A MODEL OF ECONOMIC AND DEMOGRAPHIC DEVELOPMENT

By Ryoshin Minami

Assistant, Institute of Economic Research

I. *Introduction*

(1) In some models of economic growth, the rate of population increase is treated as one of the determinants of growth. For instance the natural rate of growth 'G_n' in Harrod's model is composed of rates of population growth and technical progress. The rate of population growth is, however, regarded as an exogenous variable. It is generally agreed that there are interrelations between population increase and economic development. Therefore the assumption that population increase is independent of the economic variables seems to be an inappropriate approach to the theory of economic and demographic development.

(2) The purpose of this paper is to present a model in which population is treated as an endogenous variable, that is to say, to develop Malthus' population theory with the tools of the theories of economic growth. T. R. Malthus predicted that, in the longrun, the economy would converge to the so-called 'Malthusian equilibrium', which was considered to be stable in his theory. But economies developed and population increased rapidly during the course of Industrial Revolution; history showed that his equilibrium was not necessarily stable.

Our formula of Malthus' population theory is general in the sense that it can explain two cases; in the first, Malthusian equilibrium is stable, the second, unstable. In other words, we do not assume stability in the theory, but try to clarify the conditions under which the equilibrium is stable or unstable.[1] This analysis, in the writer's opinion, may throw some light on the economic and demographic problems of underdeveloped countries.

II. *Assumptions and Notation*

(1) To begin the analysis certain assumptions are set forth.
  1. The economy is closed, and consequently the rate of population growth is equal to the rate of natural increase.
  2. The economy is composed of one sector producing only one kind of good.
  3. Goods are produced with labor and capital.[2] Labor and capital are each homogeneous and fully employed at equilibrium.
  4. Constant returns to scale (linearity and homogeneity of production function) and decreasing returns to labor prevail. The latter is a revised version of the historical tendency of decreasing returns assumed in Malthus' population theory. Output/input

---

[1] The previous article [4] by the author assumed stability of this equilibrium.
[2] Land may be included in capital.

elasticity is constant.

5.   There is no technical progress.

6.   Labor force is distinguished from population.   The ratio of labor force to total population, which seems to be implicitly assumed by Malthus to be constant, has an important meaning especially in the analysis of backward economies.

7.   Following Malthus, the rate of population growth is assumed to be an increasing function of per capita income.   It is zero when per capita income is at the subsistence level and approaches an upper limit as income indefinitely increases.

8.   Saving is always equal to investment or capital accumulation.

9.   Saving per head is a linear function of per capita income with an intercept below than zero.

(2)   Notation used in our model is the following:

O: Total product=national income

P: Total population

L: Total labor force

K: Capital stock

α: Output/capital elasticity $\left(\dfrac{\Delta O}{O} \Big/ \dfrac{\Delta K}{K}, \text{ constant, } 0<\alpha<1\right)$

A: Level of technique of production (constant, $A>0$)

C: Consumption

S: Saving

s: Marginal propensity to save (constant, $0<s<1$)

a: Ratio of labor force to population (constant, $0<a<1$)

$$(1)\quad a=\frac{L}{P}$$

k: Capital intensity

$$(2)\quad k=\frac{K}{L}$$

l: Productivity of labor

$$(3)\quad l=\frac{O}{L}$$

m: Per capita income

$$(4)\quad m=\frac{O}{P}$$

Hence

$$(5)\quad m=al$$

m_1:   Subsistence level, or per capita income when the rate of population growth is zero (constant, $m_1>0$)

m_2:   Per capita income when saving is zero (constant, $m_2>0$)

$G(P)$:   Rate of population growth $\left(\dfrac{\dot{P}}{P}\right)$

$G(L)$:   Rate of labor force increase $\left(\dfrac{\dot{L}}{L}\right)$

B: Upper limit of the rate of population growth (constant, $B>0$)[3]

[3]   This concept will be discussed later.

Because 'a' is constant in the relation (1),

(6) $G(L)=G(P)$

$G(K)$: Rate of capital accumulation $\left(\dfrac{\dot{K}}{K}\right)$

$G(k)$: Rate of increase of capital intensity $\left(\dfrac{\dot{k}}{k}\right)$

(7) $G(k)=G(K)-G(L)$

Substituting (6) into (7), it becomes

(8) $G(k)=G(K)-G(P)$

III. *A Model*

(1) On these assumptions, we will develop a simple model of economic and demographic development. The production function takes the form:

$$O=AL^{1-\alpha}K^{\alpha}$$

Or

(9) $\dfrac{O}{L}=A\left(\dfrac{K}{L}\right)^{\alpha}$

This may be rewritten by substitution of (2) and (3).

(10) $l=Ak^{\alpha}$

Therefore

(11) $m=aAk^{\alpha}$

'k_1' and 'k_2' are defined by the following relations:

(12) $m_1=aAk_1^{\alpha}$

(13) $m_2=aAk_2^{\alpha}$

Or

(14) $k_1=\left(\dfrac{m_1}{aA}\right)^{\frac{1}{\alpha}}$

(15) $k_2=\left(\dfrac{m_2}{aA}\right)^{\frac{1}{\alpha}}$

'k_1' is the level of capital intensity in which per capita income is at the subsistance level, and 'k_2' is the level with zero saving.[4]

(2) A function of population growth may be formulated into the equation

(16) $G(P)=B(m-m_1)\dfrac{1}{m}$

Figure 1 depicts this function. It may be easy to find the assumption of the increasing rate of population growth in Malthus' theory. However, we have another assumption here; there is an upper limit in the rate of population growth. This concept is not from Malthus, but seems to be realistic.[5] The level 'B' may be dependent on many demographic and social

---

[4] Because $0<\alpha<1$, $k_2 \gtrless k_1$ when $m_2 \gtrless m_1$.

[5] H. Leibenstein supposes at some per capita income level the rate of population growth is at a maximum and beyond that point the rate declines ([2] p. 170). This declining process may be explained in our model by a rising 'subsistence' level m_1 (see footnote 9). It is, in my opinion, an economic interpretation of the 'demographic revolution' [5].

factors; fecundity, religeon, family system, etc., and consequently supposed to be rather stable in a particular society.

FIG. 1
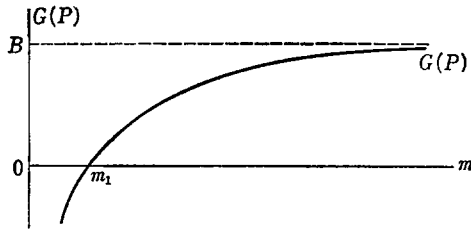
Substituting (11) (12) into (16), the latter becomes

(17) $G(P)=B(k^{\alpha}-k_1{}^{\alpha})\cdot\dfrac{1}{k^{\alpha}}$

Now the rate of population growth is expressed as a function of capital intensity.[6]
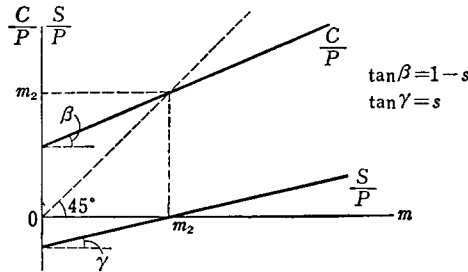
(18) $G(P)=\eta(k)$

Hence $G(P)$ is an increasing function of k, and converges to B when k becomes indefinitely large. It is zero when $k=k_1$.

(3)   The saving function may be reformulated into the equation:

(19) $\dfrac{S}{P}=s(m-m_2)$

This function is demonstrated in Figure 2.

FIG. 2

Then the rate of capital accumulation $G(K)$ is

$$G(K)=\frac{S}{K}=s\frac{O-m_2P}{K}$$

Or

(20) $G(K)=\dfrac{s}{a}(m-m_2)\dfrac{1}{k}$

---

[6]   Characteristics of this function are as follows:

$\eta(k_1)=0 \qquad \eta(\infty)=B \qquad \eta(0)=-\infty$

$\eta'(k)=\alpha Bk_1{}^{\alpha}\dfrac{1}{k^{\alpha+1}}>0 \qquad \eta''(k)=-\alpha(\alpha+1)Bk_1{}^{\alpha}\dfrac{1}{k^{\alpha+2}}<0$
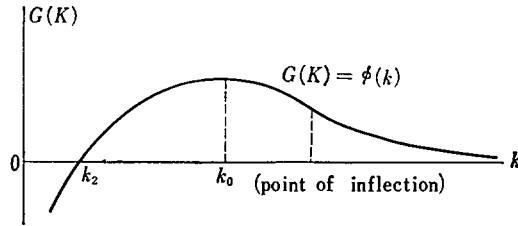
Substituting (11) and (13) into (20), the latter becomes

(21) $G(K)=sA(k^\alpha-k_2{}^\alpha)\dfrac{1}{k}$

Hence it may be rewritten as below:[7]

(22) $G(K)=\phi(k)$

FIG. 3

As demonstrated in Figure 3, the rate of capital accumulation increases and reaches a maximum at the point 'k_0' and beyond that point it begins to decrease and tends toward zero.
(4)   Now we have two fundamental equations;

(17) $G(P)=B(k^\alpha-k_1{}^\alpha)\dfrac{1}{k^\alpha}$

---

[7]   This function has the following characteristics:

$\phi(k_2)=0 \qquad \phi(\infty)=0 \qquad \phi(0)=-\infty$

$\phi'(k)=sA[(\alpha-1)k^\alpha+k_2{}^\alpha]\dfrac{1}{k^2} \qquad \phi''(k)=sA[(\alpha-1)(\alpha-2)k^\alpha-2k_2{}^\alpha]\dfrac{1}{k^3}$

Defining

$k_0=\left(\dfrac{1}{1-\alpha}\right)^{\frac{1}{\alpha}}k_2$

Then $\phi'(k_0)=0 \qquad \phi''(k_0)=sA(-\alpha k_2{}^\alpha)\dfrac{1}{k_0{}^3}<0$

Consequently $\phi(k_0)=$ max.
From (11) (15) and the definition of k_0, we get
$\quad m_2=(1-\alpha)m \qquad$ when $\qquad k=k_0$
On the other hand
$\quad \dfrac{\partial O}{\partial L}=(1-\alpha)l=\dfrac{1-\alpha}{a}m$

Therefore $m_2=\dfrac{\partial O}{\partial L} \qquad$ when $\qquad k=k_0$

If we suppose, for simplicity, that population is equal to labor ($a=1$), we can say marginal productivity of labor is equal to the level of per capita income which results in zero saving, when the rate of capital accumulation is a maximum.
   In addition

$\phi''(k)=0 \qquad$ when $\quad k=\left[\dfrac{2}{(\alpha-1)(\alpha-2)}\right]^{\frac{1}{\alpha}}k_2 \quad$ or $\quad k=\left(\dfrac{2}{2-\alpha}\right)^{\frac{1}{\alpha}}k_0$

Then we know function ϕ is inflected at the point where $k=\left(\dfrac{2}{2-\alpha}\right)^{\frac{1}{\alpha}}k_0$.

Because $0<\alpha<1$,
   (point of inflection)$>k_0>k_2$

$$(21) \quad G(K) = sA(k^\alpha - k_2{}^\alpha)\frac{1}{k}$$

Or in a simple form

$$(18) \quad G(P) = \eta(k)$$

$$(22) \quad G(K) = \phi(k)$$

They show that two determinants of economic growth, the rates of population increase and of capital accumulation, depend on the level of parameter 'k'. Capital intensity k is, however, not a parameter given from outside the system, but an endogenous variable determined by the equation already given.

$$(8) \quad G(k) = G(K) - G(P)$$

Substituting (17) and (21) into (8), we obtain the most fundamental equation of our model.

$$(23) \quad G(k) = sA(k^\alpha - k_2{}^\alpha)\frac{1}{k} - B(k^\alpha - k_1{}^\alpha)\frac{1}{k^\alpha}$$

$$(24) \quad G(k) = \phi(k) - \eta(k)$$

Let us assume for a moment that population and capital cease to increase as the same level of per capita income (or of capital intensity) ($m_1 = m_2$ or $k_1 = k_2$); in other words, total income is consumed without saving at the level of per capita income (or capital intensity), in which population is stagnant. It seems to me that this assumption is very 'Malthusian' in the sence that it is the application of 'Malthus' proposition' in his population theory to saving behavior. Hence equation (23) takes a simpler form:

$$(25) \quad G(k) = sA(k^\alpha - k_1{}^\alpha)\left(k^{\alpha-1} - \frac{B}{sA}\right)\frac{1}{k^\alpha}$$

Denoting $\dfrac{B}{sA}$ as $k_3{}^{\alpha-1}$, the expression is rewritten as the relation:

$$(26) \quad G(k) = sA(k^\alpha - k_1{}^\alpha)(k^{\alpha-1} - k_3{}^{\alpha-1})\frac{1}{k^\alpha}$$

Here

$$(27) \quad k_3 = \left(\frac{B}{sA}\right)^{\frac{1}{\alpha-1}}$$

Solutions of this equation may be found by setting the rate of change of capital intensity at zero. They are k_1 and k_3. Hence

$$(28) \quad \phi(k_1) - \eta(k_1) = 0$$

$$(29) \quad \phi(k_3) - \eta(k_3) = 0$$

(5) Consider the meanings of solutions 'k_1' and 'k_3'. Rising capital intensity increases both population growth and capital accumulation. These two opposite effects, in the states of k_1 and k_3, just cancel each other and therefore capital intensity remains unchanged.

$$(30) \quad \eta(k_1) = \phi(k_1) = 0$$

The level of capital intensity k_1 where the growth rates of population and capital are zero, or population and capital are stagnant, is nothing but the 'Malthusian equilibrium'.[8] As far as the other solution k_3 is concerned, it may be useful to suppose three cases:

---

[8] If capital stock is given as $\bar{K}$, equilibrium labor $\bar{L}$ and population $\bar{P}$ can be determined by (1) and (2).

$$\bar{L} = \frac{\bar{K}}{k_1} \qquad \bar{P} = \frac{\bar{L}}{a}$$

$\bar{P}$ is the so-called 'maximum population' of A. Sauvy, or the level of population in Malthusian equilibrium.

$$(31) \quad \begin{cases} \text{Case 1} & k_1 > k_3 \\ \text{Case 2} & k_1 < k_3 \\ \text{Case 3} & k_1 = k_3 \end{cases}$$

Because η is a monotonic increasing function of k and $\eta(k_1) = 0$,

$$(32) \quad \begin{cases} \text{Case 1} & \eta(k_3) = \phi(k_3) < 0 \\ \text{Case 2} & \eta(k_3) = \phi(k_3) > 0 \\ \text{Case 3} & \eta(k_3) = \phi(k_3) = 0 \end{cases}$$

When equilibrium k_3 is less than Malthusian equilibrium k_1, population and capital decrease at a constant rate. The economy may disappear sooner or later. Should k_3 be larger than k_1, population and capital can attain steady growth; the economy is now in a development process. Then the equilibrium k_3 in Case 2 may be named as 'development equilibrium' Case 3 where $k_1 = k_3$ needs no further explanation.

IV. *On the Stability of Malthusian Equilibrium*

(1) We are now in a position to investigate the stability of these equilibrium. We can obtain the following relations from (26).

$$(33) \quad \begin{cases} \text{Case 1} & G(k) \geqq 0 & \text{when} & k_3 \leqq k < k_1 \\ & G(k) < 0 & \text{when} & k < k_3, \ k > k_1 \\ \text{Case 2} & G(k) \geqq 0 & \text{when} & k_1 \leqq k \leqq k_3 \\ & G(k) < 0 & \text{when} & k < k_1, \ k > k_3 \\ \text{Case 3} & G(k) < 0 & \text{when} & k \neq k_1, \ k_3 \\ & G(k) = 0 & \text{when} & k = k_1, \ k_3 \end{cases}$$

FIG. 4
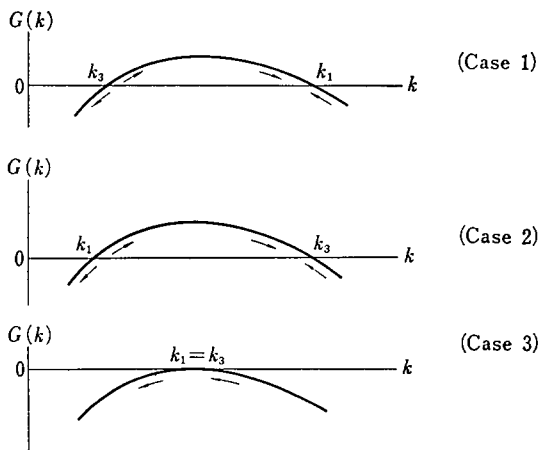
These relations are explained in Figure 4, which demonstrates the rate of growth of capital intensity as a function of k. At a glance we may easily know that equilibrium k_1 is stable and k_3 unstable in Case 1, k_1 unstable and k_3 stable in Case 2, and $k_1 = k_3$ unstable in Case 3.

$$(34) \quad \begin{cases} \text{Case 1} & k_1 > k_3 & k_1 = \text{stable}, \quad k_3 = \text{unstable} \\ \text{Case 2} & k_1 < k_3 & k_1 = \text{unstable}, \ k_3 = \text{stable} \\ \text{Case 3} & k_1 = k_3 & k_1 = k_3 = \text{unstable} \end{cases}$$

In short, Malthusian equilibrium is not always stable; stable in Case 1, unstable in other cases. On the other hand, development equilibrium is always stable. Relations (34) may be rewritten from equations (14) and (27).

$$(35) \quad \begin{cases} \text{Case 1} & 1 > \dfrac{a}{m_1} \left(\dfrac{s}{B} \right)^{\frac{\alpha}{1-\alpha}} A^{\frac{\alpha}{1-\alpha}} & k_1 = \text{stable}, \quad k_3 = \text{unstable} \\[2em] \text{Case 2} & 1 < \dfrac{a}{m_1} \left(\dfrac{s}{B} \right)^{\frac{\alpha}{1-\alpha}} A^{\frac{\alpha}{1-\alpha}} & k_1 = \text{unstable}, \ k_3 = \text{stable} \\[2em] \text{Case 3} & 1 = \dfrac{a}{m_1} \left(\dfrac{s}{B} \right)^{\frac{\alpha}{1-\alpha}} A^{\frac{\alpha}{1-\alpha}} & k_1 = k_3 = \text{unstable} \end{cases}$$

In other words the stability of Malthusian equilibrium depends on the level of parameters m_1 $(=m_2)$, a, s, α, A and B. The larger m_1 and B, or smaller a, s, α, A, because of $0 < \alpha < 1$,

the larger $\dfrac{a}{m_1} \left(\dfrac{s}{B} \right)^{\frac{\alpha}{1-\alpha}} A^{\frac{\alpha}{1-\alpha}}$, and therefore Malthusian equilibrium will be stable (Case 1 prevails).

Malthusian theory can be explained in our model as a special case, namely Case 1, where his equilibrium is stable. His persistence on this stability may suggest that in his days parameter B was large, a, s, α, A small.

(2) Our model above is illustrated in Figure 5, where capital intensity is shown on the horizontal axis, output and growth rate on the vertical axis (the directions are respectively upwards and downwards). We draw the per capita income function (11) in the first quadrant, and the functions of population growth and capital accumulation (18) (22) in the fourth quardrant, which are the same as the curves in Figures 2, 3. When per capita income m is at the subsistence level m_1 $(=m_2)$, capital intensity is k_1 $(=k_2)$, where the curves $G(P)$, $G(K)$ cross or are tangent to each other. When the product of the labor/population ratio and marginal productivity of labor $a\dfrac{\partial O}{\partial L}$ is equal to m_1, capital intensity is k_0 and the rate of capital accumulation is a maximum. Capital intensity moves in this figure in the direction as shown by sighns: In Case 1, capital intensity tends to converge to the stable 'Malthusian equilibrium' k_1. This may be the case that is implicitly suggested by Malthus' population theory. In Case 2, k converges to the stable 'development equilibrium' above defined, and Malthusian equilibrium loses its stability. In Case 3, there is only one equilibrium, so to speak, one-sided equilibrium.

(3) We have assumed $m_1 = m_2$ (or $k_1 = k_2$) in the above analysis. If $m_1 \neq m_2$ (or $k_1 \neq k_2$), the rate of population growth and that of capital accumulation cannot both be zero at the same time, or in short there will be no Malthusian equilibrium. Accordingly it may be safely deduced that Malthus implicity assumed saving is zero when per capita income is at the subsistence level.

FIG. 5
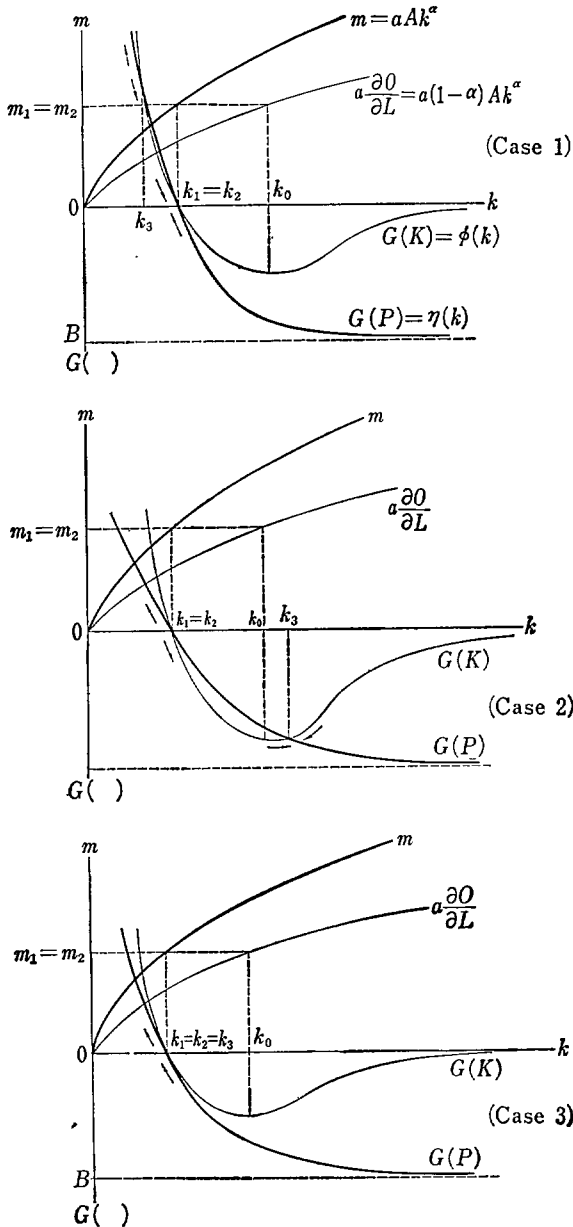
In conclusion we can state:

1. 'Malthusian equilibrium' exists when saving is zero at the subsistence level. This assumption seems to be implicit in Malthus' population theory.

2. Malthusian equilibrium is not necessarily stable. Its stability depends on the level of some parameters: the subsistence level, the ratio of population to labor, the marginal propensity to save, output/capital elasticity, the state of technical knowledge, and the maximum rate of population increase.

3. Malthus' theory may be explained in our model as a special case in which Malthusian equilibrium is stable.

4. In the case Malthusian equilibrium is unstable, there is another state of equilibrium, named as 'development equilibrium' that is stable and warrants steady growth of population and capital.

V. *Some Implications*

(1) Suppose the economy in Malthusian equilibrium. If Case 1 dominates as above explained, the economy converges to the equilibrium sooner or later. In other words, even if capital intensity is raised, for instance with importing some capital equipment from other economies, rising per capita income will be accompanied with the higher rate of population increase than that of capital accumulation. It follows that labor becomes surplus to capital, capital intensity and per capita income decrease, and consequently capital accumulation will be much more retarded. This process seems to correspond to the 'vicious circle of poverty' of R. Nurkse and the 'circular and cumulative causation' of G. Myrdal. Along this process, the economy comes back to Malthusian equilibrium. In short, it may be said that the economy in Case 1 has failed in 'economic take-off'.

Now, assume the economy changing its position from Case 1 to Case 2. With increasing capital intensity and per capita income, capital begins to grow at a higher rate than that of population growth. Labor shortage appears and capital intensity rises. The economy escapes from the Malthusian equilibrium and begins its steady growth; i.e. the economy just now succeeded in 'economic take-off'.

(2) 'Economic take-off' in our sense depends on some of the conditions of the stability or instability of Malthusian equilibrium, or the conditions that distinguish Case 1 from Case 2. They are easily seen from the relation (35).

  1. A decline in the 'subsistence' level[9]
  2. An increase in the marginal propensity to save
  3. Increasing output/capital elasticity
  4. Improvement in technical knowledge
  5. Increasing ratio of population to labor
  6. A declining maximum rate of population increase

(3) Here our discussion will be concerned with main demographic aspects of these conditions. Finding 5 means high ratio of labor to population favours 'economic take-off', because in that case higher per capita income can be enjoyed at the same level of productivity. This ratio depends on the age composition of the population and the degree of participation of labor.

Finding 6 means 'economic take-off' may be easier, if the maximum rate of population growth is low. The reason is that increasing per capita income brings less population growth in that case.

(4) Underdeveloped countries have high fertility, because of prevailing family system, religion and other social factors, and at the same time low mortality, owing to medical improvement. High fertility brings large potentiality of population growth 'B', and increases the ratio of children to total population decreases the ratio of labor to population 'a'.[10] If these economies

---

  [9] Malthus generally regarded the subsistence level as constant, but J. S. Mill supposed it to be rising historically.

  [10] J. J. Spengler says, if the age composition of underdeveloped countries were Westernized, per capita income might rise 20 to 30 or more per cent above current levels, other circumstances remaining unchanged ([6] p. 306)

keep their high fertility in the future, the potentiality of population increase remains high and ratio of labor to population low, which is unfavourable to their 'economic take-off'. In conclusion, as A. J. Coale and E. M. Hoover assist [1], the possibility of 'economic take-off' of underdeveloped countries depends mainly on the future trends of fertility.[11]

REFERENCES

[1] Coale, A. J. and Hoover, E. M., *Population Growth and Economic Development in Low-income Countries—A Case Study of India's Prospects*, Princeton, 1958.

[2] Leibenstein, H., *Economic Backwardness and Economic Growth*, New York and London, 1957.

[3] Minami Ryoshin, "Economic Potentiality and Related Demographic Factors in Underdeveloped Economies", (in Japanese), *Economic Studies Quarterly (Kikan Riron Keizai-Gaku)*, March 1961.

[4] —————, "An Analysis of Malthus' Population Theory", *Journal of Economic Behavior*, April 1961.

[5] —————, "Economic Analysis of Population Increase", (in Japanese), *Economic Studies Quarterly*, Sep. 1961.

[6] Spengler, J. J., "The Population Obstacle to Economic Betterment", *American Economic Review*, May 1951. Reprinted in, *Population Theory and Policy—Selected Readings* (ed. by J. J. Spengler and O. D. Duncan), Glencoe, Illinois, 1956.

---

[11] They suppose that population growth depends on the tendency of fertility, as mortality seems to remain unchanged. Three alternatives of fertility trends are assumed;

A. High fertility: Fertility assumed unchanged between 1951 and 1986.

B. Medium fertility: Fertility assumed to decline linearly by a total of 50% between 1956 and 1981.

C. Low fertility: Fertility assumed to decline linearly by a total of 50% between 1966 and 1981.

The results of estimation are as follows ([1] pp. 33-38):

Year	Crude Birth Rate %			Crude Death Rate %			Natural Rate of Increase %			Ratio of Productive Age Popu. to Total Popu. %		
	A	B	C	A	B	C	A	B	C	A	B	C
1951	4.32	4.32	4.32	3.10	3.10	3.10	1.22	1.22	1.22	58.4	58.4	58.4
1986	4.00	2.40	2.34	1.43	1.39	1.43	2.57	1.01	0.91	53.8	64.8	65.5

This projection shows that the rate of population growth will increase and the ratio of productive age population will decline, if fertility remains constant.

The author fully discussed the relations between economic development and demographic factors in the underdeveloped countries [3].