FORMAL POWER SERIES AND ADDITIVE NUMBER THEORY

By SETSUO ONARI*

I. *Introduction*

In this paper we shall consider some applications of formal power series to number theory. But as we shall use only elementary methods, results which we shall get in this paper are not deep ones in number theory.

At first let us collect a few results on formal power series without proof. Let I be an integral domain (we shall use only the formal power series over rational integral ring). A formal power series over I is an expression

$$a_0+a_1x+a_2x^2+a_3x^3+\cdots, \qquad a_n\in I$$

where the symbol x is an indeterminate symbol. Consequently, all questions of convergence are irrelevant. Let $I\{x\}$ be the set of all formal power series on I. $I\{x\}$ has a structure of commutative ring by defining addition and multiplication in the following way; if

$$A=\sum_{n=0}^{\infty} a_nx^n, \qquad B=\sum_{n=0}^{\infty} b_nx^n,$$

we define

$$A+B=C \qquad \text{where} \qquad C=\sum_{n=0}^{\infty} c_nx^n$$

$$AB=D \qquad \text{where} \qquad D=\sum_{n=0}^{\infty} d_nx^n$$

with the stipulation that we perform these operations in such a way that these equations are true modulo x^n whatever n be. Therefore we get

$$c_n=a_n+b_n, \qquad d_n=\sum_{\nu=0}^{n} a_\nu b_{n-\nu}.$$

It is clear that $I\{x\}$ is an integral domain too, i.e. $I\{x\}$ contains no zero-divisors. Therefore we can use the cancellation law freely.

We can give a meaning to infinite sums and infinite products very well in certain cases. Thus

$$A_1+A_2+A_3+\cdots=B$$
$$C_1C_2C_3\cdots=D$$

both equations are understood in the sense modulo x^n, so that only a finite of A's or C's can contribute as far as x^n.

We add, now, one more formal procedure, that of formal differentiation. Let

$$A=\sum_{n=0}^{\infty} a_nx^n.$$

* Assistant Professor (*Jokyōju*) in Mathematics.

The derivation A' of A is by definition

$$A'=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n.$$

This is, again, a formal power series in our sense.

Let us add one more remark. Let us consider a special case where A and B have reciprocals. Then AB has a reciprocal too, since the set of all units in I forms a group. In this case we have

$$\frac{(AB)'}{AB}=\frac{A'}{A}+\frac{B'}{B}$$

which is the rule for logarithmic differentiation. In general we have

$$\frac{(\prod_{k=1}^{n}A_k)'}{\prod_{k=1}^{n}A_k}=\sum_{k=1}^{n}\frac{A_k'}{A_k}.$$

We must remark that we can also do this for infinite products

$$\frac{(\prod_{n=1}^{\infty}A_n)'}{\prod_{n=1}^{\infty}A_n}=\sum_{n=1}^{\infty}\frac{A_n'}{A_n},$$

if the products are permissible.

In this paper we shall use various infinite products, but we shall not explain in detail. We shall use infinite product of formal power series in the above sense. Particularly we must use repeatedly the above logarithmic differentiation of infinite product of formal power series.

Now let us fix the subset S of the set N of all natural numbers. In this paper for arbitrary natural number m we shall call the number of methods decomposing m into the sum of elements in S, the solution of m with respect to S. And we shall denote this solution $S(m)$.

In the case in which it is permitted to use the same elements of S in arbitrary times we shall call the problem looking for $S(m)$ the unrestricted type problem, and in the case in which it is not permitted to use the same elements of S in arbitrary times we shall call the problem looking for $S(m)$ the restricted type problem. In the latter case we denote $S(m)$ $S^*(m)$. If the set S is a finite set, we shall call the problem finite type problem and if the set is an infinite set we shall call the problem infinite type problem.

In section 2, we shall deal with unrestricted finite type problem, in section 3 restricted finite type problem, in section 4 unrestricted infinite type problem, and in section 5 restricted infinite type problem.

The special notations which are used in this paper are as follows,

$N=\{0, 1, 2, 3, \cdots\}$ = the set of all natural numbers,

$\sigma_S(n)=\sum_{d\in S,\ d\mid n} d,$

$\tau_S(n)=\sum_{d\in S,\ d\mid n}(-1)^{\frac{n}{d}}d,$

and $[\ \ ]$ = Gaussian symbol.

II. *Unrestricted Finite Type Problem*

In this section we shall be occupied with the problem of unrestricted finite type. Let us adopt $\{a_1, a_2, \cdots, a_n\}$ as the set S explained in the section I, where we shall assume $0<a_1<a_2<\cdots<a_n$. In this case it is clear that the solution $S(m)$ of $m\in N$ with respect to S coincides with the number of non-negative integral solutions of linear equation

$$\sum_{j=1}^{n} a_j x_j = m.$$

We can prove the following theorem about this $S(m)$.

Theorem 1. *$S(0)=1$ and for all m such that $m>0$,*

$$S(m)=\frac{1}{m!}\det\begin{Vmatrix} \sigma_S(1) & -1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & -2 & \ddots & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & & \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3)\cdots & \sigma_S(1) & -(m-1) \\ \sigma_S(m) & \sigma_S(m-1) & \sigma_S(m-2)\cdots & \sigma_S(2) & \sigma_S(1) \end{Vmatrix}. \tag{2.1}$$

Proof. It is clear that $S(0)=1$ and

$$\sum_{m=0}^{\infty} S(m)x^m = \prod_{j=1}^{n}\sum_{m=0}^{\infty} x^{ma_j} = \prod_{j=1}^{n}(1-x^{a_j})^{-1} \tag{2.2}$$

Taking the logarithmic derivative of (2.2), we get

$$\sum_{m=1}^{\infty} mS(m)x^{m-1} \Big/ \sum_{m=0}^{\infty} S(m)x^m = \sum_{j=1}^{n}\frac{-(1-x^{a_j})^{-2}(-a_j x^{a_j-1})}{(1-x^{a_j})^{-1}}.$$

Multiplying both sides by x,

$$\sum_{m=1}^{\infty} mS(m)x^m \Big/ \sum_{m=0}^{\infty} S(m)x^m = \sum_{j=1}^{n}\frac{a_j x^{a_j}}{1-x^{a_j}}. \tag{2.3}$$

Let us transform the right side of (2.3) into formal power series; it becomes

$$\sum_{j=1}^{n} a_j x^{a_j}\sum_{m=0}^{\infty} x^{ma_j} = \sum_{j=1}^{n}\sum_{m=1}^{\infty} a_j x^{ma_j}$$
$$= \sum_{m=1}^{\infty}\Big(\sum_{d\in S,\ d\mid m} d\Big)x^m = \sum_{m=1}^{\infty}\sigma_S(m)x^m,$$

where $\sigma_S(m)$ denotes the sum of divisors of m, which belong to S. Therefore the equation (2.3) can be rewritten as

$$\sum_{m=1}^{\infty} mS(m)x^m = \sum_{m=1}^{\infty}\sigma_S(m)x^m \cdot \sum_{m=0}^{\infty} S(m)x^m.$$

Let us look for the coefficient of x^m $(m\geqq 1)$ on both sides. Then we get

$$mS(m) = \sum_{\nu=0}^{m-1}\sigma_S(m-\nu)S(\nu).$$

Remembering $S(0)=1$, we get

$$\sum_{\nu=1}^{m-1}\sigma_S(m-\nu)S(\nu) - mS(m) = -\sigma_S(m),$$

i.e.

$$\begin{Vmatrix} -1 & & & & 0 \\ \sigma_S(1) & -2 & & & \\ \sigma_S(2) & \sigma_S(1) & -3 & \ddots & \\ \vdots & \vdots & \vdots & & \\ \sigma_S(m-2) & \sigma_S(m-3) & \sigma_S(m-4)\cdots & -(m-1) & \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3)\cdots & \sigma_S(1) & -m \end{Vmatrix} \cdot \begin{Vmatrix} S(1) \\ S(2) \\ S(3) \\ \vdots \\ S(m-1) \\ S(m) \end{Vmatrix} = \begin{Vmatrix} -\sigma_S(1) \\ -\sigma_S(2) \\ -\sigma_S(3) \\ \vdots \\ -\sigma_S(m-1) \\ -\sigma_S(m) \end{Vmatrix}.$$

Therefore we get the result (2.1) of this theorem.

Corollary 1. *If we define*

$$D_k=\frac{(-1)^k}{k!}\det\begin{Vmatrix} \sigma_S(1) & 1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & 2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \sigma_S(k-1) & \sigma_S(k-2) & \sigma_S(k-3)\cdots & \sigma_S(1) & k-1 \\ \sigma_S(k) & \sigma_S(k-1) & \sigma_S(k-2)\cdots & \sigma_S(2) & \sigma_S(1) \end{Vmatrix} \tag{2.4}$$

for all k such that $1\leqq k\leqq a_1+a_2+\cdots+a_n$, we get

$$\det\begin{Vmatrix} D_1 & 1 & & & 0 \\ D_2 & D_1 & 1 & & \\ D_3 & D_2 & D_1 & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ D_{m-1} & D_{m-2} & D_{m-3}\cdots & D_1 & 1 \\ D_m & D_{m-1} & D_{m-2}\cdots & D_2 & D_1 \end{Vmatrix}$$

$$=\frac{(-1)^m}{m!}\det\begin{Vmatrix} \sigma_S(1) & -1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & -2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3)\cdots & \sigma_S(1) & -(m-1) \\ \sigma_S(m) & \sigma_S(m-1) & \sigma_S(m-2)\cdots & \sigma_S(2) & \sigma_S(1) \end{Vmatrix} \tag{2.5}$$

for all m such that $1\leqq m\leqq a_1+a_2+\cdots+a_n$.

Proof. Let us define c_k $(0\leqq k\leqq a_1+a_2+\cdots+a_n)$ by the following equation.

$$\prod_{j=1}^{n}(1-x^{a_j})=\sum_{k=0}^{a_1+a_2+\cdots+a_n} c_k x^k \tag{2.6}$$

It is clear $c_0=1$. Taking the logarithmic derivative of (2.6), we get

$$\sum_{j=1}^{n}\frac{-a_j x^{a_j-1}}{1-x^{a_j}}=\frac{\sum_{k=1}^{a_1+a_2+\cdots+a_n} k c_k x^{k-1}}{\sum_{k=0}^{a_1+a_2+\cdots+a_n} c_k x^k}.$$

Multiplying both sides by x

$$-\sum_{j=1}^{n}\frac{a_j x^{a_j}}{1-x^{a_j}}=\sum_{k=1}^{a_1+a_2+\cdots+a_n} k c_k x^k \Big/ \sum_{k=0}^{a_1+a_2+\cdots+a_n} c_k x^k. \tag{2.7}$$

Let us transform the left side of (2.7) into formal power series; it becomes

$$-\sum_{j=1}^{n} a_j x^{a_j}\sum_{m=0}^{\infty} x^{ma_j}=-\sum_{j=1}^{n}\sum_{m=1}^{\infty} a_j x^{ma_j}$$

$$=-\sum_{m=1}^{\infty}\Big(\sum_{d\in S,\ d\mid m} d\Big)x^m=-\sum_{m=1}^{\infty}\sigma_S(m)x^m.$$

Therefore the equation (2.7) can be rewritten as

$$\sum_{m=1}^{\infty}\sigma_S(m)x^m\cdot\sum_{m=0}^{a_1+a_2+\cdots+a_n} c_m x^m=-\sum_{m=1}^{a_1+a_2+\cdots+a_n} m c_m x^m.$$

Let us look for the coefficients of x^m $(1\leqq m\leqq a_1+a_2+\cdots+a_n)$ on both sides. Then we get

$$\sum_{\nu=0}^{m-1}\sigma_S(m-\nu)c_\nu=-m c_m.$$

Remembering that $c_0=1$, we get

$$\sum_{\nu=1}^{m-1}\sigma_S(m-\nu)c_\nu+m c_m=-\sigma_S(m),$$

i.e.
$$\begin{Vmatrix} 1 & & & & 0 \\ \sigma_S(1) & 2 & & & \\ \sigma_S(2) & \sigma_S(1) & 3 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_S(m-2) & \sigma_S(m-3) & \sigma_S(m-4) & \cdots & m-1 \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3) & \cdots\sigma_S(1) & m \end{Vmatrix} \cdot \begin{Vmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{m-1} \\ c_m \end{Vmatrix} = \begin{Vmatrix} -\sigma_S(1) \\ -\sigma_S(2) \\ -\sigma_S(3) \\ \vdots \\ -\sigma_S(m-1) \\ -\sigma_S(m) \end{Vmatrix}.$$

From this equation and (2.4), we get
$$c_k = D_k, \qquad 1 \leqq k \leqq a_1 = a_2 + \cdots + a_n. \tag{2.8}$$

On the other hand, by (2.2), we get
$$\sum_{j=1}^{n} (1 - x^{a_j}) \cdot \sum_{m=0}^{\infty} S(m) x^m = 1,$$

i.e.
$$\sum_{m=0}^{a_1+a_2+\cdots+a_n} D_m x^m \cdot \sum_{m=0}^{\infty} S(m) x^m = 1,$$

where we put $D_0 = 1$. Let us look for the coefficients of x^m $(1 \leqq m \leqq a_1 + a_2 + \cdots + a_n)$ on both sides. Then we get
$$\sum_{\nu=0}^{m} D_{m-\nu} S(\nu) = 0, \qquad 1 \leqq m \leqq a_1 + a_2 + \cdots + a_n.$$

Remembering $S(0) = 1$, we get
$$\sum_{\nu=1}^{m} D_{m-\nu} S(\nu) = -D_m,$$

i.e.
$$\begin{Vmatrix} D_0 & & & & 0 \\ D_1 & D_0 & & & \\ D_2 & D_1 & D_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ D_{m-2} & D_{m-3} & D_{m-4} & \cdots & D_0 \\ D_{m-1} & D_{m-2} & D_{m-3} & \cdots D_1 & D_0 \end{Vmatrix} \cdot \begin{Vmatrix} S(1) \\ S(2) \\ S(3) \\ \vdots \\ S(m-1) \\ S(m) \end{Vmatrix} = \begin{Vmatrix} -D_1 \\ -D_2 \\ -D_3 \\ \vdots \\ -D_{m-1} \\ -D_m \end{Vmatrix}.$$

Therefore from (2.1) we get the result (2.5).

Corollary 2. *If we define*
$$D'_k = \frac{1}{k!} \det \begin{Vmatrix} \sigma_S(1) & -1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & -2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \sigma_S(k-1) & \sigma_S(k-2) & \sigma_S(k-3) & \cdots\sigma_S(1) & -(k-1) \\ \sigma_S(k) & \sigma_S(k-1) & \sigma_S(k-2) & \cdots\sigma_S(2) & \sigma_S(1) \end{Vmatrix} \tag{2.9}$$

for all k such that $1 \leqq k \leqq a_1 + a_2 + \cdots + a_n$, we get
$$\det \begin{Vmatrix} D'_1 & 1 & & & 0 \\ D'_2 & D'_1 & 1 & & \\ D'_3 & D'_2 & D'_1 & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ D'_{m-1} & D'_{m-2} & D'_{m-3} & \cdots D_1' & 1 \\ D'_m & D'_{m-1} & D'_{m-2} & \cdots D'_2 & D'_1 \end{Vmatrix}$$
$$= \frac{1}{m!} \det \begin{Vmatrix} \sigma_S(1) & 1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & 2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3) & \cdots\sigma_S(1) & m-1 \\ \sigma_S(m) & \sigma_S(m-1) & \sigma_S(m-2) & \cdots\sigma_S(2) & \sigma_S(1) \end{Vmatrix} \tag{2.10}$$

for all m such that $1 \leqq m \leqq a_1+a_2+\cdots+a_n$.

Proof. From (2.2) and (2.6), we get

$$\sum_{m=0}^{a_1+a_2+\cdots+a_n} c_m x^m \cdot \sum_{m=0}^{\infty} S(m) x^m = 1$$

Let us look for the coefficients of $x^m (m \geqq 1)$ on both sides. Then we get

$$\sum_{\nu=0}^{m} c_\nu S(m-\nu) = 0, \qquad 1 \leqq m \leqq a_1+a_2+\cdots+a_n.$$

Remembering that $c_0=1$, we get

$$\sum_{\nu=1}^{m} c_\nu S(m-\nu) = -S(m), \qquad 1 \leqq m \leqq a_1+a_2+\cdots+a_n,$$

i.e.

$$\begin{Vmatrix} S(0) & & & & 0 \\ S(1) & S(0) & & & \\ S(2) & S(1) & S(0) & \ddots & \\ \vdots & \vdots & \vdots & & \\ S(m-2) & S(m-3) & S(m-4) & \cdots & S(0) \\ S(m-1) & S(m-2) & S(m-3) & \cdots & S(1) \quad S(0) \end{Vmatrix} \cdot \begin{Vmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{m-1} \\ c_m \end{Vmatrix} = \begin{Vmatrix} -S(1) \\ -S(2) \\ -S(3) \\ \vdots \\ -S(m-1) \\ -S(m) \end{Vmatrix}.$$

From this equation, we get

$$c_m = (-1)^m \det \begin{Vmatrix} S(1) & 1 & & & 0 \\ S(2) & S(1) & 1 & \ddots & \\ S(3) & S(2) & S(1) & \ddots & \ddots \\ \vdots & \vdots & \vdots & & \\ S(m-1) & S(m-2) & S(m-3) & \cdots S(1) & 1 \\ S(m) & S(m-1) & S(m-2) & \cdots S(2) & S(1) \end{Vmatrix}.$$

On the other hand from (2.4) and (2.8), we get

$$c_m = \frac{(-1)^m}{m!} \det \begin{Vmatrix} \sigma_S(1) & 1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & 2 & \ddots & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \ddots \\ \vdots & \vdots & \vdots & & \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3) & \cdots \sigma_S(1) & m-1 \\ \sigma_S(m) & \sigma_S(m-1) & \sigma_S(m-2) & \cdots \sigma_S(2) & \sigma_S(1) \end{Vmatrix}.$$

Therefore from (2.1) and (2.9) we get the result (2.10).

Corollary 3. *Let us denote the partition number of the case in which* **we** *may use only the number of* $\{1, 2, \cdots, n\}$ $P(n)$. *Then*

$$P(m) = \frac{1}{m!} \det \begin{Vmatrix} \sigma_S(1) & -1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & -2 & \ddots & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \ddots \\ \vdots & \vdots & \vdots & & \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3) & \cdots \sigma_S(1) & -(m-1) \\ \sigma_S(m) & \sigma_S(m-1) & \sigma_S(m-2) & \cdots \sigma_S(2) & \sigma_S(1) \end{Vmatrix}$$

for all m *such that* $1 \leqq m$.

Proof. Put $a_j = j (1 \leqq j \leqq n)$ in theorem 1.

III. *Restricted Finite Type Problem*

In this section we shall be occupied with the problem of restricted finite type. Let us adopt $\{a_1, a_2, \cdots, a_n\}$ as the set S explained in the section I, where we shall assume $0<a_1<a_2<\cdots<a_n$. In this case we must assume that when we decompose a natural number m into the sum of elements of S, it is not permitted to use the same a_j over r_j times for all $j; 1\leqq j\leqq n$. In this case it is clear that the solution $S^*(m)$ of $m\in N$ with respect to S coincides with the number of integral solutions of conditional linear equation

$$\sum_{j=1}^{n} a_j x_j = m, \qquad 0\leqq x_j<r_j, \qquad j=1, 2, \cdots, n.$$

We can prove the following theorem about this $S^*(m)$.

Theorem 2. *If we define* $S'=\{a_1r_1, a_2r_2, \cdots, a_nr_n\}$, *then*

$$S^*(m)=(-1)^m \det \begin{Vmatrix} 1 & 1 & & & 0 \\ S(1) & S'(1) & 1 & & \\ S(2) & S'(2) & S'(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ S(m-1) & S'(m-1) & S'(m-2) & \cdots S'(1) & 1 \\ S(m) & S'(m) & S'(m-1) & \cdots S'(2) & S'(1) \end{Vmatrix} \tag{3.1}$$

$$=\sum_{\nu=0}^{m} \frac{(-1)^\nu}{\nu!(m-\nu)!} \det \begin{Vmatrix} \sigma_{S'}(1) & 1 & & & 0 \\ \sigma_{S'}(2) & \sigma_{S'}(1) & 2 & & \\ \sigma_{S'}(3) & \sigma_{S'}(2) & \sigma_{S'}(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_{S'}(\nu-1) & \sigma_{S'}(\nu-2) & \sigma_{S'}(\nu-3) & \cdots \sigma_{S'}(1) & \nu-1 \\ \sigma_{S'}(\nu) & \sigma_{S'}(\nu-1) & \sigma_{S'}(\nu-2) & \cdots \sigma_{S'}(2) & \sigma_{S'}(1) \end{Vmatrix} \cdot$$

$$\det \begin{Vmatrix} \sigma_S(1) & -1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & -2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_S(m-\nu-1) & \sigma_S(m-\nu-2) & \sigma_S(m-\nu-3) & \cdots \sigma_S(1) & -(m-\nu-1) \\ \sigma_S(m-\nu) & \sigma_S(m-\nu-1) & \sigma_S(m-\nu-2) & \cdots \sigma_S(2) & \sigma_S(1) \end{Vmatrix}. \tag{3.2}$$

Proof. It is clear that $S^*(0)=1$ and

$$\sum_{m=0}^{a_1(r_1-1)+a_2(r_2-1)+\cdots+a_n(r_n-1)} S^*(m)x^m = \prod_{j=1}^{n} \sum_{m=0}^{r_j-1} x^{ma_j} = \prod_{j=1}^{n} (1-x^{a_jr_j})(1-x^{a_j})^{-1}$$

$$=\prod_{j=1}^{n} (1-x^{a_jr_j}) \cdot \sum_{m=0}^{\infty} S(m)x^m. \tag{3.3}$$

By theorem 1 in section II, we get

$$\prod_{j=1}^{n} (1-x^{a_jr_j})^{-1} = \sum_{m=0}^{\infty} S'(m)x^m.$$

Then

$$\sum_{m=0}^{\infty} S'(m)x^m \cdot \sum_{m=0}^{a_1(r_1-1)+a_2(r_2-1)+\cdots+a_n(r_n-1)} S^*(m)x^m = \sum_{m=0}^{\infty} S(m)x^m.$$

Let us look for the coefficients of x^m $(1\leqq m\leqq a_1r_1+a_2r_2+\cdots+a_nr_n)$ on both sides. Then we get

$$\sum_{\nu=0}^{m} S'(m-\nu)S^*(\nu) = S(m),$$

i.e.
$$\begin{Vmatrix} S'(0) & & & & 0 \\ S'(1) & S'(0) & & & \\ S'(2) & S'(1) & S'(0) & \ddots & \\ \vdots & \vdots & \vdots & & \\ S'(m-1) & S'(m-2) & S'(m-3) \cdots & S'(0) & \\ S'(m) & S'(m-1) & S'(m-2) \cdots & S'(1) & S'(0) \end{Vmatrix} \cdot \begin{Vmatrix} S^*(0) \\ S^*(1) \\ S^*(2) \\ \vdots \\ S^*(m-1) \\ S^*(m) \end{Vmatrix} = \begin{Vmatrix} S(0) \\ S(1) \\ S(2) \\ \vdots \\ S(m-1) \\ S(m) \end{Vmatrix}.$$

Therefore we get the result (3.1).

On the other hand let us put

$$\prod_{j=1}^{n}(1-x^{a_j r_j}) = \sum_{m=0}^{a_1 r_1 + a_2 r_2 + \cdots + a_n r_n} \alpha_{S'}(m) x^m.$$

In order to get $\alpha_{S'}(m)$, we use the methods explained at corollary 1 of theorem 1 in section II; it becomes

$$\alpha_{S'}(m) = \frac{(-1)^m}{m!} \det \begin{Vmatrix} \sigma_{S'}(1) & 1 & & & 0 \\ \sigma_{S'}(2) & \sigma_{S'}(1) & 2 & \ddots & \\ \sigma_{S'}(3) & \sigma_{S'}(2) & \sigma_{S'}(1) & \ddots & \ddots \\ \vdots & \vdots & \vdots & & \\ \sigma_{S'}(m-1) & \sigma_{S'}(m-2) & \sigma_{S'}(m-3) \cdots & \sigma_{S'}(1) & m-1 \\ \sigma_{S'}(m) & \sigma_{S'}(m-1) & \sigma_{S'}(m-2) \cdots & \sigma_{S'}(2) & \sigma_{S'}(1) \end{Vmatrix}$$

where $\alpha_{S'}(0)=1$. Remembering (3.3), we get

$$\sum_{m=0}^{a_1(r_1-1)+a_2(r_2-1)+\cdots+a_n(r_n-1)} S^*(m)x^m = \sum_{m=0}^{a_1 r_2 + a_2 r_2 + \cdots + a_n r_n} \alpha_{S'}(m) x^m \cdot \sum_{m=0}^{\infty} S(m) x^m. \tag{3.4}$$

Comparing the coefficients of (3.4) on both sides, we get

$$S^*(m) = \sum_{\nu=0}^{m} \alpha_{S'}(\nu) S(m-\nu), \qquad 0 \leqq m \leqq a_1 r_1 + a_2 r_2 + \cdots + a_n r_n.$$

Therefore we get the results (3.2) of the theorem.

Corollary. *If* $r_1 = r_2 = \cdots = r_n = 2$, then

$$S^*(m) = \frac{1}{m!} \det \begin{Vmatrix} \tau_S(1) & 1 & & & 0 \\ \tau_S(2) & \tau_S(1) & 2 & \ddots & \\ \tau_S(3) & \tau_S(2) & \tau(1) & \ddots & \ddots \\ \vdots & \vdots & \vdots & & \\ \tau_S(m-1) & \tau_S(m-2) & \tau_S(m-3) \cdots & \tau_S(1) & m-1 \\ \tau_S(m) & \tau_S(m-1) & \tau_S(m-2) \cdots & \tau_S(2) & \tau_S(1) \end{Vmatrix}$$
$$0 \leqq m \leqq a_1 r_1 + a_2 r_2 + \cdots + a_n r_n, \tag{3.5}$$

where $\tau_S(k) = \sum_{d \in S,\, d \mid k} (-1)^{\frac{k}{d}} d$.

Proof. Putting $r_1 = r_2 = \cdots = r_n = 2$ in the theorem 2, we get

$$\sum_{m=0}^{a_1+a_2+\cdots+a_n} S^*(m) x^m = \prod_{j=1}^{n}(1-x^{2a_j})(1-x^{a_j})^{-1} = \prod_{j=1}^{n}(1+x^{a_j}). \tag{3.6}$$

Taking the logarithmic derivative of (3.6), we get

$$\sum_{j=1}^{n} \frac{a_j x^{a_j - 1}}{1+x^{a_j}} = \sum_{m=1}^{a_1+a_2+\cdots+a_n} m S^*(m) x^{m-1} \Big/ \sum_{m=0}^{a_1+a_2+\cdots+a_n} S^*(m) x^m.$$

Multiplying both sides by x

$$\sum_{j=1}^{n} \frac{a_j x^{a_j}}{1+x^{a_j}} = \sum_{m=1}^{a_1+a_2+\cdots+a_n} m S^*(m) x^m \Big/ \sum_{m=0}^{a_1+a_2+\cdots+a_n} S^*(m) x^m. \tag{3.7}$$

Let us transform the left side of (3.7) into formal power series; it becomes

$$\sum_{j=1}^{n} a_j x^{a_j} \sum_{m=0}^{\infty} (-1)^m x^{m a_j} = \sum_{j=1}^{n} \sum_{m=1}^{\infty} (-1)^{m-1} a_j x^{m a_j}$$

$$= -\sum_{m=1}^{\infty}\Big(\sum_{d\in S,\, d\mid m}(-1)^{\frac{m}{d}}d\Big)x^m = -\sum_{m=1}^{\infty}\tau_S(m)x^m.$$

Therefore the equation (3.7) can be rewritten as

$$-\sum_{m=1}^{\infty}\tau_S(m)x^m\cdot\sum_{m=1}^{a_1+a_2+\cdots+a_n}S^*(m)x^m=\sum_{m=1}^{a_1+a_2+\cdots+a_n}mS^*(m)x^m.$$

Comparing the coefficients of x^m $(1\leqq m\leqq a_1+a_2+\cdots+a_n)$ on both sides, we get

$$-\sum_{\nu=0}^{m=1}\tau_S(m-\nu)S^*(\nu)=mS^*(m),\quad 1\leqq m\leqq a_1+a_2+\cdots+a_n.$$

Remembering that $S^*(0)=1$, we get

$$\sum_{\nu=1}^{m=1}\tau_S(m-\nu)S^*(\nu)+mS^*(m)=-\tau_S(m),\quad 1\leqq m\leqq a_1+a_2+\cdots+a_n,$$

i.e.

$$\begin{Vmatrix} 1 & & & & 0 \\ \tau_S(1) & 2 & & & \\ \tau_S(2) & \tau_S(1) & 3 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \tau_S(m-2) & \tau_S(m-3) & \tau_S(m-4)\cdots & m-1 & \\ \tau_S(m-1) & \tau_S(m-2) & \tau_S(m-3)\cdots & \tau_S(1) & m \end{Vmatrix}\cdot\begin{Vmatrix} S^*(1) \\ S^*(2) \\ S^*(3) \\ \vdots \\ S^*(m-1) \\ S^*(m) \end{Vmatrix}=\begin{Vmatrix} -\tau_S(1) \\ -\tau_S(2) \\ -\tau_S(3) \\ \vdots \\ -\tau_S(m-1) \\ -\tau_S(m) \end{Vmatrix}.$$

Therefore we get the result (3.5) of the corollary.

IV. *Unrestricted Infinite Type Problem*

In this section we shall be occupied with the problem of unrestricted infinite type. Let us adopt the countable infinite subset of N $\{a_1, a_2, a_3, \cdots\}$ as the set S explained in the section I, where we shall assume $0<a_1<a_2<a_3<\cdots$. In this case it is clear that the solution $S(m)$ of $m\in N$ with respect to S coincides with number of non-negative integral solutions of linear equation

$$\sum_{j=1}^{\infty}a_jx_j=m.$$

We can prove the following theorem about this $S(m)$.

Theorem 3. $S(0)=1$ *and for all* m *such that* $m>0$,

$$S(m)=\frac{1}{m!}\det\begin{Vmatrix} \sigma_S(1) & -1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & -2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & & \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3)\cdots & \sigma_S(1) & -(m-1) \\ \sigma_S(m) & \sigma_S(m-1) & \sigma_S(m-2)\cdots & \sigma_S(2) & \sigma_S(1) \end{Vmatrix}. \tag{4.1}$$

Proof. It is clear that $S(0)=1$ and

$$\sum_{m=0}^{\infty}S(m)x^m=\prod_{j=1}^{\infty}\sum_{m=0}^{\infty}x^{ma_j}=\prod_{j=1}^{\infty}(1-x^{a_j})^{-1}. \tag{4.2}$$

Taking the logarithmic derivative of (4.2), we get

$$\sum_{m=1}^{\infty}mS(m)x^{m-1}\Big/\sum_{m=0}^{\infty}S(m)x^m=\sum_{j=1}^{\infty}\frac{-(1-x^{a_j})^{-2}(-a_jx^{a_j-1})}{(1-x^{a_j})^{-1}}.$$

Multiplying both sides by x

$$\sum_{m=1}^{\infty}mS(m)x^m\Big/\sum_{m=0}^{\infty}S(m)x^m=\sum_{j=1}^{\infty}\frac{a_jx^{a_j}}{1-x^{a_j}}. \tag{4.3}$$

Let us transform the right side of (4.3) into formal power series; it becomes

$$\sum_{j=1}^{\infty} a_j x^{a_j} \sum_{m=0}^{\infty} x^{ma_j} = \sum_{j,m=1}^{\infty} a_j x^{ma_j}$$
$$= \sum_{m=1}^{\infty} \Big(\sum_{d \in S,\ d \mid m} d \Big) x^m = \sum_{m=1}^{\infty} \sigma_S(m) x^m$$

where $\sigma_S(m)$ denotes the sum of divisors of m, which belong to S.
Therefore the equation (4.3) can be rewritten as

$$\sum_{m=1}^{\infty} mS(m)x^m = \sum_{m=1}^{\infty} \sigma_S(m)x^m \cdot \sum_{m=0}^{\infty} S(m)x^m.$$

Let us look for the coefficients of x^m $(m \geqq 1)$ on both sides. Then we get

$$mS(m) = \sum_{\nu=0}^{m-1} \sigma_S(m-\nu)S(\nu).$$

Remembering that $S(0)=1$, we get

$$\sum_{\nu=1}^{m-1} \sigma_S(m-\nu)S(\nu) - mS(m) = -\sigma_S(m),$$

i.e.

$$\begin{Vmatrix} -1 & & & & 0 \\ \sigma_S(1) & -2 & & & \\ \sigma_S(2) & \sigma_S(1) & -3 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_S(m-2) & \sigma_S(m-3) & \sigma_S(m-4) & \cdots & -(m-1) \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3) & \cdots\ \sigma_S(1) & -m \end{Vmatrix} \cdot \begin{Vmatrix} S(1) \\ S(2) \\ S(3) \\ \vdots \\ S(m-1) \\ S(m) \end{Vmatrix} = \begin{Vmatrix} -\sigma_S(1) \\ -\sigma_S(2) \\ -\sigma_S(3) \\ \vdots \\ -\sigma_S(m-1) \\ -\sigma_S(m) \end{Vmatrix}.$$

Therefore we get the result (4.1) of the theorem.

Corollary 1. *If we define*

$$D_k = \frac{(-1)^k}{k!} \det \begin{Vmatrix} \sigma_S(1) & 1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & 2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & & \\ \sigma_S(k-1) & \sigma_S(k-2) & \sigma_S(k-3) & \cdots \sigma_S(1) & k-1 \\ \sigma_S(k) & \sigma_S(k-1) & \sigma_S(k-2) & \cdots \sigma_S(2) & \sigma_S(1) \end{Vmatrix} \tag{4.4}$$

for all $k=1,2,3,\cdots$, *we get*

$$\det \begin{Vmatrix} D_1 & 1 & & & 0 \\ D_2 & D_1 & 1 & & \\ D_3 & D_2 & D_1 & \ddots & \\ \vdots & \vdots & \vdots & & \\ D_{m-1} & D_{m-2} & D_{m-3} & \cdots D_1 & 1 \\ D_m & D_{m-1} & D_{m-2} & \cdots D_2 & D_1 \end{Vmatrix}$$
$$= \frac{(-1)^m}{m!} \det \begin{Vmatrix} \sigma_S(1) & -1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & -2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & & \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3) & \cdots \sigma_S(1) & -(m-1) \\ \sigma_S(m) & \sigma_S(m-1) & \sigma_S(m-2) & \cdots \sigma_S(2) & \sigma_S(1) \end{Vmatrix} \tag{4.5}$$

for all $m=1,2,3,\cdots$.

Proof. Let us define $c_k (k \in N)$ by the following equation,

$$\prod_{j=1}^{\infty} (1-x^{a_j}) = \sum_{k=0}^{\infty} c_k x^k. \tag{4.6}$$

It is clear $c_0=1$. Taking the logarithmic derivative of (4.6), we get

$$\sum_{j=1}^{\infty} \frac{-a_j x^{a_j-1}}{1-x^{a_j}} = \sum_{k=1}^{\infty} k c_k x^{k-1} / \sum_{k=0}^{\infty} c_k x^k.$$

Multiplying both sides by x

$$-\sum_{j=1}^{\infty} \frac{a_j x^{a_j}}{1-x^{a_j}} = \sum_{k=1}^{\infty} k c_k x^k / \sum_{k=0}^{\infty} c_k x^k. \tag{4.7}$$

Let us transform the left side of (4.7) into formal power series; it becomes

$$-\sum_{j=1}^{\infty} a_j x^{a_j} \sum_{m=0}^{\infty} x^{m a_j} = -\sum_{j,m=1}^{\infty} a_j x^{m a_j}$$
$$= -\sum_{m=1}^{\infty} \Big(\sum_{d \in S,\ d \mid m}^{\infty} d \Big) x^m = -\sum_{m=1}^{\infty} \sigma_S(m) x^m.$$

Therefore the equation (4.7) can be rewritten as

$$\sum_{m=1}^{\infty} \sigma_S(m) x^m \cdot \sum_{m=0}^{\infty} c_m x^m = -\sum_{m=1}^{\infty} m c_m x^m.$$

Let us look for the coefficients of x^m $(m \geqq 1)$ on both sides. Then we get

$$\sum_{\nu=0}^{m-1} \sigma_S(m-\nu) c_\nu = -m c_m.$$

Remembering that $c_0 = 1$, we get

$$\sum_{\nu=1}^{m-1} \sigma_S(m-\nu) c_\nu + m c_m = -\sigma_S(m),$$

i.e.

$$\begin{Vmatrix} 1 & & & & 0 \\ \sigma_S(1) & 2 & & & \\ \sigma_S(2) & \sigma_S(1) & 3 & & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_S(m-2) & \sigma_S(m-3) & \sigma_S(m-4) & \cdots & m-1 \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3) & \cdots\ \sigma_S(1) & m \end{Vmatrix} \cdot \begin{Vmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{m-1} \\ c_m \end{Vmatrix} = \begin{Vmatrix} -\sigma_S(1) \\ -\sigma_S(2) \\ -\sigma_S(3) \\ \vdots \\ -\sigma_S(m-1) \\ -\sigma_S(m) \end{Vmatrix}.$$

From this equation and (4.4), we get

$$c_k = D_k, \qquad k = 1, 2, 3, \cdots. \tag{4.8}$$

On the other hand, by (4.2) we get

$$\prod_{j=1}^{\infty} (1 - x^{a_j}) \cdot \sum_{m=0}^{\infty} S(m) x^m = 1$$

i.e.

$$\sum_{m=0}^{\infty} D_m x^m \cdot \sum_{m=0}^{\infty} S(m) x^m = 1,$$

where we put $D_0 = 1$. Let us look for the coefficients of x^m $(m \geqq 1)$ on both sides. Then we get

$$\sum_{\nu=0}^{m} D_{m-\nu} S(\nu) = 0, \qquad m \geqq 1.$$

Remembering $S(0) = 1$, we get

$$\sum_{\nu=1}^{m} D_{m-\nu} S(\nu) = -D_m,$$

i.e.

$$\begin{Vmatrix} D_0 & & & & 0 \\ D_1 & D_0 & & & \\ D_2 & D_1 & D_0 & & \\ \vdots & \vdots & \vdots & \ddots & \\ D_{m-2} & D_{m-3} & D_{m-4} & \cdots & D_0 \\ D_{m-1} & D_{m-2} & D_{m-3} & \cdots\ D_1 & D_0 \end{Vmatrix} \cdot \begin{Vmatrix} S(1) \\ S(2) \\ S(3) \\ \vdots \\ S(m-1) \\ S(m) \end{Vmatrix} = \begin{Vmatrix} -D_1 \\ -D_2 \\ -D_3 \\ \vdots \\ -D_{m-1} \\ -D_m \end{Vmatrix}.$$

Therefore from (4.1) we get the result (4.5).

Corollary 2. *If we define*

$$D_k'=\frac{1}{k!}\det\begin{Vmatrix} \sigma_S(1) & -1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & -2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_S(k-1) & \sigma_S(k-2) & \sigma_S(k-3)\cdots & \sigma_S(1) & -(k-1) \\ \sigma_S(k) & \sigma_S(k-1) & \sigma_S(k-2)\cdots & \sigma_S(2) & \sigma_S(1) \end{Vmatrix} \tag{4.9}$$

for all $k=1, 2, 3, \cdots$, *we get*

$$\det\begin{Vmatrix} D_1' & 1 & & & 0 \\ D_2' & D_1' & 1 & & \\ D_3' & D_2' & D_1' & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ D_{m-1}' & D_{m-2}' & D_{m-3}'\cdots & D_1' & 1 \\ D_m' & D_{m-1}' & D_{m-2}'\cdots & D_2' & D_1' \end{Vmatrix}=\frac{1}{m!}\det\begin{Vmatrix} \sigma_S(1) & 1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & 2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3)\cdots & \sigma_S(1) & m-1 \\ \sigma_S(m) & \sigma_S(m-1) & \sigma_S(m-2)\cdots & \sigma_S(2) & \sigma_S(1) \end{Vmatrix} \tag{4.10}$$

for all $m=1, 2, 3, \cdots$.

Proof. From (4.2) and (4.6), we get

$$\sum_{m=0}^{\infty} c_m x^m \cdot \sum_{m=0}^{\infty} S(m)x^m=1.$$

Let us look for the coefficients of x^m $(m\geqq 1)$ on both sides. Then we get

$$\sum_{\nu=0}^{m} c_\nu S(m-\nu)=0.$$

Remembering that $c_0=1$, we get

$$\sum_{\nu=1}^{m} c_\nu S(m-\nu)=-S(m),$$

i.e.

$$\begin{Vmatrix} S(0) & & & & 0 \\ S(1) & S(0) & & & \\ S(2) & S(1) & S(0) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ S(m-2) & S(m-3) & S(m-4)\cdots & S(0) & \\ S(m-1) & S(m-2) & S(m-3)\cdots & S(1) & S(0) \end{Vmatrix}\cdot\begin{Vmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{m-1} \\ c_m \end{Vmatrix}=\begin{Vmatrix} -S(1) \\ -S(2) \\ -S(3) \\ \vdots \\ -S(m-1) \\ -S(m) \end{Vmatrix}.$$

From this equation, we get

$$c_m=(-1)^m\det\begin{Vmatrix} S(1) & 1 & & & 0 \\ S(2) & S(1) & 1 & & \\ S(3) & S(2) & S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ S(m-1) & S(m-2) & S(m-3)\cdots & S(1) & 1 \\ S(m) & S(m-1) & S(m-2)\cdots & S(2) & S(1) \end{Vmatrix}.$$

On the other hand, from (4.4) and (4.8) we get

$$c_m=\frac{(-1)^m}{m!}\det\begin{Vmatrix} \sigma_S(1) & 1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & 2 & & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_S(m-1) & \sigma_S(m-2) & \sigma_S(m-3)\cdots & \sigma_S(1) & m-1 \\ \sigma_S(m) & \sigma_S(m-1) & \sigma_S(m-2)\cdots & \sigma_S(2) & \sigma_S(1) \end{Vmatrix}.$$

Therefore from (4.1) and (4.9) we get the result (4.10).

Corollary 3. *Let us denote the partition number of* m $P(m)$ *and the sum of all divisors of* m $\sigma(m)$. *Then*

$$P(m)=\frac{1}{m!}\det\begin{Vmatrix} \sigma(1) & -1 & & & 0 \\ \sigma(2) & \sigma(1) & -2 & \ddots & \\ \sigma(3) & \sigma(2) & \sigma(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma(m-1) & \sigma(m-2) & \sigma(m-3)\cdots & \sigma(1) & -(m-1) \\ \sigma(m) & \sigma(m-1) & \sigma(m-2)\cdots & \sigma(2) & \sigma(1) \end{Vmatrix}.$$

Proof. Put $a_j=j(j=1,2,3,\cdots)$ in theorem 3.

V. *Restricted Infinite Type Problems*

In this section we shall be occupied with the problem of restricted infinite type. Let us adopt the countable infinite subset of N $\{a_1, a_2, a_3, \cdots\}$ as the set S explained in the section I, where we shall assume $0<a_1<a_2<a_3<\cdots$. In this case we must assume that when we decompose a natural number m into the sum of elements of S, it is not permitted to use the same a_j over r_j times for all $j=1,2,3,\cdots$. In this case it is clear that the solution $S^*(m)$ of $m\in N$ with respect to S coincides with the number of integral solutions of conditional linear equation

$$\sum_{j=1}^{\infty} a_j x_j=m, \qquad 0\leqq x_j<r_j, \qquad j=1,2,3,\cdots.$$

We can prove the following theorem about this $S^*(m)$.

Theorem 4. *If we define* $S'=\{a_1r_1, a_2r_2, a_3r_3, \cdots\}$, *then*

$$S^*(m)=(-1)^m\det\begin{Vmatrix} S(0) & 1 & & & 0 \\ S(1) & S'(1) & 1 & \ddots & \\ S(2) & S'(2) & S'(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ S(m-1) & S'(m-1) & S'(m-2)\cdots & S'(1) & 1 \\ S(m) & S'(m) & S'(m-1)\cdots & S'(2) & S'(1) \end{Vmatrix} \tag{5.1}$$

$$=\sum_{\nu=0}^{m}\frac{(-1)^\nu}{\nu!(m-\nu)!}\det\begin{Vmatrix} \sigma_{S'}(1) & 1 & & & 0 \\ \sigma_{S'}(2) & \sigma_{S'}(1) & 2 & \ddots & \\ \sigma_{S'}(3) & \sigma_{S'}(2) & \sigma_{S'}(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_{S'}(\nu-1) & \sigma_{S'}(\nu-2) & \sigma_{S'}(\nu-3)\cdots & \sigma_{S'}(1) & \nu-1 \\ \sigma_{S'}(\nu) & \sigma_{S'}(\nu-1) & \sigma_{S'}(\nu-2)\cdots & \sigma_{S'}(2) & \sigma_{S'}(1) \end{Vmatrix}\cdot$$

$$\det\begin{Vmatrix} \sigma_S(1) & -1 & & & 0 \\ \sigma_S(2) & \sigma_S(1) & -2 & \ddots & \\ \sigma_S(3) & \sigma_S(2) & \sigma_S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_S(m-\nu-1) & \sigma_S(m-\nu-2) & \sigma_S(m-\nu-3)\cdots & \sigma_S(1) & -(m-\nu) \\ \sigma_S(m-\nu) & \sigma_S(m-\nu-1) & \sigma_S(m-\nu-2)\cdots & \sigma_S(2) & \sigma_S(1) \end{Vmatrix}. \tag{5.2}$$

Proof. It is clear that $S^*(0)=1$ and

$$\sum_{m=0}^{\infty} S^*(m)x^m=\prod_{j=1}^{\infty}\prod_{m=0}^{r_j-1} x^{ma_j}=\prod_{j=1}^{\infty}(1-x^{a_jr_j})(1-x^{a_j})^{-1}$$
$$=\prod_{j=1}^{\infty}(1-x^{a_jr_j})\sum_{m=0}^{\infty} S(m)x^m. \tag{5.3}$$

By theorem 3 in section IV, we get

$$\prod_{j=1}^{\infty}(1-x^{a_j r_j})^{-1}=\sum_{m=0}^{\infty}S'(m)x^m.$$

Then

$$\sum_{m=0}^{\infty}S'(m)x^m\cdot\sum_{m=0}^{\infty}S^*(m)x^m=\sum_{m=0}^{\infty}S(m)x^m.$$

Let us look for the coefficients of x^m $(m\geqq 1)$ on both sides. Then we get

$$\sum_{\nu=0}^{m}S'(m-\nu)S^*(\nu)=S(m),$$

i.e.

$$\begin{Vmatrix} S'(0) & & & & 0 \\ S'(1) & S'(0) & & & \\ S'(2) & S'(1) & S'(0) & & \\ \vdots & \vdots & \vdots & \ddots & \\ S'(m-1) & S'(m-2) & S'(m-3) & \cdots S'(0) & \\ S'(m) & S'(m-1) & S'(m-2) & \cdots S'(1) & S'(0) \end{Vmatrix}\cdot\begin{Vmatrix} S^*(0) \\ S^*(1) \\ S^*(2) \\ \vdots \\ S^*(m-1) \\ S^*(m) \end{Vmatrix}=\begin{Vmatrix} S(0) \\ S(1) \\ S(2) \\ \vdots \\ S(m-1) \\ S(m) \end{Vmatrix}.$$

Therefore we get the result (5.1) of this theorem.

On the other hand, let us put

$$\prod_{j=1}^{\infty}(1-x^{a_j r_j})=\sum_{m=0}^{\infty}\alpha_{S'}(m)x^m.$$

In order to get $\alpha_{S'}(m)$, we use the methods explained at corollary 2 of theorem 3 in section IV; it becomes

$$\alpha_{S'}(m)=\frac{(-1)^m}{m!}\det\begin{Vmatrix} \sigma_{S'}(1) & 1 & & & 0 \\ \sigma_{S'}(2) & \sigma_{S'}(1) & 2 & & \\ \sigma_{S'}(3) & \sigma_{S'}(2) & \sigma_{S'}(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \sigma_{S'}(m-1) & \sigma_{S'}(m-2) & \sigma_{S'}(m-3) & \cdots\sigma_{S'}(1) & m-1 \\ \sigma_{S'}(m) & \sigma_{S'}(m-1) & \sigma_{S'}(m-2) & \cdots\sigma_{S'}(2) & \sigma_{S'}(1) \end{Vmatrix}$$

where $\alpha_{S'}(0)=1$. Remembering (5.3) we get

$$\sum_{m=0}^{\infty}S^*(m)x^m=\sum_{m=0}^{\infty}\alpha_{S'}(m)x^m\cdot\sum_{m=0}^{\infty}S(m)x^m. \tag{5.4}$$

Comparing the coefficient of (5.4) on both sides, we get

$$S^*(m)=\sum_{\nu=0}^{\infty}\alpha_{S'}(\nu)S(m-\nu).$$

Therefore we get the result (5.2) of the theorem.

Corollary 1. *If* $r_1=r_2=r_3=\cdots=2$, *then*

$$S^*(m)=\frac{1}{m!}\det\begin{Vmatrix} \tau_S(1) & -1 & & & 0 \\ \tau_S(2) & \tau_S(1) & -2 & & \\ \tau_S(3) & \tau_S(2) & \tau_S(1) & \ddots & \\ \vdots & \vdots & \vdots & \ddots & \\ \tau_S(m-1) & \tau_S(m-2) & \tau_S(m-3) & \cdots\tau_S(1) & -(m-1) \\ \tau_S(m) & \tau_S(m-1) & \tau_S(m-2) & \cdots\tau_S(2) & \tau_S(1) \end{Vmatrix} \tag{5.5}$$

where $\tau_S(k)=\sum_{d\in S,\, d\mid k}(-1)^{\frac{k}{d}}d.$

Proof. Putting $r_1=r_2=r_3=\cdots=2$ in the theorem 4, we get

$$\sum_{m=0}^{\infty}S^*(m)x^m=\prod_{j=1}^{\infty}(1-x^{2a_j})(1-x^{a_j})^{-1}=\prod_{j=1}^{\infty}(1+x^{a_j}). \tag{5.6}$$

Taking the logarithmic derivative of (5.6), we get

$$\sum_{j=1}^{\infty}\frac{a_j x^{a_j-1}}{1+x^{a_j}}=\sum_{m=1}^{\infty}mS^*(m)x^{m-1}/\sum_{m=0}^{\infty}S^*(m)x^m$$

Multiplying both sides by x

$$\sum_{j=1}^{\infty}\frac{a_j x^{a_j}}{1+x^{a_j}}=\sum_{m=1}^{\infty} mS^*(m)x^m \Big/ \sum_{m=0}^{\infty} S^*(m)x^m \tag{5.7}$$

Let us transform the left sides of (5.7) into formal power series; it becomes

$$\sum_{j=1}^{\infty} a_j x^{a_j}\sum_{m=0}^{\infty}(-1)^m x^{ma_j}=\sum_{j,m=1}^{\infty}(-1)^m a_j x^{ma_j}$$

$$=\sum_{m=1}^{\infty}\Big(\sum_{d\in S,\, d\mid m}(-1)^{\frac{m}{d}}d\Big)x^m=\sum_{m=1}^{\infty}\tau_S(m)x^m.$$

Therefore the equation (5.7) can be rewritten as

$$\sum_{m=1}^{\infty}\tau_S(m)x^m\cdot\sum_{m=0}^{\infty}S^*(m)x^m=\sum_{m=1}^{\infty}mS^*(m)x^m.$$

Comparing the coefficients of x^m $(m\geqq 1)$ on both sides, we get

$$\sum_{\nu=0}^{m-1}\tau_S(m-\nu)S^*(\nu)=mS^*(m),\qquad m=1,2,3,\cdots.$$

Remembering that $S^*(0)=1$, we get

$$\sum_{\nu=1}^{m-1}\tau_S(m-\nu)S^*(\nu)-mS^*(m)=-\tau_S(m),\qquad m=1,2,3,\cdots,$$

i.e.

$$\begin{Vmatrix} -1 & & & & & 0 \\ \tau_S(1) & -2 & & & & \\ \tau_S(2) & \tau_S(1) & -3 & \ddots & & \\ \vdots & \vdots & \vdots & & & \\ \tau_S(m-2) & \tau_S(m-3) & \tau_S(m-4) & \cdots & -(m-1) & \\ \tau_S(m-1) & \tau_S(m-2) & \tau_S(m-3) & \cdots & \tau_S(1) & -m \end{Vmatrix}\cdot\begin{Vmatrix} S^*(1) \\ S^*(2) \\ S^*(3) \\ \vdots \\ S^*(m-1) \\ S^*(m) \end{Vmatrix}=\begin{Vmatrix} -\tau_S(1) \\ -\tau_S(2) \\ -\tau_S(3) \\ \vdots \\ -\tau_S(m-1) \\ -\tau_S(m) \end{Vmatrix}.$$

Therefore we get the result (5.5) of the corollary.

Corollary 2. *The number of methods of decomposing natural number m into the sum of mutally different natural numbers is*

$$\frac{1}{m!}\det\begin{Vmatrix} \tau(1) & -1 & & & & 0 \\ \tau(2) & \tau(1) & -2 & \ddots & & \\ \tau(3) & \tau(2) & \tau(1) & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & & \\ \tau(m-1) & \tau(m-2) & \tau(m-3) & \cdots\tau(1) & & -(m-1) \\ \tau(m) & \tau(m-1) & \tau(m-2) & \cdots\tau(2) & & \tau(1) \end{Vmatrix},$$

where $\tau(k)=\sum_{d\mid k}(-1)^{\frac{k}{d}}d$.

Proof. Put $S=N$ in the corollary 1 of theorem 4.

VI. *Some Applications*

In this section we consider some applications of theorem 3 and 4. Let $r\,(\geqq 1)$ be a given natural number and $S=\{n^r;n=1,2,3,\cdots\}$ be an infinite subset of N. Let us apply theorem 3 to this S, and use the following notations,

$P_r(m)$=the number of method of decomposing m into r-power natural numbers,

$$\sigma_S(m)=\sigma_r(m),$$
$$\alpha_S(m)=\alpha_r(m).$$

Then

$$\prod_{n=1}^{\infty}(1-x^{nr})^{-1}=\sum_{n=0}^{\infty}P_r(n)x^n=\sum_{n=0}^{\infty}\frac{1}{n!}\det\begin{Vmatrix} \sigma_r(1) & -1 & & & 0 \\ \sigma_r(2) & \sigma_r(1) & -2 & & \\ \sigma_r(3) & \sigma_r(2) & \sigma_r(1) & \ddots & \\ \vdots & \vdots & \vdots & & \\ \sigma_r(n-1) & \sigma_r(n-2) & \sigma_r(n-3)\cdots & \sigma_r(1) & -(n-1) \\ \sigma_r(n) & \sigma_r(n-1) & \sigma_r(n-2)\cdots & \sigma_r(2) & \sigma_r(1) \end{Vmatrix} x^n,$$

$$\prod_{n=1}^{\infty}(1-x^{nr})=\sum_{n=0}^{\infty}\alpha_r(n)x^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\det\begin{Vmatrix} \sigma_r(1) & 1 & & & 0 \\ \sigma_r(2) & \sigma_r(1) & 2 & & \\ \sigma_r(3) & \sigma_r(2) & \sigma_r(1) & \ddots & \\ \vdots & \vdots & \vdots & & \\ \sigma_r(n-1) & \sigma_r(n-2) & \sigma_r(n-3)\cdots & \sigma_r(1) & n-1 \\ \sigma_r(n) & \sigma_r(n-1) & \sigma_r(n-2)\cdots & \sigma_r(2) & \sigma_r(1) \end{Vmatrix} x^n.$$

Now we prepare a lemma in order to consider the relation between $P_{r+1}(n)$ and $P_r(n)$, and between $\alpha_{r+1}(n)$ and $\alpha_r(n)$.

Lemma.

$$\prod_{n=1}^{\infty}\Big(\sum_{m=0}^{n-1}x^{mnr}\Big)=\sum_{n=0}^{\infty}\beta_r(n)x^n \tag{6.1}$$

where $\beta_r(n)=\dfrac{1}{n!}$ ·

$$\det\begin{Vmatrix} \sigma_r(1)-\sigma_{r+1}(1) & -1 & & & 0 \\ \sigma_r(2)-\sigma_{r+1}(2) & \sigma_r(1)-\sigma_{r+1}(1) & -2 & & \\ \sigma_r(3)-\sigma_{r+1}(3) & \sigma_r(2)-\sigma_{r+1}(2) & \sigma_r(1)-\sigma_{r+1}(1) & \ddots & \\ \vdots & \vdots & \vdots & & \\ \sigma_r(n-1)-\sigma_{r+1}(n-1) & \sigma_r(n-2)-\sigma_{r+1}(n-2) & \sigma_r(n-3)-\sigma_{r+1}(n-3)\cdots & \sigma_r(1)-\sigma_{r+1}(1) & -(n-1) \\ \sigma_r(n)-\sigma_{r+1}(n) & \sigma_r(n-1)-\sigma_{r+1}(n-1) & \sigma_r(n-2)-\sigma_{r+1}(n-2)\cdots & \sigma_r(2)-\sigma_{r+1}(2) & \sigma_r(1)-\sigma_{r+1}(1) \end{Vmatrix}.$$

Proof. Taking the logarithmic derivative of (6.1), we get

$$\sum_{n=2}^{\infty}\Big\{\sum_{m=1}^{n-1}mn^rx^{mnr-1}\Big/\sum_{m=0}^{n-1}x^{mnr}\Big\}=\sum_{n=1}^{\infty}n\beta_r(n)x^{n-1}\Big/\sum_{n=0}^{\infty}\beta_r(n)x^n.$$

Multiplying both sides by x

$$\sum_{n=2}^{\infty}\Big\{\sum_{m=1}^{n-1}mn^rx^{mnr}\Big/\sum_{m=0}^{n-1}x^{mnr}\Big\}=\sum_{n=1}^{\infty}n\beta_r(n)x^n\Big/\sum_{n=0}^{\infty}\beta_r(n)x^n. \tag{6.2}$$

Let us transform the left side of (6.2) into formal power series; it becomes

$$\begin{aligned}
&\sum_{n=2}^{\infty}\Big\{\sum_{m=1}^{n-1}mn^rx^{mnr}\Big/\sum_{m=0}^{n-1}(x^{nr})^m\Big\}\\
&=\sum_{n=2}^{\infty}\Big\{n^r\Big(\sum_{m=1}^{n-1}m(x^{nr})^m\Big)\cdot\frac{1-x^{nr}}{1-(x^{nr})^n}\Big\}\\
&=\sum_{n=2}^{\infty}\Big\{\frac{n^r}{1-x^{nr+1}}\Big(\sum_{m=1}^{n-1}m(x^{nr})^m-\sum_{m=1}^{n-1}m(x^{nr})^{m+1}\Big)\Big\}\\
&=\sum_{n=2}^{\infty}\Big\{\frac{n^r}{1-x^{nr+1}}\Big(x^{nr}+\sum_{m=2}^{n-1}(x^{nr})^m-(n-1)(x^{nr})^n\Big)\Big\}\\
&=\sum_{n=2}^{\infty}\Big\{\frac{n^r}{1-x^{nr+1}}\Big(x^{nr}\sum_{m=0}^{n-2}(x^{nr})^m-(n-1)x^{nr+1}\Big)\Big\}\\
&=\sum_{n=2}^{\infty}\frac{n^r}{1-x^{nr+1}}\Big\{\frac{x^{nr}(1-(x^{nr})^{n-1})}{1-x^{nr}}-(n-1)x^{nr+1}\Big\}\\
&=\sum_{n=2}^{\infty}\frac{n^r}{(1-x^{nr})(1-x^{nr+1})}\{x^{nr}-nx^{nr+1}+(n-1)x^{nr+nr+1}\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=2}^{\infty} n^r \left\{ \frac{x^{n^r}}{1-x^{n^r}} - \frac{n x^{n^{r+1}}}{1-x^{n^{r+1}}} \right\} \\
&= \sum_{n=2}^{\infty} \{ n^r x^{n^r} \sum_{m=0}^{\infty} x^{mn^r} - n^{r+1} x^{n^{r+1}} \sum_{m=0}^{\infty} x^{mn^{r+1}} \} \\
&= \sum_{n=1}^{\infty} \{ \sum_{m=1}^{\infty} n^r x^{mn^r} - \sum_{m=1}^{\infty} n^{r+1} x^{mn^{r+1}} \} \\
&= \sum_{m,n=1}^{\infty} n^r x^{mn^r} - \sum_{m,n=1}^{\infty} n^{r+1} x^{mn^{r+1}} \\
&= \sum_{n=1}^{\infty} (\sum_{d^r \mid n} d^r) x^n - \sum_{n=1}^{\infty} (\sum_{d^{r+1} \mid n} d^{r+1}) x^n \\
&= \sum_{n=1}^{\infty} \{ \sigma_r(n) - \sigma_{r+1}(n) \} x^n .
\end{aligned}
$$

Hence the equation (6.2) can be rewritten as

$$\sum_{n=1}^{\infty} \{ \sigma_r(n) - \sigma_{r+1}(n) \} x^n \cdot \sum_{n=0}^{\infty} \beta_r(n) x^n = \sum_{n=1}^{\infty} n \beta_r(n) x^n .$$

Let us look for the coefficients of x^n on both sides. Then we get

$$\sum_{\nu=0}^{n-1} \{ \sigma_r(n-\nu) - \sigma_{r+1}(n-\nu) \} \beta_r(\nu) = n \beta_r(n), \qquad n=1, 2, 3, \cdots .$$

Remembering that $\beta_r(0)=1$, we get

$$1 \cdot \beta_r(1) = -\{ \sigma_r(1) - \sigma_{r+1}(1) \}$$

and

$$\sum_{\nu=1}^{n-1} \{ \sigma_r(n-\nu) - \sigma_{r+1}(n-\nu) \} \beta_r(\nu) - n \beta_r(n) = -\{ \sigma_r(n) - \sigma_{r+1}(n) \}, \qquad n=2, 3, 4, \cdots,$$

i.e.

$$
\begin{Vmatrix}
-1 & & & & 0 \\
\sigma_r(1)-\sigma_{r+1}(1) & -2 & \cdots & \cdots & \\
\sigma_r(2)-\sigma_{r+1}(2) & \sigma_r(1)-\sigma_{r+1}(1) & \cdots & \cdots & \\
\vdots & \vdots & & & \\
\sigma_r(n-2)-\sigma_{r+1}(n-2) & \sigma_r(n-3)-\sigma_{r+1}(n-3) & \cdots & \sigma_r(1)-\sigma_{r+1}(1) & -(n-1) \\
\sigma_r(n-1)-\sigma_{r+1}(n-1) & \sigma_r(n-2)-\sigma_{r+1}(n-2) & \cdots & \sigma_r(2)-\sigma_{r+1}(2) & \sigma_r(1)-\sigma_{r+1}(1) \quad -n
\end{Vmatrix}
\cdot
\begin{Vmatrix}
\beta_r(1) \\ \beta_r(2) \\ \beta_r(3) \\ \vdots \\ \beta_r(n-1) \\ \beta_r(n)
\end{Vmatrix}
$$

$$
=
\begin{Vmatrix}
-\{\sigma_r(1)-\sigma_{r+1}(1)\} \\
-\{\sigma_r(2)-\sigma_{r+1}(2)\} \\
-\{\sigma_r(3)-\sigma_{r+1}(3)\} \\
\vdots \\
-\{\sigma_r(n-1)-\sigma_{r+1}(n-1)\} \\
-\{\sigma_r(n)-\sigma_{r+1}(n)\}
\end{Vmatrix}.
$$

Therefore we get the result (6.1) of the lemma.

By the above lemma we can prove the following theorem.

Theorem 5.

$$\alpha_{r+1}(n) = \sum_{\nu=0}^{n} \beta_r(n-\nu) \alpha_r(\nu) \tag{6.3}$$

$$\sum_{\nu=0}^{\infty} \beta_r(n-\nu) P_{r+1}(\nu) = P_r(n). \tag{6.4}$$

Proof. Let us prove (6.3) at first.

$$
\begin{aligned}
\sum_{n=0}^{\infty} \alpha_{r+1}(n) x^n &= \prod_{n=1}^{\infty} (1-x^{n^{r+1}}) = \prod_{n=1}^{\infty} (1-(x^{n^r})^n) \\
&= \prod_{n=1}^{\infty} (1-x^{n^r}) \cdot \sum_{m=0}^{n-1} x^{mn^r} = \prod_{n=1}^{\infty} (1-x^{n^r}) \cdot \prod_{n=1}^{\infty} \sum_{m=0}^{n-1} x^{mn} .
\end{aligned}
$$

By the lemma, it becomes

$$\sum_{n=0}^{\infty} \alpha_r(n)x^n \cdot \sum_{n=0}^{\infty} \beta_r(n)x^n$$
$$= \sum_{n=0}^{\infty} \{ \sum_{\nu=0}^{n} \alpha_r(\nu)\beta_r(n-\nu)\} x^n.$$

Comparing the coefficients of both sides, we get (6.3). Similarly we can prove (6.4) as follows. We start from the following equation,

$$\prod_{n=1}^{\infty} (1-x^{n^{r+1}}) = \prod_{n=1}^{\infty} (1-x^{n^r}) \cdot \sum_{n=0}^{\infty} \beta_r(n)x^n.$$

Then

$$\sum_{n=0}^{\infty} P_r(n)x^n = \sum_{n=0}^{\infty} P_{r+1}(n)x^n \cdot \sum_{n=0}^{\infty} \beta_r(n)x^n$$
$$= \sum_{n=0}^{\infty} \{ \sum_{\nu=0}^{n} \beta_r(n-\nu) \cdot P_{r+1}(\nu)\} x^n.$$

Comparing the coefficients of both sides, we get (6.4).

Corollary. *If we define B_r by*

$$B_r = \begin{Vmatrix} \beta_r(0) & \beta_r(1) & \beta_r(2) \cdots \\ 0 & \beta_r(0) & \beta_r(1) \cdots \\ 0 & 0 & \beta_r(0) \cdots \\ \vdots & \vdots & \vdots \end{Vmatrix},$$

then $(\alpha_{r+1}(0), \alpha_{r+1}(1), \alpha_{r+1}(2), \cdots) = (\alpha_r(0), \alpha_r(1), \alpha_r(2), \cdots)B_r$

and $(P_{r+1}(0), P_{r+1}(1), P_{r+1}(2), \cdots) = (P_r(0), P_r(1), P_r(2), \cdots)B_r^{-1}$.

Proof. It is clear.

VII. *Generating Function of $\sigma_s(n)$, $\tau_s(n)$*

The definition of the function $\sigma_r(n)$ and $\tau_r(n)$ is

$$\sigma_r(n) = \sum_{d^r \mid n} d^r, \qquad \tau_r(n) = \sum_{d^r \mid n} (-1)^{\frac{n}{d^r}} d^r$$

as we saw it in the previous section. Let us consider the generating function of $\sigma_r(n)$ and $\tau_r(n)$ at first.

After that we shall treat of the case of the generating function of $\sigma_s(n)$ and $\tau_s(n)$, which is more general than the formar case. Throughout this section $\zeta(s)$ denotes Riemann's Zeta function.

Theorem 6.

$$\zeta(s) \cdot \zeta(rs-r) = \sum_{n=1}^{\infty} \sigma_r(n)n^{-s}, \tag{7.1}$$

$$(2^{1-s}-1)\zeta(s) \cdot \zeta(rs-r) = \sum_{n=1}^{\infty} \tau_r(n)n^{-s}. \tag{7.2}$$

Hence

$$\sum_{n=1}^{\infty} \tau_r(n)n^{-s} = (2^{1-s}-1) \sum_{n=1}^{\infty} \sigma_r(n)n^{-s}. \tag{7.3}$$

Proof. Let us prove (7.1) at first. It is clear that

$$\zeta(s) \cdot \zeta(rs-r) = \sum_{n_1=1}^{\infty} n_1^{-s} \cdot \sum_{n_2=1}^{\infty} n_2^r n_2^{-rs} = \sum_{n_1, n_2=1}^{\infty} n_2^r (n_1 n_2^r)^{-s}$$
$$= \sum_{n=1}^{\infty} (\sum_{d^r \mid n} d^r) n^{-s} = \sum_{n=1}^{\infty} \sigma_r(n)n^{-s}.$$

Next let us prove (7.2). Since it is clear that

$$\sum_{n=1}^{\infty}(-1)^n n^{-s}+\zeta(s)=2^{1-s}\zeta(s), \tag{7.4}$$

then we get the following relation.

$$(2^{1-s}-1)\zeta(s)\cdot\zeta(rs-r)=\sum_{n_1=1}^{\infty}(-1)^{n_1}n_1^{-s}\cdot\sum_{n_2=1}^{\infty}n_2^{r}n_2^{-rs}$$

$$=\sum_{n_1,n_2=1}^{\infty}(-1)^{n_1}n_2^{r}(n_1n_2^{r})^{-s}$$

$$=\sum_{n=1}^{\infty}\Big(\sum_{d^r|n}(-1)^{\frac{n}{d^r}}d^r\Big)n^{-s}=\sum_{n=1}^{\infty}\tau_r(n)n^{-s}.$$

Therefore we get the result (7.2). (7.3) is clear.

Let us consider more general case. Let $S=\{a_1, a_2, a_3, \cdots\}$ be a countable infinite subset of N, where $0<a_1<a_2<a_3<\cdots$; and $\sigma_S(n)$ and $\tau_S(n)$ are the function defined as follows,

$$\sigma_S(n)=\sum_{d\in S,\ d|n}d, \qquad \tau_S(n)=\sum_{d\in S,\ d|n}(-1)^{\frac{n}{d}}d.$$

Now let us define $\chi_S(s)$ and $\zeta_S(s)$ as follows,

$$\chi_S(n)=\begin{cases}1\ ; & n\in S\\ 0\ ; & n\notin S,\end{cases}$$

$$\zeta_S(s)=\sum_{n=1}^{\infty}\chi_S(n)n^{-s}.$$

Then we can prove the following theorem.

Theorem 7.

$$\zeta(s)\cdot\zeta_S(s-1)=\sum_{n=1}^{\infty}\sigma_S(n)n^{-s}, \tag{7.5}$$

$$(2^{1-s}-1)\zeta(s)\cdot\zeta_S(s-1)=\sum_{n=1}^{\infty}\tau_S(n)n^{-s}. \tag{7.6}$$

Hence

$$\sum_{n=1}^{\infty}\tau_S(n)n^{-s}=(2^{1-s}-1)\sum_{n=1}^{\infty}\sigma_S(n)n^{-s}. \tag{7.8}$$

Proof. Let us prove (7.5) at first. It is clear that

$$\zeta(s)\cdot\zeta_S(s-1)=\sum_{n_1=1}^{\infty}n_1^{-s}\cdot\sum_{n_2=1}^{\infty}\chi_S(n_2)n_2^{-(s-1)}=\sum_{n_1,n_2=1}^{\infty}n_2\chi_S(n_2)(n_1n_2)^{-s}$$

$$=\sum_{n=1}^{\infty}\Big(\sum_{d|n}d\chi_S(d)\Big)n^{-s}=\sum_{n=1}^{\infty}\Big(\sum_{d\in S,\ d|n}d\Big)n^{-s}=\sum_{n=1}^{\infty}\sigma_S(n)n^{-s}.$$

Next let us prove (7.6). By (7.4), we get the following relation.

$$(2^{1-s}-1)\zeta(s)\cdot\zeta_S(s-1)=\sum_{n_1=1}^{\infty}(-1)^{n_1}n_1^{-s}\cdot\sum_{n_2=1}^{\infty}\chi_S(n_2)n_2^{-(s-1)}$$

$$=\sum_{n_1,n_2=1}^{\infty}(-1)^{n_1}n_2\chi_S(n_2)(n_1n_2)^{-s}=\sum_{n=1}^{\infty}\Big(\sum_{d|n}(-1)^{\frac{n}{d}}d\chi_S(d)\Big)n^{-s}$$

$$=\sum_{n=1}^{\infty}\Big(\sum_{d\in S,\ d|n}(-1)^{\frac{n}{d}}d\Big)n^{-s}=\sum_{n=1}^{\infty}\tau_S(n)n^{-s}.$$

Therefore we get the result (7.6). (7.7) is clear.

VIII. *Some Other Results*

In this last section we consider supplementally some other results which are proved by

elementary methods as in the previous sections. At first let us define some notations;

$$p=\text{fixed rational prime number,}$$

for all natural number $n\in N$ $(n\geqq 1)$

$$P(n)=\text{partition number of } n$$
$$=\text{number of integral solutions of } \sum_{m=1}^{\infty} mx_m=n,\ 0\leqq x_m,$$
$$N_p=\{m\in N;\ m\neq 0,\ (m,p)=1\}$$
$$=\{m_1, m_2, m_3, \cdots\}$$

where $m_1<m_2<m_3<\cdots$,

for all natural number $n\in N$ $(n\geqq 1)$

$$R_p(n)=\text{number of integral solutions of } \sum_{k=1}^{\infty} m_k x_k=n,\ 0\leqq x_k$$

where we shall promise $P(0)=R_p(0)=1$. Now we can prove the following theorem.

Theorem 8. *For all natural number $n\in N$,*

$$R_p(n)=(-1)^{[\frac{n}{p}]}\det\begin{Vmatrix} P(n-[\frac{n}{p}]p) & 1 & & & 0 \\ P(n-([\frac{n}{p}]-1)p) & P(1) & 1 & & \\ P(n-([\frac{n}{p}]-2)p) & P(2) & P(1) & \ddots & \\ \vdots & \vdots & \vdots & & \\ P(n-p) & P([\frac{n}{p}]-1) & P([\frac{n}{p}]-2) & \cdots P(1) & 1 \\ P(n) & P([\frac{n}{p}]) & P([\frac{n}{p}]-1) & \cdots P(2) & P(1) \end{Vmatrix} \tag{8.1}$$

Proof. Let us consider an infinite product

$$\prod_{r=0}^{\infty}\Big(\sum_{m=0}^{p-1} x^{mp^r}\Big).$$

For all natural number n $(\geqq 1)$, we put $r_0=\left[\dfrac{\log n}{\log p}\right]$. Then we get $p^{r_0}\leqq n<p^{r_0+1}$. Therefore by decomposing the above infinite product into two parts such as

$$\prod_{r=0}^{r_0}\Big(\sum_{m=0}^{p-1} x^{mp^r}\Big)\cdot\prod_{r=r_0+1}^{\infty}\Big(\sum_{m=0}^{p-1} x^{mp^r}\Big),$$

it is clear that the second part of the infinite product does not contribute to x^n. So by developing the first part of the infinite product, we can get the coefficients of x^n. It is clear that the coefficient coincides with the number of integral solutions of

$$\sum_{k=0}^{r_0} x_k p^k=n, \qquad 0\leqq x_k<p.$$

But this linear conditional equation has always only one integral solution. Hence we get

$$\prod_{r=0}^{\infty}\Big(\sum_{m=0}^{p-1} x^{mp^r}\Big)=\sum_{n=0}^{\infty} x^n=(1-x)^{-1}.$$

Here replacing x with x^l, where $l\in \mathrm{N}_p$, we get

$$\prod_{r=0}^{\infty}\Big(\sum_{m=0}^{p-1} x^{lmp^r}\Big)=\sum_{n=0}^{\infty} x^{ln}=(1-x^l)^{-1}.$$

Now let us consider the infinite product, where l runs about in the set N_p, so we get

$$\prod_{l\in Np}\prod_{r=0}^{\infty}\Big(\sum_{m=0}^{p-1} x^{lmp^r}\Big)=\prod_{l\in Np}\sum_{n=0}^{\infty} x^{ln}=\prod_{l\in Np}(1-x^l)^{-1}. \tag{8.2}$$

Remembering that p is a fixed rational prime number, we get

$$\prod_{l\in Np}\prod_{r=0}^{\infty}\Big(\sum_{m=0}^{p-1} x^{lmp^r}\Big)=\prod_{n=1}^{\infty}\sum_{m=0}^{p-1} x^{mn},$$

and it is clear that

$$\prod_{l\in Np}(1-x^l)^{-1}=\prod_{n=1}^{\infty}(1-x^n)^{-1}/\prod_{n=1}^{\infty}(1-x^{pn})^{-1}.$$

Then we can rewrite the equation (8.2) as follows,

$$\prod_{n=1}^{\infty}\sum_{m=0}^{p-1}x^{mn}=\prod_{l\in Np}\sum_{n=0}^{\infty}x^{ln}=\prod_{n=1}^{\infty}(1-x^{n})^{-1}/\prod_{n=1}^{\infty}(1-x^{pn})^{-1}.$$

Therefore we get

$$\prod_{n=1}^{\infty}\sum_{m=0}^{p-1}x^{mn}=\prod_{l\in Np}\sum_{n=0}^{\infty}x^{ln} \tag{8.3}$$

and

$$\prod_{n=1}^{\infty}(1-x^{pn})^{-1}\cdot\prod_{n=1}^{\infty}\sum_{m=0}^{p-1}x^{mn}=\prod_{n=1}^{\infty}(1-x^{n})^{-1} \tag{8.4}$$

Now let us define

$$Q_p(n)=\text{number of integral solutions of } \sum_{m=1}^{\infty}mx_m=n,\ 0\leqq x_m<p$$

the equation (8.3) and (8.4) can be rewritten as follows,

$$\sum_{n=0}^{\infty}Q_p(n)x^n=\sum_{n=0}^{\infty}R_p(n)x^n,$$

$$\sum_{n=0}^{\infty}P(n)x^{pn}\cdot\sum_{n=0}^{\infty}Q_p(n)x^n=\sum_{n=0}^{\infty}P(n)x^n.$$

Therefore we get

$$Q_p(n)=R_p(n) \tag{8.5}$$

and

$$\sum_{\nu=0}^{[n/p]}P(\nu)Q_p(n-\nu p)=P(n). \tag{8.6}$$

Since we can rewrite (8.6) as follows,

$$\left\|\begin{matrix} P(0) & & & & 0 \\ P(1) & P(0) & & & \\ P(2) & P(1) & P(0) & & \\ \vdots & \vdots & \vdots & \ddots & \\ P([\frac{n}{p}]-1) & P([\frac{n}{p}]-2) & P([\frac{n}{p}]-3) & \cdots P(0) & \\ P([\frac{n}{p}]) & P([\frac{n}{p}]-1) & P([\frac{n}{p}]-2) & \cdots P(1) & P(0) \end{matrix}\right\|\cdot\left\|\begin{matrix} Q_p(n-[\frac{n}{p}]p) \\ Q_p(n-([\frac{n}{p}]-1)p) \\ Q_p(n-([\frac{n}{p}]-2)p) \\ \vdots \\ Q_p(n-p) \\ Q_p(n) \end{matrix}\right\|=\left\|\begin{matrix} P(n-[\frac{n}{p}]p) \\ P(n-([\frac{n}{p}]-1)p) \\ P(n-([\frac{n}{p}]-2)p) \\ \vdots \\ P(n-p) \\ P(n) \end{matrix}\right\|,$$

remembering (8.5), we get the result (8.1).

Corollary 1. $Q_p(n)=R_p(n)$

Proof. We have proved this corollary in the proof of theorem 8.

Corollary 2. (*Euler*) *The number of method decomposing natural number n into mutually different natural numbers coincides with the number of method decomposing natural number n into odd numbers.*

Proof. Put $p=2$ in corollary 1.

References

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