quad \Omega_{f,t}^* = (1 - \mu) \left( \frac{P_{f,t}^*}{P_{f,t}} \right)^{-\zeta} + \mu \left( \frac{p_{f,t}^*}{p_{f,t}^*} \right)^{\zeta} \left( \frac{\gamma_{t,1}}{\gamma_{t,1}} \right)^{\zeta} \Omega_{f,t-1}^*; \]

(A.21)

the optimal price setting rules of the foreign firm
\[ \sum_{i=0}^{\infty} (\beta \mu)^i \frac{p_{f,t+i}}{s_{t+i} p_{f,t+i}} \phi_{f,t+i}^{\zeta-1} \left( \frac{c_{f,t+i}}{c_{t+i}^*} \right) = \frac{\zeta}{\zeta - 1} E_t \sum_{i=0}^{\infty} (\beta \mu)^i m_{c_{t+i}}^* a_{t+i} \phi_{f,t+i}^{\zeta} \left( \frac{c_{f,t+i}}{c_{t+i}^*} \right), \quad \text{and} \]

(A.22)

\[ \sum_{i=0}^{\infty} (\beta \mu)^i \frac{p_{f,t+i}^*}{s_{t+i} p_{f,t+i}} \phi_{f,t+i}^* \phi_{f,t+i}^{\zeta-1} \left( \frac{c_{f,t+i}^*}{c_{t+i}^*} \right) = \frac{\zeta}{\zeta - 1} E_t \sum_{i=0}^{\infty} (\beta \mu)^i m_{c_{t+i}}^* \phi_{f,t+i}^* \phi_{f,t+i}^{\zeta} \left( \frac{c_{f,t+i}}{c_{t+i}^*} \right), \]

(A.23)
where cumulative inflations $\phi_t^*, \phi_{f,t}$, and $\phi_{f,t}^*$ are defined as

$$
\phi_{t+1}^* \equiv \Pi_{s=1}^i \left( \frac{P_{s+1}^*}{P_t^*} \right) = \frac{p_{t+1}}{p_t} \Pi_{s=1}^i \left( \frac{\gamma_{t+s}^*}{\gamma_{A,s}^*} \right),
$$

and

$$
\phi_{f,t+1}^* \equiv \Pi_{s=1}^i \left( \frac{P_{f,t+s}^*}{P_{f,t}^*} \right) = \frac{p_{f,t+1}}{p_{f,t}} \Pi_{s=1}^i \left( \frac{\gamma_{t+s}^*}{\gamma_{A,s}^*} \right),
$$

the demand functions for the foreign goods

$$
c_{f,t} = \left( \frac{p_{f,t}}{p_t} \right)^{-\gamma} c_t, \quad \text{and} \quad c_{f,t}^* = \left( \frac{p_{f,t}^*}{p_t} \right)^{-\gamma} c_t^*;
$$

the foreign CPI

$$
p_{f,t}^{1-\gamma} = p_{h,t}^* + p_{f,t}^{1-\gamma};
$$

the laws of motion of the foreign final good prices at the home and foreign countries

$$
p_{f,t}^{1-\gamma} = (1-\mu)p_{f,t}^{1-\gamma} + \mu p_{f,t-1}^{1-\gamma} \left( \frac{\gamma_{A,t}}{\gamma_{\tau,t}} \right)^{1-\gamma}, \quad \text{and} \quad p_{f,t}^{1-\gamma} = (1-\mu)p_{f,t}^{1-\gamma} + \mu p_{f,t-1}^{1-\gamma} \left( \frac{\gamma_{A,t}}{\gamma_{\tau,t}} \right)^{1-\gamma};
$$

the real marginal cost the foreign firm faces

$$
m c_t = \frac{w_t^*}{p_{f,t}^*};
$$

and the Taylor rule the foreign central bank follows

$$
(1 + \tilde{i}_{f,t}^*) = (1 + \tilde{i}_t)^{1-\rho_g}(1 + \tilde{i}_{f,t-1}^*)^{\rho_g} \left( \frac{p_t^*}{p_{t-1}^*} \right)^{\gamma_{A,t}} y_t^{\alpha_g} \exp(c_{t,\epsilon})\epsilon_t.
$$

*Log-linear approximation of the stochastically detrended system*

I now take the log-linear approximation of the simultaneous equation system consisting of the stochastically detrended FONCs to derive the corresponding linear rational expectations (LRE) model. Given the above stochastically detrended FONCs (A.1)-(A.29), the corresponding log-linearized FONCs around the deterministic steady state are derived as follows. The log-linear approximation of the home budget constraint is

$$
\tilde{b}_{h,t} + \tilde{b}_{f,t} + sA\hat{s}_t = \beta^{-1} \tilde{d} \left[ (1 + \hat{i}_{h,t}) - \hat{\gamma}_{\tau,t} \right] + \beta^{-1} \tilde{b}_{h,t-1} + \beta^{-1} \tilde{d} \left[ (1 + \hat{i}_{f,t}) - \hat{\gamma}_{\tau,t} + \hat{s}_t \right] + \beta^{-1} sA\hat{s}_{t-1} + p^c p_h^{1-\gamma} c [(1 - \zeta)\hat{p}_h + \zeta \hat{p}_t + \hat{c}_t] + sp^c p_h^{1-\gamma} c^* \left[ \hat{s}_t + (1 - \zeta)\hat{p}_h + \zeta \hat{p}_t + \hat{c}_t \right] - pc(\hat{p}_h + \hat{c}_t);
$$

(A.30)
the home intratemporal optimality condition

$$\eta \hat{N}_t = \hat{w}_t - \hat{p}_t - \hat{c}_t; \quad \text{(A.31)}$$

the home Euler equation

$$\hat{p}_t + \hat{c}_t + (1 + \hat{i}_{ht}) = E_t(\hat{p}_{h,t+1} + \hat{c}_{t+1} + \hat{\gamma}_{r,t+1}); \quad \text{(A.32)}$$

the home UIP condition

$$E_t \hat{s}_{t+1} - \hat{s}_t = (1 + \hat{i}_{ht}) - (1 + \hat{i}_{ft}) - E_t(\hat{\gamma}_{r,t+1} - \hat{\gamma}_{r,t+1}^*); \quad \text{(A.30)}$$

the home risk premium

$$(1 + \hat{i}_{ht}) = (1 + \hat{i}_{ht}^*) - \psi(1 - \kappa)\hat{b}_{h,t}, \quad \text{and} \quad (1 + \hat{i}_{ft}) = (1 + \hat{i}_{ft}^*) - \psi(1 - \kappa)\hat{b}_{f,t}; \quad \text{(A.31)}$$

where $\kappa \equiv \beta / \gamma_r$; the optimal price setting rules of the home firm for the home market

$$\hat{p}_h + \hat{p}_{h,t} + (1 - \varphi_1)\sum_{i=0}^{\infty} \varphi_1^i E_t(\hat{c}_{h,t+i} - \hat{c}_{t+i}) + (1 - \varphi_1)(\zeta - 1)\sum_{i=1}^{\infty} \varphi_1^i E_t\hat{\phi}_{h,t+i} =$$

$$\sum_{i=0}^{\infty} \varphi_2^i E_t(\hat{c}_{h,t+i} - \hat{c}_{t+i}) + (1 - \varphi_2)\sum_{i=0}^{\infty} \varphi_2^i E_t\hat{\phi}_{h,t+i} + \sum_{i=1}^{\infty} \varphi_2^i E_t\hat{\phi}_{h,t+i} \quad \text{(A.32)}$$

and for the foreign market

$$\hat{p}_h^* - \hat{p}_{h,t} + (1 - \varphi_1)\sum_{i=0}^{\infty} \varphi_1^i E_t(\hat{c}_{h,t+i}^* - \hat{c}_{t+i}) + (1 - \varphi_1)\sum_{i=0}^{\infty} \varphi_1^i E_t(\hat{s}_{t+i} + \hat{p}_{h,t+i} - \hat{p}_{h,t+i}) +$$

$$\sum_{i=1}^{\infty} \varphi_1^i E_t\hat{\phi}_{h,t+i} = (1 - \varphi_2)\sum_{i=0}^{\infty} \varphi_2^i E_t(\hat{c}_{h,t+i} - \hat{c}_{t+i}) + (1 - \varphi_2)\sum_{i=0}^{\infty} \varphi_2^i E_t(\hat{m}_{c,t+i} - \hat{a}_{t+i}) +$$

$$\sum_{i=1}^{\infty} \varphi_2^i E_t\hat{\phi}_{h,t+i} \quad \text{(A.33)}$$
where \( \varphi_1 \equiv \beta \mu (\gamma_t / \gamma_A) \zeta^{-2} \) and \( \varphi_2 \equiv \beta \mu (\gamma_t / \gamma_A) \zeta^{-1} \); the home cumulative inflations

\[
\hat{\phi}_{t+i} - \hat{\phi}_t = \sum_{s=1}^{\infty} \left( \hat{\gamma}_{r,t+s} - \hat{\gamma}_{A,t+s} \right) = \sum_{s=1}^{\infty} \hat{\chi}_{t+s},
\]

\[
\hat{\phi}_{h,t+i} = \hat{p}_{h,t+i} - \hat{p}_{h,t} + \sum_{s=1}^{\infty} \left( \hat{\gamma}_{r,t+s} - \hat{\gamma}_{A,t+s} \right) = \sum_{s=1}^{\infty} \hat{\chi}_{h,t+s},
\]

\[
\hat{\phi}_{h,t+i} = \hat{p}_{h,t+i}^* - \hat{p}_{h,t}^* + \sum_{s=1}^{\infty} \left( \hat{\gamma}_{r,t+s} - \hat{\gamma}_{A,t+s} \right) = \sum_{s=1}^{\infty} \hat{\chi}_{h,t+s}^*,
\]

with \( \hat{p}_t \equiv \hat{p}_t - \hat{p}_{t-1}, \hat{p}_{h,t} \equiv \hat{p}_{h,t} - \hat{p}_{h,t-1}, \hat{p}_{h,t}^* \equiv \hat{p}_{h,t}^* - \hat{p}_{h,t-1}, \hat{\gamma}_{r,t} \equiv \hat{\gamma}_{r,t} - \hat{\gamma}_{A,t}, \hat{\gamma}_{h,t} \equiv \hat{\gamma}_{h,t} - \hat{\gamma}_{A,t} \), and \( \hat{\chi}_{h,t} \equiv \hat{\chi}_{h,t} - \hat{\chi}_{A,t} \); the market clearing condition for the home country

\[
\hat{N}_t = \hat{\Omega}_{h,t} + \hat{c}_{h,t} + \hat{\Omega}_{h,t}^* + \hat{c}_{h,t}^* - \hat{\theta}; \quad (A.35)
\]

the transition equations of price dispersions \( \hat{\Omega}_{h,t} \) and \( \hat{\Omega}_{h,t}^* \)

\[
\hat{\Omega}_{h,t} = \zeta \mu \gamma^\zeta \Theta (\hat{\gamma}_{t+h} + \hat{\gamma}_{A,t} - \hat{\gamma}_{A,t}) + \mu \gamma^\zeta \hat{\Omega}_{h,t-1}, \quad \text{and} \quad \hat{\Omega}_{h,t}^* = \zeta \mu \gamma^\zeta \Theta (\hat{\gamma}_{t+h}^* + \hat{\gamma}_{A,t}^* - \hat{\gamma}_{A,t}^*) + \mu \gamma^\zeta \hat{\Omega}_{h,t-1}^*,
\]

where \( \Theta \equiv \gamma - \frac{1 - \mu \gamma^\zeta}{1 - \mu \gamma^\zeta} \); the demand functions for the home goods

\[
\hat{c}_{h,t} = \hat{c}_t - \zeta (\hat{p}_{h,t} - \hat{p}_t), \quad \text{and} \quad \hat{c}_{h,t}^* = \hat{c}_t^* - \zeta (\hat{p}_{h,t}^* - \hat{p}_t^*);
\]

the home CIP

\[
2\hat{p}_t = \hat{p}_{h,t} + \hat{p}_{f,t}; \quad (A.38)
\]

the laws of motion of the home good prices in the home and foreign countries

\[
\hat{p}_{h,t} = (1 - \mu) \left( \frac{p_n}{p_h} \right)^{1-\zeta} \hat{p}_{h,t} + \mu \left( \frac{\gamma}{\gamma_A} \right)^{1-\zeta} \left[ \hat{p}_{h,t-1} + \hat{\gamma}_{A,t} - \hat{\gamma}_{A,t} \right], \quad \text{and} \quad \hat{p}_{h,t}^* = (1 - \mu) \left( \frac{p_n}{p_h} \right)^{1-\zeta} \hat{p}_{h,t}^* + \mu \left( \frac{\gamma}{\gamma_A} \right)^{1-\zeta} \left[ \hat{p}_{h,t-1}^* + \hat{\gamma}_{A,t}^* - \hat{\gamma}_{A,t}^* \right];
\]

the real marginal cost of the home firm

\[
\hat{m}e_t = \tilde{w}_t - \hat{p}_{h,t}; \quad (A.40)
\]

the Taylor rule of the home central bank

\[
(1 + \hat{i}_{h,t}) = \rho_1 (1 + \hat{i}_{h,t-1}) + \alpha (\hat{\pi}_t - \hat{\gamma}_{A,t}) + \alpha y_t + \epsilon_{i,t}.
\]
The foreign country’s counterparts are as follows. The log-linear approximation of the foreign country’s budget constraint is

\[-\hat{b}_{h,t} - \hat{b}_{f,t} - s\hat{d}_t = -\beta^{-1}d\left[(1 + \hat{t}_{h,t}^*) - \gamma_{r,t}\right] - \beta^{-1}\hat{b}_{h,t-1} - \beta^{-1}d\left[(1 + \hat{t}_{f,t}^*) - \gamma_{r,t}^* + \hat{s}_t\right] - \beta^{-1}s\hat{b}_{f,t-1} + \hat{p}_t^{1-\zeta} c_1[(1 - \zeta)\hat{p}_{f,t} + \hat{\zeta}_t + \hat{\bar{c}}_t] + s p^* \hat{c}_1^{1-\zeta} c^* \left[\hat{s}_t + (1 - \zeta)\hat{p}_{f,t} + \hat{\zeta}_t + \hat{\bar{c}}_t\right] - sp^* c^*(\hat{s}_t + \hat{p}_t^* + \hat{\bar{c}}_t); \quad (A.42)\]

the foreign intratemporal optimality condition

\[\eta \hat{N}_t^* = \hat{w}_t^* - \hat{p}_t^* - \hat{\bar{c}}_t^*; \quad (A.43)\]

the foreign Euler equation

\[\hat{p}_t + \hat{c}_t^* + \hat{q}_t - \hat{a}_t - \hat{s}_t + (1 + \hat{t}_{f,t}^*) = E_t(\hat{p}_{t+1} + \hat{c}_{t+1} + \hat{q}_{t+1} - \hat{a}_{t+1} - \hat{s}_{t+1} + \gamma_{r,t+1}^*); \quad (A.44)\]

the foreign UIP condition

\[E_t \hat{s}_{t+1} - \hat{s}_t = (1 + \hat{t}_{h,t}^*) - (1 + \hat{t}_{f,t}^*) - E_t(\gamma_{r,t+1}^* - \gamma_{r,t+1}^*); \quad (A.45)\]

the optimal price setting rules of the foreign firm

\[\hat{p}_{f,t} - \hat{p}_{f,t}^* + (1 - \varphi_1) \sum_{i=0}^{\infty} \varphi_i^1 E_i(\hat{c}_{f,t+i} - \hat{c}_{t+i}) + (1 - \varphi_1) \sum_{i=0}^{\infty} \varphi_i^1 E_i(\hat{p}_{f,t+i} - \hat{s}_{t+i} - \hat{p}_{f,t+i}) + (1 - \varphi_1)(1 - 1) \sum_{i=1}^{\infty} \varphi_i E_i \hat{p}_{f,t+i} = (1 - \varphi_2) \sum_{i=0}^{\infty} \varphi_i^2 E_i(\hat{c}_{f,t+i} - \hat{c}_{t+i}) + (1 - \varphi_2) \sum_{i=0}^{\infty} \varphi_i^2 E_i(\hat{m}\hat{c}_{t+i} - \hat{a}_{t+i}) + (1 - \varphi_2) \sum_{i=1}^{\infty} \varphi_i^2 E_i \hat{p}_{f,t+i}, \quad (A.48)\]

and

\[\hat{p}_{f,t}^* - \hat{p}_{f,t}^* + (1 - \varphi_1) \sum_{i=0}^{\infty} \varphi_i^1 E_i(\hat{c}_{f,t+i}^* - \hat{c}_{t+i}^*) + (1 - \varphi_1)(1 - 1) \sum_{i=1}^{\infty} \varphi_i^1 E_i \hat{p}_{f,t+i}^* = (1 - \varphi_2) \sum_{i=0}^{\infty} \varphi_i^2 E_i(\hat{c}_{f,t+i}^* - \hat{c}_{t+i}^*) + (1 - \varphi_2) \sum_{i=0}^{\infty} \varphi_i^2 E_i \hat{m}\hat{c}_{t+i}^* + (1 - \varphi_2) \sum_{i=1}^{\infty} \varphi_i^2 E_i \hat{p}_{f,t+i}^*; \quad (A.49)\]
the foreign cumulative inflations

\[ \hat{\phi}_{t+i} = \hat{p}_{t+i} - \hat{p}_t + \sum_{s=1}^{i} (\hat{\gamma}_{t,s} - \gamma_{A,t+s}) = \sum_{s=1}^{i} (\hat{\pi}_{t+s} + \hat{\gamma}_{t+s} - \gamma_{A,t+s}) = \sum_{s=1}^{i} \chi_{t+s}, \]

\[ \hat{\phi}_{f,t+i} = \hat{p}_{f,t+i} - \hat{p}_{f,t} + \sum_{s=1}^{i} (\hat{\gamma}_{f,t+s} - \gamma_{A,t+s}) = \sum_{s=1}^{i} (\hat{\pi}_{f,t+s} + \hat{\gamma}_{f,t+s} - \gamma_{A,t+s}) = \sum_{s=1}^{i} \chi_{f,t+s}, \]

\[ \hat{\phi}_{f,t+i} = \hat{p}_{f,t+i} - \hat{p}_{f,t} + \sum_{s=1}^{i} (\hat{\gamma}_{f,t+s} - \gamma_{A,t+s}) = \sum_{s=1}^{i} (\hat{\pi}_{f,t+s} + \hat{\gamma}_{f,t+s} - \gamma_{A,t+s}) = \sum_{s=1}^{i} \chi_{f,t+s}, \] (A.50)

where \( \hat{\pi}_t \equiv \hat{p}_t - \hat{p}_{t-1}, \hat{\pi}_{f,t} \equiv \hat{p}_{f,t} - \hat{p}_{f,t-1}, \hat{\gamma}_t \equiv \hat{p}_t - \hat{p}_{f,t}, \chi_t \equiv \hat{\pi}_t + \hat{\gamma}_t - \gamma_{A,t}, \chi_{f,t} \equiv \hat{\pi}_{f,t} + \hat{\gamma}_{f,t} - \gamma_{A,t}, \) and \( \chi_{f,t} \equiv \hat{\pi}_{f,t} + \hat{\gamma}_{f,t} - \gamma_{A,t}; \) the market clearing condition for the foreign country

\[ \hat{N}_t^* = \hat{\Omega}_{f,t} + \hat{c}_{f,t} + \hat{\hat{\phi}}_{f,t} + \hat{\hat{c}}_{f,t} + \hat{\hat{a}}_t; \] (A.51)

the transition equations of price dispersions \( \hat{\Omega}_{f,t} \) and \( \hat{\Omega}_{f,t}^* \)

\[ \hat{\Omega}_{f,t} = \zeta \mu \gamma^{\zeta-1} \Omega \left( \hat{\pi}_{f,t} + \hat{\gamma}_{f,t} - \gamma_{A,t} \right) + \mu \gamma^{\zeta} \hat{\Omega}_{f,t-1}, \quad \text{and} \quad \hat{\Omega}_{f,t}^* = \zeta \mu \gamma^{\zeta-1} \Theta \left( \hat{\pi}_{f,t} + \hat{\gamma}_{f,t} - \gamma_{A,t}^* \right) + \mu \gamma^{\zeta} \hat{\Omega}_{f,t-1}, \] (A.52)

the demand functions for the foreign goods

\[ \hat{c}_{f,t} = \hat{c}_t - \xi \left( \hat{\pi}_{f,t} - \hat{\pi}_t \right), \quad \text{and} \quad \hat{c}_{f,t}^* = \hat{c}_t^* - \xi \left( \hat{\pi}_{f,t}^* - \hat{\pi}_t^* \right); \] (A.50)

the foreign CIP

\[ 2\hat{\pi}_t = \hat{\pi}_{h,t} + \hat{\hat{\phi}}_{f,t}; \] (A.53)

the laws of motion of the home good prices in the home and foreign countries

\[ \hat{p}_{f,t} = (1 - \mu) \left( \frac{p_H}{p_f} \right)^{1-\zeta} \hat{p}_{f,t} + \mu \left( \frac{\gamma_{f,t+1}}{\gamma_f} \right)^{1-\zeta} \left[ \hat{p}_{f,t-1} + \hat{\gamma}_{A,t} - \gamma_{A,t} \right], \quad \text{and} \quad \hat{p}_{f,t}^* = (1 - \mu) \left( \frac{p_H^*}{p_f^*} \right)^{1-\zeta} \hat{p}_{f,t}^* + \mu \left( \frac{\gamma_{f,t+1}^*}{\gamma_f^*} \right)^{1-\zeta} \left[ \hat{p}_{f,t-1}^* + \hat{\gamma}_{A,t}^* - \gamma_{A,t}^* \right]; \] (A.54)

the marginal cost of the foreign firm

\[ m\hat{c}_t^* = \hat{\omega}_t - \hat{\hat{\phi}}_{f,t}; \] (A.55)

the Taylor rule of the foreign central bank

\[ (1 + \hat{\pi}_{f,t}) = \rho_i (1 + \hat{\pi}_{f,t-1}) + a_x \left( \hat{\pi}_{t} - \hat{\gamma}_{A,t} \right) + a_y \hat{\gamma}_t + \varepsilon_{i,t}. \] (A.56)
Comparing the home and foreign UIPs (A.30) and (A.45) reveals that the two countries must face the same interest rate differential: 
\[(1 + \tilde{i}_{h,t}) - (1 + \tilde{i}_{f,t}) = (1 + \tilde{i}_{h,t}^*) - (1 + \tilde{i}_{f,t}^*).\] 
The home risk premium equations (A.31) then implies that the home holdings of the home issued bond must be equal to the home holdings of the foreign issued bond: 
\[\hat{b}_t = \hat{b}_{h,t} = \hat{b}_{f,t}.\]

**GNKPCs with trend inflation**

Now I derive the generalized new Keynesian Phillips curves (GNKPCs). In doing so, I assume that 
\[0 < \varphi_1 \equiv \beta \mu (\gamma_A / \gamma_t)^{\zeta - 1} < 1\] and 
\[0 < \varphi_2 \equiv \beta \mu (\gamma_t / \gamma_A)^{\zeta} < 1.\] 
At the deterministic steady state, the law of motion of the home good price at the home market (A.13) implies that 
\[
\left( \frac{p_h}{p_h} \right)^{1-\zeta} = \frac{1 - \mu \left( \frac{\gamma_A}{\gamma_t} \right)^{\zeta - 1}}{1 - \mu}.
\]  
(A.57)

Substituting the above equation (A.57) into the log-linear approximation of the law of motion of the home good price at the home market, eq. (A.39), gives 
\[
\chi_{h,t} \equiv \hat{\pi}_{h,t} - \hat{\gamma}_{A,t} + \hat{\gamma}_{t,t} = \varphi_0(\hat{p}_{h,t} - \hat{p}_{h,t}).
\]  
(A.58)

Cumulative inflation processes (A.34) rewrite the optimal price setting of the home firm for the home market (A.32) as 
\[
\hat{p}_{h,t} - \hat{p}_{h,t} = (1 - \varphi_2)D_{h,t} - (1 - \varphi_1)Z_{h,t},
\]  
(A.59)

where random variables \(D_{h,t}\) and \(Z_{h,t}\) follow the stochastic difference equations 
\[
D_{h,t} = \hat{c}_{h,t} - \hat{c}_t + \tilde{m}c_t + \frac{\zeta \varphi_2}{1 - \varphi_2} E_t \chi_{h,t+1} + \varphi_2 E_t D_{h,t+1},
\]
\[
Z_{h,t} = \hat{c}_{h,t} - \hat{c}_t + \frac{(\zeta - 1) \varphi_1}{1 - \varphi_1} E_t \chi_{h,t+1} + \varphi_1 E_t Z_{h,t+1}.
\]

After several calculations, it is easy to show that eqs. (A.57) – (A.59) provide 
\[
\chi_{h,t} = \frac{\varphi_0 (\varphi_1 - \varphi_2)}{\varphi_1} \left[ \hat{c}_{h,t} - \hat{c}_t - (1 - \varphi_1) \sum_{i=0}^{\infty} \varphi_1^i E_t (\hat{c}_{h,t+i} - \hat{c}_{t+i}) \right] + \varphi_0 (1 - \varphi_2) \tilde{m}c_t
\]
\[
+ \varphi_2 (1 + \varphi_0) E_t \chi_{h,t+1} - \frac{\varphi_0 (\varphi_1 - \varphi_2)}{\varphi_1} (\zeta - 1) \sum_{i=1}^{\infty} \varphi_1^i E_t \chi_{h,t+i}.
\]  
(A.60)
Further, the demand function for the home good (A.37) implies

\[ \hat{c}_{h,t} - \hat{c}_t - (1 - \varphi_1) \sum_{i=0}^{\infty} \varphi_i^1 E_t(\hat{c}_{h,t+i} - \hat{c}_{t+i}) = -\sum_{i=1}^{\infty} \varphi_i^1 E_t \Delta(\hat{c}_{h,t+i} - \hat{c}_{t+i}), \]

\[ = -\varsigma \sum_{i=1}^{\infty} \varphi_i^1 E_t(\hat{p}_{t+i} - \hat{p}_{h,t+i}). \]

Substituting this relation into eq. (A.60) yields the GNKPC for the home good price at the home market

\[ \chi_{h,t} = \varphi_2(1 + \varphi_0)E_t \chi_{h,t+1} + \varphi_0(1 - \varphi_2)\hat{m}c_t - \varphi_0(\varphi_1 - \varphi_2)\hat{s}_t - \hat{p}h_{t+i} - \hat{p}_h. \]

Noting that \( \chi_{h,t} = \Delta \ln P_{h,t} \) and \( \chi_{h,t} = \Delta \ln P_t \) in eq (A.61) provides eq. (3).

It is straightforward to show that the same steps of calculation is applicable to derive the GNKPCs for the home good price inflation at the foreign market \( \chi_{h,t}^* \), the foreign good price inflation at the home market \( \chi_{f,t} \), and the foreign good price inflation at the foreign market \( \chi_{f,t}^* \) as follows

\[ \chi_{h,t}^* = \varphi_2(1 + \varphi_0)E_t \chi_{h,t+1}^* + \varphi_0(1 - \varphi_2)(\hat{m}c_t - \hat{a}_t) - \frac{\varphi_0\varphi_2(1 - \varphi_1)}{\varphi_1} (\hat{s}_t + \hat{p}h_{t+i} - \hat{p}_h), \]

\[ = \varphi_0(\varphi_1 - \varphi_2)\frac{\varsigma \sum_{i=0}^{\infty} \varphi_i^1 E_t(\chi_{t+i}^* - \chi_{h,t+i}) - \varphi_0(\varphi_1 - \varphi_2)\varphi_1^1 E_t \Delta(\hat{c}_{t+i} - \hat{c}_{t+i}^*)}{\varphi_1} \]

\[ - \frac{\varphi_0(\varphi_1 - \varphi_2)(1 - \varphi_1)}{\varphi_1} \sum_{i=0}^{\infty} \varphi_i^1 E_t(\hat{s}_{t+i} + \hat{p}h_{t+i} - \hat{p}_h) - \frac{\varphi_0(\varphi_1 - \varphi_2)}{\varphi_1}(\varsigma - 1) \sum_{i=1}^{\infty} \varphi_i^1 E_t \chi_{h,t+i}. \]

(A.62)

\[ \chi_{f,t} = \varphi_2(1 + \varphi_0)E_t \chi_{f,t+1} + \varphi_0(1 - \varphi_2)(\hat{m}c_t^* + \hat{d}_t) + \frac{\varphi_0\varphi_2(1 - \varphi_1)}{\varphi_1} (\hat{s}_t + \hat{p}f_{t+i} - \hat{p}_f), \]

\[ - \frac{\varphi_0(\varphi_1 - \varphi_2)}{\varphi_1} \sum_{i=0}^{\infty} \varphi_i^1 E_t(\chi_{t+i} - \chi_{f,t+i}) - \frac{\varphi_0(\varphi_1 - \varphi_2)}{\varphi_1} \sum_{i=1}^{\infty} \varphi_i^1 E_t \Delta(\hat{c}_{t+i} - \hat{c}_{t+i}) \]

\[ + \frac{\varphi_0(\varphi_1 - \varphi_2)(1 - \varphi_1)}{\varphi_1} \sum_{i=1}^{\infty} \varphi_i^1 E_t(\hat{s}_{t+i} + \hat{p}f_{t+i} - \hat{p}_f) - \frac{\varphi_0(\varphi_1 - \varphi_2)}{\varphi_1}(\varsigma - 1) \sum_{i=1}^{\infty} \varphi_i^1 E_t \chi_{f,t+i}. \]

(A.63)
\[ \chi_{f,t}^* = \varphi_2 (1 + \varphi_0) E_t \chi_{f,t+1}^* + \varphi_0 (1 - \varphi_2) \hat{m} e_t^* \]

\[ - \frac{\varphi_0 (\varphi_1 - \varphi_2)}{\varphi_1} \zeta \sum_{i=1}^{\infty} \varphi_i E_t (\chi_{i+1}^* - \chi_{f,t+i}^*) - \frac{\varphi_0 (\varphi_1 - \varphi_2)}{\varphi_1} (\zeta - 1) \sum_{i=1}^{\infty} \varphi_i E_t \chi_{f,t+i}^*. \] (A.64)