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Additive Representation of Preference Relation

Norio Hirasawa

§1. Introduction

"Additivety" that is used in this note is as follows. Let us suppose objects to be measured are conjoint effect of \( n \)-factors (\( n \) kinds of commodities). Each factor is measured by its own function (or utility scale). Total effect is represented with additive form of factor-functions (or subutilities). And if a situation (or commodity bundle) \( x \) is, under certain ordering, preferred to \( y \), then we attach the greater value to the former, that is, total effect-function is order-preserving. Each factor-function is order-preserving, too.

\[
U : X \rightarrow R, \quad X = X_1 \times X_2 \times \cdots \times X_n
\]

for all \( x, y \) in \( X \)

\[
U(x) = (x_1, x_2, \ldots, x_n) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n),
\]

\[
\sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i) \quad \text{if and only if } x \succeq y,
\]

\[
u_i(x_i) \geq u_i(y_i) \quad \text{if and only if } x_i \succeq_i y_i \quad \text{for any } i,
\]

\( \succeq \) : ordering on \( X \),

\( \succeq_i \) : ordering on \( X_i \), for any \( i \).

This is the definition of additive utility function, which had been frequently used in the context of classical development of consumer behavior until Edgeworth (1881) and Parato (1906) noticed that there was no need to assume this for the theory of consumer behavior.
§2. Axioms

Now, we will state Axioms.1)

Axiom 1. (Ordering axiom)

口头 is a weak ordering, i.e.;

a) reflexive: for all \( x \in X, \ x \geq x, \)

b) transitive: for all \( x, y, z \in X, \)

\[ x \geq y \text{ and } y \geq z \text{ imply } x \geq z, \]

c) complete: for all \( x, y \in X, \)

either \( x \geq y \) or \( y \geq x. \)

Definition

1) for all \( x, y \in X, \ x \sim y \) if and only if \( x \geq y \) and \( y \geq x. \)

2) for all \( x, y \in X, \ x > y \) if and only if \( x \geq y \) and not \( y \geq x. \)

Axiom 2. (Solution axiom)2)

For all \( x \in X, \) and \( y_\sim \in X_1 \times X_2 \times \cdots \times X_{i-1} \times X_{i+1} \cdots \times X_n, \) i.e., \( y_\sim = (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n), \) there exists \( y_i \in x; \) such that

\[ x \sim (y_\sim, y_i) \text{ for all } i \in I = \{1, 2, \ldots, n'. \} \]

where, we assume, components of \( (y_\sim, y_i) \) are rearranged according to their suffixes.

Axiom 3. (Cancellation axiom)

Let \( I = \{1, 2, \ldots, n\}, \)

\[ J = \{j_1, j_2, \ldots, j_t\} \neq \phi, \]

\[ J \cap J' = \phi, \ J \cup J' = I. \]

If \( x^1, x^2, x^3, x^4 \in X, \ x^1 \geq x^2, x^3 \geq x^4, \) and if

\[ (x_{j^1}, x_{j^2}) \sim (x_{j^1'}, x_{j^4}), \]

1) The basic idea we adopt is similar to those of Adams and Fagot (1959), Luce and Tukey (1964) and Krantz (1964).

2) If you consider the meaning of this axiom, you may say that we had better call this axiom "compensation axiom."

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then \((x_{i'}, x_i^3) \succeq (x_j^2, x_j^4)\)

where \(x_j^k = (x_{j_1}^k, x_{j_2}^k, \ldots x_{j_y}^k)\),

\(x_i^k = (x_{i_1}^k, x_{i_2}^k, \ldots x_{i_y}^k) = (x_{i_1}^k, x_{i_2}^k, \ldots x_{i_y}^k)\),

\(x = x_i^k \setminus x_{i'}, k = 1, 2, 3, 4\).

Axiom 3 means that if he prefers \(x^1\) to \(x^2\), and \(x^3\) to \(x^4\), and if he prefers neither the bundle composed of the elements of \(x^1\) with the suffixes \(j_k \in J\) and those of \(x^3\) with the suffixes \(j_{k'} \in J'\) nor the bundle composed of the elements of \(x^2\) with the suffixes \(j_{k'} \in J'\) and those of \(x^4\) with the suffixes \(j_k \in J\), then the bundle composed of the elements \(x^1\) with the suffixes \(j_{k'} \in J'\) and those of \(x^3\) with the suffixes \(j_k \in J\) is preferred to the bundle composed of the elements of \(x^2\) with the suffixes \(j_{k'} \in J'\) and those of \(x^4\) with the suffixes \(j_k \in J\) by him.

Axiom 4. (Archimedian axiom)
This axiom will be explained in §3.

§3. Additive representation of preference relation

Theorem 1\(^1\)

If axioms 1 to 3 hold, then

\((x_1, \ldots x_i, \ldots, x_n) \succeq (x_1, \ldots x_i, \ldots, x_n)\)

implies

\((x_1', \ldots x_i, \ldots, x_n') \succeq (x_1', \ldots, x_i', \ldots x_n')\)

for all \(i \in I\).

Corollary

If axioms 1 to 3 hold, then

\((x_1, \ldots, x_i, \ldots, x_n) \succeq (x_1, \ldots, x_i', \ldots, x_n)\)

implies

\((x_1', \ldots, x_i, \ldots, x_n') \succeq (x_1', \ldots, x_i', \ldots, x_n')\)

\(^1\) For the sake of space, in the following, proofs easy to show are omitted.
Theorem 1 and its Corollary lead us to the following definition.

**Definition**

Let axioms 1 to 3 hold true. Then, define \( x_i \geq y_i \) if there exists \( z_{-i} \in (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) \) such that \( (z_{-i}, x_i) \geq (z_{-i}, y_i) \) for all \( i \in I \).

Similarly, define \( x_i \sim y_i \) if there exists \( z_{J'} \cap J = \emptyset, J \cup J' = I \), such that \( (z_{J'}, x_i) \geq (z_{J'}, y_i) \) for all \( J \subset I \). Also, define \( x_i \sim y_i \) if and only if \( x_i \geq y_i \) and \( y_i \geq x_i \) for all \( i \in I \). Define \( x_i \sim y_i \) if and only if \( x_i \geq y_i \) and \( y_i \geq x_i \) for all \( J \subset I \).\(^1\)

**Theorem 2**

If axioms 1 to 3 hold, then preference relation \( \geq_i \) on \( X_i \) is weak ordering for all \( i \in I \).

**Corollary**

If axioms 1 to 3 hold, then \( \geq_J \) is weak ordering for all \( J \subset I \).

**Theorem 3**

If axioms 1 to 3 hold, and if \( x \geq y \), then \( y_i \succ x_i, J = I - (i) \) implies \( x_i \succ y_i \).

**Theorem 4**

If axioms 1 and 3 hold, then \( x_i \geq y_i \) for all \( i \in I \) implies \( x_i \geq y_i \) for all \( J \subset I \).

But \( x \geq y \) and \( x_i \geq y_i \) does not imply \( x_{J'} \geq y_{J'} \), \( J \cap J' = \emptyset, J \cup J' = I \).

For example, in the case of \( n=2 \), \( (x_1, x_2) \geq (y_1, y_2) \) does not necessarily imply \( x_i \geq y_i, i = 1, 2 \). In fact, it is possible that \( (x_1, x_2) \geq (y_1, y_2), x_1 \geq y_1 \) and \( y_2 \geq x_2 \).

---

1) In this definition, we can replace the words "there exists" with "for all \( z_{i-1} \) or \( z_i \)" because of Theorem 1 and its Corollary.
Next, we will give a definition of "lattice."

**Definition**

Fix a point in $X$ and denote it with $x_i^0=(x_1^0, \ldots, x_n^0)$. And change the amount of one component of $x_i^0$ to get $(x_i^1, x_{i-1}^0)$. We call these two points "basis of lattice", the former point "origin" and the latter point "unit point," respectively. Next, find points which satisfy the following lattice construction rules.

**rule 1**: $x_i^{k}, \ldots, x_i^{k+1}, \ldots, x_n^{k}) \sim (x_1^{k}, \ldots, x_j^{k+1}, \ldots x_n^{k})$,

**rule 2**: $(x_i^{k}, \ldots x_n^{k}) \sim (x_i^{k}, \ldots x_i^{k+1}, \ldots x_j^{k-1}, \ldots x_n^{k})$

for all $i, j \in I$ and integer $k$.

Here, we make an important assumption, that is the assumption of hexagonal web, which is implicitly built in the above two rules. In the case of $n=2$, this says the indifference between the point $A$ and the point $B$ of the graph below.

![Graph](image)

Then, "lattice $A$ on $X$" is defined with all points founded on these rules.

$A = \{x_i^1, \ldots, x_i^m \mid i, j \in I \text{ is integer}\}$

An element of $A$ is designated "lattice point".\(^{1)}

---

1) There is a mathematical theorem that the necessary and sufficient condition for the existence of topological mapping which maps 3-web of curves into normal 3-web is that the former is the hexagonal web.
Theorem 5

If axioms 1 to 3 hold, then there uniquely exists a lattice on $X$ generated by (or, whose basis are) the two points $x_i^0$ and $(x_i^1, x^0_{i-e(i)})$.

In the following, $A[x_i^0, x_i^1]$ stands for the lattice with origin $x_i^0 = (x_i^0, \cdots, x_n^0)$ and unit point $(x_i^1, x_i^0, \cdots, x_n^0)$. When the origin and unit point is unquestionable in the context, we adopt simply $A$ as a symbol of the lattice, without mentioning to its basis.

Theorem 6

If axioms 1 to 3 hold, and $A$ is a lattice on $X$, and $i_k, j_k (k \in I)$ are integers, then

$$(x_i^{i_1}, \cdots, x_n^{i_n}) \sim (x_i^{j_1}, \cdots, x_n^{j_n})$$

whenever

$$\sum_{k=1}^n i_k = \sum_{k=1}^n j_k.$$

Theorem 7

If axioms 1 to 3 hold, $A_1$ and $A_2$ are lattices on $X$, i.e., $A_1[x_i^0, x_i^1]$, $A_2[x_i^0, y_i^1]$, and if

$$(x_i^{i_1}, \cdots, x_n^{i_n}) \succeq (y_i^{j_1}, \cdots, y_n^{j_n})$$

then $$(x_i^{i_1+k_1}, \cdots, x_n^{i_n+k_n}) \succeq (y_i^{j_1+l_1}, \cdots, y_n^{j_n+l_n})$$

for all integers $k_m, l_m (m = 1, \cdots, n)$; $\sum_{m=1}^n k_m = \sum_{m=1}^n l_m$.

Theorem 8

Suppose axioms 1 to 3 hold, and $A_1[x_i^0, x_i^1]$ and $A_2[x_i^0, x_i^a]$ are lattices on $X$, where $a$ is an integer other than zero. Then, lattice $A_2$ is embedded into $A_1$ in the sense that for all $((x_i^a)^{i_1}, \cdots, (x_n^a)^{i_n}) \in A_2$, there exists $(x_i^{a^2i_1}, \cdots, x_n^{a^2i_n}) \in A_1$ such that

$$(x_i^{0^2})^i_k = x_i^{0^a}$$

for all $k \in I$.

Theorem 9

Suppose axioms 1 to 3 hold, and $A_1[x_i^0, x_i^1]$, $A_2[x_i^0, y_i^1]$ are lattices on $X$ with the same origin and different unit points. Then, $x_i^1 \cdot y_i^1$ implies $x_i^{k} \cdot y_i^{k}$ for all positive integers $k$ and $i \in I$.  

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Theorem 10
Suppose axioms 1 to 3 hold. Choose two points $x^1, y^1$ in $X$ arbitrarily; $x^1 = (x_1^1, \ldots, x_n^1)$, $y^1 = (y_1^1, \ldots, y_n^1)$. And consider lattices $\mathcal{A}_i(\lambda = 1, 2, \ldots, 2n)$ on $X$ with same origin and different (possibly indifferent) unit point.

$$ \mathcal{A}_i = \begin{cases} \mathcal{A}_i[x^0, x^1] & \lambda = i \in I, \\ \mathcal{A}_i[x^0, y^1] & \lambda = n + j, j \in I. \end{cases} $$

If $x^1 \geq y^1$, then $x^k \geq y^k$ for all positive integer $k$,

where $x^k = (x_1^k, x_2^k, \ldots, x_n^k)$ $x_i^k \in \mathcal{A}_i$,

$y^k = (y_1^k, y_2^k, \ldots, y_n^k)$ $y_i^k \in \mathcal{A}_{n+i}, i, j \in I$

Theorem 11
Suppose axiom 1 to 3 hold, $\mathcal{A}_1[x^0, x^1], \mathcal{A}_2[y^0, y^1]$ be lattices on $X$ with the same origin and different unit point.

If $x_i^j \geq y_i^k$ and $y_i^l \geq x_i^m$, where $j, k, l, m$ are integers and $k \geq 0, 1 \geq 0$, then

$$ j/k \geq m/1. $$

Theorem 12
Suppose axioms 1 to 3 hold, $\mathcal{A}[x^0, x^1]$ be a lattice, and $w \cdot z$.

If these exist integers $a_i, b_j, i, j \in I$ such that

$$ x_i^{a_i+1} \geq w_i \geq x_i^{a_i}, $$

$$ x_i^{b_i+1} \geq z_i \geq x_i^{b_i}, $$

then

$$ \sum_{i=1}^{n} a_i + n > \sum_{i=1}^{n} b_i. $$

The conclusion of Theorem 12; $\sum_{i=1}^{n} a_i + n > \sum_{i=1}^{n} b_i$ will be actually used for the construction of the sub-utility function.

For this purpose, however, it is necessary that, as well as axioms 1 to 3, there exist such integers $a_i, b_j, i, j \in I$ as required in Theorem
12. Therefore, we set axiom 4 so as to assure the existence of such integers.

**Axiom 4. (Archimedean Axiom)**

If \( d[x^0, x^1] \) is a lattice, and \( x=(x^k, \ldots, x^n) \in d \) is given arbitrarily, then there exist integers \( a \) and \( b \) such that

\[
x^a \geq x \geq x^b
\]

where \( x=(x_1^a, \ldots, x_n^a) \) and \( x^b=(x_1^b, \ldots, x_n^b) \).

**Theorem 13**

If axioms 1 to 4 hold, \( d[x^0, x^1] \) is a lattice, and \( y \) is arbitrarily given, then there exist such integers \( a_i, b_i, i, j \in I \) as

\[
x_i^{a_i} \geq y_i \geq x_i^{b_i}, \text{ for all } i \in I.
\]

**Theorem 14. (Existence theorem of sub-utility function)**

If axioms 1 to 4 hold, and there is a lattice \( d[x^0, x^1] \), then there exists a real-valued function \( u_i \) defined on \( X_i \), and

for all \( x_i^k, x_i^1 \in x_i \); \( x_i^1 \geq x_i^1 \) implies

\[
u_i(x_i^k) \geq u_i(x_i^1).
\]

(Proof) (i) Proof of existence; Without loss of generality, we assume \( x_i^1 > \cdot x_i^0 \). Suppose we are given \( x_i^p \in x_i \). Then, we can construct a lattice \( d'[x^0, x^1] \), which necessarily exists by Theorem 5. And there also exists \( x_i^{m} \in d'(m>0) \). Axiom 4 assures that there exist \( x_i^a \), \( x_i^b \in X_i \) such that \( x_i^a \geq x_i^m \geq x_i^b \), where \( x_i^a, x_i^b \in d \). Then \( a/m \geq b/m \), by Theorem 11. This inequality divide the rational number into two sets. That is, the set \( A_1=\{a/m \mid x_i^a \geq x_i^m, m>0\} \) and the set \( A_2=\{b/m \mid x_i^m \geq x_i^b, m>0\} \). For the pair \((A_1, A_2)\), we have known that (1) for \( s \in A_1, t \in A_2 \); \( s \geq t \), (2) \( A_1 \neq \emptyset, A_2 \neq \emptyset \), and (3) \( A_1 \cup A_2 = Q \); \( Q \) is the set of all rational numbers. Here, if we could replace the one of the \( \geq \)'s in \( x_i^a \geq x \geq x_i^b \) of Axiom 4 with \( > \), then (1) changes into \( s > t \). Thus, \((A_1, A_2)\) is nothing but the Dedekind Cut. Therefore, by
the connectedness of real numbers, there exists either the minimum of \( A_1 \) or maximum of \( A_2 \). In other words, we can find a single real number which simultaneously the supremum (least upperbound) of the lower set \( A_2 \) and the infimum (greatest lower bound) of the upper set \( A_1 \). Now, define the sub-utility function of commodity \( i \) as follows:

**Definition**

For all \( i \in I \), \( u_i \) is a sub-utility function if

\[
u_i(x_i^p) = \inf \{ a/m | x_i^a \geq x_i^{pm}, \ m > 0 \} = \sup \{ b/m | x_i^{pm} \geq x_i^b, \ m > 0 \}.
\]

Thus, there necessarily exists \( u_i \) which is a real valued function defined on \( X_i \), for all \( i \in I \).

(ii) Proof of monotonicity: By axiom 4, there exist \( x_i^r, x_i^{r+1}, x_i^s, x_i^{s+1} \in X_i \) such that

\[
x_i^{r+1} \geq x_i^k \geq x_i^r, \quad x_i^{s+1} \geq x_i^l \geq x_i^s.
\]

Then, \( x_i^{(r+1)} \geq x_i^k \geq x_i^{r+1} \geq x_i^l \geq x_i^s \).

By transitivity of \( \geq_i \),

\[
x_i^{(r+1)} \geq x_i^k.
\]

Whereas, by the definition of \( u_i \),

\[
u_i(x_i^k) \geq r + 1 \geq (sk) / l \geq s \geq u_i(x_i^l),
\]

for \( x_i^k \geq x_i^l \) implies \( k \geq l \).

This completes the proof of monotonicity. (q.e.d.)

Our main theorem is as follows.

**Theorem 15.** (Existencetheorem of additive utility function)

Suppose axioms 1 to 4 hold, and \( \mathcal{A}[x_i^0, x_i^1] \) be a lattice. There exist real-valued functions \( u_i \) for all \( i \in I \), and \( U \), Which are defined on \( X_i \) for all \( i \in I \), and on \( X \), respectively, such that

\[
U(x) = \sum_{i=1}^{n} u_i(x_i) = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n),
\]
\[ \begin{aligned} &x = (x_1, x_2, \ldots, x_n), \\
&U(x) \geq U(y) \text{ if and only if } x \succeq y, \\
&u_i(x_i) \geq u_i(y_i) \text{ if and only if } x_i \succeq y_i \text{ for all } i \in I. \end{aligned} \]

(Proof) (i) Proof of existence; Without loss of generality, we may assume \( x, y \succ x_i. \) Define \( u_i \) with the sub-utility function which appeared in the proof of Theorem 14. But, when \( x_i \succ x_i \), find \( x_i \) as a solution of \((x_i, x_i^0) \sim (x_i^0, x_i, x_i \sim (i))\) and find \( x_i \) as a solution of \((x_i, x_i, x_i \sim (i)) \sim (x_i^0)\). Since \( x_i \succ x_i \), we may let \( u_i(x_i) = -u_i(x_i) \), for all such \( i \in I \) as \( x_i \succeq x_i \). Let \( U(x) = \sum_{i=1}^{n} u_i(x_i) \). Then, the existences of functions \( u_i, i \in I \) and \( U \) are direct consequences of Theorem 14.

(ii) Proof of the "if" part of the monotonicity of \( U \); Suppose \( x \succeq y \).
There exist \( 2n \) lattices with the same origin and different unit points, which we have considered in Theorem 10. The conclusion of Theorem 10 states that \( x^m \succeq y^m \) for \( m > 0 \). By axiom 4, for any \( m > 0 \), there exist integers \( s_i \) and \( t_i \) such that all \( i \in I \),
\[
\begin{aligned} &x_i^t, t_i \geq x_i^m, y_i^m \geq x_i^m, \\
&\text{where } x_i^m \in \mathcal{J}_i, y_i^m \in \mathcal{J}_{i+1}, x_i^t, x_i^t, x_i^t, x_i^t, x_i^t \in \mathcal{J}. \end{aligned}
\]
Therefore, by Theorem 12,
\[
\sum_{i=1}^{n} (s_i + 1) > \sum_{i=1}^{n} t_i.
\]
But, from the definition of \( u_i \),
\[
U(x) = \sum_{i=1}^{n} u_i(x_i) \geq (\sum_{i=1}^{n} s_i)/m = (\sum_{i=1}^{n} (s_i + 1))/m - n/m
\]
\[
> \sum_{i=1}^{n} t_i - n/m \geq \sum_{i=1}^{n} u_i(y_i) - n/m
\]
\[
= U(y) - n/m, \quad \text{for all } i \in I.
\]
Since \( m \) is arbitrary, let \( m \) increase to \( \infty \). Then, we have
\[
U(x) = \sum_{i=1}^{n} u_i(x_i) \geq \sum_{i=1}^{n} u_i(y_i) = U(y)
\]
Thus, we have proved that \( x \cdot y \) implies \( U(x) \geq U(y) \).

(iii) Proof of the "if" part of the monotonicity of \( u_i \); In Theorem
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14, it has already been shown that \( x_i \geq y_i \) implies \( u_i(x_i) \geq u_i(y_i) \).

(iv) Proof of the "only if" part of the monotonicity of \( u_i \); If \( u_i(x_i) > u_i(y_i) \), then \( x_i > y_i \) for any \( i \in I \). For, suppose \( y_i > x_i \). Then, by the proof of Theorem 14, we have \( u_i(y_i) \geq u_i(x_i) \), which contradicts our assumption. Therefore, it suffices to show that \( u_i(x_i) = u_i(y_i) \) implies \( x_i \sim y_i \) for all \( i \in I \). Suppose that \( u_i(x_i) = u_i(y_i) \) implies \( x_i \neq y_i \), that is, either \( x_i > y_i \) or \( y_i > x_i \). With no loss of generality, we may assume \( y_i' > x_i' \). Since we are given at least two definite points, we can construct a new lattice \( \mathcal{A}' \) \([(x_i', x^{0}_{i-l(i)}) , (y_i)]] \) by Theorem 5. By the lattice construction rule 1,

\[
((y_i')^1, x^{0}_{i-l(i)}) \sim (x_i', x^1, x^{0}_{l-(i,i)})
\]

Whereas, by the assumption,

\[ u_i(x_i') = u_i(y_i'). \]

And since the part (ii) of these proof says

\[
U((y_i')^1, x^{0}_{i-l(i)}) = u_i((y_i')^1 + \sum_{j \neq i} u_j(x_j^0)) = u_i(x_i') + u_j(x_j^0) + \sum_{k \neq i, j} u_k(x_k^0) = U(x_i', x^1, x^{0}_{l-(i,i)}),
\]

we have

\[ \sum u_j(x_j^0) = u_j(x_j^0) + \sum_{k \neq i, j} u_k(x_k^0), \]

that is,

\[ u_j(x_j^0) = u_j(x_j^1) \text{ for all } j \in I, \ j \neq i. \]

Similarly, by the lattice construction rule 2,

\[
((y_i')^1, x^1, x^{0}_{l-(i,i)}) \sim ((y_i')^2, x^{0}_{l-(i,i)})
\]

\[
\sim (x_i', x^2, x^{0}_{l-(i,i)}).
\]

Then, \( U((y_i')^1, x^1, x^{0}_{l-(i,i)}) = u_i((y_i')^1 + u_j(x_j^1) + \sum_{k \neq i, j} u_k(x_k^0)) = U((y_i')^2, x^{0}_{l-(i,i)}) \]

\[
\sim u_i(x_i') + u_j(x_j^2) + \sum_{k \neq i, j} u_k(x_k^0) = U(x_i', x^2, x^{0}_{l-(i,i)}).\]
Thus, we have
\[ u_i(x_i') = u_i((y_i')^t) = u_i((y_i')^s) \] for all \( i \in I \), and
\[ u_j(x_j^0) = u_j(x_j^t) = u_j(x_j^s) \] for all \( j \in I, j \neq i \).

Proceeding intuitively,
\[ u_i(x_i') = u_i((y_i')^m) \] for all integer \( m \), and \( i \in I \).

But, for any \( x_i' \in X_i \), axiom 4 asserts that there exist some integer \( h \) such that
\[ x_i^h \geq x_i' \geq x_i^{h-1}. \]

Then, by Theorem 14,
\[ u_i(x_i^h) \geq u_i(x_i^t) \geq u_i(x_i^{h-1}) \]
while, from the argument above,
\[ u_i(x_i^h) = u_i(x_i^{h-1}) \] for all \( i \in I \), and integer \( h \).

Hence,
\[ u_i(x_i^t) = u_i(x_i^0). \]

However, by the definition of \( u_i(x_i^t) = s \) where \( s \) is an arbitrary integer, we have a contradiction. Thus,
\[ u_i(x_i) = u_i(y_i) \] implies \( x_i \sim y_i \),
which completes our proof.

(v) Proof of the "only if" part of the monotonicity of \( U \); Next, we should prove that \( U(x) \geq U(y) \) implies \( x \geq y \). If \( U(x) > U(y) \), then \( x > y \). For, \( y \geq x \) implies \( U(y) \geq U(x) \) by the part (ii) of this proof, which contradicts our assumption.

Suppose \( U(x) = U(y) \), and let \( y_i'' \) such that
\[ x \sim (y_i'', y_i \,(\neg i)) \] for all \( i \in I \).

\( U(x) = U(y) \) implies
\[ u_i(x_i) + \sum_{j \neq i} u_j(x_j) = u_i(y_i) + \sum_{j \neq i} u_j(y_j), \]
and \( x \sim (y_i'', y_i \,(\neg i)) \) implies
\[ u_i(x_i) + \sum_{j \neq i} u_j(x_j) = u_i(y_i'') + \sum_{j \neq i} u_j(y_j). \]
Thus, 
\[ u_i(y_i'') = u_i(y_i), \]
whence, 
\[ y_i'' \sim y_i. \]
Therefore, 
\[ (y_i'', y_{l-\ell(i)}) \sim (y_i, y_{l-\ell(i)}) = y. \]
Thus, we have \( x \sim y \), for \( x \sim (y_i, y_{l-\ell(i)}) \). This completes our proof. 
(q. e. d.)

**Theorem 16. (Cardinality theorem)**

If axiom 1 to 4 hold and \( \mathcal{L} \) be a lattice, then for any function \( u \), which has obtained in Theorem 15
\[ u_i(x^n_i) - u_i(x^0_i) = n[u_i(x^1_i) - u_i(x_i)] \text{ for all } i \in I. \]

(_Proof_) When \( n=0 \) and 1, Theorem is trivial.

Suppose that the statement is true for \( n \). By Theorem 6, 
\[ (x_i^{n+1}, x_{l-\ell(i)}^0) \sim (x_i^n, x_i^1, x_{l-\ell(i)}^0), \] and 
\[ (x_i^1, x_{l-\ell(i)}^0) \sim (x_i^1, x_{l-\ell(i)}^0). \]
Thus by Theorem 15
\[ u_i(x_i^{n+1}) + \sum_{j \neq i} u_j(x_j^0) = u_i(x_i^n) + u_j(x_j^1) + \sum_{k \neq i, j} u_k(x_k^0), \] and 
\[ u_i(x_i^1) = u_j(x_j^1) \text{ for } u_i(x^0_i) = u_j(x^0_i). \]
Then, 
\[ u_i(x_i^{n+1}) + u_j(x_j^0) = u_i(x_i^n) + u_j(x_j^1) = u_i(x_i^n) + u_i(x_i^1). \]
Then, 
\[ u_i(x_i^{n+1}) - u_i(x_i^0) = u_i(x_i^n) + u_i(x_i^1) - u_j(x_j^0) - u_i(x_i^0) \]
\[ = n[u_i(x_i^n) - u_i(x_i^0)] + [u_i(x_i^1) - u_i(x_i^0)] \]
\[ = (n+1)[u_i(x_i^n) - u_i(x_i^0)]. \] (q. e. d.)

**Theorem 17. (Uniqueness)**

Suppose axioms 1 to 4 hold, and there is a lattice. For two sets of functions, \( U, \ u_i (i \in I) \) and \( V, \ v_i (i \in I) \), satisfying the conditions of Theorem 15, we have 
\[ V(x) = \alpha U(x) + \gamma, \]
\[ v_i(x_i) = \alpha u_i(x_i) + \beta_i, \]

1) This is a direct corollary of Theorem 16.
Proof) Without loss of generality, we may assume that $v_i$ is a function defined in the proof of Theorem 14, and that $v_i(x_i^0) = 0$ for all $i \in I$.

It suffices to show that $u_i(x_i') - u_i(x_i^0)/v_i(x_i)$ is constant for any $x_i'$ in $x_i^{13}$.

Suppose that there exists $x_i''$ in $X_i$ such that

$$\frac{u_i(x_i') - u_i(x_i^0)}{v_i(x_i')} = \frac{u_i(x_i'') - u_i(x_i^0)}{v_i(x_i'')}$$

Furthermore, we can assume $x_i', x_i'' > x_i^0$. This assumption does not restrict our proof, for if $x_i^0 > x_i'$ then there exist $x_i' \in x_i$ and $x_i'' \in X_i$ such that $x_i'' > x_i^0$ and

$$(x_i', x_i^{0_{\leftarrow i(i)})} \sim (x_i^0, x_i', x_i^{0_{\rightarrow i(i)})}),$$

$$(x_i^0) \sim (x_i^-, x_i', x_i^{0_{\leftarrow i(i)})}).$$

By the property of $U$, $V$, $u_i$, and $v_i$ $(i \in I)$,

1. $u_i(x_i') + \sum_{j \neq i} u_j(x_i^0) = u_i(x_i^0) + u_j(x_i') + \sum_{k \neq i, j} u_k(x_i^0)$,
2. $\sum_{i=1}^n u_i(x_i^0) = u_i(x_i^-') + u_j(x_i') + \sum_{k \neq i, j} u_k(x_i^0)$,
3. $v_i(x_i') + \sum_{j \neq i} v_j(x_i^0) = v_i(x_i^0) + v_j(x_i') + \sum_{k \neq i, j} v_k(x_i^0)$,
4. $\sum_{i=1}^n v_i(x_i^0) = v_i(x_i^0) + v_j(x_i') + \sum_{k \neq i, j} v_k(x_i^0)$.

1) See figure below. In order to prove that $u$ is an interval scale, it is sufficient to show that the ratio of two distances, one is measured with $u_i$ and another is measured with $v_i$, is constant. That is $[u_i(x_i') - u_i(x_i^0)]/[v_i(x_i') - v_i(x_i^0)]$ is independent from the selection of $x_i'$. The value of this ratio stands for that of the intervals of unit, for example, imagine cm, km, mile and so on.

\[\begin{array}{ccc}
\text{scale } u_i & u_i(x_i^0) & u_i(x_i') \\
\text{scale } v_i & v_i(x_i^0) & v_i(x_i') \\
\end{array}\]
By the assumption on V and \( v_i \)

(5) \( v_i(x_i') = v_j(x_j') \),

(6) \( v_i(x_i^\sim) + v_j(x_j) = 0 \).

(2) - (1);

(7) \( u_i(x_i') - u_i(x_i^0) = u_i(x_i^0) - u_i(x_i^\sim) = -[u_i(x_i^\sim) - u_i(x_i^0)] \).

From (5) and (6),

(8) \( v_i(x_i) = -v_i(x_i^\sim) \).

(7) ÷ (8);

\[
\frac{u_i(x_i') - u_i(x_i^0)}{v_i(x_i') } = \frac{u_i(x_i^\sim) - u_i(x_i^0)}{v_i(x_i^\sim)}.
\]

Thus, if we replace \( x_i' \) with \( x_i^\sim \), then \( x_i^\sim > x_i^0 \). Similarly there is no loss if we assume \( x_i'' > x_i^0 \).

Thus we get from (4), for some \( n > 0 \)

\[
\frac{v_i(x_i'')}{v_i(x_i')} - \frac{u_i(x_i'') - u_i(x_i^0)}{u_i(x_i') - u_i(x_i^0)} > \frac{1}{n}.
\]

By axiom 4, there exists \( m > 0 \) such that

\( (x_i')^m > (x_i'')^m \geq (x_i')^{m-1} \).

Then by Theorem 16,

\[
m v_i(x_i') = m[v_i(x_i') - v_i(x_i^0)]
\]

\[
= v_i((x_i')^m)
\]

\[
> v_i((x''')^m)
\]

\[
= v_i((x''')^m) - v_i(x_i^0)
\]

\[
= n[v_i(x_i'') - v_i(x_i^0)]
\]

\[
= n v_i(x_i''),
\]

Therefore

\[
\frac{v_i(x_i'')}{v_i(x_i')} < \frac{m}{n}.
\]

Similarly,

\[
n[u_i(x_i'') - u_i(x_i^0)] = u_i((x_i'')^m) - u_i(x_i^0)
\]
Therefore,
\[
\frac{u_i(x_i') - u_i(x_i^0)}{m - 1} \geq \frac{m - 1}{n}.
\]
Thus,
\[
\frac{1}{n} < \frac{u_i(x_i'') - u_i(x_i^0)}{v_i(x_i')} < \frac{m - m - 1}{n} = \frac{1}{n},
\]
which contradicts to (4). That is, there never exists such \(x_i'' \in X_i\).
Therefore \(u_i\) is linearly related to \(v_i\) for all \(i \in I\).

Furthermore,
\[
V(x_i^0) = \sum_{i=1}^{n} v_i(x_i^0) = 0,
\]
\[
\alpha u_i(x_i) + \beta_i = v(x_i) = V(x_i, x_i^0) = \alpha U(x_i, x_i^0) + \gamma
\]
\[
= \alpha u_i(x_i) + \alpha \sum_{j \neq i} u_j(x_j^0) + \gamma.
\]
Then, \(\alpha_i = \alpha\) for all \(i \in I\), and
\[
\sum_{j=1}^{n} \beta_i = \gamma. \quad (q.e.d)
\]

References